THE UNIVERSITY OF MANITOBA

THE ESTIMATION OF PARAMETERS OF MIXED WEIBULL DISTRIBUTION WITH TIME-CENSORED DATA

by

HOK-KUI LEE

A THESIS

SUBMITTED TO THE FACULTY OF

GRADUATE STUDIES IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

Department of Statistics

WINNIPEG, MANITOBA

MAY, 1976

"THE ESTIMATION OF PARAMETERS OF MIXED WEIBULL DISTRIBUTION WITH TIME-CENSORED DATA"

by

HOK-KUI LEE

A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

MASTER OF SCIENCE

TABLE OF CONTENTS

ACKNOWLEDGEMENT

I am indebted to my advisor, Professor S.K. Sinha, for suggesting the problem and for assistance at all stages of writing of the thesis.

H Lee

H. Lee

SUMMARY

The problem treated here is concerned with the mixed Weibull distribution which is widely used in life testing. Suppose that the failure population can be divided into subpopulations, each might have a different type of failure. For estimating the parameters of such mixed population model, the moment and the maximum likelihood estimates have been considered. The corresponding asymptotic variance-covariance matrix is given.

Three types of the models have been discussed in Chapters 3, 4 and 5. The methods of obtaining the moment estimators and the maximum likelihood estimators have been derived. Illustrative examples of these three cases (by generating data) are included. Some of the computer flowcharts and programs have been enclosed in the appendices.

CHAPTER 1.

INTRODUCTION

It is our everyday experience that many of the items, systems, or materials which we use suddenly go 'out of order'. For instance, the steel beam under a load may crack or break; the fuse inserted into a circuit may burn out; the wing of the airplane under influence of forces may buckle, or the electronic device may fail to function. Suppose that for any such component (or system) a state we denote as 'failure' can be defined. Each has its own statistical assessment of the life characteristic. Assuming a single homogeneous population of units, the life characteristics have been well developed during the last thirty or more years. When items are subjected to test certain complex situations arise. It is necessary to study such situations, particularly the underlying failure time distribution of items under test.

Reliability theory establishes the regularity of occurrence of defects in devices and methods of prediction. In general, it is a method of evaluating the quality of a unit or a system and is defined by $R(t|\theta) = P(X \geq t)$, where θ is the parameter of the underlying life distribution $f(x|\theta)$. If θ is unknown, the reliability is estimated by different methods.

In order to assess the reliability, we usually start from the observed data — the recorded lives of some or all items subjected to test, or to performance under either actual or simulated conditions.

Assume that the component success or failure data which are used for reliability estimation are governed by some parametric probability distribution. There are certain well-known families of failure time distributions which have been successfully used in life testing problems. So one may find it necessary to identify the family — (a group of related distribution), justify them by experiments, statistical tests or by certain well-defined assumptions. For instance, if the device is a valve which is supposed to open or to close upon demand, we might identify the binomial distribution as our parametric probability distribution. Let us consider the pressure in a boiler; the boiler does not operate if the pressure is too low and it bursts if the pressure is too high, so perhaps the normal distribution should be used. The exponential and Poisson distributions are used when failures occur at a constant level of intensity which does not vary with the accumulated service.

A great deal of modern statistical literature on the exponential distribution is concerned with estimation under conditions of censoring or truncation. It is commonly encountered in the statistical analysis and assessment of data arising from life-tests under laboratory or service condition because it carries strong implication with regard to the underlying model.

The family of exponential distributions is the best known and most thoroughly explored, largely through the work of B. Epstein and his associates (Epstein, 1958). The exponential distribution has a number of desirable mathematical properties. One important property of exponential distribution is that it is 'forgetful', i.e. under the exponential failure time model, if a unit has survived t hours, then the probability of surviving an additional h hours is exactly the same as the probability of surviving h hours of a brand new item. There are structures which have this property. For example, at any hour of time, the future life of an electric fuse (assuming it cannot melt partially) is practically unchanged as long as it does not fail.

The Weibull distribution was suggested by the Swedish physicist Weibull and first used in a paper (1939) dealing with the breaking strength of materials. It is interesting that he proposed this distribution, apparently without recognizing that it is a particular case of the extreme value distribution, considering only the class of failure distributions of the form $F(X) = 1 - \exp\{-\phi(x)\}$, where $\phi(x)$ is a positive non-decreasing function.

The Weibull distribution provides a very general family of distribution in which certain other well known distributions figure as special cases. It is an important model for life-testing problem.

Weibull distributions are closely related to exponential functions. It has one additional parameter called the shape parameter. This maybe the reason why in the past Weibull distributions have been widely used in procedures for the analysis of life-testing data.

During the last few years, Weibull estimation has been greatly aided by the development of linear weighing technique for mixtures of distributions which enables one to estimate Weibull parameters from observed data.

Kao (1959) Mann (1968), Lawless (1972), Mann, Schafer and Singpurwala (1974) have done extensive work on the Weibull distribution. Sinha (1976) studied the Weibull distribution from the Bayesian viewpoint. It is of practical importance to study a situation where the underlying failure time distribution is a mixture of two or more distributions. But it is not easy to estimate the parameters of the mixed distributions, particularly for the mixed Weibull distributions. In 1958, Mendenhall and Hader gave a method for obtaining the estimates of the parameters of mixed exponentially distributed failure time distribution from censored life test data. It is in fact a particular case of the mixed Weibull distributions when the shape parameter is known to be 1. In 1965, Cohen gave a method for estimating Weibull parameters based on complete and censored samples. For the mixed Weibull distribution, there is difficulty in estimating its parameters. Not much work has been done on this problem. It may be useful to try to develop a method for estimating the parameters of a mixed Weibull distribution, although it follows that the computations involved would not be simple.

A study done in this paper is concerned with moment and the maximum likelihood estimation in censored samples from the mixed Weibull distribution. The technique of such procedures has been discussed under the three different types of the model, treated in Chapters 3, 4 and 5. At first, we would like to give a general model for this mixture-distribution.

CHAPTER 2.

THE MODEL AND ITS LIKELIHOOD FUNCTION

2.1 The Model

Assume the parent failure distribution is made up of two subpopulations, each having a cumulative probability distribution defined by:

$$F_i(t) = 1 - \exp\left\{-\left(\frac{t^{p_i}}{\theta_i}\right)\right\} \quad ; \qquad 0 \le t \le \infty \tag{2.1}$$

and the density function:

$$f_i(t|\theta_i,p_i) = \left(\frac{p_i}{\theta_i}\right) t^{p_i-1} \exp\left\{-\left(\frac{t^{p_i}}{\theta_i}\right)\right\} \tag{2.2}$$

$$(\theta_i > 0, \quad p_i > 0 \qquad i = 1,2)$$

Let the two subpopulations be mixed in proportions $\alpha:\beta$ $(\beta = 1 - \alpha)$, $\alpha, \beta \ge 0$. Then the cumulative distribution of the parent population is given by

$$F(t) = \alpha F_1(t) + \beta F_2(t) \tag{2.3}$$

and the density function

$$f(t|\alpha,\theta_1,\theta_2,p_1,p_2) = \alpha\left(\frac{p_1}{\theta_1}\right) t^{p_1-1} \exp\left\{-\left(\frac{t^{p_1}}{\theta_1}\right)\right\} +$$

$$(1-\alpha)\left(\frac{p_2}{\theta_2}\right) t^{p_2-1} \exp\left\{1-\left(\frac{t^{p_2}}{\theta_2}\right)\right\} \tag{2.4}$$

Let $$G_i(t) = 1 - F_i(t) \tag{2.5}$$

and $$G(t) = 1 - F(t) \tag{2.6}$$

The probability function $G(t)$ is the probability that a unit will survive to time t and is called the survival function.

Assume that, as soon as an item fails, the cause of failure is known and the subpopulation from which the item belongs is identified.

Suppose n items are chosen randomly from the model (2.4) and subjected to the life-test, which terminates at a fixed time, T. During such time r units have failed, r_i from subpopulation (i) and $r_1 + r_2 = r$.

Let T_j be the time passed since the jth item was put to test and t_j be the length of life of that time. If $t_j < T_j$, the jth item is said to have failed, otherwise it is said to have survived. The probability of survival of the jth item is given by

$$Q_j = p(t_j > T_j) = \int_{T_j}^{\infty} f(t|\alpha,\theta_1,\theta_2,p_1,p_2)dt$$

$$= \alpha \exp\left\{-\left(\frac{T_j^{p_1}}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T_j^{p_2}}{\theta_2}\right)\right\} \tag{2.7}$$

For convenience, assume that all measurements of time are in units of

T, the test termination time.

So $$Q = Q_j = \alpha \exp\left\{-\left(\frac{T}{\theta_1}\right)^{p_1}\right\} + \beta \exp\left\{-\left(\frac{T}{\theta_2}\right)^{p_2}\right\}; \quad j = 1,2,\ldots,n \; . \tag{2.8}$$

Let $x = t/T$ then (2.6) becomes

$$G(xT) = \alpha \exp\left\{-\frac{(xT)^{p_1}}{\theta_1}\right\} + \beta \exp\left\{-\frac{(xT)^{p_2}}{\theta_2}\right\}$$

Particularly, for $x = 1$, we get $G(T) = Q$,

and $$F_i(T) = 1 - \exp\left\{-\left(\frac{T}{\theta_i}\right)^{p_i}\right\}; \qquad i = 1,2 \; .$$

2.2 Likelihood Function

For a given random sample of n units, the probability of r_1 units failing due to cause (1), r_2 units failing due to cause (2) and $(n-r)$ units surviving is the multinomial:-

$$p(r_1,r_2,n-r|n) = \frac{n!}{r_1!r_2!(n-r)!} [\alpha F_1(T)]^{r_1} [\beta F_2(T)]^{r_2} [G(T)]^{n-r} \tag{2.9}$$

Hence the conditional density of obtaining the ordered observations $x_{i1}, x_{i2},\ldots, x_{ir_i}$, for given r_i and $x_{ij} \le 1$ is

$$p(x_{i1}, x_{i2},\ldots, x_{ir_i}|r_i,x_{ij} \le 1) = \frac{(r_i!) \prod_{j=1}^{r_i} f_i(x_{ij})}{[F_i(T)]^{r_i}} \tag{2.10}$$

The likelihood function L of the sample is given by

$$L = \frac{n!}{(n-r)!}\,[G(T)]^{n-r}\,\alpha^{r_1}\,\beta^{r_2}\,\prod_{j=1}^{r_1} f_1(x_{1j})\,\prod_{j=1}^{r_2} f_2(x_{2j})$$

or

$$L = \frac{n!}{(n-r)!}\left[\alpha\,\exp\left\{-\left(\frac{T^{p_1}}{\theta_1}\right)\right\} + \beta\,\exp\left\{-\left(\frac{T^{p_2}}{\theta_2}\right)\right\}\right]^{n-r}\alpha^{r_1}\,\beta^{r_2}$$

$$\times\left[\prod_{j=1}^{r_1}\left(\frac{p_1}{\theta_1}\right)x_{1j}^{p_1-1}\,T^{p_1}\quad\exp\left\{\left[\frac{(x_{1j}.T)^{p_1}}{\theta_1}\right]\right\}\right]$$

$$\times\left[\prod_{j=1}^{r_2}\left(\frac{p_2}{\theta_2}\right)x_{2j}^{p_2-1}\,T^{p_2}\quad\exp\left\{\left[\frac{(x_{2j}.T)^{p_2}}{\theta_2}\right]\right\}\right] \qquad (2.11)$$

The method of moments and the method of maximum likelihood will be used to estimate the parameters of the probability density function (p.d.f.) (2.4) in the following cases

(1) The mixture parameter α known; $p_1=p_2=p$, $\theta_1 \neq \theta_2$ unknown.

(2) α unknown; $p_1=p_2=p$, $\theta_1 \neq \theta_2$ unknown.

(3) α unknown; $p_1 \neq p_2$, $\theta_1 \neq \theta_2$ unknown.

The asymptotic variance-covariance matrices for the maximum likelihood estimates (MLE) will also be obtained.

CHAPTER 3.

THE MIXTURE PARAMETER α IS KNOWN, THE COMMON SHAPE PARAMETER p AND θ_1, θ_2 UNKNOWN

3.1 The Moment Estimation of the Parameters

We consider the density function

$$f(x|\alpha,p,\theta_1,\theta_2) = \alpha(\frac{p}{\theta_1})x^{p-1} \exp\left\{-\left(\frac{x^p}{\theta_1}\right)\right\} + \beta(\frac{p}{\theta_2})x^{p-1} \exp\left\{-\left(\frac{x^p}{\theta_2}\right)\right\} \qquad (3.1)$$

Then the sth non-central moment is:

$$\mu'_s = \frac{\alpha p}{\theta_1} \int_0^\infty x^{s+p-1} \exp\left\{-\left(\frac{x^p}{\theta_1}\right)\right\} dx$$

$$+ \frac{(1-\alpha)p}{\theta_2} \int_0^\infty x^{s+p-1} \exp\left\{-\left(\frac{x^p}{\theta_2}\right)\right\} dx$$

$$= \alpha\theta_1^{\frac{s}{p}} \Gamma(\frac{s}{p} + 1) + (1 - \alpha)\theta_2^{\frac{s}{p}} \Gamma(\frac{s}{p} + 1)$$

$$= (\frac{s}{p}) \Gamma(\frac{s}{p}) \left(\alpha\theta_1^{\frac{s}{p}} + \beta\theta_2^{\frac{s}{p}}\right) \qquad (3.2)$$

Since the mean and variance are equal to $\mu = \mu'_1$ and $\text{Var}(x) = \mu'_2 - (\mu'_1)^2$, respectively,

$$\mu = (\frac{1}{p}) \Gamma(\frac{1}{p}) \left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right) \qquad \text{and}$$

$$\text{Var}(x) = \left[(\frac{2}{p}) \Gamma(\frac{2}{p}) \left(\alpha\theta_1^{\frac{2}{p}} + \beta\theta_2^{\frac{2}{p}}\right)\right] - \left[(\frac{1}{p}) \Gamma(\frac{1}{p}) \left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right)\right]^2$$

So $$\frac{\mathrm{Var}(x)}{\mu^2} = \frac{(\frac{2}{p})\,\Gamma(\frac{2}{p})\left(\alpha\theta_1^{\frac{2}{p}} + \beta\theta_2^{\frac{2}{p}}\right)}{\frac{1}{p^2}\,\Gamma^2(\frac{1}{p})\left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right)^2} - 1 \tag{3.3}$$

On taking the square root of (3.3), we have for the coefficient of variation

$$cv = \left(\frac{2p\,\Gamma(\frac{2}{p})\left(\alpha\theta_1^{\frac{2}{p}} + \beta\theta_2^{\frac{2}{p}}\right)}{\Gamma^2(\frac{1}{p})\left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right)^2} - 1\right)^{\frac{1}{2}} \tag{3.4}$$

As an example we take α known and equal to $\frac{1}{3}$. Then (3.4) can be reduced as

$$cv = \left(\frac{2p\,\Gamma(\frac{2}{p})\left(\frac{\theta_1^{\frac{2}{p}}}{3} + \frac{2\theta_2^{\frac{2}{p}}}{3}\right)}{\Gamma^2(\frac{1}{p})\left(\frac{\theta_1^{\frac{1}{p}}}{3} + \frac{2\theta_2^{\frac{1}{p}}}{3}\right)^2} - 1\right)^{\frac{1}{2}} \tag{3.5}$$

Table 3.1 is an abridged table for the coefficient of variation with p, λ_1 and λ_2, where $\lambda_i = \frac{\theta_i}{T^p}$, with known parameter $\alpha = \frac{1}{3}$. (Other ranges for p, λ_1 and λ_2 can be obtained by using the subroutine TABLE which is enclosed).

For a given sample coefficient of variation, we may obtain an approximation to the estimate p^* with corresponding estimates θ_1^* and θ_2^* by comparing the value of the coefficient of variation from table 3.1. Such estimates p^*, θ_1^* and θ_2^* will be called moment estimates.

There is another technique which might be used to obtain the moment estimates. One might let s = 1, 2, 3 successively in equation (3.2) to set up three equations and use them to solve for the three

TABLE 3.1 THE MIXED WEIBULL COEFFICIENT OF VARIATION ($\lambda_i = \theta_i/T^p$)

$\alpha = \frac{1}{3}$

Coefficient of variation	p	λ_1	λ_2
0.50069	2.10	0.600	0.650
0.50100	2.10	0.500	0.450
0.50570	2.10	0.600	0.450
0.51375	2.10	0.400	0.650
0.52315	2.00	0.600	0.650
0.52715	2.00	0.500	0.650
0.53174	2.00	0.500	0.350
0.53712	2.00	0.400	0.650
0.54422	1.95	0.420	0.615
0.55222	1.90	0.500	0.650
0.55725	1.90	0.420	0.615
0.55826	1.90	0.415	0.620
0.57098	1.85	0.420	0.615
0.57537	1.80	0.600	0.650
0.58004	1.80	0.500	0.650
0.58550	1.80	0.500	0.350
0.59157	1.80	0.400	0.650
0.60001	1.80	0.300	0.550
0.61111	1.70	0.500	0.650
0.64745	1.70	0.300	0.650

cv was computed for assigned values of p, θ_1 and θ_2. Table 3.1 was set up in such a way that the coefficients of variation are monotonic.

unknowns p, θ_1 and θ_2. Theoretically, these three unknowns can be obtained, but there are a lot of practical problems when one tries to solve three non-linear equations. Even though it is not hard to eliminate θ_i (i = 1 or 2), the problem is that a unique solution for θ_i (i = 1,2) will not be available for any starting value p. In fact, it is hard to say which of the starting points for p is reasonable to use.

3.2 Maximum Likelihood Estimation of the Parameters

The logarithm of the likelihood function is

$$\ell nL = \ell n\left(\frac{n!}{(n-r)!}\right) + (n-r)\ell n\left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]$$

$$+ r_1 \ell n\alpha + r_2 \ell n\beta$$

$$+ \sum_{j=1}^{r_1}\left[\ell np - \ell n\theta_1 + (p-1)\ell nx_{1j} + p\ell nT - \frac{(x_{1j}T)^p}{\theta_1}\right]$$

$$+ \sum_{j=1}^{r_2}\left[\ell np - \ell n\theta_2 + (p-1)\ell nx_{2j} + p\ell nT - \frac{(x_{2j}T)^p}{\theta_2}\right] \tag{3.6}$$

Taking the first partial derivative of ℓnL with respect to θ_1, θ_2 and p, we have

$$\frac{\partial \ell nL}{\partial \theta_1} = \frac{\alpha(n-r)\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}\left(\frac{T^p}{\theta_1^2}\right) - \frac{r_1}{\theta_1} + \frac{\sum_{j=1}^{r_1}(x_{1j}T)^p}{\theta_1^2}$$

$$= \frac{(n-r)kT^p}{\theta_1^2} - \frac{r_1}{\theta_1} + \frac{\sum_{j=1}^{r_1} (x_{1j}T)^p}{\theta_1^2} \tag{3.7}$$

$$\frac{\partial \ell nL}{\partial \theta_2} = \frac{\beta(n-r)\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}\left(\frac{T^p}{\theta_2^2}\right) - \frac{r_2}{\theta_2} + \frac{\sum_{j=1}^{r_2} (x_{2j}T)^p}{\theta_2^2}$$

$$= \frac{(n-r)(1-k)T^p}{\theta_2^2} - \frac{r_2}{\theta_2} + \frac{\sum_{j=1}^{r_2} (x_{2j}T)^p}{\theta_2^2} \tag{3.8}$$

$$\frac{\partial \ell nL}{\partial p} = \left[\alpha(n-r)\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}\left(-\frac{T^p}{\theta_1}(\ell nT)\right) + (n-r)\beta\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right.$$

$$\left.\times \left(-\frac{T^p}{\theta_2}(\ell nT)\right)\right] \times \frac{1}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}$$

$$+ \frac{r_1}{p} + \sum_{j=1}^{r_1} [\ell n(x_{1j}T)] - \left(\frac{1}{\theta_1}\right) \sum_{j=1}^{r_1} (x_{1j}T)^p [\ell n(x_{1j}T)]$$

$$+ \frac{r_2}{p} + \sum_{j=1}^{r_2} [\ell n(x_{2j}T)] - \left(\frac{1}{\theta_2}\right) \sum_{j=1}^{r_2} (x_{2j}T)^p [\ell n(x_{2j}T)] \quad .$$

$$= -(n-r)T^{p}(\ell n T)\left(\frac{k}{\theta_1} + \frac{1-k}{\theta_2}\right) + \frac{r_1}{p} + \frac{r_2}{p} + \sum_{j=1}^{r_1} [\ell n(x_{1j};T)]$$

$$+ \sum_{j=1}^{r_2} [\ell n(x_{2j};T)] - \frac{1}{\theta_1} \sum_{j=1}^{r_1} (x_{1j};T)^{p} [\ell n(x_{1j};T)]$$

$$- \frac{1}{\theta_2} \sum_{j=1}^{r_2} (x_{2j};T)^{p} [\ell n(x_{2j};T)] \tag{3.9}$$

where
$$k = \frac{\alpha \exp\left\{-\left(\frac{T^{p}}{\theta_1}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^{p}}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^{p}}{\theta_2}\right)\right\}}$$

$$= \frac{1}{1 + \left(\frac{\beta}{\alpha}\right)\exp\left\{T^{p}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\right\}} \tag{3.10}$$

When the partial derivatives are equated to zero, the estimating equations are

$$\hat{\theta}_1 = \frac{(n-r)T^{\hat{p}}}{r_1}\hat{k} + \frac{\sum_{j=1}^{r_1} (x_{1j};T)^{\hat{p}}}{r_1} \tag{3.11}$$

$$\hat{\theta}_2 = \frac{(n-r)T^{\hat{p}}}{r_2}(1-\hat{k}) + \frac{\sum_{j=1}^{r_2} (x_{2j};T)^{\hat{p}}}{r_2} \tag{3.12}$$

and $\hat{p}$ is the solution of

$$g(\hat{p}) \equiv -(n-r)T^{\hat{p}}(\ell nT)\left(\frac{\hat{k}}{\hat{\theta}_1}+\frac{1-\hat{k}}{\hat{\theta}_2}\right)+\frac{r_1}{\hat{p}}+\frac{r_2}{\hat{p}}+\sum_{j=1}^{r_1}[\ell n(x_{1j}T)]$$

$$+\sum_{j=1}^{r_2}[\ell n(x_{2j}T)]-\frac{1}{\hat{\theta}_1}\sum_{j=1}^{r_1}(x_{1j}T)^{\hat{p}}[\ell n(x_{1j}T)]$$

$$-\frac{1}{\hat{\theta}_2}\sum_{j=1}^{r_2}(x_{2j}T)^{\hat{p}}[\ell n(x_{2j}T)] = 0 \tag{3.13}$$

where

$$\hat{k} = \frac{1}{1+(\frac{\beta}{\alpha})\exp\left\{T^{\hat{p}}\left(\frac{1}{\hat{\theta}_1}-\frac{1}{\hat{\theta}_2}\right)\right\}} \tag{3.14}$$

If $\alpha = \frac{1}{3}$,

$$\hat{k} = \frac{1}{1+2\exp\left\{T^{\hat{p}}\left(\frac{1}{\hat{\theta}_1}-\frac{1}{\hat{\theta}_2}\right)\right\}}$$

By setting the expression (3.7) - (3.9) equal to zero, we have three equations with three unknown. Theoretically, these three unknowns can be solved, which will maximize the likelihood, at least locally.

We would like to point out that when $\alpha = 1$, the sample becomes a censored sample from a single two parameter Weibull distribution. In this case, equation (3.4) can be simplified as equation (27) in A.C. Cohen's paper (1965). Equation (3.11) and (3.13) are the same as the MLE (11) and (12) in the same paper. So it is reasonable to try to use similar methods to iterate p in equation (3.9). Substituting the moment estimators θ_1^*, θ_2^* into equation (3.9), we have

$$g(p) \equiv -(n-r)T^p\left[\ln\left(\frac{k^*}{\theta_1^*}+\frac{(1-k^*)}{\theta_2^*}\right)\right]+\frac{r_1}{p}+\frac{r_2}{p}$$

$$+\sum_{j=1}^{r_1}[\ln(x_{1j}\cdot T)]+\sum_{j=1}^{r_2}[\ln(x_{2j}\cdot T)]$$

$$-\frac{1}{\theta_1^*}\sum_{j=1}^{r_1}(x_{1j}\cdot T)^p[\ln(x_{1j}\cdot T)]-\frac{1}{\theta_2^*}\sum_{j=1}^{r_2}(x_{2j}\cdot T)^p[\ln(x_{2j}\cdot T)]$$

$$= 0 \qquad (3.15)$$

where

$$k^* = \frac{1}{1+\left(\frac{(1-\alpha)}{\alpha}\right)\exp\left\{T^{p^*}\left(\frac{1}{\theta_1^*}-\frac{1}{\theta_2^*}\right)\right\}} \qquad (3.16)$$

and here θ_1^*, θ_2^* (θ_i^* already obtained) and α ($=\frac{1}{3}$, known) are treated as constants. The value of p^* read from table (3.1) should provide a starting point to use in the iterative solution of the equation (3.15).

After we obtained the value $\hat{p}_0$ (say), the iterative process of Mendenhall and Hader (1958) can be used. We obtain $\hat{\theta}_1$ and $\hat{\theta}_2$ from (3.11) and (3.12) in terms of k, and substitute them in (3.14). (3.14) is now of the from $h(\hat{k}, \hat{p}) = \hat{k}$, where $h(\hat{k}, \hat{p})$ is essentially a non-negative function of $(\hat{p}, \hat{k})$. Also, for any fixed $\hat{p} = \hat{p}_0$, $\hat{k}$ is bounded between 0 and 1 . The correct value of $\hat{k}$ is the solution of the equation $h(\hat{k}, \hat{p}_0) - \hat{k} = 0$.

Let $\hat{\lambda}_i = \hat{\theta}_i / T^{\hat{p}}$, and $\bar{y}_i = \sum_{j=1}^{r_1} (x_{ij}^{\hat{p}}/r_i)$. So the equation (3.11), (3.12) and (3.14) become:

$$\hat{\lambda}_1 = \frac{(n-r)\hat{k}}{r_1} + \bar{y}_1 \tag{3.17}$$

$$\hat{\lambda}_2 = \frac{(n-r)(1-\hat{k})}{r_2} + \bar{y}_2 \tag{3.18}$$

$$\hat{k} = \frac{1}{1 + (\frac{\beta}{\alpha}) \exp\left\{ \frac{1}{\hat{\lambda}_1} - \frac{1}{\hat{\lambda}_2} \right\}} \tag{3.19}$$

Then by using similar techniques as Deemer and Votaw (1955) the good approximation to $\hat{k}$ (for which $\hat{p} = \hat{p}_0$) can be obtained. Since the distribution is assumed to be censored at time T, and

$$f(t|p, \theta_i) = (\frac{p}{\theta_i})t^{p-1} \exp\left\{-\left(\frac{t^p}{\theta_i}\right)\right\}$$ we have

$$F(t) = \frac{p}{\theta_i} \int_0^T t^{p-1} \exp\left\{-\left(\frac{t^p}{\theta_i}\right)\right\} dt$$

$$= 1 - \exp\left\{-\left(\frac{T^p}{\theta_i}\right)\right\}$$

Then $$f(t|t \le T) = \frac{\frac{p}{\theta_i} t^{p-1} \exp\left\{-\left(\frac{t^p}{\theta_i}\right)\right\}}{1 - \exp\left\{-\left(\frac{T^p}{\theta_i}\right)\right\}}$$

Put $y = x^p$ and $\lambda_i = \theta_i/T^p$ where $x = \frac{t}{T}$. We get

$$f(t|t \le T) = \frac{\frac{1}{\lambda_i} \exp\{-(\frac{y}{\lambda_i})\}}{1 - \exp\{-(\frac{1}{\lambda})\}} .$$

For this model the likelihood

$$L \propto \frac{\exp\left\{-\left(\frac{r_i \bar{y}_i}{\lambda_i}\right)\right\}}{\lambda_i^{r_i} \left(1 - \exp\{-(\frac{1}{\lambda_i})\}\right)^{r_i}}$$

where $$\bar{y}_i = \frac{\sum_{j=1}^{r_i} y_{ij}}{r_i} = \frac{\sum_{j=1}^{r_i} x_{ij}^p}{r_i}$$

$$\ell nL = \text{constant} - r_i(\ell n\lambda_i) - \frac{r_i \bar{y}_i}{\lambda_i}$$

$$- r_i \ell n\left(1 - \exp\{-(\frac{1}{\lambda_i})\}\right)$$

and $$\frac{\partial \ell nL}{\partial \lambda_i} = -\frac{r_i}{\lambda_i} + \frac{r_i \bar{y}_i}{\lambda_i^2} + \frac{r_i}{1 - \exp\{-(\frac{1}{\lambda_i})\}} \frac{1}{\lambda_i^2} \exp\{-(\frac{1}{\lambda_i})\}$$

setting this expression equal to zero, we get

$$1 = \frac{\bar{y}_i}{\lambda_i} + \frac{1}{\lambda_i \exp\{+(\frac{1}{\lambda_i})\}-1}$$

i.e. the MLE of λ_i is the solution of the equation

$$(\lambda_i - \bar{y}_i)(\exp\{\frac{1}{\lambda_i}\}-1) = 1 \qquad (3.20)$$

So we may consider $\hat{\lambda}_i$ as a function of $\bar{y}_i$, for a given value $\hat{p} = \hat{p}_0$.

Table 3.2 gives the value of $\bar{y}_i$'s for arbitrary λ_i's . Let $\bar{y} = \min\{\bar{y}_1, \bar{y}_2\}$ and for convenience let us label it as sub-population (1). From our sample we pick $\bar{y}$ as defined above and obtain the corresponding λ graphically from fig. 3.1. Let us represent this λ by $\hat{\lambda}_{01}$. We now solve for $\hat{k}_0$, the starting value of k from (3.17), viz.,

$$\hat{\lambda}_{01} = \bar{y} + \hat{k}_0 \frac{n-r}{r_1} \qquad (3.21)$$

If $D_0 = h(\hat{k}_0, \hat{p}_0) - \hat{k}_0$ equals zero, we are done. If not, consider the sign of D_0. If $D_0 < 0$, then the value of $\hat{k}$ which satisfies $D \equiv h(\hat{k}, \hat{p}) - \hat{k} = 0$ and provides a solution to equations (3.17), (3.18) and (3.19) must be such that $\hat{k} < \hat{k}_0$; similarly $\hat{k} > \hat{k}_0$ if $D_0 > 0$.

Letting $h(k, p_0) = \frac{1}{1+v}$

where $$v = (\frac{\beta}{\alpha}) \exp\{\frac{1}{\hat{\lambda}_1} - \frac{1}{\hat{\lambda}_2}\} \qquad (3.22)$$

Since $v = \frac{1}{h(\hat{k},\hat{p}_0)} - 1$, so $\frac{dv}{d\hat{k}} = - \frac{1}{h^2(\hat{k}, \hat{p}_0)} \times \frac{d\left(h(\hat{k}, \hat{p}_0)\right)}{d\hat{k}}$

TABLE 3.2 THE VALUE OF λ_i AND $\bar{y}_i$ FOR THE EQUATION:

$$(\lambda_i - \bar{y}_i)\,(\exp\{\frac{1}{\lambda_i}\}-1) = 1$$

λ_i-value	$\bar{y}_i$-value	λ_i-value	$\bar{y}_i$-value	λ_i-value	$\bar{y}_i$-value
0.01	0.01000	0.29	0.25715	0.49	0.34068
0.07	0.07000	0.31	0.26863	0.51	0.34620
0.13	0.12954	0.33	0.27925	0.56	0.35854
0.15	0.14873	0.35	0.28907	0.61	0.36914
0.17	0.16720	0.37	0.29816	0.66	0.37832
0.19	0.18479	0.39	0.30659	0.71	0.38633
0.21	0.20138	0.41	0.31441	0.81	0.39964
0.23	0.21090	0.43	0.32169	0.91	0.41022
0.25	0.23134	0.45	0.32846	1.01	0.41881
0.27	0.24475	0.47	0.33478	1.21	0.43190

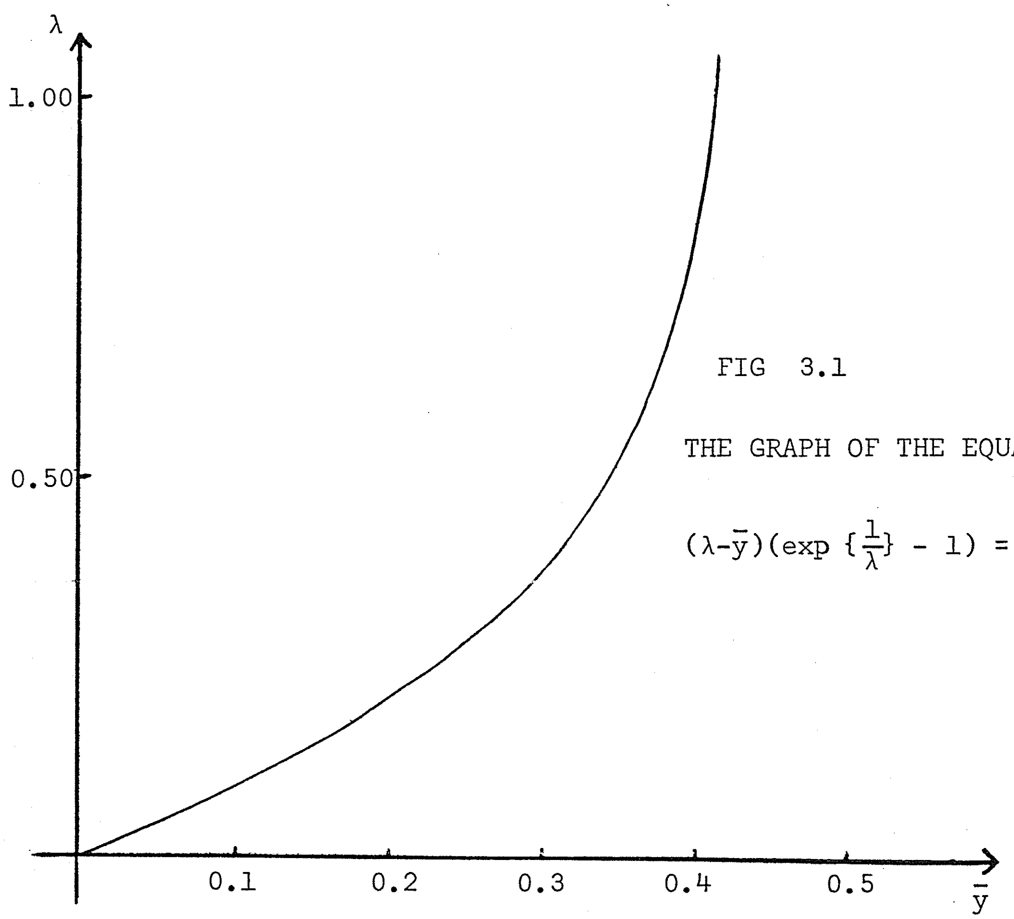


FIG 3.1

THE GRAPH OF THE EQUATION:

$$(\lambda-\bar{y})(\exp\{\frac{1}{\lambda}\} - 1) = 1$$

Then $$\frac{dD}{d\hat{k}} = -h^2(\hat{k}, \hat{p}_0)\frac{dv}{d\hat{k}} - 1$$

i.e. $$d\hat{k} = -\frac{dD}{1 + h^2(\hat{k}, \hat{p}_0)(\frac{dv}{d\hat{k}})} \qquad (3.23)$$

where $$\frac{dv}{d\hat{k}} = -v(n-r)\left[\frac{1}{r_1\hat{\lambda}_1^2} + \frac{1}{r_2\hat{\lambda}_2^2}\right] \qquad (3.24)$$

Choosing $$dD = -D_0 \qquad (3.25)$$

$$\hat{k}_1 = \hat{k}_0 + d\hat{k}_0$$
$$= \hat{k}_0 + \frac{D_0}{1 + h^2(\hat{k}_0, \hat{p}_0)\left(\frac{dv}{d\hat{k}_0}\right)} \qquad (3.26)$$

This iterative process can be repeated until the desired degree of accuracy is attained. The estimates of λ_1 and λ_2 are then obtained by substituting the solution for $\hat{k}$ into estimation (3.17) and (3.18) and then $\hat{\theta}_1$ and $\hat{\theta}_2$ are obtained from $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

In practical work, we expect the probability of $r_1 = 0$ or $r_2 = 0$ to be very small, since, when $r_i = 0$, no definite inference can be drawn about the parameters, so it may be necessary to choose n and T large enough to avoid such a situation.

Once we got $\hat{\theta}_{01}$ and $\hat{\theta}_{02}$ for a given value $\hat{p}_0$, compare $\hat{\theta}_{01}$ and $\hat{\theta}_{02}$ with θ^*_{01} and θ^*_{02} (which had been read from table 3.1), respectively. If $\hat{\theta}_{01} - \theta^*_{01} = 0$ and $\hat{\theta}_{02} - \theta^*_{02} = 0$ then $\hat{\theta}_{01}$,

$\hat{\theta}_{02}$ and $\hat{p}_0$ are the required maximum likelihood estimators. Otherwise, substitute $\hat{\theta}_{01}$ and $\hat{\theta}_{02}$ thus obtained in (3.15) and (3.16) to get a new $\hat{p} = \hat{p}_1$ (say). Using the same method as discussed earlier, we get new estimates $\hat{\theta}_{11}$ and $\hat{\theta}_{12}$ (say) and compare them with the previous results. If they agree, we stop and we get the MLE $\hat{p} = \hat{p}_1$, $\hat{\theta}_1 = \hat{\theta}_{11}$ and $\hat{\theta}_2 = \hat{\theta}_{12}$. If not, put $\hat{\theta}_{11}$ and $\hat{\theta}_{12}$ into equation (3.15) and (3.16) again and repeat the same procedures until the process converges.

3.3 Variances and Covariances of the MLE Estimators

Taking the second derivatives of the equation (3.6) with respect to θ_1, θ_2 and p, respectively, we have (see (3.7), (3.8) and (3.9)):

$$\frac{\partial^2 \ell nL}{\partial \theta_1^2} = \left[\left(\alpha \exp\left\{-\left(\frac{T^P}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^P}{\theta_2}\right)\right\}\right)\cdot\alpha\cdot (n-r) \exp\left\{-\left(\frac{T^P}{\theta_1}\right)\right\}\left(\frac{T^P}{\theta_1^2}\right)\right.$$

$$\left. - \alpha (n-r) \exp\left\{-\left(\frac{T^P}{\theta_1}\right)\right\} \alpha \exp\left\{-\left(\frac{T^P}{\theta_1}\right)\right\}\left(\frac{T^P}{\theta_1^2}\right)\right]$$

$$\times \frac{T^P}{\theta_1^2} \times \frac{1}{\left[\alpha \exp\left\{-\left(\frac{T^P}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^P}{\theta_2}\right)\right\}\right]^2}$$

$$- \frac{\alpha (n-r) \exp\left\{-\left(\frac{T^P}{\theta_1}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^P}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^P}{\theta_2}\right)\right\}}\left(\frac{2T^P}{\theta_1^3}\right) + \frac{r_1}{\theta_1^2} - \frac{2\sum_{j=1}^{r_1} (x_{1j}T)^P}{\theta_1^3}$$

$$= \left[\frac{(n-r)kT^p}{\theta_1^2} - \frac{(n-r)k^2T^p}{\theta_1^2}\right]\left(\frac{T^p}{\theta_1^2}\right) - \frac{2(n-r)kT^p}{\theta_1^3} + \frac{r_1}{\theta_1^2} - \frac{2\sum_{j=1}^{r_2}(x_{1j};T)^p}{\theta_1^3}$$

$$\text{i.e.} \quad \frac{\partial^2 \ell nL}{\partial\theta_1^2} = \frac{k(1-k)(n-r)T^{2p}}{\theta_1^4} - \frac{2k(n-r)T^p}{\theta_1^3} + \frac{r_1}{\theta_1^2} - \frac{2\sum_{j=1}^{r_1}(x_{1j};T)^p}{\theta_1^3} \qquad (3.27)$$

$$\frac{\partial^2 \ell nL}{\partial\theta_2^2} = \left[\alpha\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\beta\,(n-r)\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\left(\frac{T^p}{\theta_2^2}\right)\right.$$

$$\left. - \beta^2(n-r)\exp\left\{-\frac{2T^p}{\theta_2}\right\}\left(\frac{T^p}{\theta_2^2}\right)\right]$$

$$\times\left(\frac{T^p}{\theta_2^2}\right)\times\frac{1}{\left[\alpha\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]^2}$$

$$- \frac{\beta(n-r)\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}{\alpha\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}\left(\frac{2T^p}{\theta_2^3}\right) + \frac{r_2}{\theta_2^2} - \frac{2\sum_{j=1}^{r_2}(x_{2j};T)^p}{\theta_2^2}\,.$$

$$= \left[\frac{(n-r)(1-k)T^p}{\theta_2^2} - \frac{(n-r)(1-k)^2T^p}{\theta_2^2}\right]\left(\frac{T^p}{\theta_2^2}\right)$$

$$- \frac{2(n-r)(1-k)T^{p}}{\theta_2^3} + \frac{r_2}{\theta_2^2} - \frac{2 \sum_{j=1}^{r_2} (x_{2j}.T)^{p}}{\theta_2^3}$$

i.e.

$$\frac{\partial^2 \ell nL}{\partial \theta_2^2} = \frac{k(1-k)(n-r)T^{2p}}{\theta_2^4} - \frac{2(1-k)(n-r)T^{p}}{\theta_2^3}$$

$$+ \frac{r_2}{\theta_2^2} - \frac{2 \sum_{j=1}^{r_2} (x_{2j}.T)^{p}}{\theta_2^3} \qquad (3.28)$$

$$\frac{\partial^2 \ell nL}{\partial p^2} = \left\{ \left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\} \right] \right.$$

$$\times \left[- \frac{\alpha(n-r)T^{p}(\ell nT)^2}{\theta_1} \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} \right.$$

$$+ \frac{\alpha(n-r)T^{2p}(\ell nT)^2}{\theta_1^2} \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}$$

$$- \frac{\beta(n-r)T^{p}(\ell nT)^2}{\theta_2} \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}$$

$$\left. + \frac{\beta(n-r)T^{2p}(\ell nT)^2}{\theta_2^2} \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\} \right]$$

$$+ \left[\frac{\alpha(n-r)T^{p}(\ell nT)}{\theta_1} \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \frac{\beta(n-r)T^{p}(\ell nT)}{\theta_2^2} \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]$$

$$\times \left[-\frac{\alpha T^{p}(\ell nT)}{\theta_1} \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} - \frac{\beta T^{p}(\ell nT)}{\theta_2} \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}\right] \Bigg\}$$

$$\times \frac{1}{\left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]^2}$$

$$- \frac{r_1}{p^2} - \left(\frac{1}{\theta_1}\right) \sum_{j=1}^{r_1} [\ell n(x_{1j}T)]^2 (x_{1j}T)^p$$

$$- \frac{r_2}{p^2} - \left(\frac{1}{\theta_2}\right) \sum_{j=1}^{r_2} [\ell n(x_{2j}T)]^2 (x_{2j}T)^p$$

$$= \left[\frac{\alpha(n-r)T^{p}(\ell nT)^2 \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\theta_1} \left(\frac{T^p}{\theta_1} - 1\right)\right.$$

$$\left. + \frac{\beta(n-r)T^{p}(\ell nT)^2 \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}{\theta_2} \left(\frac{T^p}{\theta_2} - 1\right)\right]$$

$$\times \frac{1}{\alpha \cdot \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}$$

$$- \left[\frac{\alpha(n-r)T^p(\ell nT)\ \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\theta_1} + \frac{\beta(n-r)T^p(\ell nT)\ \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}{\theta_2} \right]$$

$$\times \left[\frac{\alpha T^p(\ell nT)\ \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\theta_1} + \frac{\beta T^p(\ell nT)\ \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}{\theta_2} \right]$$

$$\times \frac{1}{\left[\alpha\ \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta\ \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]^2}$$

$$- \frac{r_1}{p^2} - \left(\frac{1}{\theta_1}\right) \sum_{j=1}^{r_1} [\ell n(x_{1j}.T)]^2\ (x_{1j}.T)^p - \frac{r_2}{p^2} - \left(\frac{1}{\theta_2}\right) \sum_{j=1}^{r_2} [\ell n(x_{2j}.T)]^2\ (x_{2j}.T)^p$$

$$= (n-r)(\ell nT)^2 T^p \left[\frac{k(T^p - \theta_1)}{\theta_1^2} + \frac{(1-k)(T^p - \theta_2)}{\theta_2^2} \right]$$

$$- (n-r)(\ell nT)^2 T^{2p} \left(\frac{k}{\theta_1} + \frac{(1-k)}{\theta_2}\right)^2 - \frac{r_1 + r_2}{p^2}$$

$$- \left(\frac{1}{\theta_1}\right) \sum_{j=1}^{r_1} [\ell n(x_{1j}.T)]^2 (x_{1j}.T)^p - \frac{1}{\theta_2} \sum_{j=1}^{r_2} [\ell n(x_{2j}.T)]^2\ (x_{2j}.T)^p$$

$$= (n-r)(\ell nT)^2 T^{2p}\ k(1-k) \left[\frac{1}{\theta_1^2} - \frac{2}{\theta_1\theta_2} + \frac{1}{\theta_2^2} \right]$$

$$- (n-r)(\ell nT)^2 T^p \left[\frac{k}{\theta_1} + \frac{1-k}{\theta_2} \right] - \frac{r_1 + r_2}{p^2}$$

$$- (\frac{1}{\theta_1}) \sum_{j=1}^{r_1} [\ell n(x_{1j}T)]^2 (x_{1j}T)^p - (\frac{1}{\theta_2}) \sum_{j=1}^{r_2} [\ell n(x_{2j}T)]^2 (x_{2j}T)^p$$

So
$$\frac{\partial^2 \ell nL}{\partial p^2} = k(1-k)(n-r)T^{2p} (\ell nT)^2 (\frac{1}{\theta_1} - \frac{1}{\theta_2})^2$$

$$- (n-r)(\ell nT)^2 T^p \left(\frac{k}{\theta_1} + \frac{(1-k)}{\theta_2} \right) - \frac{r_1 + r_2}{p^2}$$

$$- \frac{1}{\theta_1} \sum_{j=1}^{r_1} [\ell n(x_{1j}T)]^2 (x_{1j}T)^p - \frac{1}{\theta_2} \sum_{j=1}^{r_2} [\ell n(x_{2j}T)]^2 (x_{2j}T)^p \tag{3.29}$$

$$\frac{\partial^2 \ell nL}{\partial \theta_2 \partial \theta_1} = \frac{\alpha(n-r) \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]^2}$$

$$\times \left[\beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\} \left(-\frac{T^p}{\theta_2^2}\right) \right] \left(\frac{T^p}{\theta_1^2}\right)$$

i.e.
$$\frac{\partial^2 \ell nL}{\partial \theta_2 \partial \theta_1} = - \frac{(n-r)\ k(1-k)T^{2p}}{\theta_1^2\ \theta_2^2} = \frac{\partial^2 \ell nL}{\partial \theta_1 \partial \theta_2} \tag{3.30}$$

$$\frac{\partial^2 \ell nL}{\partial p \partial \theta_1} = \left(\left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right] \alpha(n-r)\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}\right.$$

$$\times\left(-\frac{T^p}{\theta_1} \ell nT\right) - \left[\alpha^2(n-r)\left(\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}\right)^2\left(-\frac{T^p}{\theta_1} \ell nT\right)\right]$$

$$\left.- \left[\alpha\beta(n-r)\left(\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}\right)^2 \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]\right) \times \left(\frac{T^p}{\theta_1^2}\right)$$

$$\times \frac{1}{\left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]^2}$$

$$+ \frac{\alpha(n-r)\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}\left(\frac{T^p}{\theta_1^2}\right)\ell nT + \frac{1}{\theta_1^2}\sum_{j=1}^{r_1}(x_{1j}T)^p[\ell n(x_{1j}T)]$$

$$= \left[\frac{-k(n-r)T^p \ell nT}{\theta_1} + \frac{k^2(n-r)T^p \ell nT}{\theta_1} + \frac{k(1-k)(n-r)T^p \ell nT}{\theta_2}\right]\left(\frac{T^p}{\theta_1^2}\right)$$

$$+ \frac{k(n-r)T^p \ell nT}{\theta_1^2} + \frac{\sum_{j=1}^{r_1}(x_{1j}T)^p\,[\ell n(x_{1j}T)]}{\theta_1^2}$$

i.e. $$\frac{\partial^2 \ell nL}{\partial p \partial \theta_1} = \frac{k(1-k)(n-r)T^{2p} \ell nT}{\theta_1^2}\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) + \frac{k(n-r)T^p \ell nT}{\theta_1^2}$$

$$+\frac{\sum_{j=1}^{r_1} (x_{1j}T)^p \, [\ln(x_{1j}T)]}{\theta_1^2} = \frac{\partial^2 \ln L}{\partial\theta_1 \partial p} \tag{3.31}$$

$$\frac{\partial^2 \ln L}{\partial p \partial \theta_2} = \left\{\left(\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right)\beta(n-r) \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\left(-\frac{T^p}{\theta_2}\ln T\right)\right.$$

$$- \alpha\beta(n-r) \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\} \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} \left(-\frac{T^p}{\theta_2}\ln T\right)$$

$$\left.- \beta^2(n-r) \left[\exp\left\{-\left(\frac{T}{\theta_2}\right)\right\}\right]^2 \left(-\frac{T^p}{\theta_2}\ln T\right)\right\} \times \left(\frac{T^p}{\theta_2^2}\right)$$

$$\times \frac{1}{\left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}+\beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]^2}$$

$$+ \frac{\beta(n-r) \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}+\beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}} \left(\frac{T^p}{\theta_1^2}\right) \ln T$$

$$+ \frac{1}{\theta_2^2} \sum_{j=1}^{r_2} (x_{2j}T)^p \, [\ln(x_{2j}T)]$$

$$= \left[\frac{-(1-k)(n-r)T^p \ln T}{\theta_2} + \frac{k(1-k)(n-r)T^p \ln T}{\theta_1}\right.$$

$$+ \frac{(1-k)^2(n-r)T^p \ln T}{\theta_2} \Bigg] \times \left(\frac{T^p}{\theta_2^2}\right)$$

$$+ \frac{(1-k)(n-r)T^p \ln T}{\theta_2^2} + \frac{\sum_{j=1}^{r_2} (x_{2j}T)^p \, [\ln(x_{2j}T)]}{\theta_2^2}$$

i.e. $$\frac{\partial^2 \ln L}{\partial p \partial \theta_2} = \frac{k(1-k)(n-r)T^{2p} \ln T}{\theta_2^2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right) + \frac{(1-k)(n-r)T^p \ln T}{\theta_2^2}$$

$$+ \frac{\sum_{j=1}^{r_2} (x_{2j}T)^p \ln(x_{2j}T)}{\theta_2^2} = \frac{\partial^2 \ln L}{\partial \theta_2 \partial p} \qquad (3.32)$$

The asymptotic variance-covariance matrix of $(\hat{\theta}_1, \hat{\theta}_2, \hat{p})$ can be obtained by inverting Fisher's information matrix:

$$\begin{bmatrix} -E\,\dfrac{\partial^2 \ln L}{\partial \theta_1^2} & -E\,\dfrac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2} & -E\,\dfrac{\partial^2 \ln L}{\partial \theta_1 \partial p} \\ & -E\,\dfrac{\partial^2 \ln L}{\partial \theta_2^2} & -E\,\dfrac{\partial^2 \ln L}{\partial \theta_2 \partial p} \\ & & -E\,\dfrac{\partial^2 \ln L}{\partial p^2} \end{bmatrix}^{-1}$$

Cohen (1965) approximates it by

$$\begin{bmatrix} -\left.\dfrac{\partial^2 \ell nL}{\partial\theta_1^2}\right|_{\hat\theta_1,\hat\theta_2,\hat p} & -\left.\dfrac{\partial^2 \ell nL}{\partial\theta_1\partial\theta_2}\right|_{\hat\theta_1,\hat\theta_2,\hat p} & -\left.\dfrac{\partial^2 \ell nL}{\partial\theta_1\partial p}\right|_{\hat\theta_1,\hat\theta_2,\hat p} \\ & -\left.\dfrac{\partial^2 \ell nL}{\partial\theta_2^2}\right|_{\hat\theta_1,\hat\theta_2,\hat p} & -\left.\dfrac{\partial^2 \ell nL}{\partial\theta_2\partial p}\right|_{\hat\theta_1,\hat\theta_2,\hat p} \\ & & -\left.\dfrac{\partial^2 \ell nL}{\partial p^2}\right|_{\hat\theta_1,\hat\theta_2,\hat p} \end{bmatrix}^{-1}$$

$$\doteqdot \begin{bmatrix} v(\hat\theta_1) & cov(\hat\theta_1,\hat\theta_2) & cov(\hat\theta_1,\hat p) \\ & v(\hat\theta_2) & cov(\hat\theta_2,\hat p) \\ & & v(\hat p) \end{bmatrix} \qquad (3.33)$$

The above results are valid only if the samples are large. The bias diminishes as the sample size becomes large.

3.4 Illustration

We suppose that the subpopulation (1) and the subpopulation (2) have the distribution functions:

$$y_1 = F_1(x) = 1 - \exp\left\{-\left(\frac{x^{p_1}}{\theta_1}\right)\right\} \tag{3.34}$$

$$y_2 = F_2(x) = 1 - \exp\left\{-\left(\frac{x^{p_2}}{\theta_2}\right)\right\} \tag{3.35}$$

respectively. Let the two subpopulations be mixed in proportion $\alpha:\beta(= 1 - \alpha)$. We generate 50 observations from the subpopulation (1) by using the function

$$x_1 = \left[-\theta_1 \ell n(1-y_1)\right]^{\frac{1}{p_1}} \tag{3.36}$$

and 100 observations from the subpopulations (2) by using the function

$$x_2 = \left[-\theta_1 \ell n(1-y_1)\right]^{\frac{1}{p_2}} \tag{3.37}$$

where y_1 and y_2 are random $U(0,1)$ variates.

Suppose the population parameters are assigned as $\alpha = 1/3$, $\beta = 2/3$, $\theta_1 = 4000$, $\theta_2 = 6000$, $p=p_1=p_2=2$, and the test termination time, T, is 97 hours.

The subprogram GEN provided in the appendix gives these 150 observations, in which subroutine RANS for random numbers from U(0,1) has been used.

<u>EXAMPLE 1</u> : ($\alpha = 1/3$ is known, $p_1=p_2$, θ_1, θ_2 unknown)

Use the subprogram GEN for $\theta_1 = 4000$, $\theta_2 = 6000$ and $p = 2$. We have

TABLE 3.3 THE SAMPLE FROM SUBPOPULATION (1) (n_1=50)

141.9301	11.1163	24.5941	52.7174	27.0414
12.4836	72.8493	66.7386	137.3727	86.0745
36.8422	24.0082	76.2810	48.0882	56.9093
43.9238	29.3049	56.4898	70.8291	73.6127
69.9857	110.0193	47.4235	40.5259	49.4165
79.3363	118.4618	59.4898	59.5717	16.2287
45.2993	34.4604	65.0193	52.5409	35.4843
35.7572	81.5988	63.0705	69.1037	13.8501
10.9996	56.7194	97.7687	69.1311	87.1832
71.0011	8.7451	77.7436	97.3814	67.6381

TABLE 3.4 THE SAMPLE FROM SUBPOPULATION (2) (n_2=100)

61.4761	74.4209	39.9106	26.4205	120.4062
58.9189	93.3233	169.3876	178.8535	58.5754
58.0472	33.4782	106.9338	155.8303	56.8428
76.6547	28.9321	34.8444	129.4308	73.0130
112.6527	41.3338	58.7548	96.6894	87.9453
80.0557	47.6291	149.7707	45.1910	29.9543
12.0040	80.3243	123.0466	76.7872	47.4747
40.1066	27.6235	89.0096	32.0990	77.1125
36.3229	96.3542	27.6430	47.6793	136.1780
20.5436	79.1817	27.7697	61.9176	20.5321
77.3403	99.0928	79.0902	176.9316	100.8320
33.9316	186.8052	98.1420	87.1532	45.0725
52.5389	89.1316	143.7369	51.4669	53.2425
78.4084	59.7743	75.7505	82.6031	82.0281
67.0875	16.6700	155.1094	109.3131	110.6927
55.0424	58.1224	50.8214	20.4790	47.6058
77.1881	150.9445	42.6953	77.5957	18.4598
37.5142	167.5791	57.5187	35.8555	64.4958
73.0030	30.7973	35.0464	60.4849	64.8205
79.3711	139.3956	45.8876	4.3977	80.0522

Now for $n=n_1+n_2 = 150$, $T = 97$, $r_1 = 44$, $r_2 = 78$ and $n-r_1-r_2 = 28$ the data are summarized by:

$$\bar{x} = \sum_{i=1}^{150} x_i/150 = 0.7059803, \quad s^2 = \left(\sum_{i=1}^{150} x_i^2 - \frac{\left(\sum_{i=1}^{150} x_i\right)^2}{150} \right) \Bigg/ 149$$

$= 0.1617389$. It follows that $(cv)^2 = \frac{s^2}{\bar{x}^2} = 0.324511$ and the coefficient of variation of the mixed sample with 150 data is $cv = 0.5696586$. Reading from table 3.1, we have the moment estimates: $p^* = 1.875$, $\theta_1^* = \lambda_1^* T^{p^*} = 0.4175\times(97)^{1.875} = 2217.4190$ and $\theta_2^* = \lambda_2^* T^{p^*} = 0.6150\times(97)^{1.875} = 3662.1860$. Keep θ_1^* and θ_2^* as constants, substitute them in the equations (3.13) and (3.14) and use $p^* = 1.875$ as a starting point, iterating p with equation (3.13). We get $\hat{p}_0 = 1.8530$. Substitute p_0 in equation (3.11), (3.13) and (3.14). Using the technique that we mentioned in section 3.2, the first and the final iterations are given in the following

TABLE 3.5(a) FIRST ITERATIONS (WITH $\hat{p}_0 = 1.8530$)

u	$\hat{k}_u$	$\hat{\theta}_{u1}$	$\hat{\theta}_{u2}$	v_u	$h(\hat{k}_u,\hat{p}_0)$	D_u
0	0.2717	2509.4160	7417.7850	7.0962	0.1235	-0.1482
1	0.0692	1890.1760	3526.1450	6.5013	0.1333	0.0642
2	0.2123	2327.9560	3279.1030	3.6386	0.2156	0.0033
3	0.2262	2352.1270	3265.5570	3.5406	0.2202	0.0000

TABLE 3.5(b) FINAL ITERATIONS (WITH $\hat{p}_0 = 2.0257$)

u	$\hat{k}_u$	$\hat{\theta}_{u1}$	$\hat{\theta}_{u2}$	v_u	$h(\hat{k}_u,\hat{p}_0)$	D_u
0	0.1766	4629.9920	17671.9000	10.8050	0.0847	-0.0919
1	0.0511	3784.4860	7584.1710	8.1184	0.1097	0.0586
2	0.1825	4669.1990	7085.1010	4.3318	0.1876	0.0051
3	0.1959	4759.4330	7034.2030	4.1050	0.1959	0.0000
4	0.1959	4759.8470	7033.9640	4.1040	0.1959	0.0000

Since $|\hat{\theta}_{16,1} - \theta_1^*|$ and $|\hat{\theta}_{16,2} - \theta_2^*|$ are not equal to zero, we substitute $\hat{\theta}_{16,1}$ and $\hat{\theta}_{16,2}$ in equations (3.13) and (3.14), use $\hat{p}_0 = 1.8530$ as a starting point and do the same iteration; we get $\hat{p}_1 = 1.8562$ and $\hat{\theta}_1 = 2383.2250$, $\hat{\theta}_2 = 3312.1140$. Continue the iterations until the estimators satisfy the equations (3.11), (3.12) and (3.13). The results are listed below with corresponding moment estimates, MLE along with the population values.

	Population Values	Moment Estimates	MLE
p	2	1.8750	2.0257
θ_1	4000	2217.4460	4759.8470
θ_2	6000	3279.6950	7033.9640

And the asymptotic variance-covariance can be obtained by changing the signs and substituting the MLE $\hat{p} = 2.0257$, $\hat{\theta}_1 = 4759.8160$ and $\hat{\theta}_2 = 7033.9840$ in equation (3.27) — (3.32) . The desired matrix is given by:

$$\begin{bmatrix} 0.98\times10^{-6} & 0.441\times10^{-6} & -0.3274806\times10^{-1} \\ & 0.1375\times10^{-5} & -0.5163375\times10^{-1} \\ & & 0.231231\times10^{4} \end{bmatrix}^{-1}$$

$$
\doteq \begin{bmatrix} 6601412 & 8398373 & 279.9475 \\ & 1510531 & 454.4892 \\ & & 0.01449 \end{bmatrix}
$$

In the results we obtained before, there are large gaps between the estimates and the accurate values of θ_i $(i = 1,2)$. It appears that $\hat{\theta}_i$ is quite sensitive even to a very small change in $\hat{p}$. Even though $\hat{p}$ is just increased (or decreased) by 10^{-4}, the value of $\hat{\theta}_i$ will increase (or decrease) at least by 1.0000 (This is what the output shows). Since $\hat{\theta}$ is a function of $T^{\hat{p}}$, $|\hat{\theta}_i - \theta_i|$ is large. This also explains the large variances and covariances for $\hat{\theta}_1$ and $\hat{\theta}_2$.

CHAPTER 4

α IS UNKNOWN, THE COMMON SHAPE PARAMETER p AND θ_1, θ_2 UNKNOWN

As Mendenhall and Hader have discussed in their paper (1958), there may be a case where the two subpopulations may be mixed with unknown proportions α and $(1 - \alpha)$. We therfore, extend our study from three parameters to four parameters. For solving the estimations of the parameters α, θ_1, θ_2 and p, the procedures are essentially the same as in Chapter 3.

4.1 The Moment Estimation of the Parameters

The coefficient of variation is the same as given by (3.4) except that α is not known. So in the table of the coefficient of variation, we will have 5 columns instead of 4.

A similar technique is used here for getting the moment estimations of α, θ_1, θ_2 and p .

TABLE 4.1 THE MIXED WEIBULL COEFFICIENT OF VARIATION $(\lambda_i = \theta_i/T^p)$

Coefficient of variation	p	α	λ_1	λ_2
0.48029	2.20	0.200	0.450	0.500
0.48095	2.20	0.400	0.350	0.400
0.48306	2.20	0.200	0.450	0.600
0.49695	2.20	0.400	0.350	0.600

0.54360	1.95	0.300	0.420	0.615
0.54406	1.95	0.305	0.420	0.620
0.55670	1.90	0.305	0.420	0.615
0.55726	1.90	0.305	0.415	0.615
0.55762	1.90	0.305	0.415	0.620
0.57127	1.85	0.300	0.415	0.620
0.60712	1.70	0.400	0.350	0.400
0.61177	1.70	0.300	0.450	0.600
0.61495	1.70	0.300	0.350	0.500
0.62021	1.70	0.200	0.350	0.600
0.63015	1.70	0.400	0.350	0.600
0.83904	1.20	0.300	0.450	0.400
0.84492	1.20	0.200	0.450	0.600
0.87946	1.20	0.400	0.350	0.600
1.46597	0.70	0.200	0.450	0.500
1.49440	0.70	0.200	0.350	0.500
1.52330	0.70	0.200	0.350	0.600
1.57829	0.70	0.400	0.350	0.600

cv was computed for assigned values of p, θ_1, θ_2 and α. The table 4.1 was set up in such a way that coefficients of variation are monotonic.

4.2 Maximum Likelihood Estimation of the Parameters

Most of the maximum likelihood estimating equations here are the same as in section 3.2. The only thing we have to find is the MLE of α. We take the first partial derivative of equation (3.6) with respect to α.

$$\frac{\partial \ln L}{\partial \alpha} = (n-r)\left[\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} - \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]$$

$$\times \frac{1}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}} + \frac{r_1}{\alpha} - \frac{r_2}{\beta}$$

$$= \frac{k(n-r) + r_1}{\alpha} - \frac{(1-k)(n-r) + r_2}{\beta} \tag{4.1}$$

where

$$k = \frac{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}$$

$$= \frac{1}{1 + (\frac{\beta}{\alpha}) \exp \{T^p(\frac{1}{\theta_1} - \frac{1}{\theta_2})\}} \tag{4.2}$$

Rewrite the MLE of θ_1, θ_2, p and together with the MLE of α, obtained by setting expression (4.1) equal to zero, we have:

$$\hat{\alpha} = \frac{r_1}{n} + \hat{k}\,\frac{(n-r)}{n} \tag{4.3}$$

$$\hat{\theta}_1 = \frac{\hat{k}(n-r)T^{\hat{p}}}{r_1} + \frac{\sum_{j=1}^{r_1} (x_{1j}.T)^{\hat{p}}}{r_1} \tag{4.4}$$

$$\hat{\theta}_2 = \frac{(1-\hat{k})(n-r)T^{\hat{p}}}{r_2} + \frac{\sum_{j=1}^{r_2} (x_{2j}.T)^{\hat{p}}}{r_2} \tag{4.5}$$

and $\hat{p}$ is the solution of

$$g(\hat{p}) \equiv -(n-r)\ T^{\hat{p}}\ell nT \left(\frac{\hat{k}}{\hat{\theta}_1} + \frac{1-\hat{k}}{\hat{\theta}_2}\right) + \frac{r_1}{\hat{p}} + \frac{r_2}{\hat{p}} + \sum_{j=1}^{r_1} \ell n(x_{1j}.T)$$

$$+ \sum_{j=1}^{r_2} \ell n(x_{2j}T) - \frac{1}{\hat{\theta}_1} \sum_{j=1}^{r_1} (x_{1j}T)^{\hat{p}} [\ell n(x_{1j}T)]$$

$$- \frac{1}{\hat{\theta}_2} \sum_{j=1}^{r_2} (x_{2j}T)^{\hat{p}} [\ell n(x_{2j}T)] = 0 \tag{4.6}$$

where

$$\hat{k} = \frac{1}{1 + \frac{\hat{\beta}}{\hat{\alpha}} \exp\left\{ T^{\hat{p}} \left(\frac{1}{\hat{\theta}_1} - \frac{1}{\hat{\theta}_2}\right)\right\}} \tag{4.7}$$

We will use the moment estimators of the parameters as the starting values for the iterative solution of the equations (4.3) — (4.7).

In fact, when $p = 1$ is known, the equations (4.3), (4.4) (4.5) and 4.7) are the same as equations (4.10), (4.11), (4.12) and (4.13) in Mendenhall and Hader's paper (1958) setting $\lambda_i = \theta_i/T^p = \theta_i/T$, $i = 1, 2$. So it might be reasonable for us to use a similar technique to find out the estimators for θ_1, θ_2, α and p which we discussed above. Note that in this case we have

$$\frac{dv}{d\hat{k}} = -(n-r)v \left[\frac{\left(\frac{1}{\hat{\alpha}} + \frac{1}{\hat{\beta}}\right)}{n} + \frac{1}{r_1\hat{\lambda}_1^2} + \frac{1}{r_2\hat{\lambda}_2^2}\right] \quad \text{corresponding to (3.24)}$$

4.3 <u>The Asymptotic Variance-covariance of Estimations</u>

We need to take the second derivative of equation (3.6) with respect to α only, as the others are the same as in section 3.3. Using equations (3.7), (3.8), (3.9) and (4.1) we obtain the following results:

$$\frac{\partial^2 \ell nL}{\partial\theta_1^2} \quad : \quad \text{the same as in (3.27)}$$

$$\frac{\partial^2 \ell nL}{\partial\theta_2^2} \quad : \quad \text{the same as in (3.28)}$$

$$\frac{\partial^2 \ell nL}{\partial p^2} \quad : \quad \text{the same as in (3.29)}$$

$$\frac{\partial^2 \ell nL}{\partial\theta_1 \partial\theta_2} \quad : \quad \text{the same as in (3.30)}$$

$$\frac{\partial^2 \ell nL}{\partial\theta_1 \partial p} \quad : \quad \text{the same as in (3.31)}$$

$$\frac{\partial^2 \ell nL}{\partial\theta_2 \partial p} \quad : \quad \text{the same as in (3.32)}$$

$$\frac{\partial^2 \ell nL}{\partial\alpha^2} = \frac{(n-r)k(1-k)}{\alpha^2\beta^2} - \frac{k(n-r) + r_1}{\alpha^2} - \frac{(1-k)(n-r) + r_2}{\beta^2} \tag{4.8}$$

$$\frac{\partial^2 \ell nL}{\partial\theta_1 \partial\alpha} = \frac{k(1-k)(n-r)T^p}{\alpha\beta\theta_1^2} = \frac{\partial^2 \ell nL}{\partial\alpha\partial\theta_1} \tag{4.9}$$

$$\frac{\partial^2 \ell nL}{\partial\theta_2 \partial\alpha} = -\frac{k(1-k)(n-r)T^p}{\alpha\beta\theta_2^2} = \frac{\partial^2 \ell nL}{\partial\alpha\partial\theta_2} \tag{4.10}$$

$$\frac{\partial^2 \ell nL}{\partial\alpha\partial p} = k(1-k)(n-r)T^p(\ell nT)\left(\frac{\theta_1 - \theta_2}{\alpha\beta\theta_1\theta_2}\right) = \frac{\partial^2 \ell nL}{\partial p\partial\alpha} \qquad (4.11)$$

So the asymptotic variance-covariance matrix of $(\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{p})$ may be estimated by

$$\begin{bmatrix} -\left.\frac{\partial^2 \ell nL}{\partial\theta_1^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} & -\left.\frac{\partial^2 \ell nL}{\partial\theta_1\partial\theta_2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} & -\left.\frac{\partial^2 \ell nL}{\partial\theta_1\partial\alpha}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} & -\left.\frac{\partial^2 \ell nL}{\partial\theta_1\partial p}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} \\ & -\left.\frac{\partial^2 \ell nL}{\partial\theta_2^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} & -\left.\frac{\partial^2 \ell nL}{\partial\theta_2\partial\alpha}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} & -\left.\frac{\partial^2 \ell nL}{\partial\theta_2\partial p}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} \\ & & -\left.\frac{\partial^2 \ell nL}{\partial\alpha^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} & -\left.\frac{\partial^2 \ell nL}{\partial\alpha\partial p}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} \\ & & & -\left.\frac{\partial^2 \ell nL}{\partial p^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}} \end{bmatrix}^{-1}$$

$$\doteqdot \begin{bmatrix} v(\hat{\theta}_1) & cov(\hat{\theta}_1,\hat{\theta}_2) & cov(\hat{\theta}_1,\hat{\alpha}) & cov(\hat{\theta}_1,\hat{p}) \\ & v(\hat{\theta}_2) & cov(\hat{\theta}_2,\hat{\alpha}) & cov(\hat{\theta}_2,\hat{p}) \\ & & v(\hat{\alpha}) & cov(\hat{\alpha},\hat{p}) \\ & & & v(\hat{p}) \end{bmatrix}$$

4.4 Illustration

EXAMPLE 2 (α, $p=p_1=p_2$, θ_1, θ_2 unknown)

Using the same samples as in example 1, by treating α as unknown, we have $n = 150$, $T = 97$, $r_1 = 44$, $r_2 = 78$, $n-r_1-r_2 = 28$, $\bar{x} = 0.7059803$, $s^2 = 0.1617389$ and $cv = 0.5696586$. Reading from table 4.1, we have the moment estimates $p^* = 1.875$, $\alpha^* = 0.3025$, $\theta_1^* = \lambda_1^* \times T^{p^*} = 0.4175\times(97)^{1.875} = 2217.4460$ and $\theta_2^* = \lambda_2^* = T^{p^*} = 0.6175\times(97)^{1.875} = 3279.6950$. Using the method we mentioned in section 3.2 and 4.2, we replace $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\alpha}$ in equation (4.6) and (4.7) by θ_1^*, θ_2^* and α^*. Taking p^* as a starting point and iterating p with equation (4.6), we get $\hat{p}_0 = 1.8540$. Substituting it in equations (4.3), (4.4), (4.5) and (4.7), we have $\hat{\theta}_{01} = 2371.5840$, $\hat{\theta}_{02} = 3274.4970$ and $\hat{\alpha}_0 = 0.3350$. By successive iteration as discussed in Chapter 3, we obtain several tables similar to table 3.5. One such table is represented below.

TABLE 4.2 RECORD OF ITERATIONS WITH $\hat{p} = 1.854$

u	$\hat{k}_u$	$\hat{\theta}_{u1}$	$\hat{\theta}_{u2}$	$\hat{\alpha}_u$	v_u	$h(\hat{k}_u,\hat{p})$	D_u
0	0.2719	2520.9040	7449.2570	0.4231	4.8375	0.1713	-0.1006
1	0.0853	1947.8750	3513.5120	0.3093	6.7350	0.1293	0.0440
2	0.1993	2297.7920	3316.1230	0.3305	3.8597	0.2058	0.0065
3	0.2226	2369.3400	3275.7610	0.3349	3.4891	0.2228	0.0002
4	0.2233	2371.5840	3274.4970	0.3350	3.4783	0.2233	0.0000

Proceeding as in example (1), we obtain the final results tabulated below:

	Population values	Moment Estimates	MLE
p	2	1.8750	2.0284
α	0.3333	0.3025	0.3280
θ_1	4000	2217.4460	4746.0850
θ_2	6000	3279.6950	7156.5582

And the asymptotic variance-covariance matrix is given by

$$\begin{bmatrix} 0.995\times10^{6} & 0.421\times10^{-6} & -0.91406\times10^{-2} & -0.32592\times10^{-2} \\ & 0.1338\times10^{-5} & 0.4020\times10^{-2} & -0.50796\times10^{-1} \\ & & 0.59340\times10^{3} & 0.66844\times10^{2} \\ & & & 0.23169\times10^{4} \end{bmatrix}^{-1}$$

$$\doteq \begin{bmatrix} 6636591 & 8322003 & 14.82886 & 275.3815 \\ & 1621338 & -34.99499 & 473.5393 \\ & & 0.002221 & -0.000623 \\ & & & 0.014705 \end{bmatrix}$$

The variance-covariance matrix of the MLE in Mendenhall and Hader (1958) (i.e. $p = 1$ (known), $\hat{\alpha} = 0.3098$, $\hat{\theta}_1 = 234.7$ and $\hat{\theta}_2 = 355.2$) is

$$\begin{bmatrix} v(\hat{\theta}_1) & cov(\hat{\theta}_1,\hat{\theta}_2) & cov(\hat{\theta}_1,\hat{\alpha}) \\ & v(\hat{\theta}_2) & cov(\hat{\theta}_2,\hat{\alpha}) \\ & & v(\hat{\alpha}) \end{bmatrix}$$

$$= \begin{bmatrix} 1025.765 & -254.1632 & 0.236181 \\ & 607.6025 & -0.116986 \\ & & 0.000688 \end{bmatrix}$$

Assuming p unknown and $\hat{p} = 1$, the asymptotic variance-covariance matrix of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\alpha}$ and $\hat{p}$ turns out to be

$$\begin{bmatrix} v(\hat{\theta}_1) & cov(\hat{\theta}_1,\hat{\theta}_2) & cov(\hat{\theta}_1,\hat{\alpha}) & cov(\hat{\theta}_1,\hat{p}) \\ & v(\hat{\theta}_2) & cov(\hat{\theta}_2,\hat{\alpha}) & cov(\hat{\theta}_2,\hat{p}) \\ & & v(\hat{\alpha}) & cov(\hat{\alpha},p) \\ & & & v(\hat{p}) \end{bmatrix}$$

$$= \begin{bmatrix} 4773.125000 & 5957.4680 & 0.100379 & 3.034480 \\ & 10944.03000 & -0.342091 & 5.029965 \\ & & 0.000693 & -0.000110 \\ & & & 0.002457 \end{bmatrix}$$

Compared to our result in examples 3.1 and 4.1, the variances of the estimators appear to be small and this is due to the small value of $\hat{p}$ (see the comments on page 38).

CHAPTER 5

α, THE SHAPE PARAMETERS p_1, p_2 AND θ_1, θ_2 ALL UNKNOWN

In a previous study, the mixed Weibull distribution had been treated as

$$f(t|\theta_1,\theta_2,\alpha,p) = \alpha(\frac{p}{\theta_1})\ t^{p-1}\ \exp\left\{-\left(\frac{t^p}{\theta_1}\right)\right\}$$

$$+\ (1-\alpha)(\frac{p}{\theta_1})\ t^{p-1}\ \exp\left\{-\left(\frac{t^p}{\theta_2}\right)\right\}$$

In fact, one might consider that the shape parameter p may not be the same for the two subpopulations. Now suppose that $p_1 \neq p_2$, so the Weibull distribution may be presented as

$$f(t|\theta_1,\theta_2,\alpha,p_1,p_2) = \alpha\left(\frac{p_1}{\theta_1}\right) t^{p_1-1}\ \exp\left\{-\left(\frac{t^{p_1}}{\theta_1}\right)\right\}$$

$$+\ (1-\alpha)\left(\frac{p_2}{\theta_2}\right) t^{p_2-1}\ \exp\left\{-\left(\frac{t^{p_2}}{\theta_2}\right)\right\} \qquad (5.1)$$

5.1 The Moment Estimate of the Parameters

Since the s^{th} non-central moment is

$$\mu_s' = \frac{\alpha p_1}{\theta_1}\int_0^\infty x^{s+p_1-1}\ \exp\left\{-\left(\frac{x^{p_1}}{\theta_1}\right)\right\} dx$$

$$+ \frac{(1-\alpha)p_2}{\theta_2} \int_0^{\infty} x^{s+p_2-1} \exp\left\{-\left(\frac{x}{\theta_2}\right)^{p_2}\right\} dx$$

$$= \frac{\alpha s}{p_1} \Gamma(\frac{s}{p_1})(\theta_1)^{\frac{s}{p_1}} + \frac{(1-\alpha)s}{p_2} \Gamma(\frac{s}{p_2})(\theta_2)^{\frac{s}{p_2}} \quad (5.2)$$

That is, we have the first and second non-central moments:

$$\mu_1' = (\frac{\alpha}{p_1}) \Gamma(\frac{1}{p_1}) \theta_1^{\frac{1}{p_1}} + \frac{(1-\alpha)}{p_2} \Gamma(\frac{1}{p_2}) \theta_2^{\frac{1}{p_2}}$$

and

$$\mu_2' = (\frac{2\alpha}{p_1}) \Gamma(\frac{2}{p_2}) \theta_1^{\frac{2}{p_1}} + \frac{2(1-\alpha)}{p_2} \Gamma(\frac{2}{p_2}) \theta_2^{\frac{2}{p_2}}$$

So the variance is

$$v(x) = \mu_2' - (\mu_1')^2 = \frac{2\alpha}{p_1} \Gamma(\frac{2}{p_1}) \theta_1^{\frac{2}{p_1}} + \frac{2(1-\alpha)}{p_2} \Gamma(\frac{2}{p_2}) \theta_2^{\frac{2}{p_2}}$$

$$- \left[\frac{\alpha}{p_1} \Gamma(\frac{1}{p_1}) \theta_1^{\frac{1}{p_1}} + \frac{(1-\alpha)}{p_2} \Gamma(\frac{1}{p_2}) \theta_2^{\frac{1}{p_2}}\right]^2$$

Then the coefficient of variation is

$$cv = \left(\frac{v(x)}{\mu^2}\right)^{\frac{1}{2}} = \left(\frac{\frac{2\alpha}{p_1}\Gamma(\frac{2}{p_1})\,\theta_1^{\frac{2}{p_1}} + \frac{2(1-\alpha)}{p_2}\Gamma(\frac{2}{p_2})\,\theta_2^{\frac{2}{p_2}}}{\left[\frac{\alpha}{p_1}\Gamma(\frac{1}{p_1})\,\theta_1^{\frac{1}{p_1}} + \frac{(1-\alpha)}{p_2}\Gamma(\frac{1}{p_2})\,\theta_2^{\frac{1}{p_2}}\right]^2} - 1 \right)^{\frac{1}{2}}$$

The abridged table of the coefficient of variation corresponding to $p_1, p_2, \alpha, \theta_1$ and θ_2 is given in Table 5.1.

When the sample coefficient of variation is obtained, the corresponding values $p_1^*, p_2^*, \alpha^*, \theta_1^*$ and θ_2^* can be obtained by interpolation.

TABLE 5.1 THE MIXED WEIBULL COEFFICIENT OF VARIATION

$(\lambda_i = \theta_i/T^{p_i})$

coefficient of variation	p_1	p_2	α	λ_1	λ_2
0.92733	1.1	1.1	0.300	0.400	0.56
0.94875	1.1	1.0	0.300	0.400	0.56
0.95048	1.1	1.0	0.305	0.395	0.56
0.97501	1.1	0.9	0.305	0.400	0.56
0.99844	1.0	1.1	0.305	0.400	0.56
1.00003	1.0	1.1	0.300	0.395	0.56
1.02055	1.0	1.0	0.305	0.400	0.56
1.02197	1.0	1.0	0.305	0.395	0.56
1.04570	1.0	0.9	0.300	0.400	0.56
1.04770	1.0	0.9	0.305	0.395	0.56
1.08996	0.9	1.1	0.300	0.400	0.56
1.09176	0.9	1.1	0.300	0.395	0.56
1.09255	0.9	1.1	0.305	0.400	0.57
1.09306	0.9	1.1	0.305	0.400	0.57
1.09441	0.9	1.1	0.305	0.395	0.57
1.11193	0.9	1.0	0.305	0.400	0.56

1.11209	0.9	1.00	0.300	0.400	0.56
1.11374	0.9	1.00	0.305	0.395	0.56
1.11477	0.9	1.00	0.305	0.400	0.57
1.11662	0.9	1.00	0.305	0.395	0.57
1.13776	0.9	0.90	0.300	0.400	0.56
1.13945	0.9	0.90	0.300	0.395	0.56
1.16015	0.9	0.90	0.300	0.400	0.57
1.14189	0.9	0.90	0.300	0.395	0.57
1.14226	0.9	0.90	0.305	0.395	0.57

cv was computed for assigned values of p_1, p_2, θ_1, θ_2, and α . The table 5.1 was set up in such a way that coefficients of variation are monotonic.

5.2 Maximum Likelihood Estimation of the Parameters p_1, p_2, α, θ_1 and θ_2

$$\ell n L = \ell n \frac{n!}{(n-r)!} + (n-r)\,\ell n\left[\alpha \exp\left\{-\left(\frac{T^{p_1}}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^{p_2}}{\theta_2}\right)\right\}\right]$$

$$+ r_1 \ell n\alpha + r_2 \ell n(1-\alpha)$$

$$+ \sum_{j=1}^{r_1}\left[\ell n p_1 - \ell n\theta_1 + (p_1-1)\ell n(x_{1j}) + p_1 \ell n T - \frac{(x_{1j}T)^{p_1}}{\theta_1}\right]$$

$$+ \sum_{j=1}^{r_2}\left[\ell n p_2 - \ell n\theta_2 + (p_2-1)\ell n(x_{2j}) + p_2 \ell n T - \frac{(x_{2j}T)^{p_2}}{\theta_2}\right] \qquad (5.4)$$

Differentiating (5.2) with respect to θ_1, θ_2, α, p_1 and p_2 in turn, then we have

$$\frac{\partial \ln L}{\partial \theta_1} = \frac{k(n-r)T^{p_1}}{\theta_1^2} - \frac{r_1}{\theta_1} + \frac{\sum_{j=1}^{r_1} (x_{1j}T)^{p_1}}{\theta_1^2} \tag{5.5}$$

$$\frac{\partial \ln L}{\partial \theta_2} = \frac{(1-k)(n-r)T^{p_2}}{\theta_2^2} - \frac{r_2}{\theta_2} + \frac{\sum_{j=1}^{r_2} (x_{2j}T)^{p_2}}{\theta_2^2} \tag{5.6}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{k(n-r) + r_1}{\alpha} - \frac{(1-k)(n-r) + r_2}{\beta} \tag{5.7}$$

$$\frac{\partial \ln L}{\partial p_1} = -\frac{k(n-r)T^{p_1}\ln T}{\theta_1} + \frac{r_1}{p_1} + \sum_{j=1}^{r_1} \ln(x_{1j}T)\left[1 - \frac{(x_{1j}T)^{p_1}}{\theta_1}\right] \tag{5.8}$$

$$\frac{\partial \ln L}{\partial p_2} = \frac{-(1-k)(n-r)T^{p_2}\ln T}{\theta_2} + \frac{r_2}{p_2} + \sum_{j=1}^{r_2} \ln(x_{2j}T)\left[1 - \frac{(x_{2j}T)^{p_2}}{\theta_2}\right] \tag{5.9}$$

where
$$k = \frac{1}{1 + \left(\frac{\beta}{\alpha}\right) \exp\left\{\left(\frac{T^{p_1}}{\theta_1}\right) - \left(\frac{T^{p_2}}{\theta_2}\right)\right\}} \tag{5.10}$$

When the partial derivatives are equated to zero, the estimating equations are:

$$\hat{\alpha} = \frac{r_1}{n} + \hat{k}\,\frac{(n-r)}{n} \tag{5.11}$$

$$\hat{\theta}_1 = \frac{\hat{k}(n-r)T^{\hat{p}_1}}{r_1} + \frac{\sum_{j=1}^{r_1} (x_{1j}.T)^{\hat{p}_1}}{r_1} \tag{5.12}$$

$$\hat{\theta}_2 = \frac{(1-\hat{k})(n-r)T^{\hat{p}_2}}{r_2} + \frac{\sum_{j=1}^{r_2} (x_{2j}.T)^{\hat{p}_2}}{r_2} \tag{5.13}$$

And $\hat{p}_1$, $\hat{p}_2$ are the solution of

$$g_1(\hat{p}_1) \equiv \frac{-(n-r)\hat{k}T^{\hat{p}_1}(\ell nT)}{\hat{\theta}_1} + \frac{r_1}{\hat{p}_1}$$

$$+ \sum_{j=1}^{r_1} \ell n(x_{1j}.T) \left[1 - \frac{(x_{1j}.T)^{\hat{p}_1}}{\hat{\theta}_1}\right] = 0 \tag{5.14}$$

$$g_2(\hat{p}_2) \equiv \frac{(n-r)(1-\hat{k})T^{\hat{p}_2}(\ell nT)}{\hat{\theta}_2} + \frac{r_2}{\hat{p}_2}$$

$$+ \sum_{j=1}^{r_2} \ell n(x_{2j}.T) \left[1 - \frac{(x_{2j}.T)^{\hat{p}_2}}{\hat{\theta}_2}\right] = 0 \tag{5.15}$$

respectively, where

$$\hat{k} = \frac{1}{1 + \frac{\hat{\beta}}{\hat{\alpha}} \exp\left\{\left(\frac{T^{\hat{p}_1}}{\hat{\theta}_1}\right) - \left(\frac{T^{\hat{p}_2}}{\hat{\theta}_2}\right)\right\}} \tag{5.16}$$

We will use techniques similar to Chapter 3 and 4 for finding the MLE of α, θ_1, θ_2, p_1 and p_2. We keep p_2^*, θ_1^*, θ_2^* and α^* as constant, substitute them in the equation (5.10), and iterate for p_1 using p_1^* as a starting value. We use this iterated result $\hat{p}_{01}$, say, together with θ_1^*, θ_2^* and α^*, substituting them in the equation (5.15) and iterate for p_2 using p_2^* as a starting value. We continue in this manner until we get the values of p_1 and p_2 such that they both satisfy the equations (5.14) and (5.15). This kind of looping will take quite a lot of computer time. Once these two values are obtained, we may obtain θ_1, θ_2 and α by using the similar technique that we discussed in section 3.2 and section 4.2. It is quite difficult and time consuming to get these five estimates which satisfy equation (5.11), (5.12), (5.13), (5.14) and (5.15).

In choosing p_1 or p_2 to start first, we prefer to use p_1 in subpopulation (1) which is defined in Chapter 3 (That is, take the minimum value of these two sample means and define it as a first subpopulation).

5.3 Asymptotic Variance-covariance Matrix for $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\alpha}$, $\hat{p}_1$ and $\hat{p}_2$

The asymptotic variance-covariance matrix of $(\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{p}_1, \hat{p}_2)$ is approximated by

$$
\begin{bmatrix}
a(\hat{\theta}_1^2) & a(\hat{\theta}_1,\hat{\theta}_2) & a(\hat{\theta}_1,\hat{\alpha}) & a(\hat{\theta}_1,\hat{p}_1) & a(\hat{\theta}_1,\hat{p}_2) \\
 & a(\hat{\theta}_2^2) & a(\hat{\theta}_2,\hat{\alpha}) & a(\hat{\theta}_2,\hat{p}_1) & a(\hat{\theta}_2,\hat{p}_2) \\
 & & a(\hat{\alpha}^2) & a(\hat{\alpha},\hat{p}_1) & a(\hat{\alpha},\hat{p}_2) \\
 & & & a(\hat{p}_1^2) & a(\hat{p}_1,\hat{p}_2) \\
 & & & & a(\hat{p}_2^2)
\end{bmatrix}^{-1}
$$

$$
= \begin{bmatrix}
v(\hat{\theta}_1) & cov(\hat{\theta}_1,\hat{\theta}_2) & cov(\hat{\theta}_1,\hat{\alpha}) & cov(\hat{\theta}_1,\hat{p}_1) & cov(\hat{\theta}_1,\hat{p}_2) \\
 & v(\hat{\theta}_2) & cov(\hat{\theta}_2,\hat{\alpha}) & cov(\hat{\theta}_2,\hat{p}_1) & cov(\hat{\theta}_2,\hat{p}_2) \\
 & & v(\hat{\alpha}) & cov(\hat{\alpha},\hat{p}_1) & cov(\hat{\alpha},\hat{p}_2) \\
 & & & v(\hat{p}_1) & cov(\hat{p}_1,\hat{p}_2) \\
 & & & & v(\hat{p}_2)
\end{bmatrix}
\tag{5.17}
$$

where

$$
a(\hat{\theta}_1^2) = - \left. \frac{\partial^2 \ell nL}{\partial \theta_1^2} \right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}
$$

$$
= - \left[\frac{\hat{k}(1-\hat{k})(n-r)T^{2\hat{p}_1}}{\hat{\theta}_1^4} - \frac{2\hat{k}(n-r)T^{\hat{p}_1}}{\hat{\theta}_1^3} + \frac{r_1}{\hat{\theta}_1^2} \right.
$$

$$-\left(\frac{2}{\hat{\theta}_1^3}\right)\sum_{j=1}^{r_1}(x_{1j}T)^{\hat{p}_1}\Bigg] \qquad (5.18)$$

$$a(\hat{\theta}_2^2) = -\left.\frac{\partial^2 \ell nL}{\partial\theta_2^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= -\left[\frac{\hat{k}(1-\hat{k})(n-r)T^{2\hat{p}_2}}{\hat{\theta}_2^4} - \frac{2(1-\hat{k})(n-r)T^{\hat{p}_2}}{\hat{\theta}_2^3} + \frac{r_2}{\hat{\theta}_2^2}\right.$$

$$-\left(\frac{2}{\hat{\theta}_2^3}\right)\sum_{j=1}^{r_2}(x_{2j}T)^{\hat{p}_2}\Bigg] \qquad (5.19)$$

$$a(\hat{\alpha}^2) = -\left.\frac{\partial^2 \ell nL}{\partial\alpha^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= -\left[\frac{k(1-k)(n-r)}{\alpha^2\beta^2}\right.$$

$$-\frac{k(n-r)+r_1}{\alpha^2} - \frac{(1-k)(n-r)+r_2}{\beta^2}\Bigg] \qquad (5.20)$$

$$a(\hat{p}_1^2) = -\left.\frac{\partial^2 \ell nL}{\partial p_1^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= \frac{\hat{k}(n-r)T^{\hat{p}_1}(\ell nT)^2}{\hat{\theta}_1}\left[1 - \frac{(1-\hat{k})T^{\hat{p}_1}}{\hat{\theta}_1}\right] + \frac{r_1}{\hat{p}_1^2}$$

$$+ \left(\frac{1}{\hat{\theta}_1}\right)\sum_{j=1}^{r_1}[\ell n(x_{1j}T)]^2\,(x_{1j}T)^{\hat{p}_1} \tag{5.21}$$

$$a(\hat{p}_2^2) = -\left.\frac{\partial^2 \ell nL}{\partial p_2^2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= \frac{(1-\hat{k})(n-r)T^{\hat{p}_2}(\ell nT)^2}{\hat{\theta}_2}\left[1 - \frac{\hat{k}T^{\hat{p}_2}}{\hat{\theta}_2}\right] + \frac{r_2}{\hat{p}_2^2}$$

$$+ \left(\frac{1}{\hat{\theta}_2}\right)\sum_{j=1}^{r_2}[\ell n(x_{2j}T)]^2\,(x_{2j}T)^{\hat{p}_2} \tag{5.22}$$

$$a(\hat{\theta}_1,\hat{\theta}_2) = -\left.\frac{\partial^2 \ell nL}{\partial \theta_1 \partial \theta_2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= \frac{\hat{k}(1-\hat{k})(n-r)T^{\hat{p}_1+\hat{p}_2}}{\hat{\theta}_1^2\ \hat{\theta}_2^2}$$

$$= \left.\frac{\partial^2 \ell nL}{\partial\theta_1\partial\theta_2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2} \qquad (5.23)$$

$$a(\hat{\theta}_1,\hat{\alpha}) = -\left.\frac{\partial^2 \ell nL}{\partial\theta_1\partial\alpha}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= -\left[\frac{\hat{k}(1-\hat{k})(n-r)T^{\hat{p}_1}}{\hat{\alpha}\hat{\beta}\hat{\theta}_1^2}\right]$$

$$= -\left.\frac{\partial^2 \ell nL}{\partial\alpha\partial\theta_1}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2} \qquad (5.24)$$

$$a(\hat{\theta}_1,\hat{p}_1) = -\left.\frac{\partial^2 \ell nL}{\partial\theta_1\partial p_1}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= -\left[\frac{\hat{k}(n-r)T^{\hat{p}_1}\ell nT}{\hat{\theta}_1^2}\left(1 - \frac{(1-\hat{k})T^{\hat{p}_1}}{\hat{\theta}_1}\right)\right.$$

$$+ \left(\frac{1}{\hat{\theta}_1^2}\right) \sum_{j=1}^{r_1} (x_{1j}T)^{\hat{p}_1} \ell n(x_{1j}T) \Bigg]$$

$$= - \left. \frac{\partial^2 \ell nL}{\partial p_1 \partial \theta_1} \right|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{p}_1, \hat{p}_2} \qquad (5.25)$$

$$a(\hat{\theta}_1, \hat{p}_2) = - \left. \frac{\partial^2 \ell nL}{\partial \theta_1 \partial p_2} \right|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{p}_1, \hat{p}_2}$$

$$= - \left[\frac{(n-r)T^{(\hat{p}_1+\hat{p}_2)} (\ell nT) \hat{k}(1-\hat{k})}{\hat{\theta}_1^2 \; \hat{\theta}_2} \right]$$

$$= - \left. \frac{\partial^2 \ell nL}{\partial p_2 \partial \theta_1} \right|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{p}_1, \hat{p}_2} \qquad (5.26)$$

$$a(\hat{\theta}_2, \hat{\alpha}) = - \left. \frac{\partial^2 \ell nL}{\partial \theta_2 \partial \alpha} \right|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{p}_1, \hat{p}_2}$$

$$= \frac{\hat{k}(1-\hat{k})(n-r)T^{\hat{p}_2}}{\hat{\alpha}\hat{\beta}\hat{\theta}_2^2}$$

$$= - \left.\frac{\partial^2 \ell n L}{\partial\alpha\partial\theta_2}\right|_{\hat\theta_1,\hat\theta_2,\hat\alpha,\hat p_1,\hat p_2} \tag{5.27}$$

$$a(\hat\theta_2,\hat p_1) = - \left.\frac{\partial^2 \ell n L}{\partial\theta_2\partial p_1}\right|_{\hat\theta_1,\hat\theta_2,\hat\alpha,\hat p_1,\hat p_2}$$

$$= - \left[\frac{\hat k(1-\hat k)(n-r)T^{(\hat p_1+\hat p_2)}(\ell nT)}{\hat\theta_1\ \hat\theta_2^2}\right]$$

$$= - \left.\frac{\partial^2 \ell n L}{\partial p_1\partial\theta_2}\right|_{\hat\theta_1,\hat\theta_2,\hat\alpha,\hat p_1,\hat p_2} \tag{5.28}$$

$$a(\hat\theta_2,\hat p_2) = - \left.\frac{\partial^2 \ell n L}{\partial\theta_2\partial p_2}\right|_{\hat\theta_1,\hat\theta_2,\hat\alpha,\hat p_1,\hat p_2}$$

$$= - \left[\frac{(n-r)T^{\hat p_2}(\ell nT)(1-\hat k)}{\hat\theta_2^2}\left(1 - \frac{\hat k T^{\hat p_2}}{\hat\theta_2}\right)\right.$$

$$\left. + \left(\frac{1}{\hat\theta_2^2}\right)\sum_{j=1}^{r_2}(x_{2j}T)^{\hat p_2}\ \ell n(x_{2j}T)\right]$$

$$= - \left. \frac{\partial^2 \ell nL}{\partial p_2 \partial \theta_2} \right|_{\hat\theta_1, \hat\theta_2, \hat\alpha, \hat p_1, \hat p_2} \tag{5.29}$$

$$a(\hat\alpha, \hat p_1) = - \left. \frac{\partial^2 \ell nL}{\partial \alpha \partial p_1} \right|_{\hat\theta_1, \hat\theta_2, \hat\alpha, \hat p_1, \hat p_2}$$

$$= \frac{\hat k (1-\hat k)(n-r) T^{\hat p_1} \ell nT}{\hat\theta_1} \quad \left(\frac{1}{\hat\alpha} + \frac{1}{\hat\beta}\right)$$

$$= - \left. \frac{\partial^2 \ell nL}{\partial p_1 \partial \alpha} \right|_{\hat\theta_1, \hat\theta_2, \hat\alpha, \hat p_1, \hat p_2} \tag{5.30}$$

$$a(\hat\alpha, \hat p_2) = - \left. \frac{\partial^2 \ell nL}{\partial \alpha \partial p_2} \right|_{\hat\theta_1, \hat\theta_2, \hat\alpha, \hat p_1, \hat p_2}$$

$$= - \left[\frac{\hat k (1-\hat k)(n-r) T^{\hat p_2} \ell nT}{\hat\theta_2} \quad \left(\frac{1}{\hat\alpha} + \frac{1}{\hat\beta}\right) \right]$$

$$= - \left. \frac{\partial^2 \ell nL}{\partial p_2 \partial \alpha} \right|_{\hat\theta_1, \hat\theta_2, \hat\alpha, \hat p_1, \hat p_2} \tag{5.31}$$

$$a(\hat{p}_1,\hat{p}_2) = \left.\frac{\partial^2 \ell nL}{\partial p_1 \partial p_2}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2}$$

$$= \frac{\hat{k}(1-\hat{k})(n-r)T^{\hat{p}_1+\hat{p}_2}(\ell nT)^2}{\hat{\theta}_1\ \hat{\theta}_2}$$

$$= -\left.\frac{\partial^2 \ell nL}{\partial p_2 \partial p_1}\right|_{\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha},\hat{p}_1,\hat{p}_2} \qquad (5.32)$$

5.4 Illustration

We assigned $\theta_1 = 250$, $\theta_2 = 375$, $\alpha = 0.3333$, $p_1 = 1$, $p_2 = 1$ and $T = 650$ hours. We generated 50 observations and 100 observations from subpopulations (1) and (2) as the same in section 3.4 and 4.4 and we get the following samples:

TABLE 5.2 THE SAMPLE FROM SUBPOPULATION (1) $(n_1=50)$

211.4832	175.4812	50.1170	561.4687	617.2304
53.2834	99.7688	45.1315	83.2134	901.3647
127.8581	153.5822	393.0839	7.5080	247.0877
285.2824	20.2458	183.0842	207.1253	260.2941
547.3520	349.8488	370.9638	95.2040	27.6767
87.6436	910.9799	108.1031	251.9786	550.6755
44.9812	28.5336	121.5087	131.8994	460.7363
269.5200	40.8359	98.3575	294.4521	137.1630
548.7194	104.5854	5.1163	336.3872	1079.6620
70.8535	187.3612	353.6635	4.5376	13.2642

TABLE 5.3 THE SAMPLE FROM SUBPOPULATION (2) (n_2=100)

198.9965	198.2433	653.5061	558.2910	231.7297
12.6528	31.0559	410.5151	1096.6160	750.7640
385.8967	1337.1510	920.9221	1347.0810	53.2347
209.8570	163.8347	110.4131	119.6205	632.0517
192.7241	721.0405	55.1178	82.3419	72.5529
199.5567	446.7805	431.8437	62.2858	148.9624
168.4152	362.5832	141.8717	521.0239	239.3262
92.4667	110.2134	637.5363	381.9235	844.9895
431.2675	585.1977	347.8881	244.6254	75.5520
8.2283	124.9406	64.6036	144.1181	382.5886
258.4177	284.1938	918.0566	430.9328	14.4903
567.1623	117.1270	191.2685	94.0527	58.4981
218.8362	1991.4110	88.5051	228.2120	335.3662
62.1384	630.1501	27.5493	102.5474	71.1792
112.3546	58.7085	320.6489	514.8305	95.2729
290.7202	68.1725	257.5600	295.6079	587.4726
149.1163	704.4907	601.5539	513.1872	446.7385
71.8515	747.3177	181.0456	37.7467	17.7401
87.6151	320.8911	122.7632	111.9181	209.1385
1101.2350	70.1238	721.5310	219.8885	363.9187

EXAMPLE 3

Using the above samples, we summarized the data as below: $n = n_1 + n_2 = 150$, $T = 100$, $r_1 = 47$. $r_2 = 86$ and $n-r_1-r_2 = 17$. The mixed sample mean and variance are $\bar{x} = 0.4731393$, $s^2 = 0.2334213$. The coefficient variation is $cv = 1.021129$. Reading from table 5.1, we have the moment estimates: $p_1^* = 1.000$, $p_2^* = 1.000$, $\alpha^* = 0.3025$, $\theta_1^* = \lambda_1^* \times T^{p_1^*} = 0.3975 \times (650)^{1.000} = 258.3747$ and $\theta_2^* = \lambda_2^* \times T^{p_2^*} = 0.56 \times (650)^{1.000} = 363.9997$. We keep p_2^*, θ_1^*, θ_2^* and α^* as constants, substitue them in equations (5.14) and (5.16); using p_1^* as a starting point we iterate p_1 in equation (5.14); we then treat the solution constant, substitute it in (5.15) and (5.16), together with θ_1^*, θ_2^* and α^*; now using p_2^* as a starting point, we iterate p_2 in equation (5.15), substitute it in equations (5.14) and (5.16), use the previous solution for p_1 as a starting point and iterate p_1 again in equation (5.14). We continue this procedure until we obtain estimates of p_1, p_2 which satisfy the equations (5.14) and (5.15) (with θ_1^*, θ_2^*, α^* substituted for this corresponding parameters). After we get the values $\hat{p}_{01} = 1.010995$, $\hat{p}_{02} = 1.014502$, put them in equations (5.11), (5.12), (5.13) and (5.16). Repeating the procedure as in example (1) and (2), we have the first set of the following results.

TABLE 5.4 RECORD OF ITERATIONS WITH $\hat{p}_1$=1.010995, $\hat{p}_2$=1.014502

u	$\hat{k}_u$	$\hat{\theta}_{u1}$	$\hat{\theta}_{u2}$	$\hat{\alpha}_u$	v_u	$h(\hat{k}_u,\hat{p}_1\hat{p}_2)$	D_u
0	0.2260	270.4673	1780.0320	0.3390	17.2444	0.0548	-0.1712
1	0.0240	219.4484	389.9832	0.3160	8.3455	0.1070	0.0830
2	0.1786	285.4924	368.1550	0.3336	4.2751	0.1896	0.0110
3	0.2058	265.3657	364.3123	0.3367	3.8519	0.2061	0.0003
4	0.2065	265.5383	364.2158	0.3367	3.8420	0.2065	0.0000

We continued the same iterations until such 5 estimators satisfied the equations (5.11) — (5.15). The final results are listed below:

	Population Values	Moment Estimates	MLE
p_1	1.0000	1.0000	0.981898
p_2	1.0000	1.0000	1.031562
α	0.3333	0.3025	0.3402
θ_1	250.00	258.3747	230.6272
θ_2	375.000	363.9997	398.4226

In order to estimate the variance-covariance matrix, we evaluate the partial derivatives (5.18) — (5.32) and the desired matrix follows as:

$$\begin{bmatrix} v(\hat{\theta}_1) & cov(\hat{\theta}_1,\hat{\theta}_2) & cov(\hat{\theta}_1,\hat{\alpha}) & cov(\hat{\theta}_1,\hat{p}_1) & cov(\hat{\theta}_1,\hat{p}_2) \\ & v(\hat{\theta}_2) & cov(\hat{\theta}_2,\hat{\alpha}) & cov(\hat{\theta}_2,\hat{p}_1) & cov(\hat{\theta}_2,\hat{p}_2) \\ & & v(\hat{\alpha}) & cov(\hat{\alpha},\hat{p}_1) & cov(\hat{\alpha},\hat{p}_2) \\ & & & v(\hat{p}_1) & cov(\hat{p}_1,\hat{p}_2) \\ & & & & v(\hat{p}_2) \end{bmatrix}$$

$$= \begin{bmatrix} 3388.723 & -1995.220 & 1.074998 & -0.001988 & 0.007728 \\ & 4454.6090 & -1.43448 & 0.031931 & 0.007074 \\ & & 0.002051 & -0.000001 & -0.000012 \\ & & & 0.000000 & 0.000011 \\ & & & & 0.000000 \end{bmatrix}$$

Setting $\hat{p}_1 = 1$ and $\hat{p}_2 = 1$ in Mendenhall and Hader's paper we have the asymptotic variance-covariance matrix for $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\alpha}$, $\hat{p}_1$ and $\hat{p}_2$ as follows:

$$\begin{bmatrix} v(\hat\theta_1) & cov(\hat\theta_1,\hat\theta_2) & cov(\hat\theta_1,\hat\alpha) & cov(\hat\theta_1,\hat p_1) & cov(\hat\theta_1,\hat p_2) \\ & v(\hat\theta_2) & cov(\hat\theta_2,\hat\alpha) & cov(\hat\theta_2,\hat p_1) & cov(\hat\theta_2,\hat p_2) \\ & & v(\hat\alpha) & cov(\hat\alpha_1,\hat p_1) & cov(\hat\alpha_1,\hat p_2) \\ & & & v(\hat p_1) & cov(\hat p_1,\hat p_2) \\ & & & & v(\hat p_2) \end{bmatrix}$$

$$= \begin{bmatrix} 850.1950 & 52.00306 & 0.03741 & 0.001559 & 0.041910 \\ & 1629.6560 & -0.10827 & 0.09299 & 0.00403 \\ & & 0.000644 & -0.000003 & -0.000019 \\ & & & 0.000002 & 0.000026 \\ & & & & 0.000001 \end{bmatrix}$$

The variances of the estimators increase considerably as p_1, p_2 increase (in our case $p_1 = 1.00$, $p_2 = 1.00$. See the comments on page 38).

Note that all the procedures for solving the moment estimates, MLE and their asymptotic variance-covariance matrix have been programmed and are given in the appendix.

The data had been generated by using the cumulative distribution

$$y = F(x) = 1 - \exp\{-(\frac{x^p}{\theta})\}$$

(with no loss of generality, we may set $p = \theta = 1$) .

It follows that $x \to \infty$ as $y \to 1$ and $x \to 0$ as $y \to 0$. So in order to generate a representative sample, one should be careful that neither too many large nor too many small y's are used.

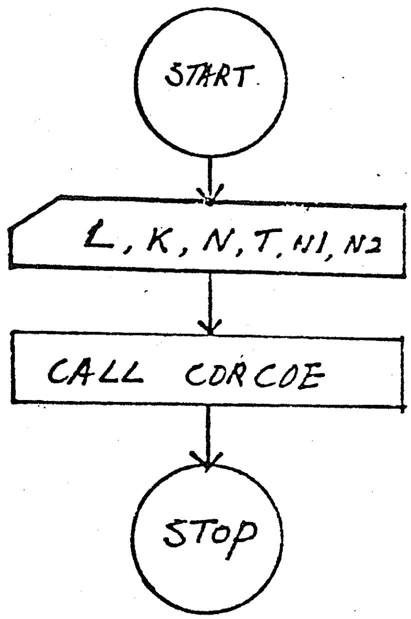
Tables for the mixed Weibull coefficient of variation (cv), tables 3.1, 4.1 and 5.1 are abridged because the value of the parameters can take on values on the whole real line. To suit the example used, and for convenience, we restrict them to closed intervals.

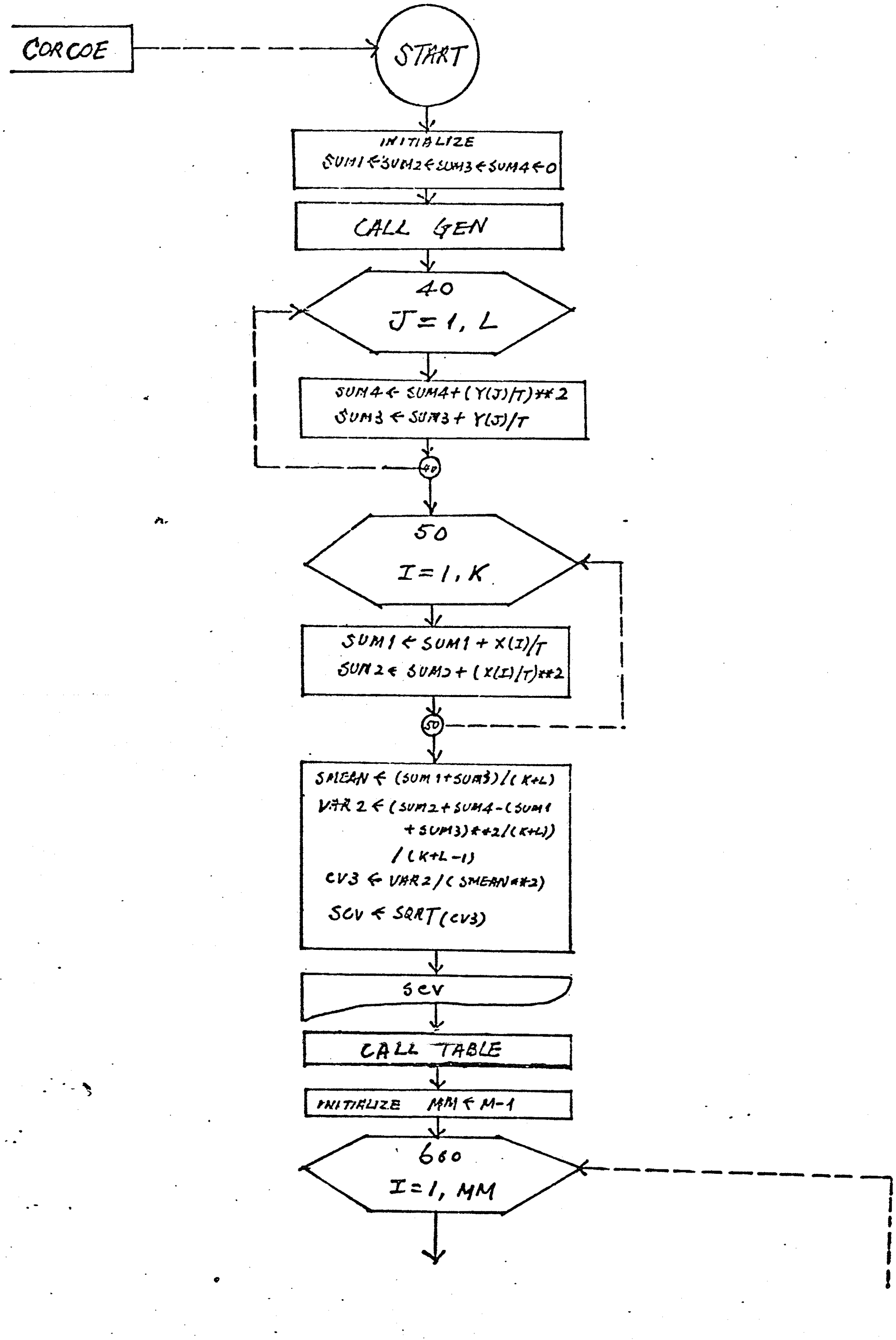
These tables give the values of the cv associated with each possible combination of various values of the parameters. To illustrate how, without any loss of generality and for simplicity, take table 3.1. Let $1.70 \le p \le 2.10$, $0.300 \le \lambda_1 \le 0.600$ and $0.350 \le \lambda_2 \le 0.650$. We first keep p and λ_1 at their respective lower bounds and increase λ_2 from 0.350 to 0.650 by 0.05 (Remark: the magnitude of the increment depends on the degree of precision desired). We then increase λ_1 again by 0.05, and start λ_2 from 0.350 to 0.650. We repeat this process until $\theta_1 = 0.6000$, and each cv is calculated. Now, we increase p by, say, 0.05 again and obtain a whole new series of cv. The whole procedure is repeated until we find the cv with $p = 0.50(0.1)2.10$.

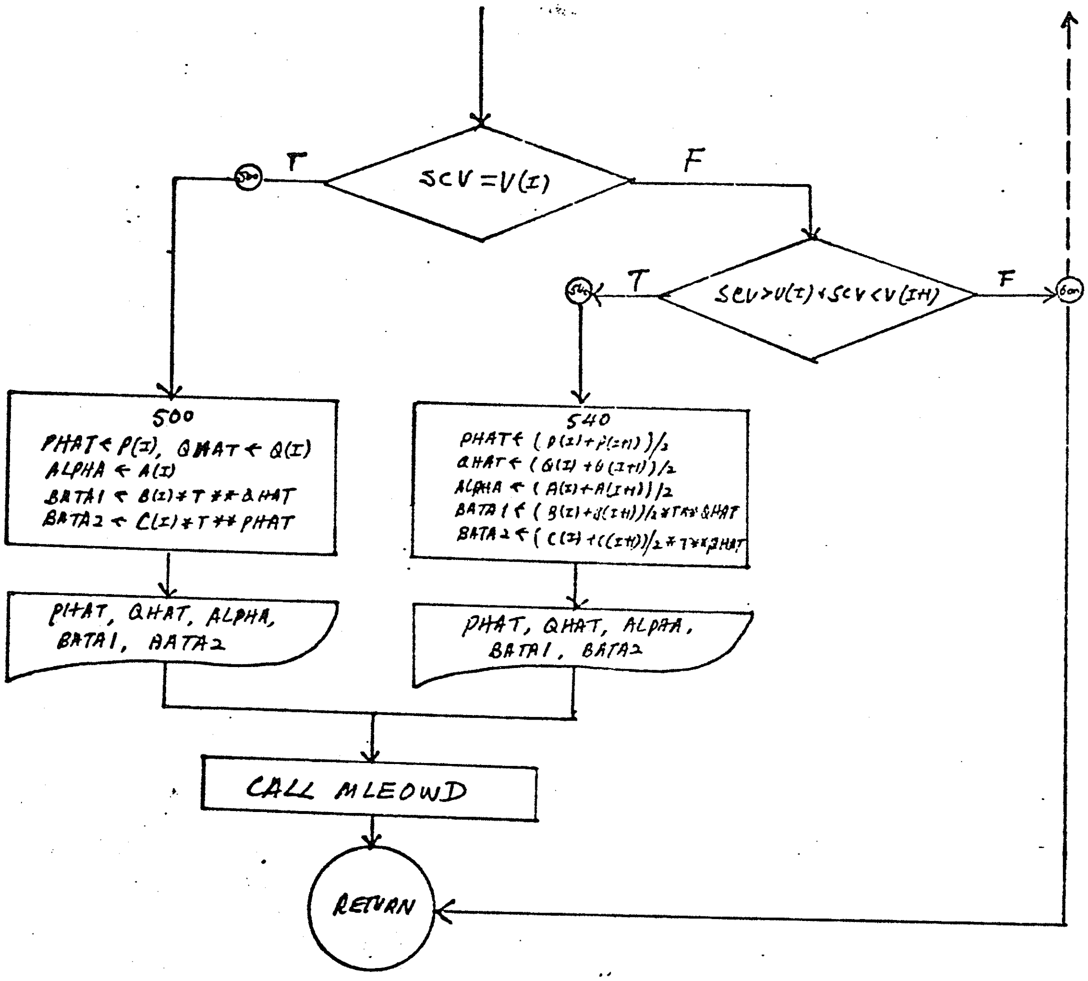
There is room for further investigation in this problem and I hope to continue working on this project for a while.

APPENDIX 1

THE FLOWCHARTS




CORCOE
START
INITIALIZE
SUM1 ← SUM2 ← SUM3 ← SUM4 ← 0
CALL GEN
40
J = 1, L
SUM4 ← SUM4 + (Y(J)/T)**2
SUM3 ← SUM3 + Y(J)/T
40
50
I = 1, K
SUM1 ← SUM1 + X(I)/T
SUM2 ← SUM2 + (X(I)/T)**2
50
SMEAN ← (SUM1+SUM3)/(K+L)
VAR2 ← (SUM2+SUM4-(SUM1 +SUM3)**2/(K+L)) /(K+L-1)
CV3 ← VAR2/(SMEAN**2)
SCV ← SQRT(CV3)
SCV
CALL TABLE
INITIALIZE MM ← M-1
660
I = 1, MM


SCV = V(I)
T
F
SCV>V(I) & SCV<V(I+1)
T
F
500
PHAT ← P(I), QHAT ← Q(I)
ALPHA ← A(I)
BATA1 ← B(I)*T**QHAT
BATA2 ← C(I)*T**PHAT
PHAT, QHAT, ALPHA,
BATA1, BATA2
540
PHAT ← (P(I)+P(I+1))/2
QHAT ← (Q(I)+Q(I+1))/2
ALPHA ← (A(I)+A(I+1))/2
BATA1 ← (B(I)+B(I+1))/2*T**QHAT
BATA2 ← (C(I)+C(I+1))/2*T**PHAT
PHAT, QHAT, ALPHA,
BATA1, BATA2
CALL MLEOWD
RETURN

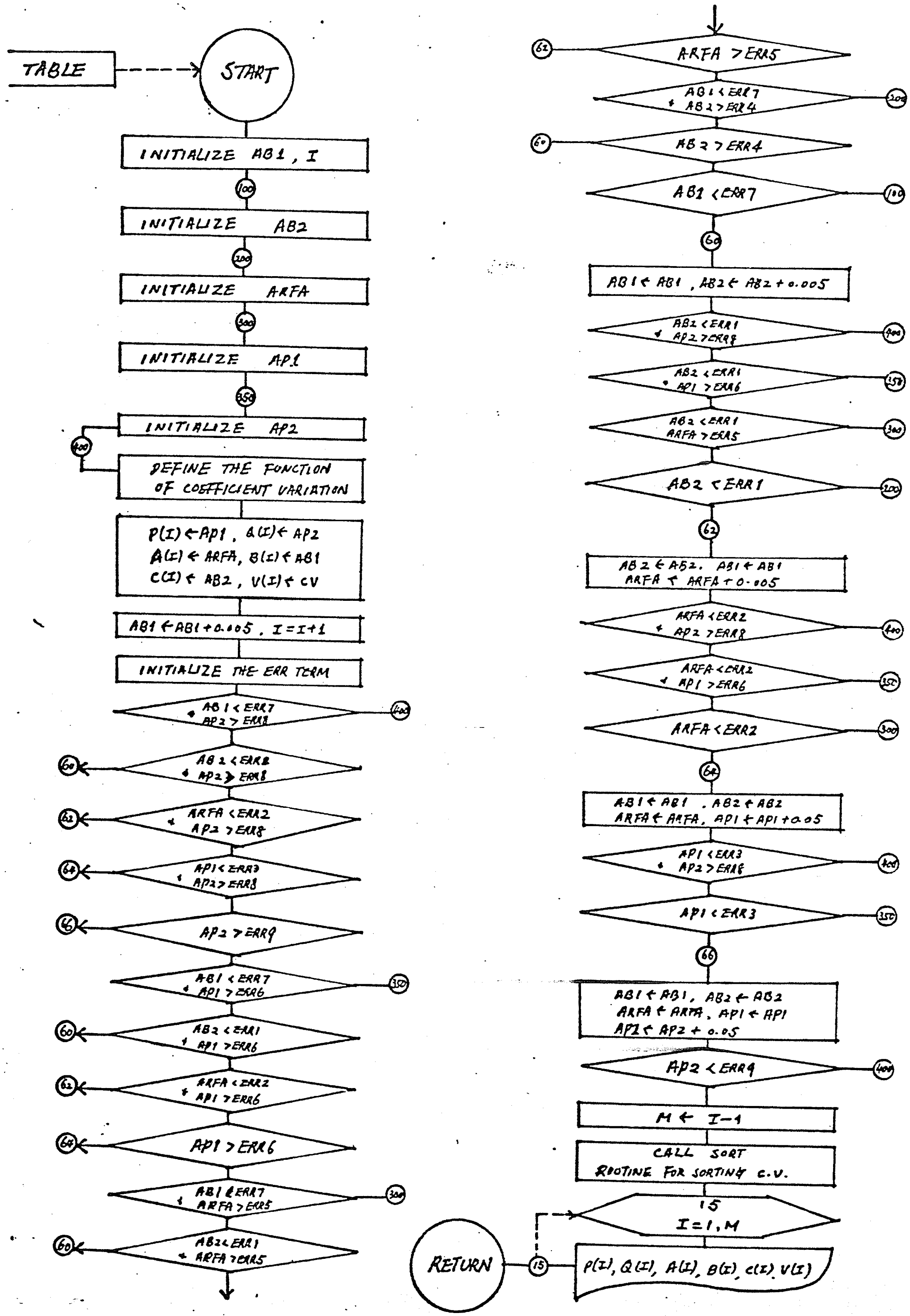
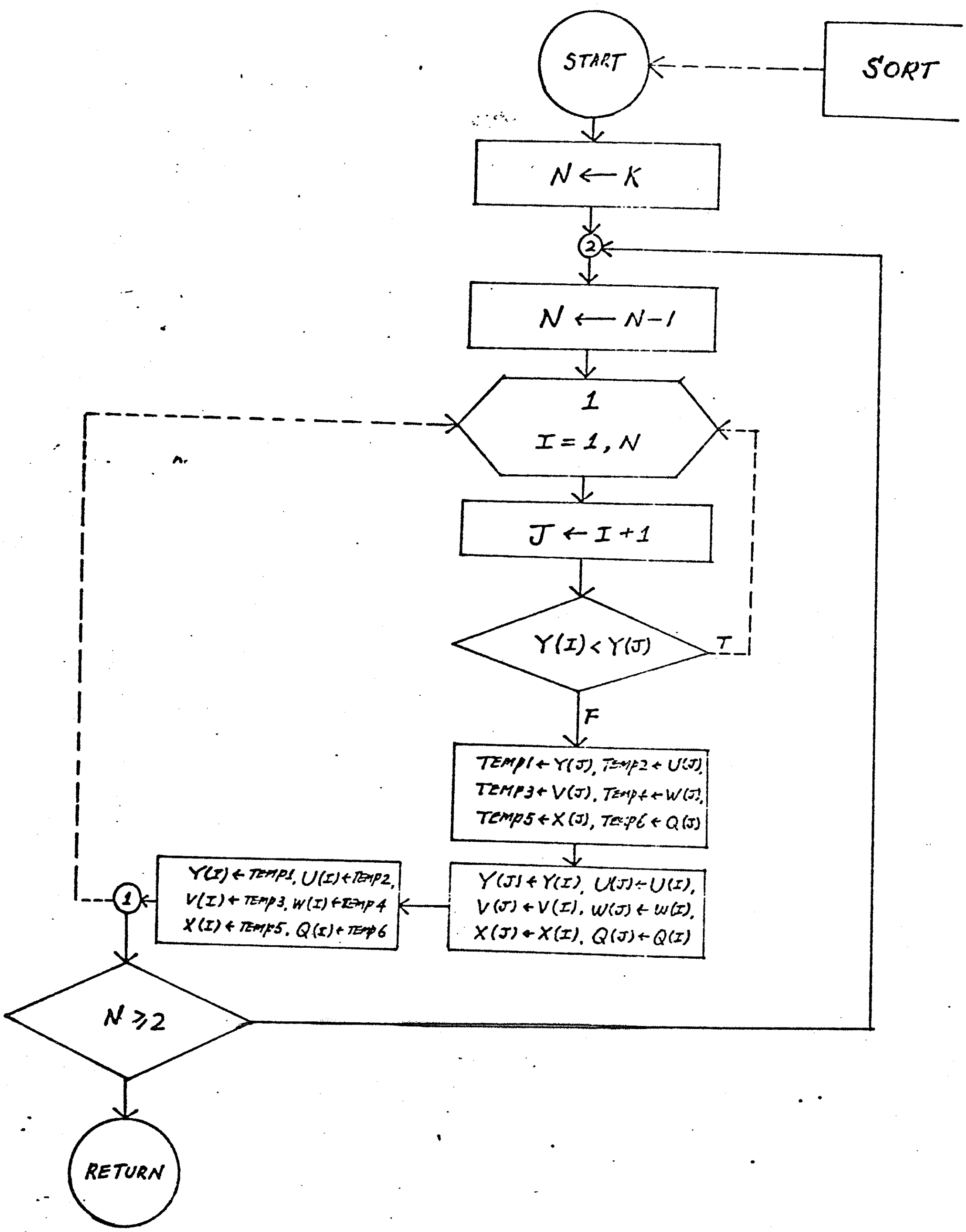
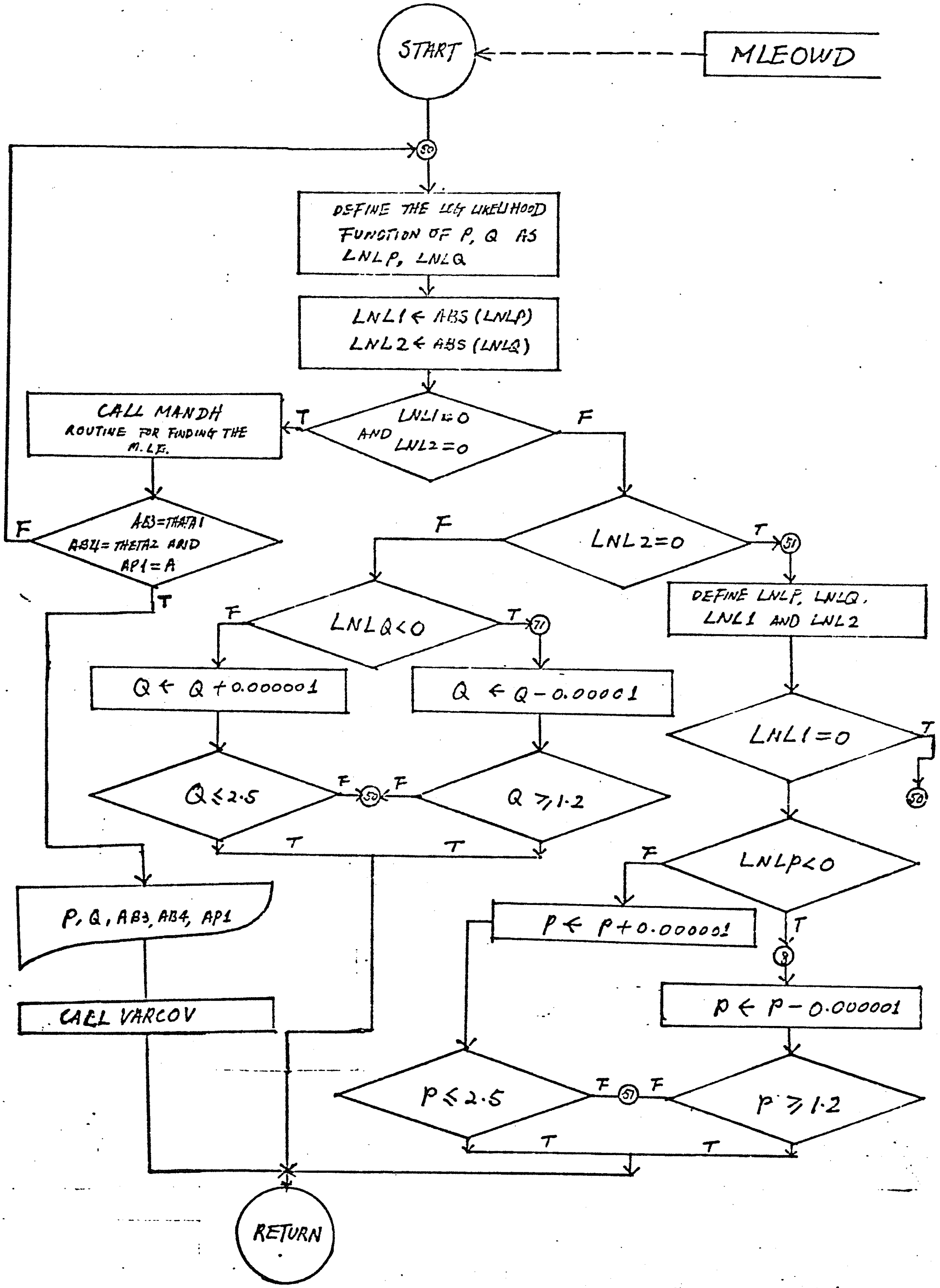
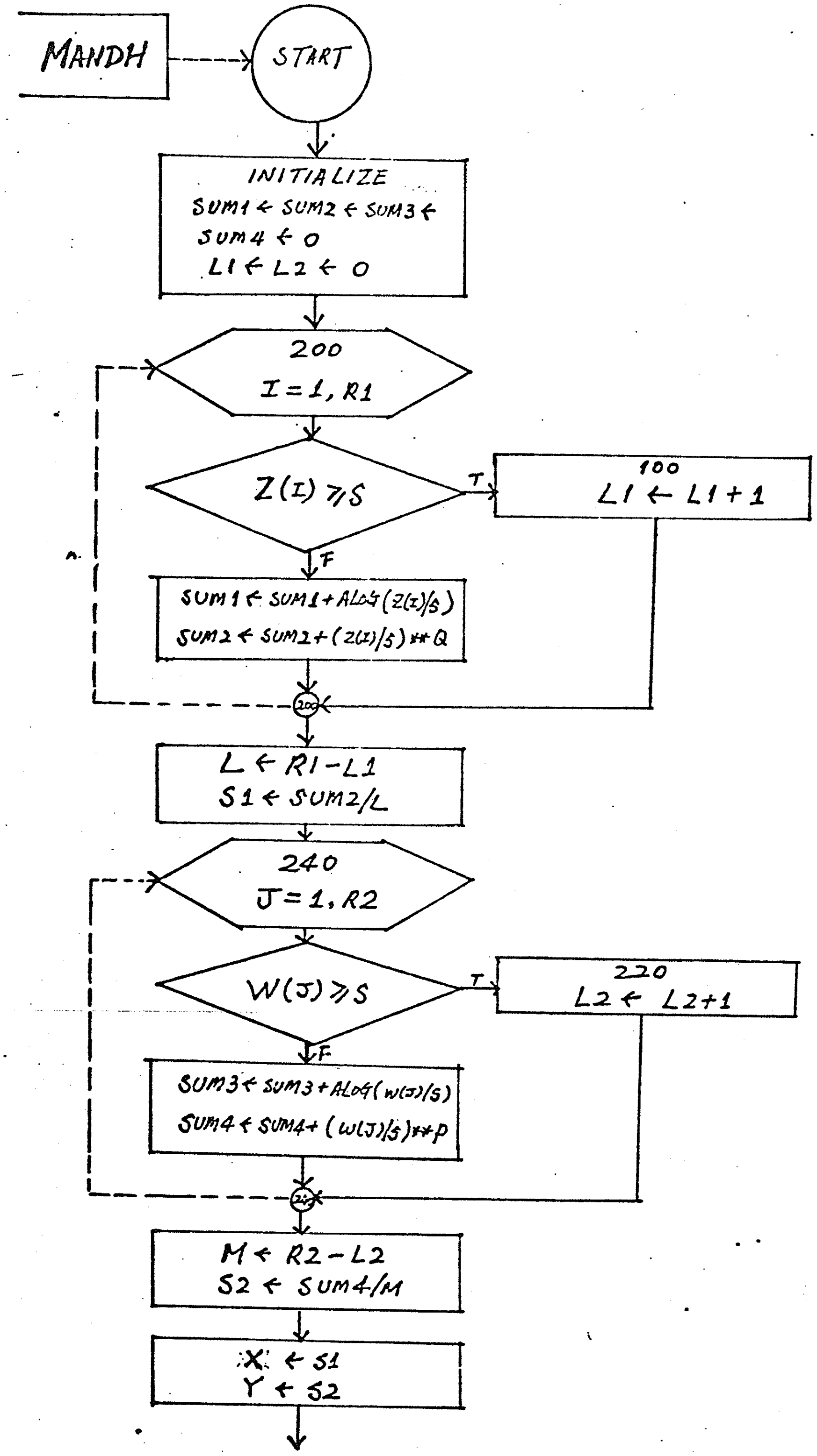


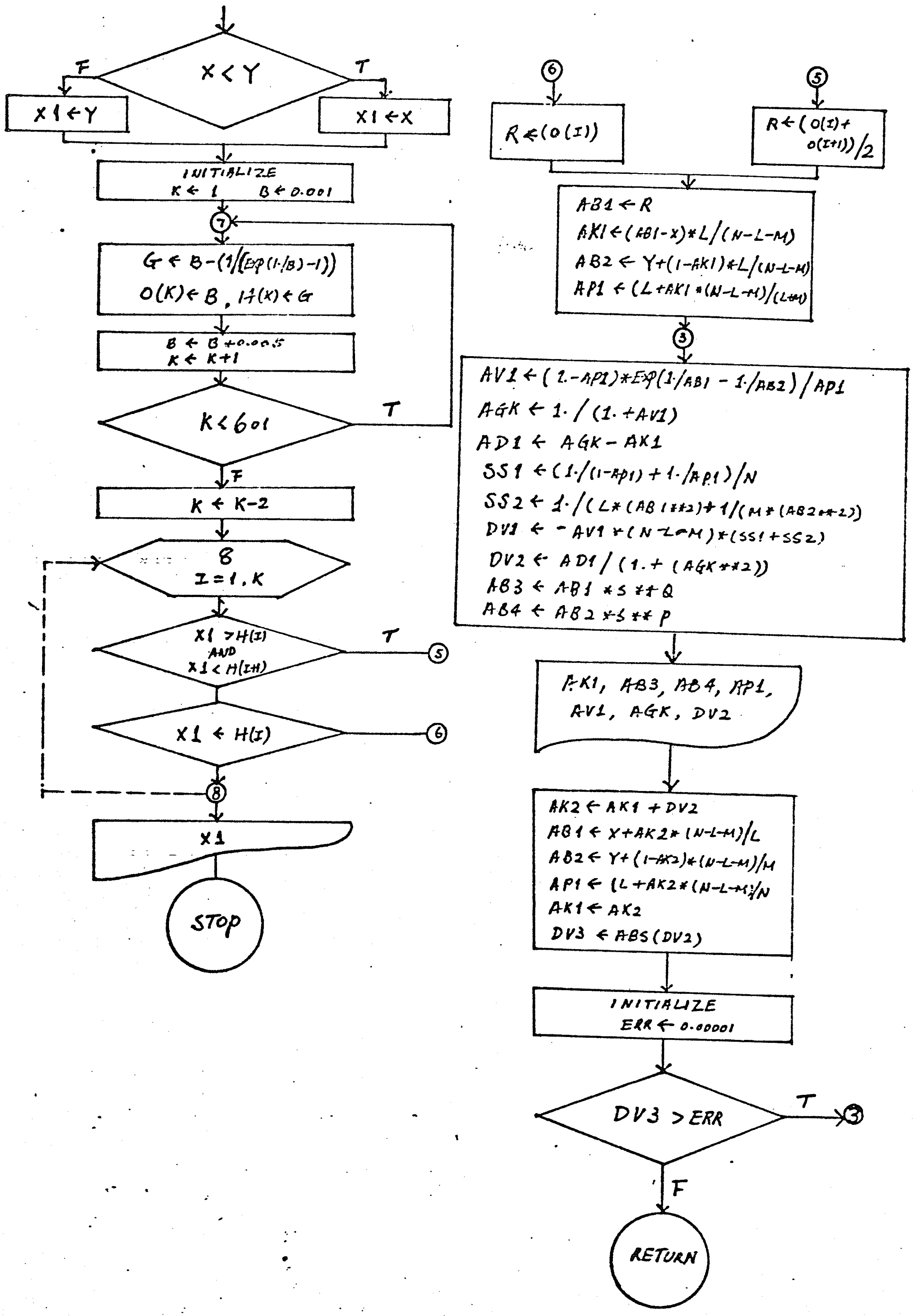
TABLE
START
INITIALIZE AB1, I
100
INITIALIZE AB2
200
INITIALIZE ARFA
300
INITIALIZE AP1
350
INITIALIZE AP2
400
DEFINE THE FUNCTION OF COEFFICIENT VARIATION
P(I) ← AP1, Q(I) ← AP2
A(I) ← ARFA, B(I) ← AB1
C(I) ← AB2, V(I) ← CV
AB1 ← AB1 + 0.005, I = I + 1
INITIALIZE THE ERR TERM
AB1 < ERR7 + AP2 > ERR8
AB2 < ERR8 + AP2 > ERR8
ARFA < ERR2 + AP2 > ERR8
AP1 < ERR3 + AP2 > ERR8
AP2 > ERR9
AB1 < ERR7 + AP1 > ERR6
AB2 < ERR1 + AP1 > ERR6
ARFA < ERR2 + AP1 > ERR6
AP1 > ERR6
AB1 < ERR7 + ARFA > ERR5
AB2 < ERR1 + ARFA > ERR5
ARFA > ERR5
AB1 < ERR7 + AB2 > ERR4
AB2 > ERR4
AB1 < ERR7
60
AB1 ← AB1, AB2 ← AB2 + 0.005
AB2 < ERR1 + AP2 > ERR8
AB2 < ERR1 + AP1 > ERR6
AB2 < ERR1 ARFA > ERR5
AB2 < ERR1
62
AB2 ← AB2, AB1 ← AB1
ARFA ← ARFA + 0.005
ARFA < ERR2 + AP2 > ERR8
ARFA < ERR2 + AP1 > ERR6
ARFA < ERR2
64
AB1 ← AB1, AB2 ← AB2
ARFA ← ARFA, AP1 ← AP1 + 0.05
AP1 < ERR3 + AP2 > ERR8
AP1 < ERR3
66
AB1 ← AB1, AB2 ← AB2
ARFA ← ARFA, AP1 ← AP1
AP2 ← AP2 + 0.05
AP2 < ERR4
M ← I − 1
CALL SORT ROOTINE FOR SORTING C.V.
15
I = 1, M
P(I), Q(I), A(I), B(I), C(I), V(I)
RETURN

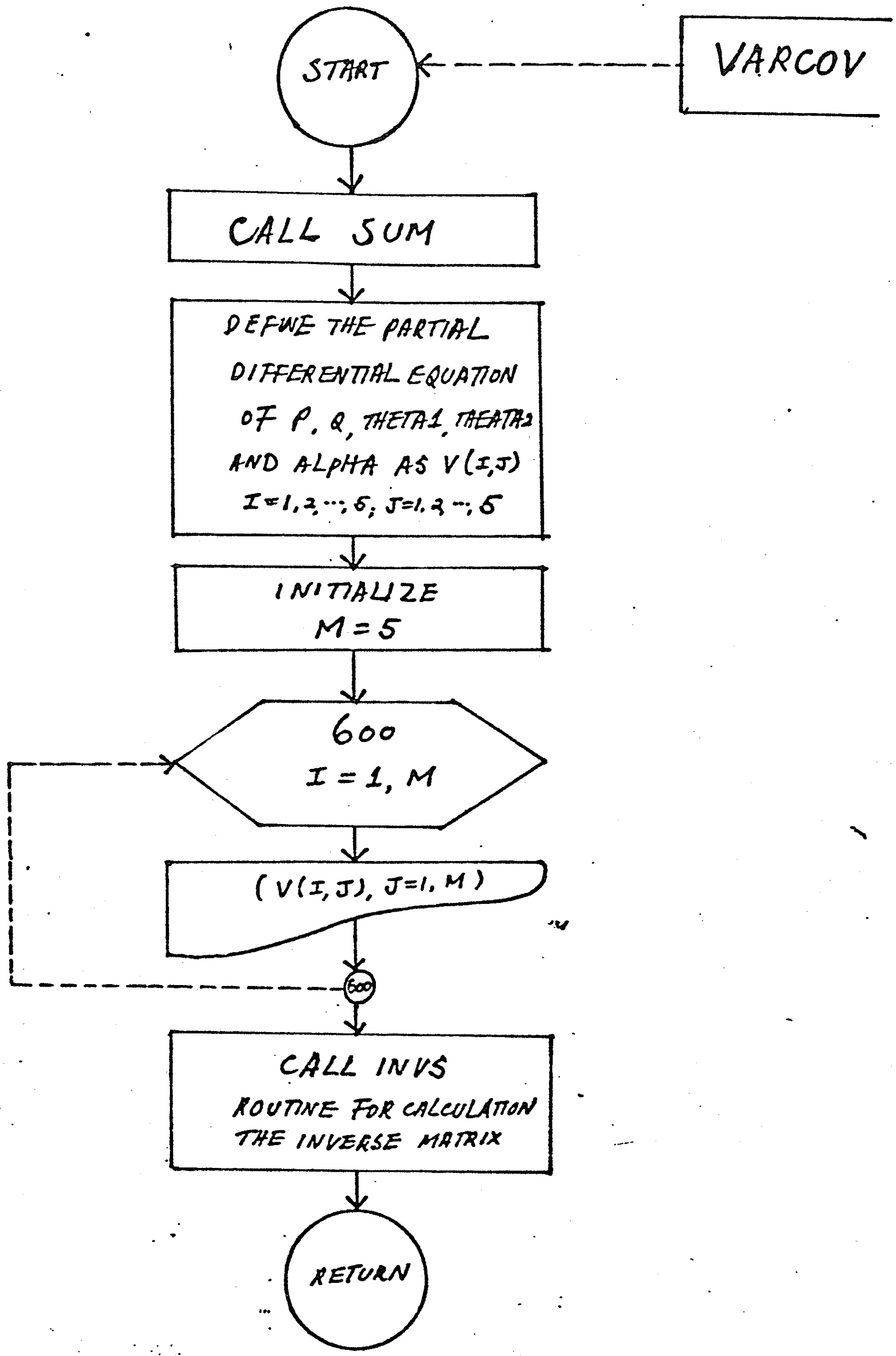


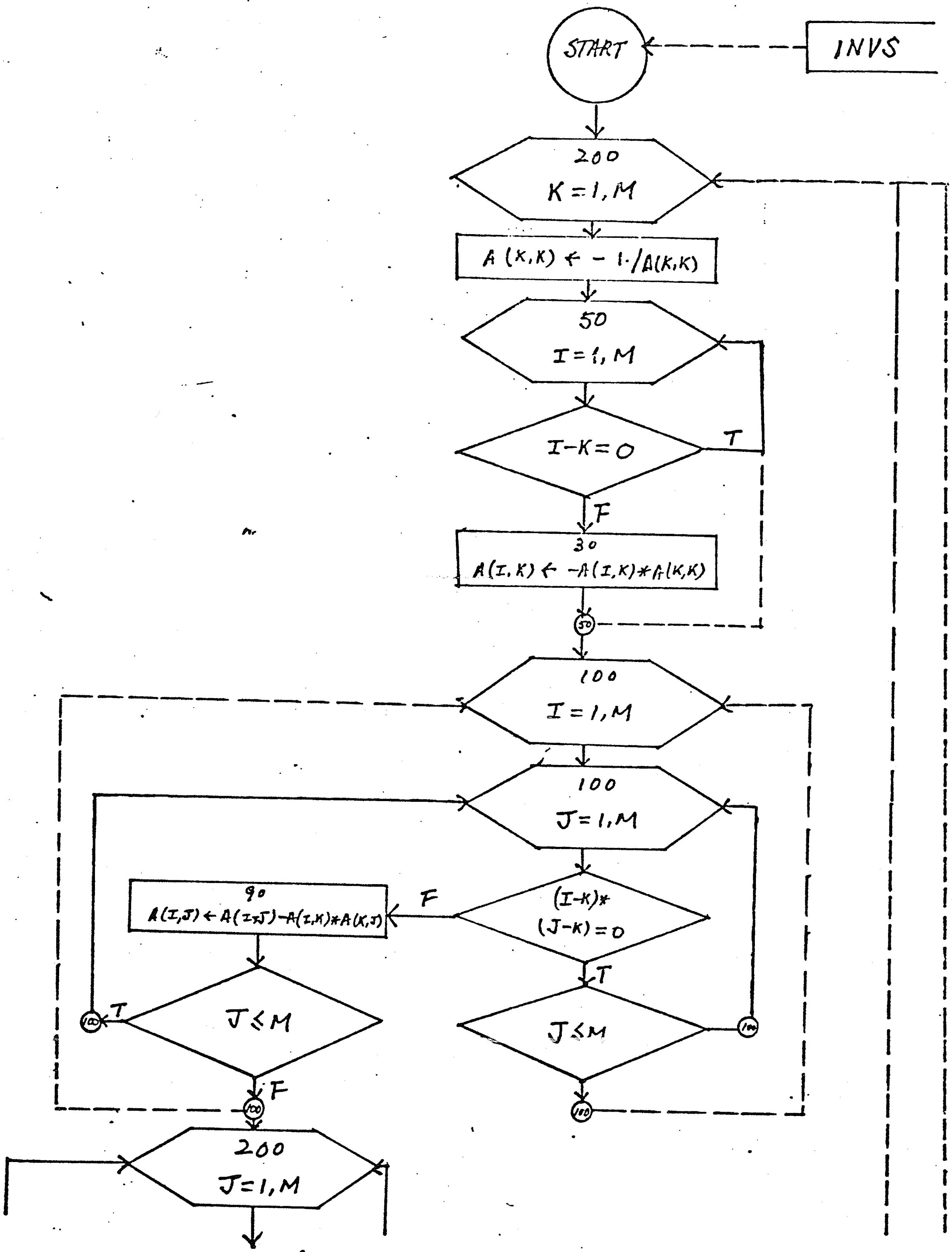
START
SORT
N ← K
2
N ← N-1
1
I = 1, N
J ← I+1
Y(I) < Y(J)
T
F
TEMP1 ← Y(J), TEMP2 ← U(J),
TEMP3 ← V(J), TEMP4 ← W(J),
TEMP5 ← X(J), TEMP6 ← Q(J)
Y(J) ← Y(I), U(J) ← U(I),
V(J) ← V(I), W(J) ← W(I),
X(J) ← X(I), Q(J) ← Q(I)
Y(I) ← TEMP1, U(I) ← TEMP2,
V(I) ← TEMP3, W(I) ← TEMP4
X(I) ← TEMP5, Q(I) ← TEMP6
1
N ≥ 2
RETURN

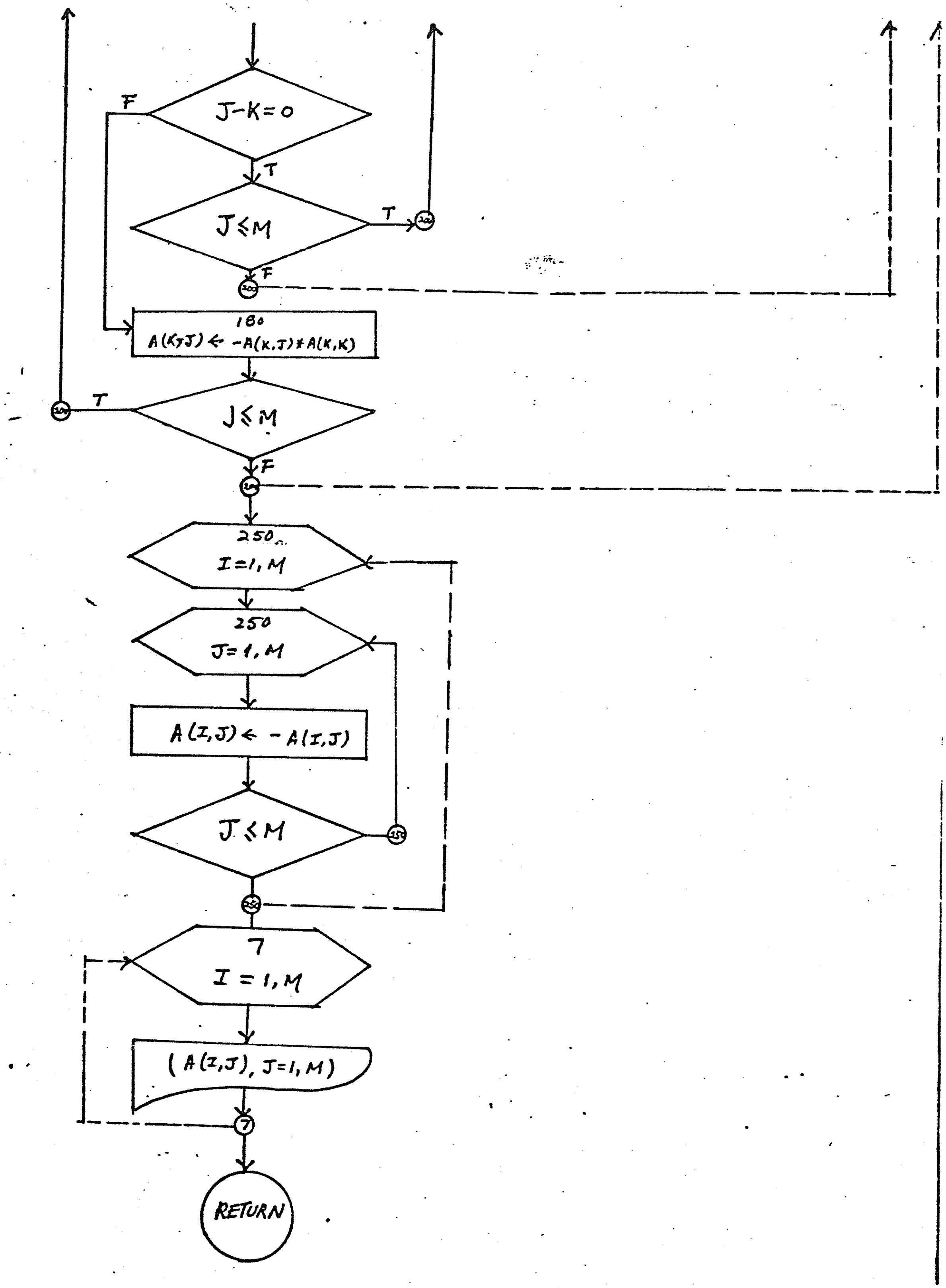

START
MLEOWD
DEFINE THE LOG LIKELIHOOD FUNCTION OF P, Q AS LNLP, LNLQ
LNL1 ← ABS (LNLP)
LNL2 ← ABS (LNLQ)
LNL1 = O AND LNL2 = O
T
F
CALL MANDH ROUTINE FOR FINDING THE M.LE.
AB3=THETA1 AB4= THETA2 AND AP1 = A
F
T
LNL2=O
T
F
DEFINE LNLP, LNLQ, LNL1 AND LNL2
LNLQ<O
T
F
Q ← Q + 0.000001
Q ← Q − 0.00001
LNL1=O
T
Q ≤ 2.5
F
F
Q ≥ 1.2
T
T
LNLP<O
F
T
P, Q, AB3, AB4, AP1
P ← P + 0.000001
P ← P − 0.000001
CALL VARCOV
P ≤ 2.5
F
F
P ≥ 1.2
T
T
RETURN

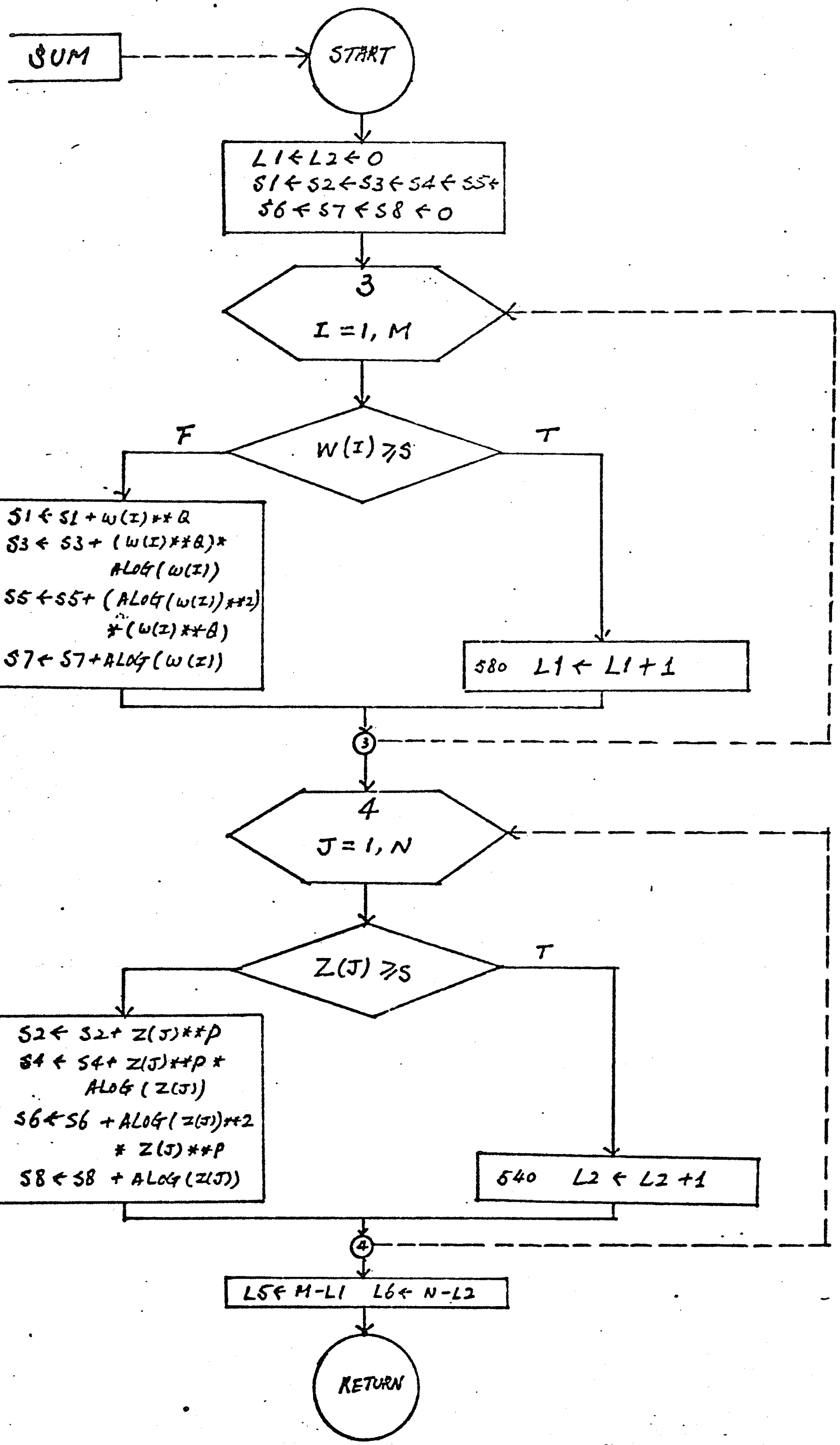

MANDH
START
INITIALIZE
SUM1 ← SUM2 ← SUM3 ←
SUM4 ← 0
L1 ← L2 ← 0
200
I = 1, R1
Z(I) ≥ S
T
100
L1 ← L1 + 1
F
SUM1 ← SUM1 + ALOG(Z(I)/S)
SUM2 ← SUM2 + (Z(I)/S) ** Q
200
L ← R1 − L1
S1 ← SUM2/L
240
J = 1, R2
W(J) ≥ S
T
220
L2 ← L2 + 1
F
SUM3 ← SUM3 + ALOG(W(J)/S)
SUM4 ← SUM4 + (W(J)/S) ** P
M ← R2 − L2
S2 ← SUM4/M
X ← S1
Y ← S2

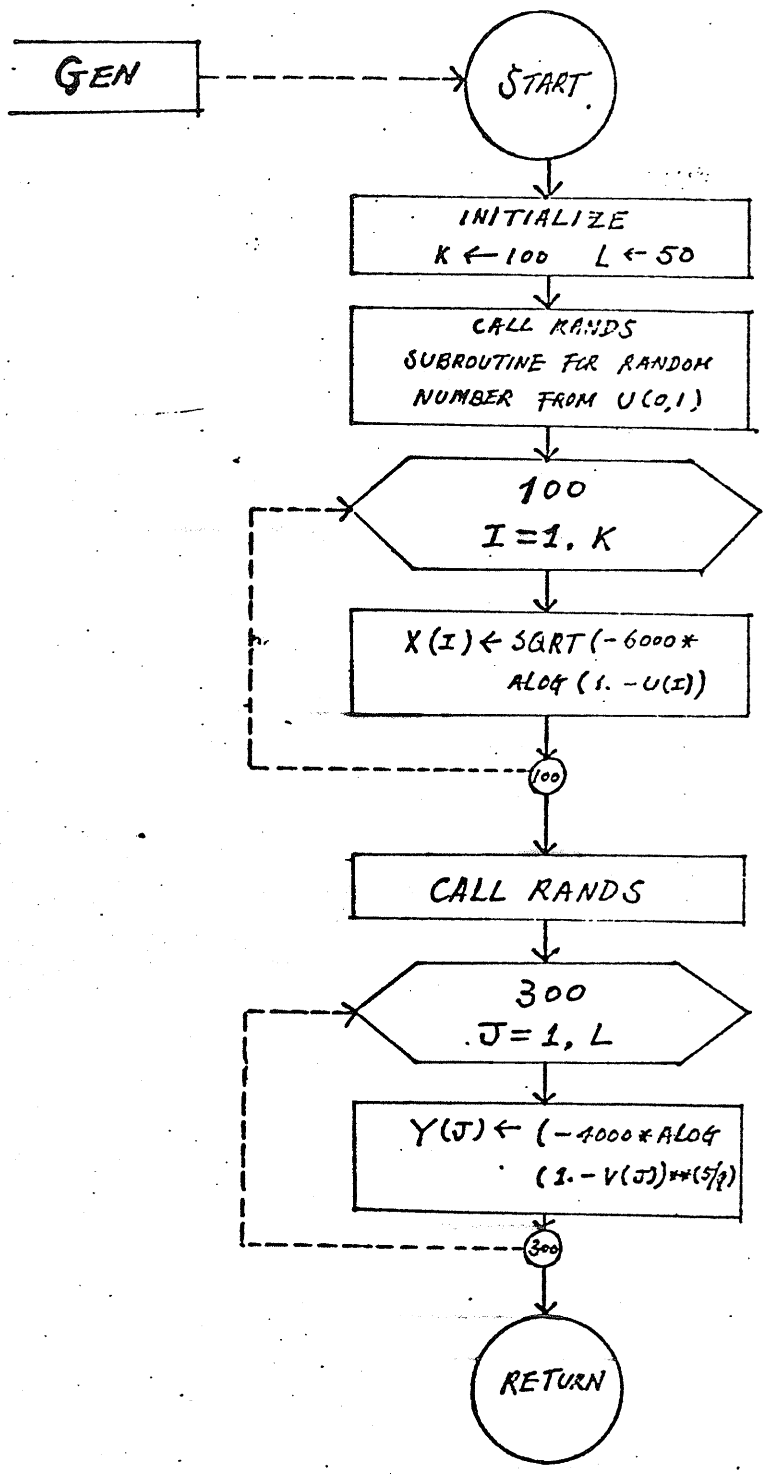

X < Y
F
T
X1 ← Y
X1 ← X
INITIALIZE
K ← 1 B ← 0.001
7
G ← B-(1/(EXP(1./B)-1))
O(K) ← B , H(K) ← G
B ← B+0.005
K ← K+1
K < 601
T
F
K ← K-2
8
I = 1, K
X1 > H(I)
AND
X1 < H(I+1)
T
5
X1 < H(I)
6
8
X1
STOP
6
R ← (O(I))
5
R ← (O(I)+
O(I+1))/2
AB1 ← R
AK1 ← (AB1-X)*L/(N-L-M)
AB2 ← Y+(1-AK1)*L/(N-L-M)
AP1 ← (L+AK1*(N-L-M))/(L+M)
3
AV1 ← (1.-AP1)*EXP(1./AB1 - 1./AB2)/AP1
AGK ← 1./(1.+AV1)
AD1 ← AGK - AK1
SS1 ← (1./(1-AP1) + 1./AP1)/N
SS2 ← 1./(L*(AB1**2)+1/(M*(AB2**2))
DV1 ← - AV1*(N-L-M)*(SS1+SS2)
DV2 ← AD1/(1.+(AGK**2))
AB3 ← AB1*S**Q
AB4 ← AB2*S**P
AK1, AB3, AB4, AP1,
AV1, AGK, DV2
AK2 ← AK1 + DV2
AB1 ← X+AK2*(N-L-M)/L
AB2 ← Y+(1-AK2)*(N-L-M)/M
AP1 ← (L+AK2*(N-L-M))/N
AK1 ← AK2
DV3 ← ABS(DV2)
INITIALIZE
ERR ← 0.00001
DV3 > ERR
T
3
F
RETURN

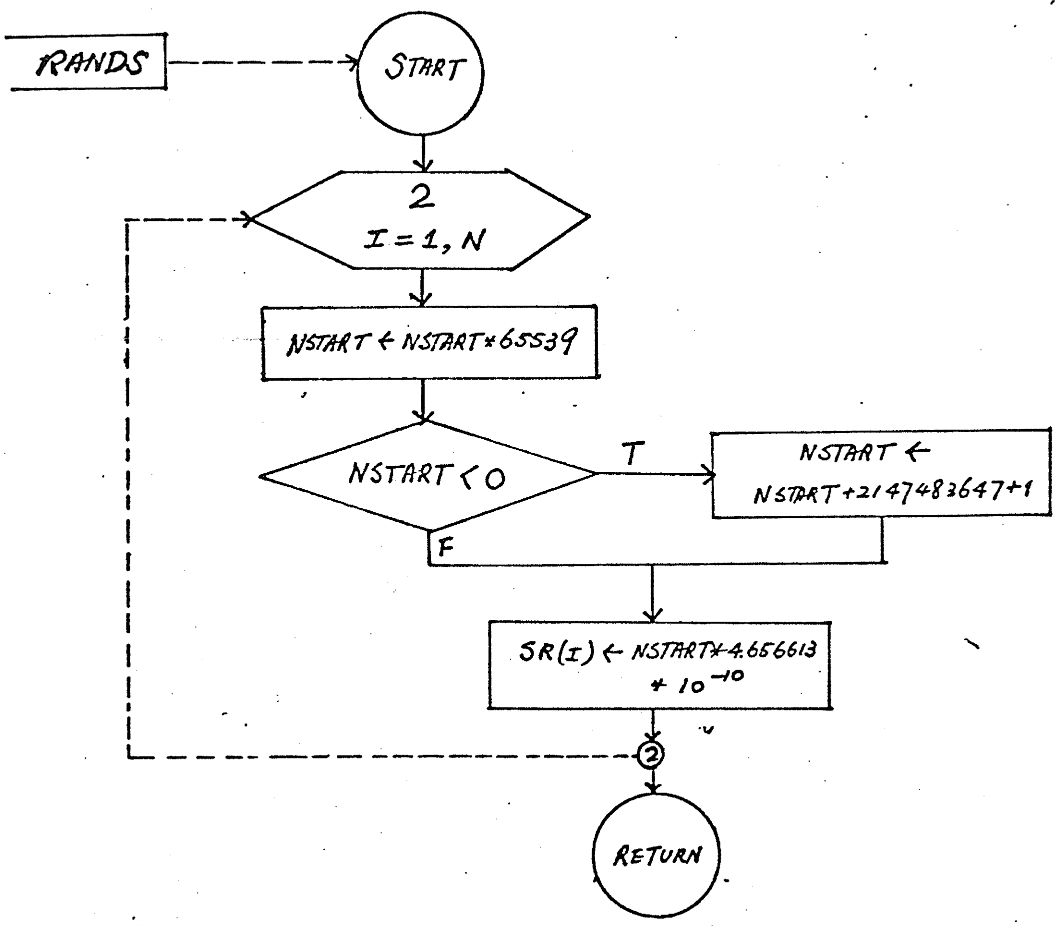

START
VARCOV
CALL SUM
DEFWE THE PARTIAL DIFFERENTIAL EQUATION OF P, Q, THETA1, THEATA2 AND ALPHA AS V(I,J) I=1,2,…,5; J=1,2,…,5
INITIALIZE M=5
600 I = 1, M
(V(I,J), J=1, M)
600
CALL INVS ROUTINE FOR CALCULATION THE INVERSE MATRIX
RETURN


INVS
START
200
K = 1, M
A (K,K) ← − 1./A(K,K)
50
I = 1, M
I−K = 0
T
F
30
A(I,K) ← −A(I,K)*A(K,K)
50
100
I = 1, M
100
J = 1, M
(I−K)*
(J−K) = 0
F
90
A(I,J) ← A(I,J)−A(I,K)*A(K,J)
T
J ≤ M
J ≤ M
T
F
100
100
100
100
200
J = 1, M


J-K=0
F
T
J≤M
T
F
180
A(K,J) ← -A(K,J)*A(K,K)
T
J≤M
F
250
I=1, M
250
J=1, M
A(I,J) ← - A(I,J)
J≤M
7
I = 1, M
(A(I,J), J=1, M)
7
RETURN


SUM
START
L1 ← L2 ← 0
S1 ← S2 ← S3 ← S4 ← S5 ←
S6 ← S7 ← S8 ← 0
3
I = 1, M
W(I) ≥ S
F
T
S1 ← S1 + W(I)**Q
S3 ← S3 + (W(I)**Q)*
ALOG(W(I))
S5 ← S5 + (ALOG(W(I))**2)
*(W(I)**Q)
S7 ← S7 + ALOG(W(I))
580 L1 ← L1 + 1
3
4
J = 1, N
Z(J) ≥ S
T
S2 ← S2 + Z(J)**P
S4 ← S4 + Z(J)**P *
ALOG (Z(J))
S6 ← S6 + ALOG(Z(J))**2
* Z(J)**P
S8 ← S8 + ALOG(Z(J))
540 L2 ← L2 + 1
4
L5 ← M-L1 L6 ← N-L2
RETURN


GEN
START
INITIALIZE
K ← 100 L ← 50
CALL RANDS
SUBROUTINE FOR RANDOM
NUMBER FROM U(0,1)
100
I=1, K
X(I) ← SQRT(-6000*
ALOG(1.-U(I))
100
CALL RANDS
300
J=1, L
Y(J) ← (-4000*ALOG
(1.-V(J))**(5/9)
300
RETURN



RANDS
START
2
I = 1, N
NSTART ← NSTART*65539
NSTART < 0
T
F
NSTART ←
NSTART+2147483647+1
SR(I) ← NSTART*4.656613
* 10^-10
2
RETURN

```
$JOB WATFIV   WEIBULL-DISTRIBUTION LINE=66 TIME=250M
C
C
C  APPENDIX 2 :
C
C
C
C     THE COMPUTER PROGRAM
C      ==================
C
C
C
C
C
C
C
C
C
C ----- PROJECT TITLE : --------------------------------------------
C
C
C
C
C
C        EATIMATION OF PARAMETERS OF MIXED WEIBULL DISTRIBUTIONS
C
C                       WITH TIME-CENSORED DATA
C
C
C
C
C
C --------------- HOMER HOK-KUI LEE ----- ADVISOR DR.S.K.SINHA ----
C
C
C
C
C --- DEPARTMENT OF STATISTICS ------- UNIVERSITY OF MANITOBA -----
C
C
C ---- ACADMIC YEAR : -------------------------------1975/1976 ----
C
C
C
C
C
C
C     ***********************
C     *                     *
C     *  MAIN LINE PROGRAM  *
C     *                     *
C     ***********************
C
C
C
C
      READ(5,*) L,K,N,T,N1,N2
C
      CALL CORCOE(L,K,N,T,N1,N2)
      STOP
      END
```

```
C
C
C        SUBROUTINE : CORCOE
C
C     ------------------------------------------------------------
C     !                                                          !
C     !  PROGRAMME FOR FINDING MOMENT ESTIMATE OF P Q LAMBDA1    !
C     !  LAMBDA2 AND ALPHA                                       !
C     !  WHERE WE LET LAMBDA1=THETA1/(T**P)  LAMBDA2=THETA2/(T**Q) !
C     !                                                          !
C     ------------------------------------------------------------
C
C
      SUBROUTINE CORCOE(L,K,N,T,N1,N2)
      REAL X(200),Y(200)
      REAL P(2000),Q(2000),A(2000),B(2000),C(2000),V(2000)
      SUM1=0
      SUM2=0
      SUM3=0
      SUM4=0
      CALL GEN(N1,N2,X,Y,T)
      DO 40 J=1,L
      SUM4=SUM4+(Y(J)/T)**2
   40 SUM3=SUM3+Y(J)/T
      DO 50 I=1,K
      SUM1=SUM1+X(I)/T
   50 SUM2=SUM2+(X(I)/T)**2
      SMEAN=(SUM1+SUM3)/(K+L)
      VAR2=(SUM2+SUM4-(SUM1+SUM3)**2/(K+L))/(K+L-1)
      CV3=VAR2/(SMEAN**2)
      SCV=SQRT(CV3)
      PRINT,SMEAN,VAR2,CV3
      PRINT9,SCV
    9 FORMAT('1','THE COEFFICIENT-VARIATION = ',F10.8)
      CALL TABLE(P,Q,A,B,C,V,M)
C
C       COMPARISION WITH THE COEFFICIENT VARIATION FROM THE MIXED
C      SAMPLE AND THE C-V TABLE
C
      MM=M-1
      DO 600 I=1,MM
      IF(SCV.EQ.V(I)) GOTO 500
      IF(SCV.GT.V(I).AND.SCV.LT.V(I+1)) GOTO 520
  600 CONTINUE
      PRINT,SCV
      GOTO 900
  500 PHAT=P(I)
      QHAT=Q(I)
      ALPHA=A(I)
      BATA1=B(I)*T**QHAT
      BATA2=C(I)*T**PHAT
      PRINT700,PHAT,QHAT,ALPHA,BATA1,BATA2
      GOTO 900
  520 PHAT=(P(I)+P(I+1))/2
      QHAT=(Q(I)+Q(I+1))/2
      ALPHA=(A(I)+A(I+1))/2
      BATA1=(B(I)+B(I+1))/2*T**QHAT
      BATA2=(C(I)+C(I+1))/2*T**PHAT
      PRINT700,PHAT,QHAT,ALPHA,BATA1,BATA2
  700 FORMAT('1',2X,'  P1=',F7.4,'  P2=',F7.4,'  ALPHA=',F7.5,
     1       '  THETA1=',F10.5,'  THETA2=',F10.5)
C
```

```
C
C      ONCE THE MLE(S) BE OBTAINED THE V-C MATRIX CAN ALSO BE FOUND
C
C
       CALL MLEOWD(L,K,N,T,PHAT,QHAT,BATA1,BATA2,ALPHA,X,Y)
C
C
       PRINT1
     1 FORMAT('1','        ')
   900 RETURN
       END
```

```
C
C
C
C        SUBROUTINE : TABLE
C
C
C
C      -----------------------------------------------------------------
C      !                                                               !
C      !  TABULATION THE COEFFICIENT VARIATION WITH PARAMETER P Q      !
C      !  LAMBDA1 LAMBDA2 AND PROB.                                    !
C      !                                                               !
C      -----------------------------------------------------------------
C
C
      SUBROUTINE TABLE(P,Q,A,B,C,V,M)
      REAL P(2000),Q(2000),A(2000),B(2000),C(2000),V(2000)
      REAL AP1,AP2,ARFA,AB1,AB2,D1,D2,E1,E2,M1,M2,CV2,CV3
      I=1
      PRINT800
  800 FORMAT('1',27X,66('_'))
      PRINT820
  820 FORMAT(27X,'|',65X,'|')
      PRINT500
  500 FORMAT(27X,'|',2X,2HP1,11X,2HP2,6X,4HPROB,8X,7HLAMBDA1,4X,
     1       7HLAMBDA2,6X,2HCV,4X,'|')
      PRINT830
  830 FORMAT(27X,'|',65X,'|')
      PRINT840
  840 FORMAT(27X,66('-'))
      PRINT850
  850 FORMAT(27X,'|',65X,'|')
      AB1=0.42
  100 AB2=0.615
  200 ARFA=0.3
  300 AP1=1.95
  350 AP2=1.90
  400 G1=GAMMA(1./AP1)
      G2=GAMMA(2./AP1)
      G3=GAMMA(1./AP2)
      G4=GAMMA(2./AP2)
      D1=ARFA*G1*AB1**(1./AP1)/AP1
      D2=(1-ARFA)*G3*AB2**(1./AP2)/AP2
      M1=D1+D2
      E1=2*ARFA*G2*AB1**(2./AP1)/AP1
      E2=2*(1-ARFA)*G4*AB2**(2./AP2)/AP2
      M2=E1+E2
      VAR1=M2-M1**2
      CV2=VAR1/(M1**2)
      CV=SQRT(CV2)
      P(I)=AP1
      Q(I)=AP2
      A(I)=ARFA
      B(I)=AB1
      C(I)=AB2
      V(I)=CV
      AB1=AB1+0.005
      I=I+1
      ERR1=0.625
      ERR2=0.31
      ERR3=2.10
      ERR4=0.615
      ERR5=0.3
```

```
      ERR6=1.95
      ERR7=0.43
      ERR8=1.90
      ERR9=2.05
      IF(AB1.LT.ERR7.AND.AP2.GT.ERR8) GOTO 400
      IF(AB2.LT.ERR2.AND.AP2.GT.ERR8) GOTO 60
      IF(ARFA.LT.ERR2.AND.AP2.GT.ERR8) GOTO 62
      IF(AP1.LT.ERR3.AND.AP2.GT.ERR8) GOTO 64
      IF(AP2.GT.ERR9) GOTO 66
      IF(AB1.LT.ERR7.AND.AP1.GT.ERR6) GOTO 350
      IF(AB2.LT.ERR1.AND.AP1.GT.ERR6) GOTO 60
      IF(ARFA.LT.ERR2.AND.AP1.GT.ERR6) GOTO 62
      IF(AP1.GT.ERR6) GOTO 64
      IF(AB1.LT.ERR7.AND.ARFA.GT.ERR5) GOTO 300
      IF(AB2.LT.ERR1.AND.ARFA.GT.ERR5) GOTO 60
      IF(ARFA.GT.ERR5) GOTO 62
      IF(AB1.LT.ERR7.AND.AB2.GT.ERR4) GOTO 200
      IF(AB2.GT.ERR4) GOTO60
      IF(AB1.LT.ERR7) GOTO 100
   60 AB1=0.42
      AB2=AB2+0.005
      IF(AB2.LT.ERR1.AND.AP2.GT.ERR8) GOTO 400
      IF(AB2.LT.ERR1.AND.AP1.GT.ERR6) GOTO 350
      IF(AB2.LT.ERR1.AND.ARFA.GT.ERR5) GOTO 300
      IF(AB2.LT.ERR1) GOTO 200
   62 AB1=0.42
      AB2=0.615
      ARFA=ARFA+0.005
      IF(ARFA.LT.ERR2.AND.AP2.GT.ERR8) GOTO 400
      IF(ARFA.LT.ERR2.AND.AP1.GT.ERR6) GOTO 350
      IF(ARFA.LT.ERR2) GOTO 300
   64 AB1=0.42
      AB2=0.615
      ARFA=0.3
      AP1=AP1+0.05
      IF(AP1.LT.ERR3.AND.AP2.GT.ERR8) GOTO 400
      IF(AP1.LT.ERR3) GOTO 350
   66 AB1=0.42
      AB2=0.615
      ARFA=0.3
      AP1=1.95
      AP2=AP2+0.05
      IF(AP2.LT.ERR9) GOTO 400
      M=I-1
      CALL SORT(M,P,Q,A,B,C,V)
      DO 15 I=1,M
      PRINT700,P(I),Q(I),A(I),B(I),C(I),V(I)
  700 FORMAT(27X,'|',1X,F5.3,9X,F5.3,3X,F7.4,5X,F9.5,2X,F9.5,4X,F7.5,1X,
     1       '|')
   15 CONTINUE
      PRINT860
  860 FORMAT(27X,'|',65X,'|')
      PRINT870
  870 FORMAT(27X,66('-'))
      RETURN
      END
```

```
C
C
C
C        SUBROUTINE : SORT
C
C
C       ------------------------------------------------------------
C       |                                                          |
C       |  REARRANGE THE COEFFICIENT VARIATION VALUES WITH P Q     |
C       |  LAMBDA1 LAMBDA2 AND ALPHA                               |
C       |                                                          |
C       ------------------------------------------------------------
C
C
C
      SUBROUTINE SORT(K,U,O,V,W,X,Y)
      REAL U(K),O(K),W(K),V(K),X(K),Y(K)
      N=K
    2 N=N-1
      DO 1 I=1,N
      J=I+1
      IF(Y(I).LT.Y(J)) GOTO 1
      TEMP1=Y(J)
      TEMP2=U(J)
      TEMP3=V(J)
      TEMP4=W(J)
      TEMP5=X(J)
      TEMP6=O(J)
      Y(J)=Y(I)
      U(J)=U(I)
      O(J)=O(I)
      V(J)=V(I)
      W(J)=W(I)
      X(J)=X(I)
      Y(I)=TEMP1
      U(I)=TEMP2
      V(I)=TEMP3
      W(I)=TEMP4
      X(I)=TEMP5
      O(I)=TEMP6
    1 CONTINUE
      IF(N.GE.2) GOTO 2
      RETURN
      END
```

```
C
C          SUBROUTINE : MLEOWD
C
C
C       ---------------------------------------------------------------
C       !                                                             !
C       !                                                             !
C       !  METHED FOR FINDING MAXIMUM LIKELIHOOD ESTIMATES OF P Q      !
C       !  THETA1 THETA2 AND ALPHA                                    !
C       !                                                             !
C       !                                                             !
C       ---------------------------------------------------------------
C
C
C
C
C
C
      SUBROUTINE MLEOWD(R1,R2,R3,T,P,Q,THETA1,THETA2,A,X,Y)
      REAL X(200),Y(200)
      REAL A1,A2,B1,B2,A4,A5,K1,K2,K,T,P,Q,EXP,ALOG,ABS
      REAL LNLP,LNLQ,LNL1,LNL2
      INTEGER R1,R2,R3
C
C
C     INTERATION FOR FINDING P WQ HERE THETA1 THETA2 AND ALPHA ARE
C     FOUND IN C-V TABLE AND KEEP IT AS CONSTANT
C
C     THE LOG LIKELIHOOD FUNCTION FOR P  AND FOR Q
C
   50 T1=-T**Q/THETA1
      T2=-T**P/THETA2
      K1=A*EXP(T1)
      K2=K1+(1.-A)*EXP(T2)
      K=K1/K2
      CALL SUM(P,Q,T,R1,R2,X,Y,S1,S2,SUM1,SUM3,S5,S6,SUM2,SUM4,L5,L6)
      A1=(-(R3-L5-L6)*K*T**Q*ALOG(T))/THETA1
      A2=(-(R3-L5-L6)*(1-K)*(T**P)*ALOG(T))/THETA2
      B1=L5/Q
      B2=L6/P
      A4=SUM1/THETA1-SUM2
      A5=SUM3/THETA2-SUM4
      LNLQ=A1+B1-A4
C
      LNLP=A2+B2-A5
      LNL1=ABS(LNLP)
      LNL2=ABS(LNLQ)
      ERR1=0.001
      IF(LNL1.LE.ERR1.AND.LNL2.LE.ERR1) GOTO 70
      IF(LNL2.LE.ERR1) GOTO 71
      PRINT,P,Q,THETA1,THETA2,A,LNLQ
      Q=Q-0.000001
      IF(Q.LE.2.5) GOTO 50
      GOTO 80
   71 IF(LNL1.LE.ERR1) GOTO 70
      PRINT,'     '
      PRINT,P,Q,THETA1,THETA2,A,LNLP,LNLQ
      PRINT,'     '
      Q=Q
      P=2.194756
   51 T1=-T**Q/THETA1
      T2=-T**P/THETA2
```

```
      K1=A*EXP(T1)
      K2=K1+(1.-A)*EXP(T2)
      K=K1/K2
      CALL SUM(P,Q,T,R1,R2,X,Y,S1,S2,SUM1,SUM3,S5,S6,SUM2,SUM4,L5,L6)
      A1=(-(R3-L5-L6)*K*T**Q*ALOG(T))/THETA1
      A2=(-(R3-L5-L6)*(1-K)*(T**P)*ALOG(T))/THETA2
      B1=L5/Q
      B2=L6/P
      A4=SUM1/THETA1-SUM2
      A5=SUM3/THETA2-SUM4
      LNLQ=A1+B1-A4
      LNLP=A2+B2-A5
      LNL1=ABS(LNLP)
      LNL2=ABS(LNLQ)
      IF(LNL1.LE.ERR1.AND.LNL2.LE.ERR1) GOTO 70
      PRINT,P,Q,THETA1,THETA2,A,LNLP
      IF(LNL1.LE.ERR1) GOTO 50
      P=P+0.000001
      IF(P.LE.2.5) GOTO 51
      GOTO 80
   70 PRINT700,P,Q,THETA1,THETA2,A
  700 FORMAT('1',2X,'  P=',F12.7,'   Q=',F12.7,'   THETA1=',F10.5,
     1       '   THETA2=',F10.5,'   PROBABILITY=',F7.5)
C
C     FOR A GIVEN P AND Q THE M.L.E.OF THETA1 THETA2 AND ALPHA CAN BE
C     FOUND BY USING THE SUBPROGRAMME MANDH
C
      CALL MANDH(P,Q,T,R1,R2,R3,Y,X,AB3,AB4,AP1)
C
      B1=ABS(THETA1-AB3)
      B2=ABS(THETA2-AB4)
      B3=ABS(A-AP1)
      ERR=0.0000001
      IF(B1.LE.ERR.AND.B2.LE.ERR.AND.B3.LE.ERR) GOTO 900
      THETA1=AB3
      THETA2=AB4
      A=AP1
      P=P
      Q=Q
      GOTO 50
   80 PRINT800,P,Q,THETA1,THETA2,A
  800 FORMAT('1',2X,'NO SOLUTION FOR P=1.2 TO P=',F7.5,' Q=1.2 TO Q=',
     1       F7.5,'  THETA1=',F10.5,'  THETA2=',F10.5,'  PROB=',F7.5)
      GOTO 850
  900 PRINT920,P,Q,AB3,AB4,AP1
  920 FORMAT('1',2X,'  P=',F7.4,'  Q=',F7.4,'  THETA1=',F10.5,'  THETA2
     1       ',F10.5,'  PROBABILITY=',F7.5)
C
C
C        CALCULATE THE ASYMPTOTIC VAR-COV FOR P,Q,THETA1,THETA2
C        WITH MLE VALUES
C
C
      M=5
C
      CALL VARCOV(R2,R3,R1,T,P,Q,AB3,AB4,A,X,Y,M)
C
C
      PRINT950
  950 FORMAT('1','     ')
  850 RETURN
      END
```

```
C
C
C
C       SUBROUTINE : MANDH
C
C
C            ---------------------------------------------------------------
C            !                                                             !
C            !  FIND THE MAXIMUM LIKELIHOOD ESTIMATES OF THETA1 THETA2     !
C            !  AND ALPHA WHERE P AND Q ARE GIVEN                          !
C            !                                                             !
C            ---------------------------------------------------------------
C
C
C
C
      SUBROUTINE MANDH(P,Q,S,R1,R2,N,Z,W,AB3,AB4,AP1)
      REAL Z(200),W(200),Q(600),H(600),A,B,C,D,E,F,AK1,AK2,AGK
      REAL AB1,AB2,S1,S2,AP1,AV1,EXP
      INTEGER R1,R2
      PRINT710
  710 FORMAT(/ 2X,110('-'))
      PRINT600
  600 FORMAT(2X,'|',108X,'|')
      PRINT700
  700 FORMAT(2X,'|',6X,4HKHAT,12X,6HTHETA1,11X,6HTHETA2,10X,4HARFA,
     1       1HV,13X,4HG(K),11X,4HD(V),4X,'|')
      PRINT720
  720 FORMAT(2X,'|',108X,'|')
      SUM1=0
      SUM2=0
      SUM3=0
      SUM4=0
      L1=0
      L2=0
      DO 200 I=1,R1
      IF(Z(I).GE.S) GOTO 100
      SUM1=SUM1+ALOG(Z(I)/S)
      SUM2=SUM2+(Z(I)/S)**Q
      GOTO 200
  100 L1=L1+1
  200 CONTINUE
      L=R1-L1
      S1=SUM2/L
      DO 240 J=1,R2
      IF(W(J).GE.S) GOTO 220
      SUM3=SUM3+ALOG(W(J)/S)
      SUM4=SUM4+(W(J)/S)**P
      GOTO 240
  220 L2=L2+1
  240 CONTINUE
      M=R2-L2
      S2=SUM4/M
      X=S1
      Y=S2
      IF(X.LT.Y) X1=X
      IF(Y.LT.X) X1=Y
C
C     THE TABLE FOR THE EQUATION :
C
C       ((LAMBDA-YBAR1)*EXP(1/LAMBDA-1)=1
      K=1
      B=0.01
```

```
    7 G=B-(1./(EXP(1./B)-1))
      O(K)=B
      H(K)=G
      B=B+0.005
      K=K+1
      IF(K.LT.601) GOTO 7
C
C
      K=K-2
      DO 8 I=1,K
      IF(X1.GT.H(I).AND.X1.LT.H(I+1)) GOTO 5
      IF(X1.EQ.H(I)) GOTO 6
    8 CONTINUE
      PRINT,X1
      STOP
    5 R=(O(I+1)+O(I))/2
      GOTO 10
    6 R=(O(I))
   10 AB1=R
      X=S1
      Y=S2
      IF(L.LT.M) GOTO 1
      IF(M.LT.L) GOTO 2
    1 L3=L
      M3=M
      GOTO 4
    2 L3=M
      M3=L
    4 L=L3
      AK1=(AB1-X)*L/(N-L-M)
      M=M3
      AB2=Y+(1-AK1)*L/(N-L-M)
      AP1=(L+AK1*(N-L-M))/N
    3 AV1=((1.-AP1)*EXP(1./AB1-1./AB2))/AP1
      AGK=1./(1+AV1)
      AD1=AGK-AK1
      SS1=(1./(1.-AP1)+1./AP1)/N
      SS2=1./(L*(AB1**2))+1./(M*(AB2**2))
      DV1=-AV1*(N-L-M)*(SS1+SS2)
      DV2=AP1/(1.+(AGK**2)*DV1)
      AB3=AB1*S**O
      AB4=AB2*S**P
      PRINT500,AK1,AB3,AB4,AP1,AV1,AGK,DV2
  500 FORMAT(2X,'|',1X,F10.4,5X,F12.4,5X,F12.4,5X,F10.4,5X,F10.4,5X,
     1       F10.4,5X,F10.4,3X,'|')
      AK2=AK1+DV2
      AB1=X+AK2*(N-L-M)/L
      AB2=Y+(1-AK2)*(N-L-M)/M
      AP1=(L+AK2*(N-L-M))/N
      AK1=AK2
      ERR=0.00001
      DV3=ABS(DV2)
      IF(DV3.GT.ERR) GOTO 3
      PRINT950
  950 FORMAT(2X,'|',108X,'|')
      PRINT520
  520 FORMAT(2X,110('-'))
      RETURN
      END
```

```
C
C
C
C       SUBROUTINE : VARCOV
C
C
C     ---------------------------------------------------------------
C     !
C     !  ASYMPTOTIC VARIANCE COVARIANCE MATRIX FOR P Q THETA1       !
C     !  THETA2 AND ALPHA                                           !
C     !                                                             !
C     ---------------------------------------------------------------
C
C
C
C
      SUBROUTINE VARCOV(R1,R2,N,T,P,Q,THETA1,THETA2,A,X,Y,M)
      REAL X(300),Y(300)
      REAL V(10,10),K1,K2,K,P,T,THETA1,THETA2,A,EXP
      INTEGER R1,R2
C
      CALL SUM(P,Q,T,R1,R2,X,Y,S1,S2,S3,S4,S5,S6,S7,S8,L5,L6)
      R1=L5
      R2=L6
      K1=A*EXP(-T**P/THETA1)
      K2=K1+(1.-A)*EXP(-T**P/THETA2)
      K=K1/K2
      A1=(N-R1-R2)*T**(2*P)*K*(1-K)/(THETA1**4)
      A2=2*(N-R1-R2)*T**P*K/(THETA1**3)
      A3=R1/(THETA1**2)
      A4=2*S1/(THETA1**3)
      V(1,1)=-(A1-A2+A3-A4)
      B1=(N-R1-R2)*(T**(2*P))*K*(1-K)/(THETA2**4)
      B2=2*(N-R1-R2)*T**P*(1-K)/(THETA2**3)
      B3=R2/(THETA2**2)
      B4=2*S2/(THETA2**3)
      V(2,2)=-(B1-B2+B3-B4)
      C1=(N-R1-R2)*K*(1-K)/(A*(1-A))**2
      C2=(K*(N-R1-R2)+R1)/A**2
      C3=((1-K)*(N-R1-R2)+R2)/(1-A)**2
      V(3,3)=-(C1-C2-C3)
      D1=(N-R1-R2)*(ALOG(T)**2)*T**(2*P)
      D2=D1*K*(1-K)*(1./THETA1-1./THETA2)**2
      D3=(N-R1-R2)*((ALOG(T))**2)*T**P
      D4=D3*(K/THETA1+(1-K)/THETA2)
      D5=(R1+R2)/P
      D6=S5/THETA1
      D7=S6/THETA2
      V(4,4)=-(D2-D4-D5-D6-D7)
      E1=-(N-R1-R2)*K*(1-K)*(T**(2*P))
      E2=(THETA1**2)*(THETA2**2)
      V(1,2)=-(E1/E2)
      F1=(N-R1-R2)*T**P*K*(1-K)
      F2=A*(1-A)*THETA1**2
      V(1,3)=-(F1/F2)
      H1=-(N-R1-R2)*(T**P)*K*(1-K)
      H2=A*(1-A)*THETA2**2
      V(2,3)=-(H1/H2)
      V(2,1)=V(1,2)
      V(3,1)=V(1,3)
      V(3,2)=V(2,3)
      PRINT950
```

```
  950 FORMAT('1',51HTHE INFORMATION MATRIX OF P THETA1 THETA2 AND ALPHA)
      M=3
      DO 600 I=1,M
      PRINT650,(V(I,J),J=1,M)
  650 FORMAT(/ 2X,4F20.9)
      PRINT,' '
  600 CONTINUE
      CALL INVS(V,M)
      RETURN
      END
```

```
C
C
C
C     SUBROUTINE : INVS
C
C
C    ---------------------------------------------------------------
C    !                                                             !
C    !  SUBPROGRAM FOR INVERSE MATRIX                              !
C    !                                                             1
C    ---------------------------------------------------------------
C
C
C
C
      SUBROUTINE INVS(A,M)
      DIMENSION A(10,10)
      DO 200 K=1,M
      A(K,K)=-1./A(K,K)
      DO 50 I=1,M
      IF(I-K)30,50,30
   30 A(I,K)=-A(I,K)*A(K,K)
   50 CONTINUE
      DO 100 I=1,M
      DO 100 J=1,M
      IF((I-K)*(J-K))90,100,90
   90 A(I,J)=A(I,J)-A(I,K)*A(K,J)
  100 CONTINUE
      DO 200 J=1,M
      IF(J-K)180,200,180
  180 A(K,J)=-A(K,J)*A(K,K)
  200 CONTINUE
      DO 250 I=1,M
      DO 250 J=1,M
  250 A(I,J)=-A(I,J)
      PRINT670
  670 FORMAT('0',66HTHE ASYMPTOTIC VARIANCE-COVARIANCE MATRIX OF P THETA
     -1 THETA2 ALPHA)
      DO 7 I=1,M
      PRINT8,(A(I,J),J=1,M)
    8 FORMAT(/ 2X,5F20.6)
    7 CONTINUE
      RETURN
      END
```

```
C
C
C
C        SUBROUTINE : SUM
C
C
C     ------------------------------------------------------------
C     !                                                          !
C     !  SUBPROGRAM FOR GETTING THE SUM                          !
C     !                                                          !
C     ------------------------------------------------------------
C
C
C
      SUBROUTINE SUM(P,Q,S,M,N,Z,W,S1,S2,S3,S4,S5,S6,S7,S8,L5,L6)
      REAL W(300),Z(300),S1,S2,S3,S4,S5,S6,S7,S8,P,Q,S
      L1=0
      L2=0
      S1=0
      S2=0
      S3=0
      S4=0
      S5=0
      S7=0
      S6=0
      S8=0
      DO 3 I=1,M
      IF(W(I).GE.S) GOTO 580
      S1=S1+W(I)**Q
      S3=S3+(W(I))**Q*ALOG(W(I))
      S5=S5+(ALOG(W(I))**2)*(W(I)**Q)
      S7=S7+ALOG(W(I))
      GOTO 3
  580 L1=L1+1
    3 CONTINUE
      DO 4 J=1,N
      IF(Z(J).GE.S) GOTO 540
      S4=S4+Z(J)**P*ALOG(Z(J))
      S2=S2+Z(J)**P
      S6=S6+(ALOG(Z(J))**2)*Z(J)**P
      S8=S8+ALOG(Z(J))
      GOTO 4
  540 L2=L2+1
    4 CONTINUE
      L5=M-L1
      L6=N-L2
      RETURN
      END
```

```
C
C
C
C          SUBROUTINE : GEN
C
C
C        -------------------------------------------------------------------
C        |                                                                 |
C        |   GENERTING THE DATA FROM SUBPOPULATION (1) AND                 |
C        |   SUBPOPULATION (2) WITH CUMULATIVE DISTRIBULATIONS :           |
C        |     Y1=1-EXP(-X**1.95/4200)                                     |
C        |     AND                                                         |
C        |     Y2=1-EXP(-X**2/6800)                                        |
C        |                                                                 |
C        -------------------------------------------------------------------
C
C
C
      SUBROUTINE GEN(N1,N2,X,Y,S)
      REAL U(200),X(200),V(200),Y(200)
      M=0
      N=0
      K=100
      CALL RANDS(N1,U,200)
      DO 100 I=1,K
  100 X(I)=(-6000*ALOG(1.-U(I)))**(1./2)
      L=50
      CALL RANDS(N2,V,200)
      DO 300 J=1,L
  300 Y(J)=(-4000*ALOG(1-V(J)))**(1./2)
      RETURN
      END
```

```
C
C
C       SUBROUTINE : RANDS
C
C
C      -----------------------------------------------------------
C      |
C      |  SUBPROGRAM FOR RANDOM VARIABLE FROM U(0,1)             |
C      |                                                         |
C      -----------------------------------------------------------
C
C
C
      SUBROUTINE RANDS(NSTART,SR,N)
      DIMENSION SR(N)
      DO 2 I=1,N
      NSTART=NSTART*65539
      IF(NSTART.LT.0) NSTART=NSTART+2147483647+1
    2 SR(I)=NSTART*4.656613E-10
      RETURN
      END

$ENTRY
```

REFERENCES

1. Barlow, R.E. and Proscham, F. (1965), "Mathematical theory of probability", John Wiley and Sons Inc.

2. Bartholomew, D.J. (1957), "A problem in life Testing", J. Amer. Stat. Assoc. 65, 350-355.

3. Buckland, W.K. (1964), "Statistical assessment of the life characteristic", Hafner Probability Company.

4. Cohen, Jr., A.C. (1965), "Maximum likelihood estimates in the Weibull distribution based on complete and censored samples", Technometrics, vol 7, 579-588.

5. Deemer, Jr., N.L. and Votaw, Jr., D.F. (1955), "Estimation of parameters of truncated or censored Exponential distribution", Ann. Math. Stat. 26, 498-504.

6. Epstein, B. (1958), "Exponential distribution and its role in life testing", Industrial Quality Control, 4-9.

7. Kao, J.H.K. (1959), "A graphical estimation of mixed Weibull parameters in life testing of electron tubes", Technometrics 1, 389-407.

8. Lawless, J. (1972), "On the estimation of safe life when the underlying life distribution is Weibull", Technometrics 15, 857-864.

9. Locks, M.O. (1973), "Reliability, maintainability, and availability assessment", Hayden Book Company, Inc.

10. Mann, R.N. (1968), "Point and interval estimation procedures for the two-parameter Weibull and extreme-value distributions", Technometrics 10, 231-256.

11. Mann, R.N., Schafer, R.E. and Singpurwala, N.D. (1974), "Methods for statistical analysis of reliability and life data", John Wiley & Sons.

12. Mendenhall, W. and Hader, R.J. (1958), "Estimation of parameters of mixed exponentially distributed failure time distribution from censored life test data", Biometrika 45, 504-519.

13. Rider, P.R. (1961), "The method of moments applied to a mixture of two exponential distributions", Ann. Math. Stat. 32, 143-147.

14. Sinha, S.K. (1976), "Robustness of posteriors under improper priors in some life testing models", Technical report #73, Department of Statistics, University of Manitoba, Winnipeg, Canada.

15. Swamy, P.S. and Doss, S.A.D.C. (1961), "On a problem of Bartholomew in life testing", Sankhyā Series A, Vol. 23 part 3, 225-230.