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CONTRIBUTIONS

TO

INDUSTRIAL STATISTICS

by

Bartholomew Ping Kei Leung

A Thesis

submitted to the Faculty of Graduate Studies

in partial fulfillment of the requirements for the Degree of

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Department of Statistics
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FACULTY OF GRADUATE STUDIES
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Contributions to Industrial Statistics

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

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ABSTRACT

The main theme of this dissertation deals with the impact and consequences of non-normal distribution on the process capability index Cpm. In this thesis, much work has been done in this area including the properties of $\hat{C}pm$, the estimate of Cpm, under normality, its sensitivity to non-normality and also the relationship of Cpm to squared error loss. Related to Cpm is the unifying measure of process capability index Cpw. Several properties of $\hat{C}pw$ are investigated. Much of the controversy surrounding the Cp index involves 6σ in the denominator. It carries particular physical meaning when the process characteristic is normally distributed. A new index Cpo is proposed which is based on the difference between two order statistics. The sampling distribution of $\hat{C}po$ is obtained for those cases where the process characteristic is uniform, exponential and normal distributions. The behavior of $\hat{C}p$, when $n = 2$, under non-normal situations such as uniform and exponential distributions is also investigated as a special case of $\hat{C}po$.

Another major issue addressed in this dissertation is the Inverted Probability Loss Functions (IPLFs). It is a modified loss function found by inverting a probability density function which was first invented by my supervisor Dr. F.A. Spiring in 1993. The first loss function I studied is the inverted beta loss function (IBLF). I have found certain interesting properties that this class of loss function possesses such as the shape, the loss function and its associated risk function of the IBLF are scale invariant under linear transformation. Finally, I have investigated a few more IPLFs satisfying the usual loss function properties and developed some theorems in this portion of the study.

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Symbols and Notations

cdf	Cumulative distribution function
pdf	Probability density function
LSL	Lower specification limit
USL	Upper specification limit
T	Target value of a process
μ	Mean of a process
σ^2	Variance of a process
Cp	$= \dfrac{\text{Allowable process spread}}{\text{Actual process spread}} = \dfrac{\text{USL} - \text{LSL}}{6\sigma}$, the process capability index
Cpm	$= \dfrac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - \text{T})^2}}$, the modified process capability index
Cpm*	$= \dfrac{\min[\text{USL} - \text{T}, \text{T} - \text{LSL}]}{3\sqrt{\sigma^2 + (\mu - \text{T})^2}}$, the generalized Cpm index
Cpo	$= \dfrac{\text{USL} - \text{LSL}}{\text{D}}$, the process capability index based on order statistics
$Y_1, Y_2, \ldots, Y_n$	Order statistics of a random sample of size n
D	$\xi_s - \xi_r$, the difference between the sth and rth population quantiles
ξ_γ	The γth population quantile, where $P(X \le \xi_\gamma) = \gamma$

Cpw	$=\dfrac{USL-LSL}{6\sqrt{\sigma^2+w(\mu-T)^2}}$, the unifying process capability index
Cpw*	$=\dfrac{\min[USL-T, T-LSL]}{3\sqrt{\sigma^2+w(\mu-T)^2}}$, the generalized process capability index
w	A non-stochastic weight
n	Sample size
$X_1, X_2, \ldots, X_n$	Random sample of size n
$\overline{X}$	$=\dfrac{\sum_{i=1}^{n} X_i}{n}$, sample mean
$\hat{\sigma}^2$	$=\dfrac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n}$
S^2	$=\dfrac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}$, sample variance, unbiased
$\hat{D}$	$= Y_s - Y_r$, the difference between the rth and sth order statistics
a	$=\dfrac{\sqrt{n}[USL-LSL]}{6\sigma}$
a*	$=\dfrac{\sqrt{n}\min[USL-T, T-LSL]}{3\sigma}$
a'	$=$ USL - LSL
a_1	$=\dfrac{\sqrt{n-1}[USL-LSL]}{6}$

a_1*	$=\frac{\sqrt{n-1}\min[USL-T, T-LSL]}{3}$
$\hat{C}pm$	$=\frac{USL-LSL}{6\sqrt{\hat{\sigma}^2+(\overline{X}-T)^2}}$, an estimate of Cpm
	$=\frac{USL-LSL}{6\sqrt{S^2+\frac{n(\overline{X}-T)^2}{n-1}}}$, another form of estimate of Cpm
$\hat{C}pm*$	$=\frac{\min[USL-T, T-LSL]}{3\sqrt{\hat{\sigma}^2+(\overline{X}-T)^2}}$, an estimate of Cpm*
	$=\frac{\min[USL-T, T-LSL]}{3\sqrt{S^2+\frac{n(\overline{X}-T)^2}{n-1}}}$, another form of estimate of Cpm*
$\hat{C}po$	$=\frac{USL-LSL}{\hat{D}}$, the estimate of Cpo
$\chi^2_{n,\lambda}$	Non-central chi-square distribution with n degrees of freedom (df) and non-centrality parameter λ
λ	$=\frac{n(\mu-T)^2}{\sigma^2}$, non-centrality parameter
λ_3	Skewness
λ_4	Kurtosis
w	Non stochastic weight
K, K_1, K_2	Maximum loss

$\pi(x, T)$	Function of the form of a pdf
$m = \sup_x \pi(x,T)$	Supremum or maximum of $\pi(x, T)$
LIR	$= \dfrac{\pi(x,T)}{m}$, loss inversion ratio
$Be(\alpha,\beta)$	Beta distribution with parameters α and β
$G(\alpha,\beta)$	Gamma distribution with parameters α and β
$N(\mu,\sigma^2)$	Normal distribution with parameters μ and σ^2
$U(\alpha,\beta)$	Uniform distribution with parameters α and β
$W(\alpha,\beta)$	Weibull distribution with parameters α and β
IBLF	Inverted Beta Loss Function
IGLF	Inverted Gamma Loss Function
INLF	Inverted Normal Loss Function
IPLF	Inverted Probability Loss Function
$L(x,T)$	$= K\left[1 - \dfrac{\pi(x,T)}{m}\right]$, IPLF
$\Gamma(\alpha)$	$= \int_0^\infty x^{\alpha-1} e^{-x}\, dx$, $\alpha > 0$, gamma function
$\Gamma[\alpha,0,z]$	$= \int_z^\infty x^{\alpha-1} e^{-x}\, dx$, $\alpha > 0$, incomplete gamma function
$B(\alpha,\beta)$	$= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, beta function
$B_T(\alpha,\beta)$	$= \int_0^T x^{\alpha-1}(1-x)^{\beta-1}\, dx$, incomplete beta function

$I_T(\alpha,\beta)$	$= \int_0^T \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$, incomplete beta function
${}_nC_i$	General binomial coefficient
μ_r'	The rth raw moment of a random variable
$\mathrm{erf}(t)$	$= \frac{2}{\sqrt{\pi}}\int_0^t \exp\left[-u^2\right]du$, $t > 0$, error function
${}_1F_1[a,b;z]$	$= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt$, $a > 0$, $b - a > 0$, confluent hypergeometric function
${}_2F_1[b,c,d;z]$	$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 (1-tz)^{-b}t^{c-1}(1-t)^{d-c-1}dt$, $c > 0$, $d - c > 0$, confluent hypergeometric function

Chapter 1

Introduction

1.1 Overview

Those measures of process capability known as process capability indices (PCIs) have been used in industry for more than 20 years. Since the introduction of process capability indices popularized by Juran (1974), second and third generation indices have been developed as well as a myriad of other measures have proliferated in both application and variety. The widespread use and abuse of process capability measurements has led to improvements in quality while also becoming topics of considerable controversy in the last few years. Many of the controversies could be avoided through better knowledge of the properties associated with the various measures of process capability indices.

1.2 Introduction of Cp Index

The process capability index, Cp, was first introduced by Juran (1974) and it has been used extensively in manufacturing during the early 80's in Japan. The Cp index has been defined to be

$$\mathrm{Cp} = \frac{\text{Allowable process spread}}{\text{Actual process spread}}.$$

The allowable process spread is generally taken to be the difference between the upper specification limit and the lower specification limit while the actual process spread is represented by 6σ where σ is the process standard deviation associated with the measurement of a specified characteristic (i.e., X). Cp is generally calculated as follows

$$Cp = \frac{USL - LSL}{6\sigma}.$$

The traditional assumptions associated with Cp include

(1) the characteristic measurements arise from a normal distribution;

(2) the measurements are taken only when the process is in control; and

(3) the target of the characteristic is the midpoint of USL and LSL.

The traditional estimator of Cp is defined as

$$\hat{C}p = \frac{USL - LSL}{6S}. \qquad [1.2.1]$$

The probability density function, expectation, and mean squared error of $\hat{C}p$ as defined in equation [1.2.1] have been developed in Chan, Cheng and Spiring (1988c) and Chou and Owen (1989). Most studies that deal with the estimation of Cp are based on the above assumptions (Kane (1986a)). The impact of non-normal processes including mixtures of two normal distributions possessing different means but similar variance (Kocherlakota, Kocherlakota and Kirmani (1992)); and a single normal distribution distorted with different values of skewness and kurtosis (Chan, Cheng and Spiring (1988c)) have been documented. While others have promoted alternative techniques including Clements (1989) which uses the difference between the upper (at the 99.865^{th}) percentile and the lower (at the $.135^{th}$) percentile as a measure of actual process spread. Chan, Cheng and Spiring (1988a) used $Cp^{*} = \frac{USL-LSL}{d}$, with d denoting the width of the interval expected to contain 99.73% of the process measurements. Their goal was to use the width of the tolerance interval with 99.73% coverage 95% of the time (i.e., w) rather then 6S as a measure of actual process spread to assess $\hat{C}p$. A similar approach was proposed by Pearn, Kotz and Johnson (1992). They use

$$C_\theta = \frac{USL - LSL}{\theta\sigma}$$

as a robust capability index developed to be as insensitive as possible to non-normal data. The constant θ is chosen such that the probability of coverage, $P(\mu - \theta\sigma < X < \mu + \theta\sigma)$, is close to one and as independent of the original distribution as possible.

1.3 The Development of Other Process Capability Indices

Many researchers (Hsiang and Taguchi, (1985), Chan, Cheng and Spiring (1988b)) point out that Cp does not incorporate a target value into its determination. Second generation process capability indices (Kane (1986a), Kane (1986b)) attempt to incorporate deviations from the target value, T into their assessment of process capability. The list of second generation PCIs include

$$Cpu = \frac{USL - \mu}{3\sigma}$$

$$Cpl = \frac{\mu - LSL}{3\sigma}$$

$$Cpk = \min[Cpl, Cpu]$$

and

$$Cpk^* = (1\text{-}k)\, Cp$$

where $k = \dfrac{2|T - \mu|}{USL - LSL}$, $0 \le k \le 1$, and $LSL < \mu < USL$.

These indices attempt to take into account process variation as well as departures from the target value in their assessment of process capability. Each of these indices involves the unknown parameters μ and σ^2 which generally must be estimated, resulting in the following estimators

$$\hat{C}pu = \frac{USL - \overline{X}}{3S}$$

$$\hat{C}pl = \frac{\overline{X} - LSL}{3S}$$

$$\hat{C}pk = \min[\hat{C}pl, \hat{C}pu]$$

$$\hat{C}pk = (1 - \hat{k})\,\hat{C}p$$

where

$$\hat{k} = \frac{2|T - \overline{X}|}{USL - LSL}.$$

These estimators provide reasonable point estimators for their respective indices, but the statistical distributions associated with them are quite complicated.

The distributions (Chow and Owen (1989)) of the estimated process capability indices including $\hat{C}pk$, $\hat{C}pl$ and $\hat{C}pu$ are determined, their means, variances and mean squared errors are given. Interval estimations and skewness of $\hat{C}pk$ (Zhang, Stenback and Wardrop (1990)), confidence bounds of $\hat{C}pk$, $\hat{C}pl$ and $\hat{C}pu$ (Kushler and Hurley (1992)) are investigated under the assumption that process measurements are independent and normally distributed. A number of authors have proposed modifications of process capability indices that take into account to some extent the centre, or the target value, of the process and others that perhaps are more appropriate for non-normal situations.

The third generation of PCIs include measures such as Cpm which is defined as

$$\mathrm{Cpm}=\frac{\mathrm{USL}-\mathrm{LSL}}{6\sqrt{\sigma^2+(\mu-\mathrm{T})^2}} \qquad [1.3.1]$$

proposed independently by Hsiang and Taguchi (1985) and Chan, Cheng and Spiring (1988b), that incorporate the proximity to the target value as well as the process variation when assessing process performance. The sampling distribution, under the normality assumption, for an estimate of Cpm ($\hat{\mathrm{C}}\mathrm{pm}$) and some of its properties (Chan, Cheng and Spiring (1988b)) are examined. Estimators, bias and mean squared error of $\hat{\mathrm{C}}\mathrm{pm}$ are investigated and various approximate confidence intervals (Subbaiah and Taam (1993)) are obtained and compared in terms of coverage probabilities, missed rate and average interval width. The robustness of $\hat{\mathrm{C}}\mathrm{pm}$ to departures from normality is studied and extended to include the PCI

$$\mathrm{Cpm}^* = \frac{\min[\mathrm{USL}-\mathrm{T},\mathrm{T}-\mathrm{LSL}]}{3\sqrt{\sigma^2+(\mu-\mathrm{T})^2}} \qquad [1.3.2]$$

so as to include asymmetric specification limits. Derivation of the distribution and its properties are studied and summarized in Chapter 2. Critical values for estimating Cpm are suggested for small sample sizes, and the control chart constants used for monitoring Cpm (Spiring (1995)) when the normal distribution is slightly distorted by skewness and kurtosis (Gayen (1949), Barton and Dennis (1952), Draper and Tierney (1972)) are tabulated.

The relationship between Cpm and the expected squared error loss provides an intuitive interpretation of Cpm. Johnson (1992) relates Cpm to squared error loss and this loss is expressed in a relative manner such that users need to specify the target and the distance from the target at which the product would have zero worth. Confidence limits for the expected relative loss are also discussed. A similar relationship between Cpm and the estimated expected loss is proposed, and the upper confidence limits for the loss function parameters and its approximation suggested.

1.4 Examination of Cp Index in Various Distributions

The sampling distribution of $\hat{C}p$ has been established under the assumption of normality. We will examine the distribution of $\hat{C}p$ when the process characteristic follows uniform and exponential distributions. However, due to the difficulties of the sampling distribution of S under different distributions, the sample size is limited to n = 2. Comparisons with the normal distribution results with respect to expectations, mean squared errors (if such exist), probabilities and the related critical values (lower, c_L, and upper, c_U) are tabulated and summarized in Chapter 3 as special cases of $\hat{C}po$ (a proposed process capability measure introduced in the following section).

1.5 Proposed Measure of Cp Index Irrespective to Normality Assumption

It is well known that the distribution of the sample standard deviation, S, is not robust to non-normal pdfs (Nelson (1992)) and departures from normality hinder the effectiveness of the estimators in drawing inferences regarding population parameters. In Chapter 3, a proposed index Cpo, which is based on order statistics, and is defined as

$$Cpo = \frac{USL - LSL}{D} \quad [1.5.1]$$

where $D = \xi_s - \xi_r$, with $P(X<\xi_\gamma) = \gamma$, $0 < \gamma < 1$, $r < s$, ξ_i the ith quantile, is investigated.

The distribution of $\hat{D}$, the difference between the rth and sth order statistics, can be obtained for various distributions of X and hence the distribution of

$$\hat{C}po = \frac{USL-LSL}{\hat{D}} \qquad [1.5.2]$$

can also be found. The pdf of $\hat{C}po$ under different distributional forms including the uniform, exponential, normal distributions are examined.

1.6 Unifying Approach of Process Capability Indices

In an attempt to summarize the process capability indices as one simple form, several authors (Pearn, Kotz and Johnson (1992); Vännman (1995); Spiring (1997)) have proposed a general form of PCI that encompasses a wide variety of existing PCIs. Vännman (1995) proposes the index

$$C_p(u,v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}},$$

where $d = \frac{USL-LSL}{2}$, $M = \frac{USL+LSL}{2}$, $u \geq 0$, $v \geq 0$.

Most existing indices are then considered as special cases of $C_p(u,v)$. For example, letting $u = 0$ and $v = 0$, results in

$$C_p(0,0) = \frac{d}{3\sigma} = Cp$$

while $u = 1$, $v = 0$, it produces

$$C_p(1,0)=\frac{d-|\mu-M|}{3\sigma}=Cpk$$

$u=0, v=1,$

$$C_p(0,1)=\frac{d}{3\sqrt{\sigma^2+(\mu-T)^2}}=Cpm$$

and $u=1, v=1,$

$$C_p(1,1)=\frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}}=Cpmk.$$

The derivation of the pdf can be found in Vännman and Kotz (1995a), expectations and mean squared errors can be seen in Vännman and Kotz (1995b) and Vännman (1995). An alternative algorithm was proposed by (Spiring (1997)) which suggested using:

$$Cpw=\frac{USL-LSL}{6\sqrt{\sigma^2+w(\mu-T)^2}} \qquad [1.6.1]$$

$$Cpw^*=\frac{\min[USL-T,T-LSL]}{3\sqrt{\sigma^2+w(\mu-T)^2}} \qquad [1.6.2]$$

where w is a non-stochastic weight. The pdf of $\hat{C}pw$ (see equation [4.2.1]) developed in Spiring (1997) is based on a mixture of central chi-squared distributions. Similar to Vännman (1995) the most common indices can be formed as special cases of Cpw by putting different weights. For example, assuming $T=\frac{USL+LSL}{2}$, by setting

(i) $w=0$, results in

$$Cpw=\frac{USL-LSL}{6\sigma}=Cp$$

(ii) for w = 1,

$$Cpw = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}} = Cpm$$

(iii) when $w = \dfrac{k(2-k)}{(1-k)^2 p^2}$, with $p = \dfrac{|\mu-T|}{\sigma}$

$$Cpw = \frac{USL-LSL}{6\sqrt{\sigma^2+\dfrac{k(2-k)}{(1-k)^2p^2}(\mu-T)^2}}$$

$$=(1-k)\frac{USL-LSL}{6\sigma}$$

$$= (1-k)\,Cp$$

$$= Cpk^*.$$

(iv) similarly for $w = \left[\left(\dfrac{d}{d-|a|}\right)^2 - 1\right]\dfrac{1}{p^2}$, with $d = \dfrac{USL-LSL}{2}$, $a = \mu - \dfrac{USL-LSL}{2}$,

$$Cpw = \frac{USL-LSL}{6\sqrt{\sigma^2+\left[\left(\dfrac{d}{d-|a|}\right)^2-1\right]\dfrac{1}{p^2}(\mu-T)^2}}$$

$$= \left[1-\frac{\left|\mu-\frac{1}{2}(USL+LSL)\right|}{d}\right]Cp$$

$$= Cpk.$$

The statistical properties of $\hat{C}pw$ including its pdf and associated confidence intervals are investigated, analogous to Spiring (1997). The relationship between Cpw and the estimated expected weighted loss is discussed and upper confidence limits for this loss function parameters are illustrated. Further study of $\hat{C}pw$ including its density under normal distribution distorted by skewness and kurtosis (Gayen (1949)) is discussed in Chapter 4.

1.7 Loss Functions

In decision theory, loss functions are used to describe the deviation of an estimator from a parameter value. Loss functions traditionally take forms such as squared error loss, absolute error loss, weighted loss and 0-1 loss. Each of these forms tacitly assumes that the larger the error made in estimating the parameter value the larger the loss incurred. Different levels of penalties are inherent to each form the loss function takes. Keeping these in mind, statisticians and practitioners make use of this concept to develop new applications in quality settings. This idea helps to stress the importance of being on target for both customers and suppliers. The use of loss functions has increased steadily in industrial applications.

1.8 Modified Loss Functions

The loss function approach for assessing quality was first proposed by Taguchi (1986) who uses a modified squared error loss (quadratic loss) function to assess and illustrate losses to society associated with departures from a process target. Taguchi's modification added a bound to the usual quadratic loss function in order to avoid an infinite penalty for those measures situated large distances from the target. Spiring (1993) proposed an Inverted Normal Loss Function which differed from the traditional quadratic loss by providing a bounded, and hence more reasonable, assessment of economic loss. Claiming that the INLF severely penalizes off-targetness, Sun, Laramee and Ramberg (1996) refined the INLF and provided nonlinear least squares estimates for the shape parameter of the modified loss function. Spiring and Yeung (1998) developed a class of loss functions based on inverted pdfs including the gamma, chi-square, Laplace and Tukey's Symmetric Lambda distributions. This general class of loss functions has nice properties and can accurately reflect symmetric and asymmetric losses incurred by the process. However the various loss functions in this class have parameters nested in their associated pdfs. In conjunction with these Inverted Probability Loss Functions there is a limited number of conjugate distributions for the loss functions to select in order to assess the average loss or the risk function associated with the process.

1.9 Another Family of Loss Function

In Chapter 5, a family of loss functions is developed based on an inverted beta pdf. The shape of the Inverted Beta Loss Function can be modified to suit the practitioner's needs, while providing all the properties of the above mentioned loss functions. By restricting this family of loss functions to those derived from the beta pdf we manage to provide a wide variety of potential loss functions while maintaining one set of parameters for the entire family. The conjugate distribution can be used to characterize the process measurements and has finite moments, allowing the risk function generated by the IBLF to be evaluated and depicting the true average loss/cost associated with off-targetness.

1.10 Some Properties of the Inverted Probability Loss Function

The invention of inverting a normal pdf to assess loss functions for off-targetness is primarily introduced by Spiring (1993). The development of this type of loss functions is further developed utilizing other density functions. However the properties of the family of Inverted Probability Loss Functions has not been fully studied. In Chapter 6 several properties of this family of IPLFs are investigated. A few particular IPLFs have unique and interesting properties that can help practitioners to assess these loss functions correctly and appropriately. For each IPLF considered, some plausible conjugate distributions are suggested and worked out for comparison purposes. The general forms of the expected value of the loss inversion ratio (i.e., $E\left[\left[\frac{\pi(X,T)}{m}\right]^r\right]$) under each IPLF are listed as theorems, followed by the associated mean (risk function) and the variation arising from different distributions of the process characteristic. The general performances of the IPLFs are compared numerically under homogeneous conditions.

Chapter 2

The Index Cpm

2.1 Introduction

The process capability index Cpm is used to provide an assessment of the ability of the process to be clustered around the target. As Cpm is not traditionally used to provide insights into the number of parts non-conforming the Cpm parameter does not require 6σ to reflect a precise number of non-conforming. As a result, unlike other capability indices including Cp, Cpu, Cpl and Cpk, the Cpm index can provide practitioners with meaningful information in non-normal settings. The robustness of an estimator of Cpm to distributional assumptions and the resulting impact on the inferences is investigated.

With the capability indices receiving increased usage in process assessments and purchasing decision in the industry, the indices Cp, Cpk and Cpm were of particular interest. These indices are easy to compute and interpret, and they are convenient for use by quality practitioners because these are based on traditional specification limits. Nevertheless, some of them are not related to the loss incurred in failing to meet customers' requirement. Taguchi (1986) emphasized the loss in a product's worth when one of its characteristics departs from the customers' target value. Johnson (1992) related the Cpm index to the symmetric squared error loss and expressed the loss in a relative manner so that the users need only to specify the target and the distance from the target where the product would have zero worth. Upper confidence limits and its approximation for this expected relative loss were illustrated. In this chapter, a similar relationship between Cpm and squared error loss is established which is based on the estimated loss other than the relative loss. Upper confidence limits and its approximation for the loss function are discussed.

2.2 Measuring Process Capability

Process Capability indices are used to assess the ability of a process to meet customer specifications. There are many indices currently available, with the most well known being Cp. Cp is often referred to as a measure of process potential rather than process capability, as it fails to consider where the process measurements are located.

Processes with small variability, but poor proximity to a target, have sparked the derivation of several indices that attempt to incorporate a target into their assessment of process capability. The most common of these measures assume T to be the midpoint of the specification limits and include Cpm and Cpk.

The process capability indices Cp, Cpl, Cpu, Cpk and Cpm belong to the family of indices that relate customer requirements to process performance in the form of a ratio. As process performance improves, either through reductions in variation and/or moving closer to the target, these indices increase in magnitude for fixed customer requirements. In each case larger index values indicate a more capable process.

2.3 Effects of Non-Normality

If the process measurements do not arise from a Normal distribution none of the indices discussed in Section 2.2 provide valid measures of the number of parts non-conforming. Each index uses a function of σ as a measure of actual process spread in its determination of process capability. But as several authors (Hoaglin, Mosteller and Tukey (1983), Mosteller and Tukey (1977), Tukey (1970) and Huber (1977)) have pointed out, that although the standard deviation has become synonymous with the term "dispersion", its physical meaning needs not be the same for different families of distributions, or for that matter, within a family of distributions. Therefore the actual process spread (a function of 6σ) does not provide a consistent meaning over various distributions. To illustrate, suppose that precisely 99.73% of the process measurements fall within the specification limits. The values of Cp are 0.5766, 0.7954, 1.0000, 1.2210 and 1.4030 respectively when the measurements arise from a uniform, triangular, normal, logistic and double exponential distribution. As long as 6σ carries some practical interpretation when assessing process capability (i.e., is translated into ppm non-conforming), none of the indices should be used if the distribution of the characteristic under investigation is not normal.

If we assume process capability assessments to be studies of the ability of the in-control process to produce product around the target, then Cpm will provide practitioners with an assessment of capability regardless of the distribution associated with the measurements. Clustering around the target, rather than a measure of non-conforming releases the physical meaning attached to 6σ. The denominator of Cpm then provides a measure of the clustering around the target and compares this with customer tolerance.

Eliminating the physical meaning allows Cpm to be used to compare the capability of various processes (or processes over time) regardless of the underlying distribution. However the underlying distribution will impact the inferences that we can make from samples gathered from the population. The effects of non-normality on an estimator of Cpm are examined.

In order to assess process capability using Cpm in the presence of non-normal process measurements we need to better understand the effect of non-normality on the behaviour of the estimated process capability (i.e., $\hat{C}pm$). Moderate departures from normality can be emulated using a modified Gayen (1949) approach where the pdf associated with $\sum_{i=1}^{n}(X_i-T)^2$ (where $X_1, X_2, \ldots, X_n$ represents n observations selected randomly from a population) is transformed to reflect the pdf associated with $\hat{C}pm$. The third and fourth moments are then varied to examine the impact of moderate departures from normality, for the characteristic under investigation, on the distribution of $\hat{C}pm$. The results provide practitioners with a graphical view of the impact of non-normality as well as providing mechanisms for analyzing and correcting for the impact of non-normality.

2.4 Generating and Estimating Cpm

Cpm was previously defined in equation [1.3.1] and its generalized form (i.e., Cpm*), which includes the original definition of Cpm (i.e., when USL-T = T-LSL) in equation [1.3.2]. Cpm* continues to reflect changes in the process analogous to other measures of the process capability while allowing T to be any value between LSL and

USL. If the process variance (σ^2) increases, the denominator in equation [1.3.2] increases and Cpm* will decrease indicating that the process is less capable. If the process drifts from its target value (i.e., if μ moves away from T), the denominator of equation [1.3.1] will again increase causing Cpm* to decline, again indicating that the process is less capable.

In this section we will examine some statistical properties of $\hat{C}pm$ and $\hat{C}pm^*$ including means and variances. For $X \sim N(\mu,\sigma^2)$, and

$$\hat{C}pm = \frac{USL-LSL}{6\sqrt{\hat{\sigma}^2+(\overline{X}-T)^2}} \qquad [2.4.1]$$

$$= \frac{USL-LSL}{6\frac{\sigma}{\sqrt{n}}\sqrt{\frac{n\hat{\sigma}^2}{\sigma^2}+\frac{n(\overline{X}-T)^2}{\sigma^2}}}$$

$$= \frac{a}{\sqrt{\theta+\eta}}$$

where $a = \dfrac{\sqrt{n}\,[USL-LSL]}{6\sigma} = \sqrt{n+\lambda}\ Cpm$

$\theta = \dfrac{n\hat{\sigma}^2}{\sigma^2}$, central chi square distribution with n - 1 df

$\eta = \dfrac{n(\overline{X}-T)^2}{\sigma^2}$, non-central chi square distribution with 1 df and non-centrality parameter $\lambda = \dfrac{n(\mu-T)^2}{\sigma^2}$, and is independent of θ

$\theta + \eta \sim \chi^2_{n,\lambda}$, non-central chi square distribution with n df and non-centrality parameter λ (Johnson, Kotz and Balakrishnan (1995)).

Similarly, $$\hat{C}pm^* = \frac{\min[USL-T, T-LSL]}{3\frac{\sigma}{\sqrt{n}}\sqrt{\frac{n\hat{\sigma}^2}{\sigma^2}+\frac{n(\overline{X}-T)^2}{\sigma^2}}} \qquad [2.4.2]$$

$$= \frac{a^*}{\sqrt{\theta+\eta}}$$

where $a^* = \dfrac{\sqrt{n}\min[USL-T, T-LSL]}{3\sigma} = \sqrt{n+\lambda}\ Cpm^*$

Theorem 2.4.1:

The rth moment of $\hat{C}pm$ when $X \sim N(\mu, \sigma^2)$ is

$$E\left(\hat{C}pm^r\right) = \left(\frac{n+\lambda}{2}Cpm^2\right)^{\frac{r}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}, \text{ for } r < n. \qquad [2.4.3]$$

Proof:

$$E\left(\hat{C}pm^r\right) = a^r E\left[(\theta+\alpha)^{-\frac{r}{2}}\right]$$

$$= a^r 2^{-\frac{r}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}$$

$$= \left(\frac{n+\lambda}{2}Cpm^2\right)^{\frac{r}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}$$

where $P_j = \dfrac{\exp\left(-\frac{\lambda}{2}\right)\left(\frac{\lambda}{2}\right)^j}{j!}$, the Poisson weights.

The mean and variance of $\hat{C}pm$ are respectively

$$E\left(\hat{C}pm\right)=\sqrt{\frac{n+\lambda}{2}}\,Cpm\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} \tag{2.4.4}$$

$$V\left(\hat{C}pm\right)=\frac{n+\lambda}{2}Cpm^2\left\{\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n-2}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)}-\left[\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)}\right]^2\right\} \tag{2.4.5}$$

The rth moment of $\hat{C}pm*$ is similarly obtained and stated without proof in the following theorem.

Theorem 2.4.2:

The rth moment of $\hat{C}pm*$ when $X \sim N(\mu,\sigma^2)$ is

$$E\left(\hat{C}pm*^r\right)=a*^r\,E\left[(\theta+\alpha)^{-\frac{r}{2}}\right]=a*^r\,2^{-\frac{r}{2}}\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} \tag{2.4.6}$$

where $a*=\dfrac{\sqrt{n}\min[USL-T,T-LSL]}{3\sigma}=\sqrt{n+\lambda}\,Cpm*$.

Now, let us examine the biases and mean squared errors of $\hat{C}pm$ and $\hat{C}pm*$. Let $B\left(\hat{\theta}\right)$ and $MSE\left(\hat{\theta}\right)$ be the bias and mean squared error respectively of an estimator $\hat{\theta}$, then

$$B(\hat{C}pm) = E(\hat{C}pm) - Cpm$$

$$= \sqrt{\frac{n+\lambda}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} Cpm - Cpm$$

$$= \left[\sqrt{\frac{n+\lambda}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} - 1 \right] Cpm \qquad [2.4.7]$$

$$MSE(\hat{C}pm) = E\left[(\hat{C}pm - Cpm)^2\right] = V(\hat{C}pm) + \left[B(\hat{C}pm)\right]^2$$

$$= \frac{n+\lambda}{2} \left\{ \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-2}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} - \left[\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} \right]^2 \right\} Cpm^2 +$$

$$\left[\sqrt{\frac{n+\lambda}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} - 1 \right]^2 Cpm^2$$

$$= \left[\frac{n+\lambda}{2} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-2}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} - 2\sqrt{\frac{n+\lambda}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} + 1 \right] Cpm^2 \qquad [2.4.8]$$

$$B(\hat{C}pm*) = E(\hat{C}pm*) - Cpm*$$

$$= \left[\sqrt{\frac{n+\lambda}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} - 1 \right] Cpm* \qquad [2.4.9]$$

$$MSE(\hat{C}pm*) = E\left[(\hat{C}pm* - Cpm*)^2\right] = V(\hat{C}pm*) + [B(\hat{C}pm*)]^2$$

$$= \left[\frac{n+\lambda}{2} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-2}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} - 2\sqrt{\frac{n+\lambda}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} + 1 \right] Cpm*^2 \quad [2.4.10]$$

Under the assumption $\mu = T$, i.e., equivalently $\lambda = 0$, hence the biases and mean squared errors of $\hat{C}pm$ and $\hat{C}pm*$ become

$$B(\hat{C}pm) = \left[\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 1 \right] Cpm$$

$$B(\hat{C}pm*) = \left[\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 1 \right] Cpm*$$

$$MSE(\hat{C}pm) = \left[\frac{n}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 2\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} + 1 \right] Cpm^2$$

$$MSE(\hat{C}pm*) = \left[\frac{n}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 2\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} + 1 \right] Cpm*^2$$

Taking limits as n approaches ∞, it can be shown that

$$\lim_{n\to\infty} \frac{n}{2}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}=1 \text{ and } \lim_{n\to\infty} \sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}=1.$$

Thus both the biases of $\hat{C}pm$ and $\hat{C}pm*$ become zero, i.e.,

$$B\left(\hat{C}pm\right)=0, \text{ and } B\left(\hat{C}pm*\right)=0.$$

These imply that both $\hat{C}pm$ and $\hat{C}pm*$ are asymptotically unbiased. Also, the mean squared errors of $\hat{C}pm$ and $\hat{C}pm*$ become zero, i.e.,

$$MSE\left(\hat{C}pm\right)=0, \text{ and } MSE\left(\hat{C}pm*\right)=0.$$

These imply that both $\hat{C}pm$ and $\hat{C}pm*$ are mean squared error consistent. These also imply that both $\hat{C}pm$ and $\hat{C}pm*$ converge in probability to Cpm and Cpm*, respectively.

2.5 Effects of Non-Normality on $\hat{C}pm$

The generalized Cpm, Cpm*, can be estimated by

$$\hat{C}pm* = \frac{\min[USL-T, T-LSL]}{3\sqrt{S^2+\frac{n(\overline{X}-T)^2}{n-1}}}. \qquad [2.5.1]$$

Defining $\sigma' = \sqrt{E\left[(X-T)^2\right]} = \sqrt{\sigma^2+(\mu-T)^2}$ and letting $X_1, X_2, \ldots, X_n$ denote a random sample of size n, an estimator of σ' is given by

$$\hat{\sigma}'=\sqrt{\frac{\sum_{i=1}^{n}(X_i-T)^2}{n-1}}=\sqrt{\frac{\sum_{i=1}^{n}\left[(X_i-\overline{X})^2+(\overline{X}-T)\right]^2}{n-1}}$$

$$=\sqrt{\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2+n(\overline{X}-T)^2}{n-1}}$$

$$=\sqrt{\frac{(n-1)S^2+n(\overline{X}-T)^2}{n-1}}$$

$$=\sqrt{S^2+\frac{n}{n-1}(\overline{X}-T)^2}$$

and

$$\hat{C}pm=\frac{USL-LSL}{6\hat{\sigma}'}=\frac{USL-LSL}{6\sqrt{\frac{Y}{n-1}}}$$

$$=\frac{a_1}{\sqrt{Y}}. \qquad [2.5.2]$$

where $Y=\sum_{i=1}^{n}(X_i-T)^2$.

Similarly, $\hat{C}pm*$ is defined as

$$\hat{C}pm*=\frac{\min[USL-T,T-LSL]}{3\sqrt{\frac{Y}{n-1}}}=\frac{a_1^*}{\sqrt{Y}}. \qquad [2.5.3]$$

Using Gayen (1949), the distribution of $\hat{C}pm$ and $\hat{C}pm*$ can be obtained from a Normal distribution distorted with nonzero values of λ_3 and λ_4, where λ_3 and λ_4 are measures of skewness and kurtosis, respectively. The resulting distribution of $\hat{C}pm$ (similarly for $\hat{C}pm*$) can be obtained through transformation.

Consider a random sample of n measurements, $X_1, X_2, \ldots, X_n$, of a characteristic taken from a process which is in-control, then the sum of squares of the measurements with respect to the target value, T, is

$$\sum_{i=1}^{n}(X_i - T)^2 = n(\overline{X} - T)^2 + (n-1)S^2$$

$$= n\left(\frac{S_1}{n} - T\right)^2 + S_2$$

where $S_1 = \sum_{i=1}^{n} X_i$,

$$S_2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}(X_i - T)^2 - n\left(\frac{S_1}{n} - T\right)^2.$$

Let the transformation be $\begin{cases} Y = \sum_{i=1}^{n}(X_i - T)^2 = n\left(\frac{S_1}{n} - T\right)^2 + S_2 \\ Z = \left(\frac{S_1}{n} - T\right)^2 \end{cases}$

and results in $\begin{cases} y = n\left(\frac{s_1}{n} - T\right)^2 + s_2 \\ z = \left(\frac{s_1}{n} - T\right)^2 \end{cases}$

with inverse transformation $\begin{cases} s_1 = n(\pm\sqrt{z} + T) \\ s_2 = y - nz \end{cases}$

and the Jacobian is $\mathbf{J} = \left|\frac{\partial(s_1, s_2)}{\partial(y, z)}\right| = \begin{vmatrix} 0 & \pm\frac{n}{2\sqrt{z}} \\ 1 & -n \end{vmatrix} = \mp\frac{n}{2\sqrt{z}}$.

If T = 0, then the inverse transformation becomes $\begin{cases} s_1 = \pm n\sqrt{z} \\ s_2 = y - nz \end{cases}$ and the Jacobian of the inverse transformation remains unchanged. Hence, following Gayen (1949) equation [2.1], we have:

$$W_1(n-1) = \frac{\exp\left(-\frac{y}{2}\right)(y-nz)^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}\sqrt{2\pi n}}$$

$$= \frac{\exp\left(-\frac{y}{2}\right)(y-nz)^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}\sqrt{n}} = W_2(n-1) = W(n-1)$$

and the Hermite polynomials become

$$\begin{aligned} H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} H_1(-\sqrt{z}) &= -\sqrt{z} \\ H_2(-\sqrt{z}) &= z^2 - 1 \\ H_3(-\sqrt{z}) &= (-\sqrt{z})^3 + 3\sqrt{z} \\ H_4(-\sqrt{z}) &= z^2 - 6z + 3 \end{aligned} \quad \text{and} \quad \begin{aligned} H_1(\sqrt{z}) &= \sqrt{z} \\ H_2(\sqrt{z}) &= z^2 - 1 \\ H_3(\sqrt{z}) &= (\sqrt{z})^3 - 3\sqrt{z} \\ H_4(\sqrt{z}) &= z^2 - 6z + 3. \end{aligned}$$

Note that
$$\begin{aligned} H_1(-\sqrt{z}) &= -H_1(\sqrt{z}) \\ H_3(-\sqrt{z}) &= -H_3(\sqrt{z}) \\ H_2(-\sqrt{z}) &= H_2(\sqrt{z}) \\ H_4(-\sqrt{z}) &= H_4(\sqrt{z}). \end{aligned}$$

The joint density of Y and Z is

$$
\begin{aligned}
h(y,z)=W_1(n-1)\Bigg\{1 &+\frac{n\lambda_3}{6}\left[H_3\left(-\sqrt{z}\right)+3\left(\frac{y}{n}-z\right)H_1\left(-\sqrt{z}\right)\right] \\
&+\frac{n\lambda_4}{24}\left[H_4\left(-\sqrt{z}\right)+6\left(\frac{y}{n}-z\right)H_2\left(-\sqrt{z}\right)+\frac{3(n-1)}{n+1}\left(\frac{y}{n}-z\right)^2\right] \\
&+\frac{n\lambda_3^2}{72}\Bigg[\, n\left(-\sqrt{z}\right)^6-3(2n+3)\left(-\sqrt{z}\right)^4+9(n+4)\left(-\sqrt{z}\right)^2-15 \\
&\qquad +6\left(\frac{y}{n}-z\right)\left(n\left(-\sqrt{z}\right)^4-3(n+3)\left(-\sqrt{z}\right)^2+6\right) \\
&\qquad +\frac{9}{(n+1)}\left(\frac{y}{n}-z\right)^2\left(n(n+1)\left(-\sqrt{z}\right)^2-3(n-1)\right) \\
&\qquad +\frac{6n(n-2)}{(n+3)(n+1)}\left(\frac{y}{n}-z\right)^3 \Bigg]\Bigg\}\left|\frac{n}{2\sqrt{z}}\right|
\end{aligned}
$$

$$
\begin{aligned}
+W_2(n-1)\Bigg\{1 &+\frac{n\lambda_3}{6}\left[H_3\left(\sqrt{z}\right)+3\left(\frac{y}{n}-z\right)H_1\left(\sqrt{z}\right)\right] \\
&+\frac{n\lambda_4}{24}\left[H_4\left(\sqrt{z}\right)+6\left(\frac{y}{n}-z\right)H_2\left(\sqrt{z}\right)+\frac{3(n-1)}{n+1}\left(\frac{y}{n}-z\right)^2\right] \\
&+\frac{n\lambda_3^2}{72}\Bigg[\, n\left(\sqrt{z}\right)^6-3(2n+3)\left(\sqrt{z}\right)^4+9(n+4)\left(\sqrt{z}\right)^2-15 \\
&\qquad +6\left(\frac{y}{n}-z\right)\left(n\left(\sqrt{z}\right)^4-3(n+3)\left(\sqrt{z}\right)^2+6\right) \\
&\qquad +\frac{9}{(n+1)}\left(\frac{y}{n}-z\right)^2\left(n(n+1)\left(\sqrt{z}\right)^2-3(n-1)\right) \\
&\qquad +\frac{6n(n-2)}{(n+3)(n+1)}\left(\frac{y}{n}-z\right)^3 \Bigg]\Bigg\}\left|\frac{n}{2\sqrt{z}}\right|
\end{aligned}
$$

$$
\begin{aligned}
h(y,z)=&\frac{\exp\left(-\frac{y}{2}\right)(y-nz)^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}\sqrt{n}}\frac{n}{2\sqrt{z}}\Bigg\{\ 2\\
&+\frac{n\lambda_3}{6}\left[H_3(-\sqrt{z})+H_3(\sqrt{z})+3\left(\frac{y}{n}-z\right)\left[H_1(-\sqrt{z})+H_1(\sqrt{z})\right]\right]\\
&+\frac{n\lambda_4}{24}\left[H_4(-\sqrt{z})+H_4(\sqrt{z})+6\left(\frac{y}{n}-z\right)\left[H_2(-\sqrt{z})+H_2(\sqrt{z})\right]+\frac{6(n-1)}{(n+1)}\left(\frac{y}{n}-z\right)^2\right]\\
&+\frac{n\lambda_3^2}{72}\Bigg[\ 2nz^3-6(2n+3)z^2+18(n+4)z-30+12\left(\frac{y}{n}-z\right)\left[nz^2-3(n+3)z+6\right]\\
&\qquad+\frac{18}{(n+1)}\left(\frac{y}{n}-z\right)^2\left[n(n+1)z-3(n-1)\right]+\frac{12n(n-2)}{(n+3)(n+1)}\left(\frac{y}{n}-z\right)^3\ \Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
h(y,z)=&\frac{\exp\left(-\frac{y}{2}\right)(y-nz)^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}\frac{\sqrt{n}}{2\sqrt{z}}\Bigg\{\ 1\\
&+\frac{n\lambda_4}{24}\left[\left(z^2-6z+3\right)+6\left(\frac{y}{n}-z\right)+\frac{3(n-1)}{(n+1)}\left(\frac{y}{n}-z\right)^2\right]\\
&+\frac{n\lambda_3^2}{72}\Bigg[\ nz^3-3(2n+3)z^2+9(n+4)z-15+6\left(\frac{y}{n}-z\right)\left[nz^2-3(n+3)z+6\right]\\
&\qquad+\frac{9}{(n+1)}\left(\frac{y}{n}-z\right)^2\left[n(n+1)z-3(n-1)\right]+\frac{6n(n-2)}{(n+3)(n+1)}\left(\frac{y}{n}-z\right)^3\ \Bigg]\Bigg\}
\end{aligned}
$$

$$0<z<\frac{y}{n},\ 0<y<\infty,\ \text{zero elsewhere.} \quad [2.5.4]$$

In order to find the marginal pdf, $h_1(y)$, of Y we need to integrate z between the limits 0 and $\frac{y}{n}$. By making a substitution $u=\frac{y-nz}{y}$ which makes the integration easier. Now,

let $u=\frac{y-nz}{y}$, then $z=\frac{y}{n}(1-u)$, $\frac{y}{n}-z=\frac{y}{n}u$, and $yu=y-nz$, with $\frac{y}{n}du=-dz$. Thus

$$h(y,u)=h(y,z)\left|\frac{y}{n}\right|,\quad \text{for } 0<u<1,\ 0<y<\infty$$

$$=h\left(y,\frac{y}{n}(1-u)\right)\frac{y}{n}$$

$$=\frac{\exp\left(-\frac{y}{2}\right)y^{\frac{n-1}{2}-1}u^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}\sqrt{n}\,y^{\frac{1}{2}-1}(1-u)^{\frac{1}{2}-1}\frac{y}{n}\Bigg\{1$$

$$+\frac{n\lambda_4}{24}\left[\frac{y}{n}(1-u)^2-6\frac{y}{n}(1-u)+3+6\frac{y}{n}u\left(\frac{y}{n}(1-u)-1\right)+\frac{3(n-1)}{(n+1)}\frac{y^2}{n^2}u^2\right]$$

$$+\frac{n\lambda_3^2}{72}\Bigg[n\frac{y^3}{n^3}(1-u)^3-3(2n+3)\frac{y^2}{n^2}(1-u)^2+9(n+4)\frac{y}{n}(1-u)-15$$

$$+6\frac{y}{n}u\left(n\frac{y^2}{n^2}(1-u)^2-3(n+3)\frac{y}{n}(1-u)+6\right)$$

$$+\frac{9}{(n+1)}\frac{y^2}{n^2}u^2\left[n(n+1)\frac{y}{n}(1-u)-3(n-1)\right]+\frac{6n(n-2)}{(n+3)(n+1)}\frac{y^3}{n^3}u^3\Bigg]\Bigg\}$$

$$
= \frac{\exp\left(-\frac{y}{2}\right) y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}} u^{\frac{n-1}{2}-1}(1-u)^{\frac{1}{2}-1} \left\{ 1 \right.
$$

$$
+ \frac{n\lambda_4}{24}\left[\frac{y^2}{n^2}(1-u)^2 - 6\frac{y}{n}(1-u) + 3 + 6\frac{y^2}{n^2}u(1-u) - 6\frac{y}{n}u + \frac{3(n-1)}{(n+1)}\frac{y^2}{n^2}u^2\right]
$$

$$
+ \frac{n\lambda_3^2}{72}\left[\frac{y^3}{n^2}(1-u)^3 - 3(2n+3)\frac{y^2}{n^2}(1-u)^2 + 9(n+4)\frac{y}{n}(1-u) - 15 \right.
$$

$$
+ 6\frac{y^3}{n^2}u(1-u)^2 - 18(n+3)\frac{y^2}{n^2}u(1-u) + 36\frac{y}{n}u
$$

$$
\left. \left. + 9\frac{y^3}{n^2}u^2(1-u) - 27\frac{(n-1)}{(n+1)}\frac{y^2}{n^2}u^2 + \frac{6n(n-2)}{(n+3)(n+1)}\frac{y^3}{n^3}u^3 \right] \right\}
$$

Hence the marginal pdf of Y is

$$h_1(y)=\int_0^1 h(y,u)\,du$$

$$=\frac{\exp\left(-\frac{y}{2}\right)y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}\left\{\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right.$$

$$+\frac{n\lambda_4}{24}\left[\frac{y^2}{n^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-6\frac{y}{n}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+3\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right.$$

$$\left.+6\frac{y^2}{n^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-6\frac{y}{n}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+\frac{3(n-1)}{(n+1)}\frac{y^2}{n^2}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\right]$$

$$+\frac{n\lambda_3^2}{72}\left[\frac{y^3}{n^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}-3(2n+3)\frac{y^2}{n^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\right.$$

$$+9(n+4)\frac{y}{n}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}-15\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}+6\frac{y^3}{n^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}$$

$$-18(n+3)\frac{y^2}{n^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+36\frac{y}{n}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+9\frac{y^3}{n^2}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}$$

$$\left.\left.-27\frac{(n-1)}{(n+1)}\frac{y^2}{n^2}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+\frac{6n(n-2)}{(n+3)(n+1)}\frac{y^3}{n^3}\frac{\Gamma\left(\frac{n+5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\right]\right\}$$

$0<y<\infty$, zero elsewhere. [2.5.5]

It can be easily shown that for any given sample size n (≥ 2), where λ_3 and λ_4 satisfy the positive definite region described by Barton and Dennis (1952) and Draper and Tierney (1972), then $\int_0^\infty h_1(y)\,dy = 1$. This can be verified by integrating the marginal pdf of Y over the region from 0 to ∞. The positive definite region is defined as follows. When the measures of shape, λ_3 and λ_4, are known a curve of the form

$$f(x)=p_\upsilon(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$$

chosen to have the same first four moments as the pdf of x, is often taken to represent it, where $p_\upsilon(x)=1+\sum_{r=1}^{\upsilon} c_r H_r(x)$ is a υth degree polynomial in x expressed as a sum of constant multiples of Hermite polynomials, $H_r(x)$. When x is in standardized form, the values $c_1 = c_2 = 0$, $c_3 = \frac{\sqrt{\lambda_3}}{3!}$, $c_4 = \frac{\lambda_4}{4!}$ and $c_r = 0$ for $r \geq 5$. Then f(x) is positive definite if $p_\upsilon(x) \geq 0$ for all x. The solution of this positive definite region is described by the co-ordinates $(c_1, c_2, \ldots, c_\upsilon)$ satisfying the following equations

$$1+\sum_{r=1}^{\upsilon} c_r H_r(x) = 0 \text{ and } \sum_{r=1}^{\upsilon} c_r\, r\, H_{r-1}(x) = 0.$$

Now, the constant term inside the curly bracket of equation [2.5.4], when multiplied by the integrand and integrated with respect to y over the region 0 to ∞, is:

$$\int_0^\infty \frac{\exp\left(-\frac{y}{2}\right) y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right) 2^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} dy = \int_0^\infty \frac{\exp\left(-\frac{y}{2}\right) y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} dy = 1$$

The coefficient associated with λ_4 when integrating y from 0 to ∞, becomes:

$$\text{(a)}\quad \frac{1}{n^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\frac{\Gamma\left(\frac{n+4}{2}\right)2^{\frac{n+4}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{3}{n^2}$$

$$\text{(b)}\quad -\frac{6}{n}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\Gamma\left(\frac{n+2}{2}\right)2^{\frac{n+2}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=-\frac{6}{n}$$

$$\text{(c)}\quad 3\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=3$$

$$\text{(d)}\quad \frac{6}{n^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\frac{\Gamma\left(\frac{n+4}{2}\right)2^{\frac{n+4}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{6(n-1)}{n^2}$$

$$\text{(e)}\quad -\frac{6}{n}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\Gamma\left(\frac{n+2}{2}\right)2^{\frac{n+2}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=-\frac{6(n-1)}{n}$$

$$\text{(f)}\quad \frac{3(n-1)}{(n+1)n^2}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\frac{\Gamma\left(\frac{n+4}{2}\right)2^{\frac{n+4}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{3(n-1)^2}{n^2}$$

combining the coefficients, from (a) to (f), gives:

$$\frac{n\lambda_4}{24}\left[\frac{3}{n^2}-\frac{6}{n}+3+\frac{6(n-1)}{n^2}-\frac{6(n-1)}{n}+\frac{3(n-1)^2}{n^2}\right]$$

$$= \frac{n\lambda_4}{24n^2}\left[3-6n+3n^2-6+6n+6n-6n^2+3-6n+3n^2\right]=0.$$

Similarly, the coefficient associated with λ_3^2 is:

(a') $$\frac{1}{n^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\frac{\Gamma\left(\frac{n+6}{2}\right)2^{\frac{n+6}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{15}{n^2}$$

(b') $$-\frac{3(2n+3)}{n^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\frac{\Gamma\left(\frac{n+4}{2}\right)2^{\frac{n+4}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=-\frac{9(2n+3)}{n^2}$$

(c') $$\frac{9(n+4)}{n}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\Gamma\left(\frac{n+2}{2}\right)2^{\frac{n+2}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{9(n+4)}{n}$$

(d') $$-15\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=-15$$

(e') $$\frac{6}{n^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\frac{\Gamma\left(\frac{n+6}{2}\right)2^{\frac{n+6}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{18(n+1)}{n^2}$$

(f') $$-\frac{18(n+3)}{n^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\frac{\Gamma\left(\frac{n+4}{2}\right)2^{\frac{n+4}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=-\frac{18(n+3)(n-1)}{n^2}$$

$$\text{(g')}\quad \frac{36}{n}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\Gamma\left(\frac{n+2}{2}\right)2^{\frac{n+2}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{36(n-1)}{n}$$

$$\text{(h')}\quad \frac{9}{n^2}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\frac{\Gamma\left(\frac{n+6}{2}\right)2^{\frac{n+6}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{9(n+1)(n-1)}{n^2}$$

$$\text{(i')}\quad -\frac{27(n-1)}{(n+1)n^2}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\frac{\Gamma\left(\frac{n+4}{2}\right)2^{\frac{n+4}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=-\frac{27(n-1)^2}{n^2}$$

$$\text{(j')}\quad \frac{6(n-2)}{(n+3)(n+1)n^2}\frac{\Gamma\left(\frac{n+5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\frac{\Gamma\left(\frac{n+6}{2}\right)2^{\frac{n+6}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}=\frac{6(n-1)(n-2)}{n^2}$$

combining the coefficients, from (a') to (j'), gives:

$$\frac{n\lambda_3^2}{72}\left[\frac{15}{n^2}-\frac{9(2n+3)}{n^2}\frac{9(n+4)}{n}-15+\frac{18(n-1)}{n^2}-\frac{18(n+3)(n-1)}{n^2}+\frac{36(n-1)}{n}+\frac{9(n+1)(n-1)}{n^2}\right.$$
$$\left.-\frac{27(n-1)^2}{n^2}+\frac{6(n-1)(n-2)}{n^2}\right]$$

$$=\frac{3n\lambda_3^2}{72n^2}\left[5-9-6n+12n+3n^2-5n^2-6+6n+18-12n-6n^2-12n+12n^2-3+3n^2-9\right.$$
$$\left.+18n-9n^2+4-6n+2n^2\right]$$

$$=0.$$

Therefore $\int_0^\infty h_1(y)dy = 1$ and hence is a proper marginal pdf associated with the joint density function h(y, z), where $Y=\sum_{i=1}^{n}(X_i-T)^2$ and $Z=(\overline{X}-T)^2$. Now,

$$\hat{C}pm=\frac{USL-LSL}{6\sqrt{\frac{Y}{n-1}}}=\frac{a_1}{\sqrt{Y}}=V,$$

for each

$$v=\frac{a_1}{\sqrt{y}} \Rightarrow y=\frac{a_1^2}{v^2}$$

with Jacobian of the inverse transformation,

$$J=\frac{dy}{dv}=-\frac{2a_1^2}{v^3}.$$

So that the pdf of $\hat{C}pm$ is:

$$g_{\hat{C}pm}(v)=h_1\left(\frac{a_l^2}{v^2}\right)\left|-\frac{2a_l^2}{v^3}\right|$$

$$=\frac{\exp\left(-\frac{a_l^2}{2v^2}\right)\left(\frac{a_l^2}{v^2}\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}\frac{2a_l^2}{v^3}\left\{\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right.$$

$$+\frac{n\lambda_4}{24}\left[\frac{a_l^4}{n^2v^4}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-6\frac{a_l^2}{nv^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+3\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right.$$

$$\left.+6\frac{a_l^4}{n^2v^4}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-6\frac{a_l^2}{nv^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+3\frac{(n-1)}{(n+1)}\frac{a_l^4}{n^2v^4}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\right]$$

$$+\frac{n\lambda_3^2}{72}\left[\frac{a_l^6}{n^2v^6}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}-3(2n+3)\frac{a_l^4}{n^2v^4}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\right.$$

$$+9(n+4)\frac{a_l^2}{nv^2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}-15\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}+6\frac{a_l^6}{n^2v^6}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}$$

$$-18(n+3)\frac{a_l^4}{n^2v^4}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+36\frac{a_l^2}{nv^2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+9\frac{a_l^6}{n^2v^6}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}$$

$$\left.\left.-27\frac{(n-1)}{(n+1)}\frac{a_l^4}{n^2v^4}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+\frac{6n(n-2)}{(n+3)(n+1)}\frac{a_l^6}{n^3v^6}\frac{\Gamma\left(\frac{n+5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\right]\right\}$$

$0<v<\infty$, zero elsewhere. [2.5.6]

It can be shown that $g_{\hat{C}pm}(v)$ is a proper pdf when λ_3 and λ_4 satisfy the positive definite region associated with the density, by making a substitution of the form $y=\frac{a_I^2}{v^2}$. Hence, it is just the marginal pdf given by equation [2.5.5] and the results follows. For $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, the pdf of $\hat{C}pm$ can be written in terms of the beta function and it is

$$
\begin{aligned}
g_{\hat{C}pm}(v)=&\frac{\exp\left(-\frac{a_I^2}{2v^2}\right)\left(\frac{a_I^2}{v^2}\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}}\frac{2a_I^2}{v^3}\Bigg\{B\left(\frac{n-1}{2},\frac{1}{2}\right)\\
&+\frac{n\lambda_4}{24}\Bigg[\frac{a_I^4}{n^2v^4}B\left(\frac{n-1}{2},\frac{5}{2}\right)-6\frac{a_I^2}{nv^2}B\left(\frac{n-1}{2},\frac{3}{2}\right)+3B\left(\frac{n-1}{2},\frac{1}{2}\right)\\
&\qquad+6\frac{a_I^4}{n^2v^4}B\left(\frac{n+1}{2},\frac{3}{2}\right)-6\frac{a_I^2}{nv^2}B\left(\frac{n+1}{2},\frac{1}{2}\right)+3\frac{(n-1)}{(n+1)}\frac{a_I^4}{n^2v^4}B\left(\frac{n+3}{2},\frac{1}{2}\right)\Bigg]\\
&+\frac{n\lambda_3^2}{72}\Bigg[\frac{a_I^6}{n^2v^6}B\left(\frac{n-1}{2},\frac{7}{2}\right)-3(2n+3)\frac{a_I^4}{n^2v^4}B\left(\frac{n-1}{2},\frac{5}{2}\right)\\
&\qquad+9(n+4)\frac{a_I^2}{nv^2}B\left(\frac{n-1}{2},\frac{3}{2}\right)-15B\left(\frac{n-1}{2},\frac{1}{2}\right)+6\frac{a_I^6}{n^2v^6}B\left(\frac{n+1}{2},\frac{5}{2}\right)\\
&\qquad-18(n+3)\frac{a_I^4}{n^2v^4}B\left(\frac{n+3}{2},\frac{1}{2}\right)+36\frac{a_I^2}{nv^2}B\left(\frac{n+1}{2},\frac{1}{2}\right)+9\frac{a_I^6}{n^2v^6}B\left(\frac{n+3}{2},\frac{3}{2}\right)\\
&\qquad-27\frac{(n-1)}{(n+1)}\frac{a_I^4}{n^2v^4}B\left(\frac{n+3}{2},\frac{1}{2}\right)+\frac{6n(n-2)}{(n+3)(n+1)}\frac{a_I^6}{n^3v^6}B\left(\frac{n+5}{2},\frac{1}{2}\right)\Bigg]\Bigg\}
\end{aligned}
$$

$0 < v < \infty$, zero elsewhere. [2.5.6a]

Theorem 2.5.1:

The rth moment of $\hat{C}pm$, for $r < n$, is

$$E\left(\hat{C}pm^r\right)=\frac{a_l^r}{2^{\frac{r}{2}}}\left\{\frac{\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}+\frac{n\lambda_4}{8}\left[\frac{\Gamma\left(\frac{n+4-r}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-2\frac{\Gamma\left(\frac{n+2-r}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+\frac{\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right.$$

$$\left.+\frac{5n\lambda_3^2}{24}\left[\frac{\Gamma\left(\frac{n+6-r}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}-3\frac{\Gamma\left(\frac{n+4-r}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+3\frac{\Gamma\left(\frac{n+2-r}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}-\frac{\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right\}$$

for $r < n$. [2.5.7]

Proof:

Using equation [2.5.4], the expectation of the rth moment of $\hat{C}pm$ is

$$E\left(\hat{C}pm^r\right)=E\left(a_l^r Y^{-\frac{r}{2}}\right)=a_l^r\int_0^\infty y^{-\frac{r}{2}}h_l(y)dy$$

and the results follow after simplification.

Lemma 2.5.2

The first, second moments and variance of $\hat{C}pm$ for $r < n$, are respectively,

$$E\left(\hat{C}pm\right)=\frac{a_l}{\sqrt{2}}\left\{\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}+\frac{n\lambda_4}{8}\left[\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-2\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right.$$

$$\left.+\frac{5n\lambda_3^2}{24}\left[\frac{\Gamma\left(\frac{n+5}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}-3\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+3\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}-\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right\} \quad [2.5.8]$$

$$E\left(\hat{C}pm^2\right)=a_l^2\left\{\frac{1}{n-2}+\frac{\lambda_4}{(n+2)(n-2)}+\frac{10\lambda_3^2}{(n+4)(n+2)(n-2)}\right\}$$

$$V\left(\hat{C}pm\right)=a_l^2\left\{\frac{1}{n-2}+\frac{\lambda_4}{(n+2)(n-2)}+\frac{10\lambda_3^2}{(n+4)(n+2)(n-2)}\right.$$

$$-\frac{1}{2}\left(\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}+\frac{n\lambda_4}{8}\left[\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-2\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right.$$

$$\left.\left.+\frac{5n\lambda_3^2}{24}\left[\frac{\Gamma\left(\frac{n+5}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}-3\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+3\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}-\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right)^2\right\} \quad [2.5.9]$$

Proof:

For the first and second moments, just direct substitute $r = 1$ and 2 into equation [2.5.7].

The second moment can be simplified as

$$E\left(\hat{C}pm^2\right)=\frac{a_l^2}{2}\left\{\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}+\frac{n\lambda_4}{8}\left[\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}-2\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right.$$

$$\left.+\frac{5n\lambda_3^2}{24}\left[\frac{\Gamma\left(\frac{n+4}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}-3\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}+3\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}-\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right\}$$

$$=\frac{a_1^2}{2}\left\{\frac{2}{n-2}+\frac{2\lambda_4}{(n+2)(n-2)}+\frac{20\lambda_3^2}{(n+4)(n+2)(n-2)}\right\}$$

$$=a_1^2\left\{\frac{1}{n-2}+\frac{\lambda_4}{(n+2)(n-2)}+\frac{10\lambda_3^2}{(n+4)(n+2)(n-2)}\right\}$$

The variance of $\hat{C}pm$ follows from $V(\hat{C}pm)=E(\hat{C}pm^2)-[E(\hat{C}pm)]^2$ after simplification. It can be seen from equations [2.5.8] and [2.5.9] that the expectation of $\hat{C}pm$ will increase when λ_4 increases and will decrease when λ_3 increases. Increasing the values of λ_3 and λ_4 causes the variance of $\hat{C}pm$ to increase and vice versa.

Following from equation [2.5.3], a proper estimator of Cpm* is

$$\hat{C}pm* = \frac{\min[USL - T, T - LSL]}{3\sqrt{\frac{Y}{n-1}}} = \frac{a_1^*}{\sqrt{Y}},$$

the pdf of $\hat{C}pm*$ can be easily obtained by substituting a_1^* to a_1 in $g_{\hat{C}pm}(v)$ as $g_{\hat{C}pm^*}(v)$. It takes the following form:

$$g_{\hat{C}pm^*}(v)=\frac{\exp\left(-\frac{a_l^{*2}}{2v^2}\right)\left(\frac{a_l^{*2}}{v^2}\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}-1}}\frac{a_l^{*2}}{v^3}\left\{ B\left(\frac{n-1}{2},\frac{1}{2}\right)\right.$$

$$+\frac{n\lambda_4}{24}\left[\frac{a_l^{*4}}{n^2v^4}B\left(\frac{n-1}{2},\frac{5}{2}\right)-6\frac{a_l^{*2}}{nv^2}B\left(\frac{n-1}{2},\frac{3}{2}\right)+3B\left(\frac{n-1}{2},\frac{1}{2}\right)\right.$$

$$\left.+6\frac{a_l^{*4}}{n^2v^4}B\left(\frac{n+1}{2},\frac{3}{2}\right)-6\frac{a_l^{*2}}{nv^2}B\left(\frac{n+1}{2},\frac{1}{2}\right)+3\frac{(n-1)}{(n+1)}\frac{a_l^{*4}}{n^2v^4}B\left(\frac{n+3}{2},\frac{1}{2}\right)\right]$$

$$+\frac{n\lambda_3^2}{72}\left[\frac{a_l^{*6}}{n^2v^6}B\left(\frac{n-1}{2},\frac{7}{2}\right)-3(2n+3)\frac{a_l^{*4}}{n^2v^4}B\left(\frac{n-1}{2},\frac{5}{2}\right)\right.$$

$$+9(n+4)\frac{a_l^{*2}}{nv^2}B\left(\frac{n-1}{2},\frac{3}{2}\right)-15B\left(\frac{n-1}{2},\frac{1}{2}\right)+6\frac{a_l^{*6}}{n^2v^6}B\left(\frac{n+1}{2},\frac{5}{2}\right)$$

$$-18(n+3)\frac{a_l^{*4}}{n^2v^4}B\left(\frac{n+1}{2},\frac{3}{2}\right)+36\frac{a_l^{*2}}{nv^2}B\left(\frac{n+1}{2},\frac{1}{2}\right)+9\frac{a_l^{*6}}{n^2v^6}B\left(\frac{n+3}{2},\frac{3}{2}\right)$$

$$\left.\left.-27\frac{(n-1)}{(n+1)}\frac{a_l^{*4}}{n^2v^4}B\left(\frac{n+3}{2},\frac{1}{2}\right)+6\frac{(n-2)}{(n+3)(n+1)}\frac{a_l^{*6}}{n^2v^6}B\left(\frac{n+5}{2},\frac{1}{2}\right)\right]\right\}$$

$a_l^* > 0$, $0<v<\infty$, zero elsewhere. [2.5.10]

For $n = 5$, $T = 0$ and $a_l = \frac{4\sqrt{n-1}}{6}=\frac{4}{3}$ (i.e. $USL-LSL=4$), the density function of $\hat{C}pm$ becomes

$$g_{\hat{C}pm}(v)=\frac{.84057}{\exp\left(\frac{.888889}{v^2}\right)v^6}\left\{1.33333+.20833\lambda_4\left[4+.361199v^{-4}-2.84444v^{-2}\right]\right.$$

$$\left.+.069444\lambda_3^2\left[-20+.35674v^{-6}-5.41799v^{-4}+21.33333v^{-2}\right]\right\}$$

$0<v<\infty$, zero elsewhere. [2.5.11]

Figures 2.5.1 a-e shows the graph of $\hat{C}pm$ using equation [2.5.11] for different values of λ_3 and λ_4.

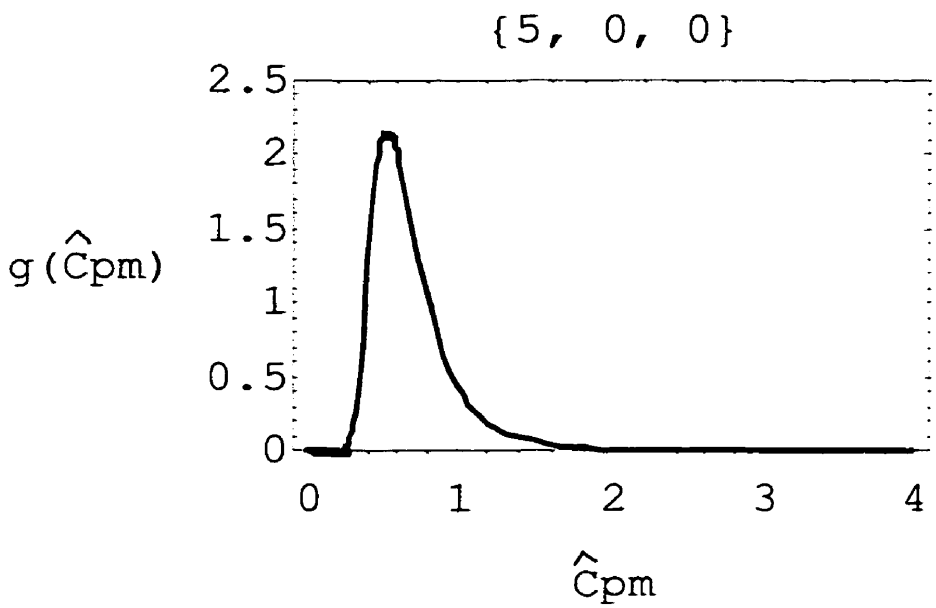


Figure 2.5.1a Graph of $\hat{C}pm$ when $n = 5$, $T = 0$, $a_1 = \frac{4}{3}$, $\lambda_3 = 0$ and $\lambda_4 = 0$

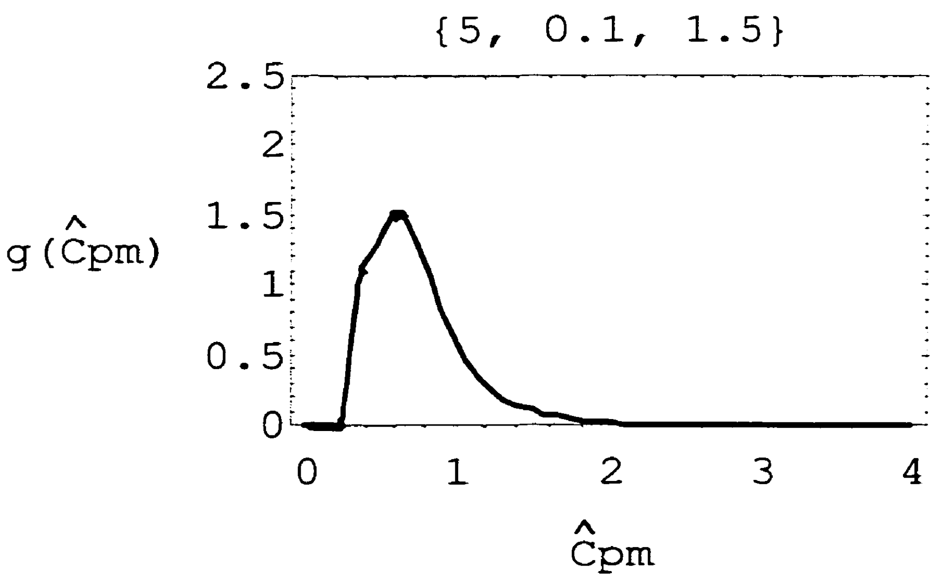


Figure 2.5.1b Graph of $\hat{C}pm$ when $n = 5$, $T = 0$, $a_1 = \frac{4}{3}$, $\lambda_3 = .1$ and $\lambda_4 = 1.5$

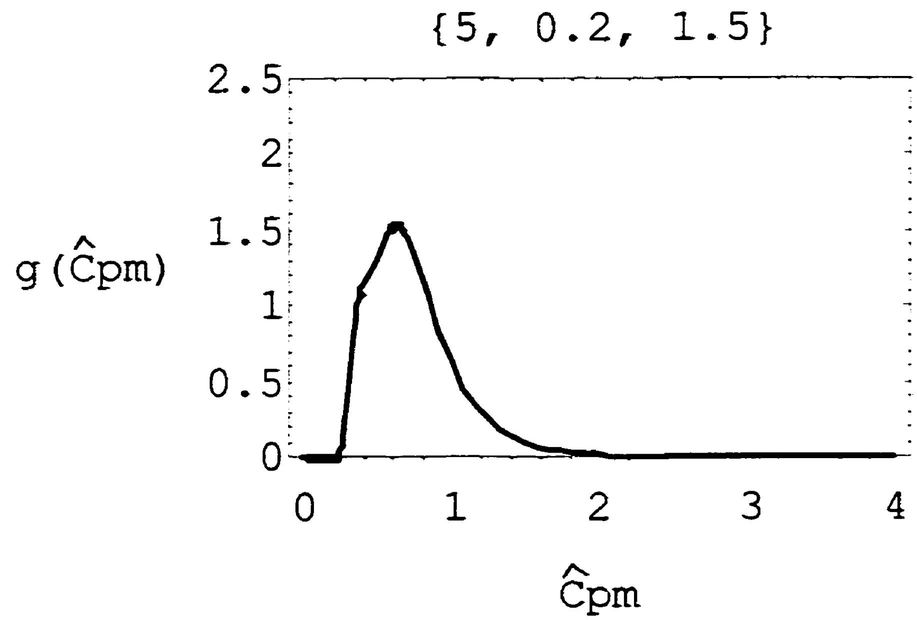


Figure 2.5.1c Graph of $\hat{C}pm$ when $n = 5, T = 0,\ a_1 = \frac{4}{3},\ \lambda_3 = .2$ and $\lambda_4 = 1.5$

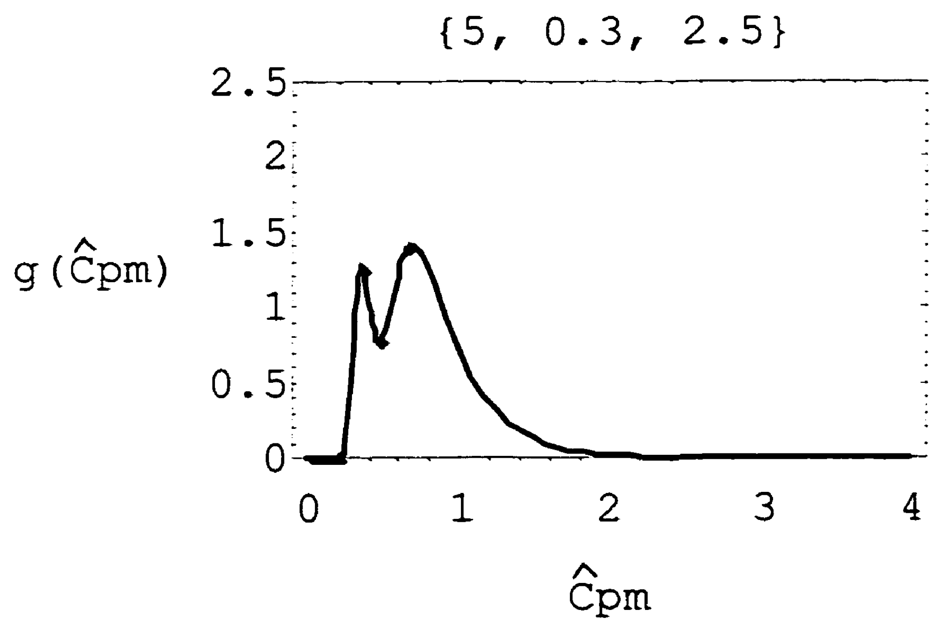


Figure 2.5.1d Graph of $\hat{C}pm$ when $n = 5, T = 0,\ a_1 = \frac{4}{3},\ \lambda_3 = .3$ and $\lambda_4 = 2.5$

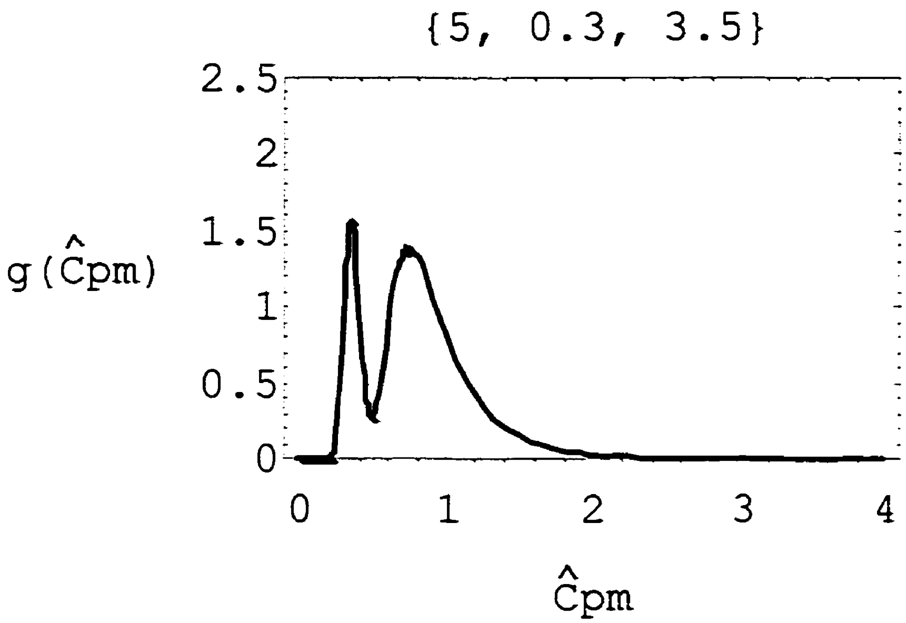


Figure 2.5.1e Graph of $\hat{C}pm$ when $n=5$, $T=0$, $a_1=\frac{4}{3}$, $\lambda_3=.3$ and $\lambda_4=3.5$

The mean and variance are, respectively

$$E(\hat{C}pm)=\frac{4}{3\sqrt{2\pi}}\left(\frac{4}{3}+\frac{\lambda_4}{21}-\frac{25\lambda_3^2}{378}\right)$$

$$V(\hat{C}pm)=\frac{16}{9}\left[\frac{1}{3}+\frac{\lambda_4}{21}+\frac{10\lambda_3^2}{189}-\frac{1}{2\pi}\left(\frac{4}{3}+\frac{\lambda_4}{21}-\frac{25\lambda_3^2}{378}\right)^2\right]$$

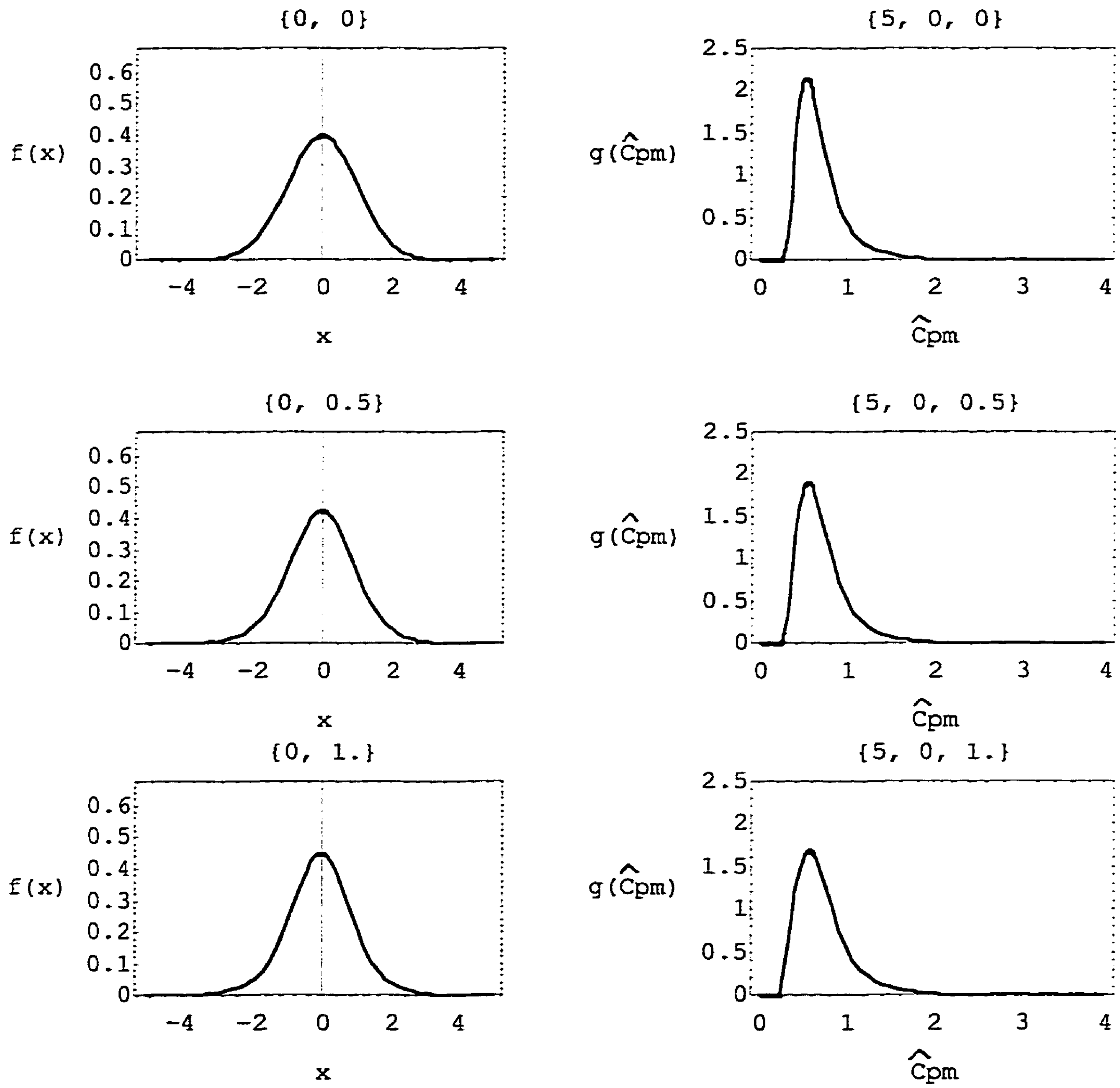
When λ_3 and λ_4 are both zero, then the mean and variance will be

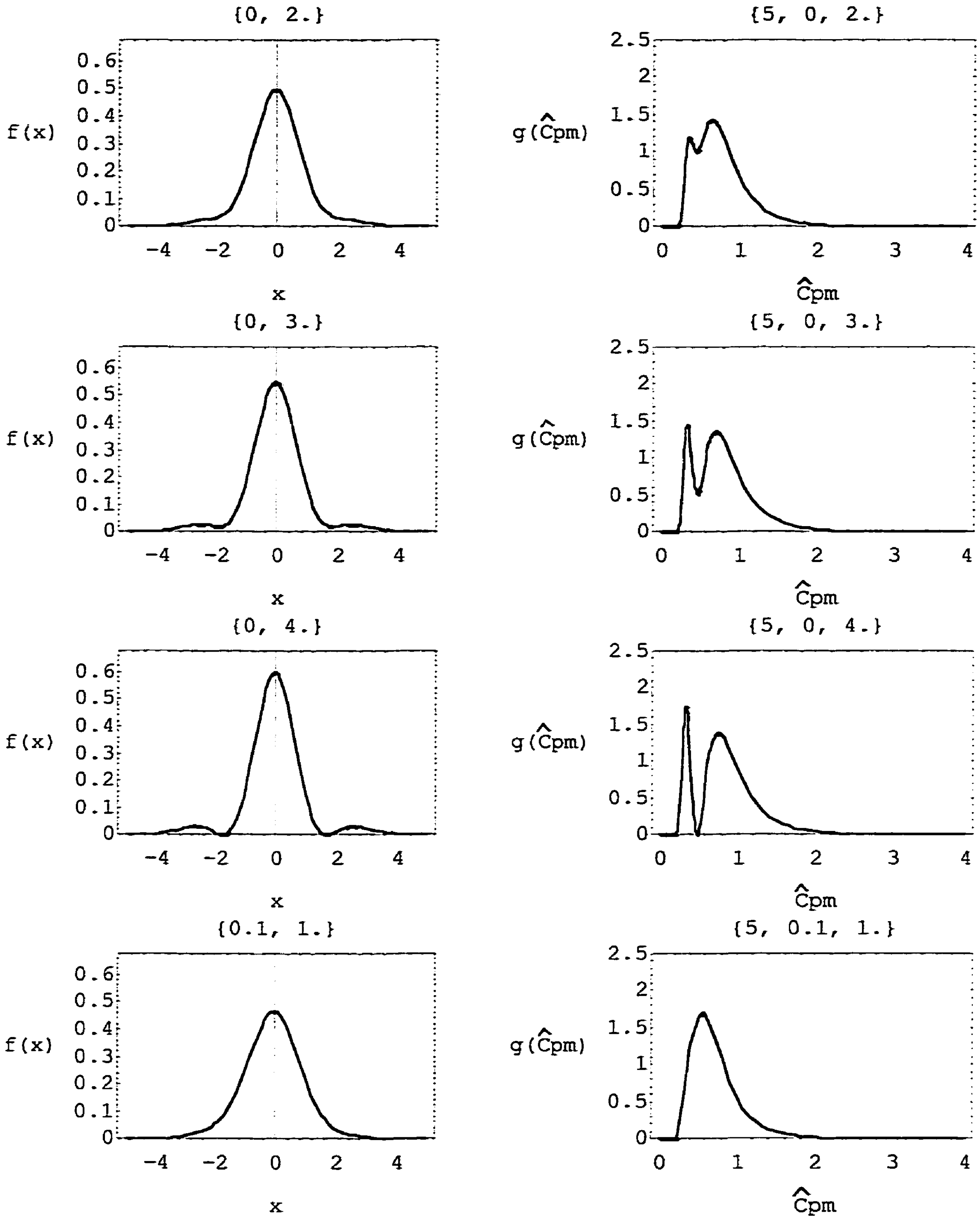
$$E(\hat{C}pm)=\frac{16}{9\sqrt{2\pi}}$$

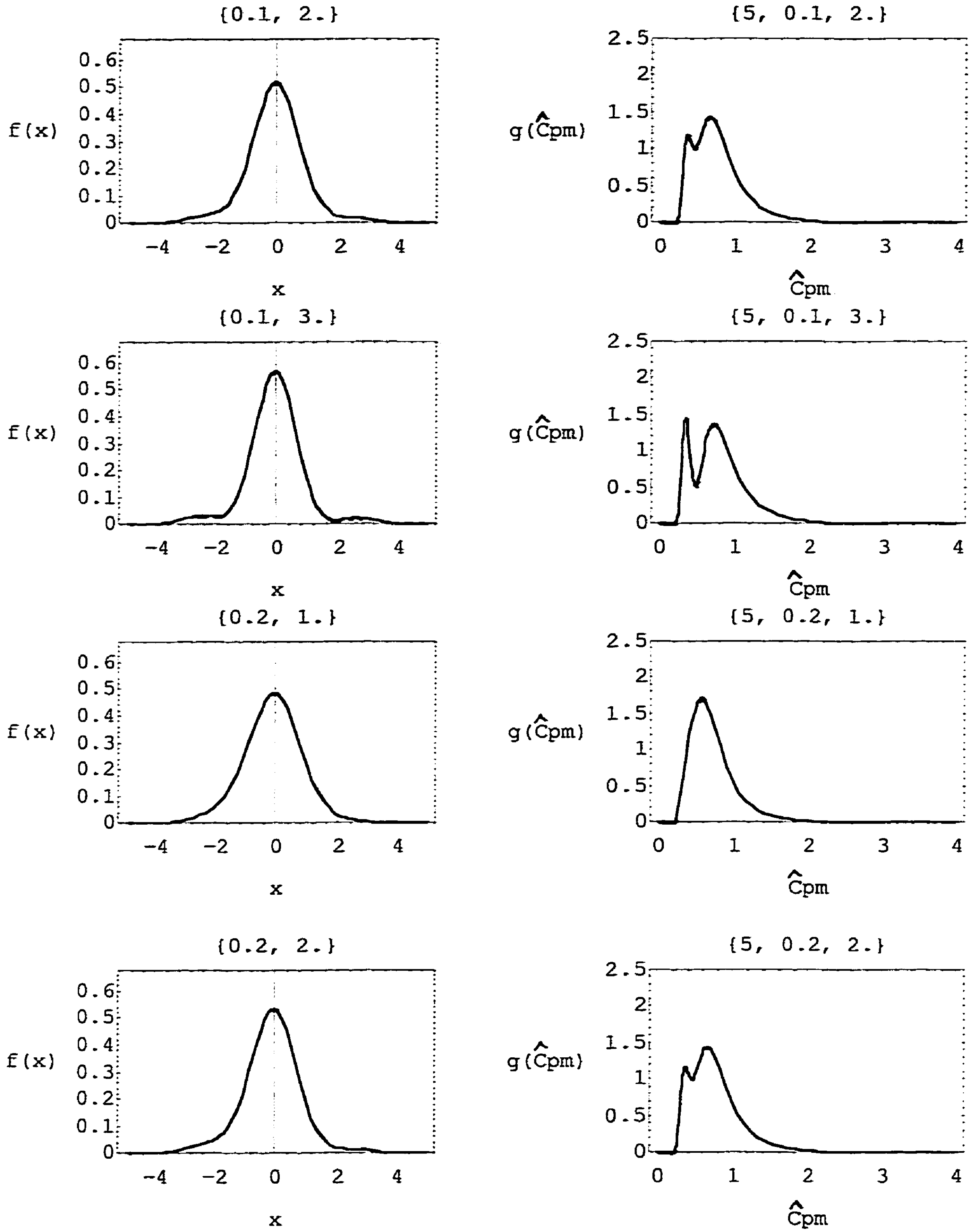
$$V(\hat{C}pm)=\frac{16(3\pi-8)}{81\pi}$$

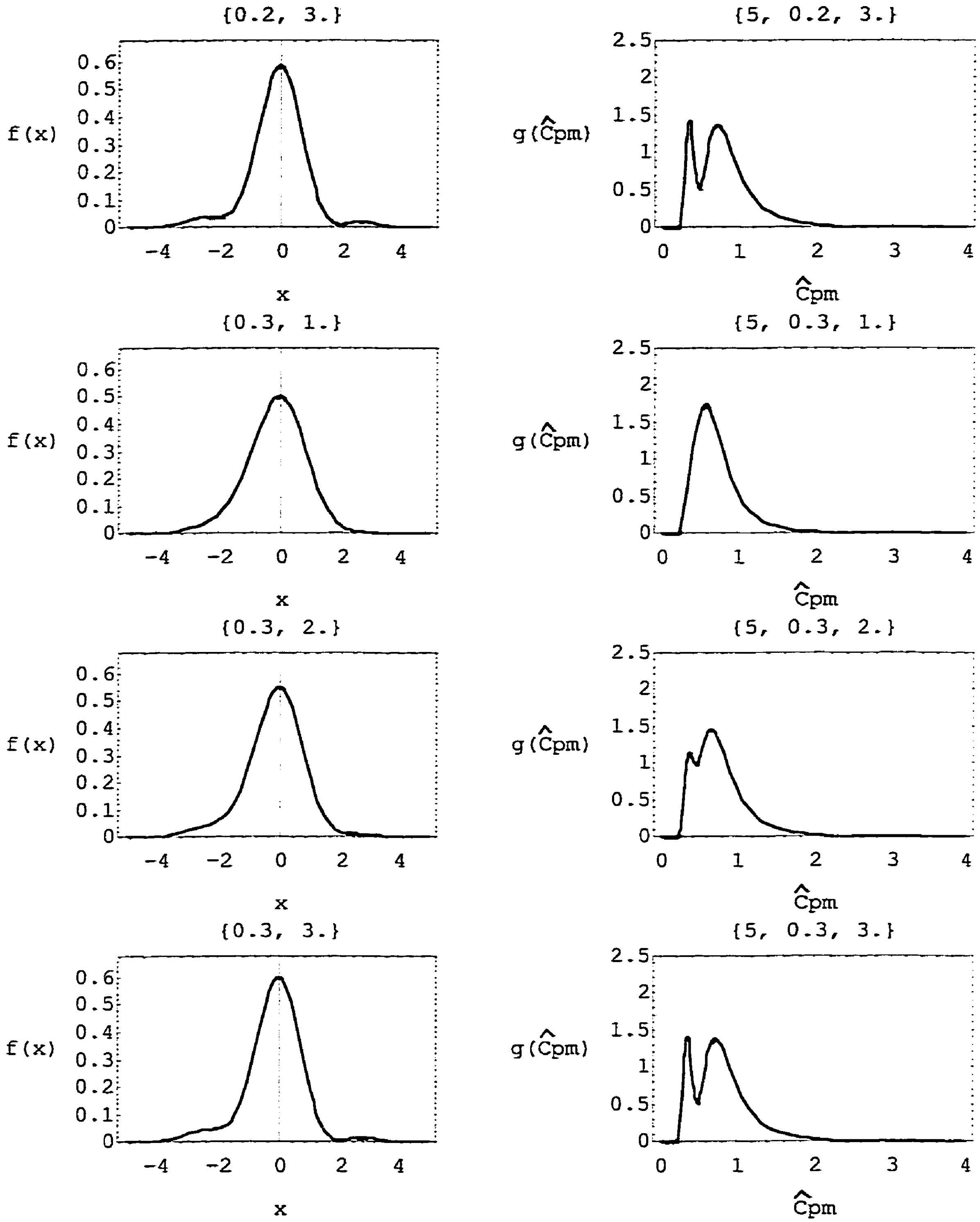
Using this result we can examine the impact of moderate departures from the normality on the density function of $\hat{C}pm$. The plots that follow, graphically depict the distorted distribution of X (the characteristic of interest) and the resulting distribution of $\hat{C}pm$ for various values of λ_3 (i.e., $\lambda_3 = 0.0(0.1)0.4$) and λ_4 (i.e., $\lambda_4 = 0.0, 0.5, 1.0(1)4.0$). Each pair of plots represents the distorted distribution of the X's (i.e., f(x)) and the resulting distribution of $\hat{C}pm$ (i.e. $g(\hat{C}pm)$) for n = 5.

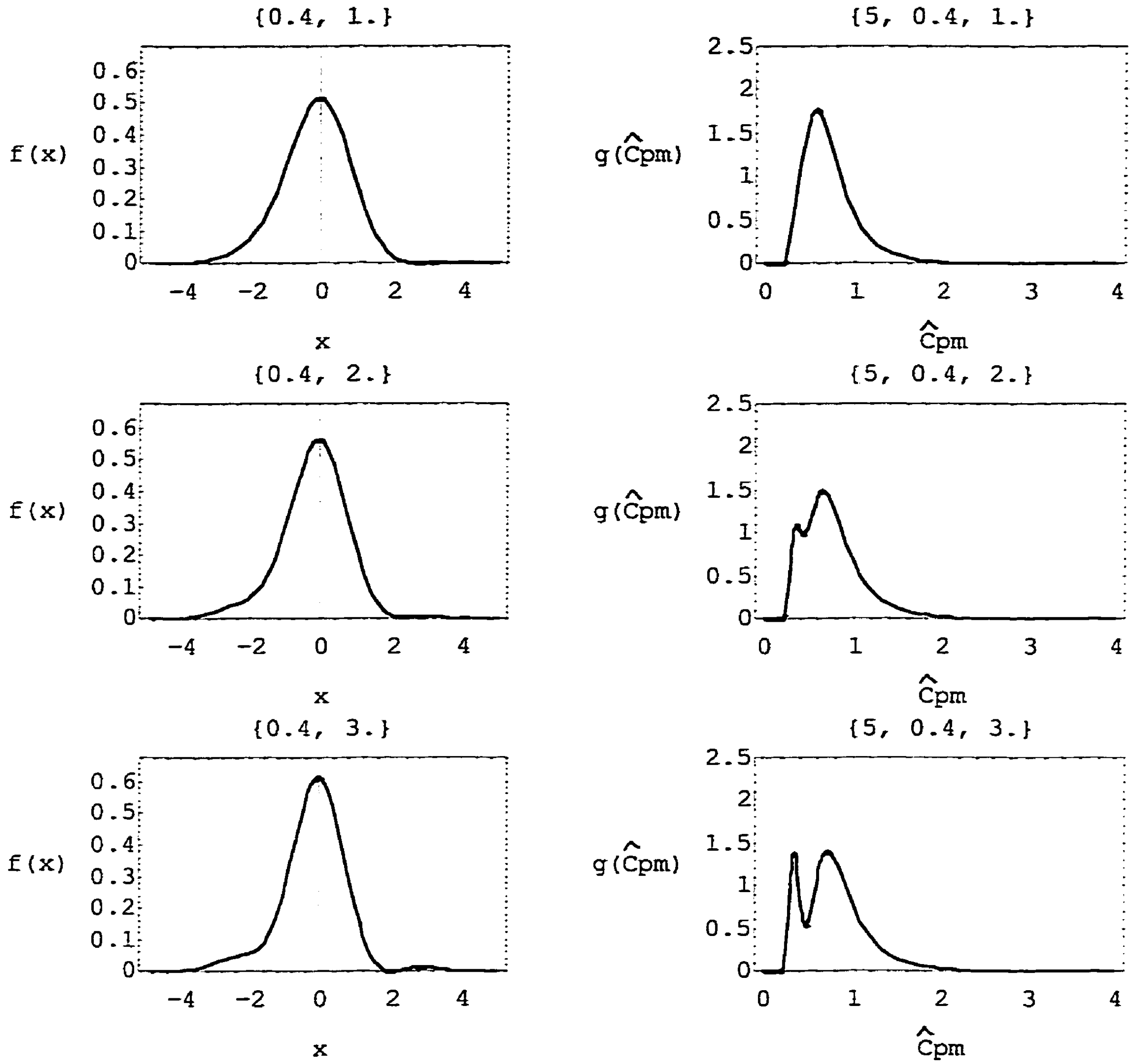
Figure 2.5.2 Series of plots showing the impact of λ_3 and λ_4 on $g_{\hat{C}pm}(\hat{C}pm)$




{0, 2.}
f(x)
x
{5, 0, 2.}
g(Ĉpm)
Ĉpm
{0, 3.}
f(x)
x
{5, 0, 3.}
g(Ĉpm)
Ĉpm
{0, 4.}
f(x)
x
{5, 0, 4.}
g(Ĉpm)
Ĉpm
{0.1, 1.}
f(x)
x
{5, 0.1, 1.}
g(Ĉpm)
Ĉpm


{0.1, 2.}
{5, 0.1, 2.}
{0.1, 3.}
{5, 0.1, 3.}
{0.2, 1.}
{5, 0.2, 1.}
{0.2, 2.}
{5, 0.2, 2.}
f(x)
x
g(Ĉpm)
Ĉpm


{0.2, 3.}
{5, 0.2, 3.}
{0.3, 1.}
{5, 0.3, 1.}
{0.3, 2.}
{5, 0.3, 2.}
{0.3, 3.}
{5, 0.3, 3.}
f(x)
x
g(Ĉpm)
Ĉpm



From the above plots it is evident that λ_3 and λ_4 have relatively small impact on the underlying distribution (i.e., f(x)) while the impact on $g(\hat{C}pm)$ appears much more dramatic. Clearly λ_4 introduces bimodality to $g(\hat{C}pm)$ and would appear to have an impact on any inferences made. To quantify the impact of skewness and kurtosis on $g(\hat{C}pm)$, the tail probabilities (i.e., $P(\hat{C}pm > c)$ associated with various values of λ_3 and λ_4 are summarized in Table 2.5.1.

TABLE 2.5.1 $P[\hat{C}pm > c = 1.5 | n = 5, T = 0, a = \frac{4}{3}, |\lambda_3|, \lambda_4]$

λ_4 / $\lvert\lambda_3\rvert$	0	0.5	1.0	2.0	3.0	4.0
.0	.0223410	.0278535	.0333660	.0443910	.0554159	.0664409
.1	*	.0276911	.0332035	.0442285	.0552535	*
.2	*	.0272038	.0327163	.0437413	.0547662	*
.3	*	*	.0319042	.0429291	.0539541	*
.4	*	*	*	.0417922	.0528172	*

* $g(\hat{C}pm)$ is not positive definite

Reading across Table 2.5.1, it can be seen that the right hand tail probability associated with a fixed constant c is about 1.25 to 1.5 times larger, for $\lambda_4 = .5$ and 1 and fixed values of $|\lambda_3|$, and about a double when λ_4 is in the range of 2 to 4. This implies that $g(\hat{C}pm)$ is flatter, thicker and heavy tailed for nonnegative values of λ_4 and $|\lambda_3|$ fixed. On the other hand, when we read vertically down Table 2.5.1, the right hand tail probability of $\hat{C}pm$ decreases gradually for a fixed λ_4, when $|\lambda_3|$ increases.

The density function of $\hat{C}pm$ for $n = 10$, $T = 0$ and $a_l = \frac{4\sqrt{n-1}}{6} = 2$ is

$$g_{\hat{C}pm}(v) = \frac{3.10428}{\exp\left(\frac{2}{v^2}\right)v^{11}} \left\{ .859029 + .416666\lambda_4 \left[2.57709 + .343612v^{-4} - 2.06167v^{-2} \right] \right.$$

$$\left. + .138888\lambda_3^2 \left[-12.8854 + .49087v^{-6} - 5.15418v^{-4} + 15.4625v^{-2} \right] \right\}$$

$$0 < v < \infty, \text{ zero elsewhere.} \qquad [2.5.12]$$

with mean and variance

$$E(\hat{C}pm)=\frac{2\sqrt{2\pi}}{3}\left(\frac{35}{128}+\frac{35\lambda_4}{4096}-\frac{125\lambda_3^2}{24576}\right)$$

$$V(\hat{C}pm)=\frac{16}{9}\left[\frac{1}{8}+\frac{\lambda_4}{96}+\frac{5\lambda_3^2}{672}-\frac{\pi}{2}\left(\frac{35}{128}+\frac{35\lambda_4}{4096}-\frac{125\lambda_3^2}{24576}\right)^2\right]$$

When λ_3 and λ_4 are both zero, then the mean and variance will be

$$E(\hat{C}pm)=\frac{35\sqrt{2\pi}}{192}$$

$$V(\hat{C}pm)=\frac{4096-1225\pi}{18432}$$

The tail probabilities $P[\hat{C}pm>c|\ n=10,\ T=0,\ a_t=2,\ |\lambda_3|,\lambda_4]$ are summarized in Table 2.5.2.

TABLE 2.5.2 $P[\hat{C}pm>c=1.5|\ n=10,\ T=0,\ |\lambda_3|,\ \lambda_4]$

λ_4 / $\|\lambda_3\|$	0	0.5	1.0	2.0	3.0	4.0
0	.0022230	.0032352	.0042474	.0062717	.0082961	.0103205
.1	*	.0032065	.0042187	.0062431	.0082675	*
.2	*	.0031207	.0041329	.0061573	.0081816	*
.3	*	*	.0039898	.0060142	.0080385	*
.4	*	*	*	.0058139	.0078382	*

* $g(\hat{C}pm)$ is not positive definite

Again reading across Table 2,5.2, the right hand tail probability of $\hat{C}pm$ at a fixed constant c and $\lambda_4=0$ is 1.5 to 2 times larger for $\lambda_4=.5$ and 1 respectively, and fixed values of $|\lambda_3|$ are about triple the probabilities for values of λ_4 in the range of 2 to 4 suggesting $g(\hat{C}pm)$ is flatter, thicker and heavy tailed when $|\lambda_3|$ is fixed with nonnegative values of λ_4. Meanwhile, when we read vertically down Table 2.5.2, the behavior of the right hand tail probability of $\hat{C}pm$ decreases gradually for a fixed λ_4, when $|\lambda_3|$ increases.

In those cases where the practitioner is monitoring process capability on a regular basis either by hand or in conjunction with a capability chart (e.g., Spiring (1995)) inferences will be affected. For those process measurements exhibiting near normal distribution characteristics with non-zero estimates of λ_3 and λ_4, corrections should be made either to the action limits (the critical values, c_L and c_U, where $P(\hat{C}pm < c_L)$ = $P(\hat{C}pm > c_U) = .00135$) or to the specified level of α (the level of significance) associated with the decision making process or capability chart.

If the practitioner can identify the amount of distortion from normality and if it is moderate, corrections can be made that will provide the practitioner with viable decision rules or action limits. In Tables 2.5.3a through 2.5.6a the critical values (i.e., c_U) associated with the upper tail of $g(\hat{C}pm)$ have been determined for various n, λ_3 and λ_4. Again it appears that the λ_4 has a substantial impact on any inferences made. In order to maintain the same confidence in the decision that the process capability has significantly improved (increased) in the presence of nonzero values of λ_4, larger values of $\hat{C}pm$ are required.

Similarly if we examine the lower tail (Tables 2.5.3b through 2.5.6b) of $g(\hat{C}pm)$ (i.e., when we want to identify significant declines in capability) the impact of λ_3 and λ_4 can be quantified and used in the decision making process.

TABLE 2.5.3a Values of c_U for $n = 4, |\lambda_3| = 0(.1).4$ and $\lambda_4 = 0(0.5)1, 2(1)4$

where $P[\hat{C}pm > c_U \mid n, T = 0, a_1 = \frac{\sqrt{n-1}\,[USL - LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\lvert\lambda_3\rvert$	0	.5	1	2	3	4
.0	3.550540	3.751788	3.926199	4.220000	4.464035	4.674280
.1	*	3.745660	3.920817	4.215630	4.460321	*
.2	*	3.727110	3.904539	4.202425	4.449120	*
.3	*	*	3.876973	4.180156	4.430270	*
.4	*	*	*	4.148400	4.403484	*

* $g(\hat{C}pm)$ is not positive definite

TABLE 2.5.3b Values of c_L for $n = 4, |\lambda_3| = 0(.1).4$ and $\lambda_4 = 0(0.5)1, 2(1)4$

where $P[\hat{C}pm < c_L \mid n, T = 0, a_1 = \frac{\sqrt{n-1}\,[USL - LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\lvert\lambda_3\rvert$	0	.5	1	2	3	4
.0	.273692	.253044	.244100	.235150	.230200	.226849
.1	*	.252530	.243718	.234980	.230080	*
.2	*	.251040	.242860	.234460	.229730	*
.3	*	*	.241399	.233640	.229158	*
.4	*	*	*	.232558	.228395	*

* $g(\hat{C}pm)$ is not positive definite

TABLE 2.5.4a Values of c_U for $n=5, |\lambda_3|=0(.1).4$ and $\lambda_4=0(0.5)1, 2(1)4$

where $P[\hat{C}pm > c_U \mid n, T=0,\ a_1 = \dfrac{\sqrt{n-1}\,[USL-LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\|\lambda_3\|$	0	.5	1	2	3	4
.0	2.73330	2.882670	3.007967	3.212000	3.376110	3.514335
.1	*	3.745660	3.004204	3.209100	3.373670	*
.2	*	3.727110	2.992814	3.200160	3.366310	*
.3	*	*	2.973469	3.185120	3.353910	*
.4	*	*	*	3.163630	3.362610	*

* $g(\hat{C}pm)$ is not positive definite

TABLE 2.5.4b Values of c_L for $n=5, |\lambda_3|, 0(.1).4$ and $\lambda_4=0(0.5)1, 2(1)4$

where $P[\hat{C}pm < c_L \mid n, T=0,\ a_1 = \dfrac{\sqrt{n-1}\,[USL-LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\|\lambda_3\|$	0	.5	1	2	3	4
.0	.299490	.278476	.269310	.260059	.254915	.251420
.1	*	.277970	.268990	.259889	.254790	*
.2	*	.276483	.268070	.259380	.254440	*
.3	*	*	.266617	.258557	.253868	*
.4	*	*	*	.257470	.253110	*

* $g(\hat{C}pm)$ is not positive definite

TABLE 2.5.5a Values of c_U for $n=6, |\lambda_3|=0(.1).4$ and $\lambda_4=0(0.5)1, 2(1)4$

where $P[\hat{C}pm > c_U \mid n, T=0,\ a_1 = \dfrac{\sqrt{n-1}\,[USL-LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\|\lambda_3\|$	0	.5	1	2	3	4
.0	2.291049	2.411792	2.510318	2.666526	2.789089	2.890586
.1	*	2.408353	2.507439	2.664338	2.787311	*
.2	*	2.397909	2.498718	2.657721	2.781940	*
.3	*	*	2.483877	2.646526	2.772884	*
.4	*	*	*	2.630490	2.759975	*

* $g(\hat{C}pm)$ is not positive definite

TABLE 2.5.5b Values of c_L for $n=6, |\lambda_3| = 0(.1).4$ and $\lambda_4 = 0(0.5)1, 2(1)4$

where $P[\hat{C}pm < c_L \mid n, T=0,\ a_l = \dfrac{\sqrt{n-1}\,[USL-LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\lvert\lambda_3\rvert$	0	.5	1	2	3	4
.0	.319724	.298679	.289428	.280030	.274766	.271190
.1	*	.298187	.289120	.279850	.274650	*
.2	*	.296717	.288206	.279349	.274300	*
.3	*	*	.286780	.278540	.273740	*
.4	*	*	*	.277469	.272990	*

* $g(\hat{C}pm)$ is not positive definite

TABLE 2.5.6a Values of c_U for $n=10, |\lambda_3| = 0(.1).4$ and $\lambda_4 = 0(0.5)1, 2(1)4$

where $P[\hat{C}pm > c_U \mid n, T=0,\ a_l = \dfrac{\sqrt{n-1}\,[USL-LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\lvert\lambda_3\rvert$	0	.5	1	2	3	4
.0	1.589231	1.662535	1.718065	1.800279	1.860998	1.909326
.1	*	1.660692	1.716605	1.799243	1.860191	*
.2	*	1.655081	1.712173	1.796109	1.857756	*
.3	*	*	1.704606	1.790797	1.853644	*
.4	*	*	*	1.7831658	1.847770	*

* $g(\hat{C}pm)$ is not positive definite

TABLE 2.5.6b Values of c_L for $n=10, |\lambda_3| = 0(.1).4$ and $\lambda_4 = 0(0.5)1, 2(1)4$

where $P[\hat{C}pm < c_L \mid n, T=0,\ a_l = \dfrac{\sqrt{n-1}\,[USL-LSL]}{6}, |\lambda_3|, \lambda_4] = 0.00135$

λ_4 / $\lvert\lambda_3\rvert$	0	.5	1	2	3	4
.0	.372771	.352621	.343510	.334058	.328700	.325028
.1	*	.352182	.343232	.333910	.328600	*
.2	*	.350918	.342443	.333468	.328290	*
.3	*	*	.341188	.332758	.327800	*
.4	*	*	*	.331800	.327128	*

* $g(\hat{C}pm)$ is not positive definite

Tables 2.5.7 a-d give the right hand tail probabilities for different values of $\hat{C}pm$.

TABLE 2.5.7 a

		$P\left(\hat{C}pm>c \mid n=4,\ a_1=\frac{2}{\sqrt{3}},\ \lvert\lambda_3\rvert,\lambda_4\right)$			
$\lvert\lambda_3\rvert$	λ_4	.5	1.0	1.5	2.0
.0	.0	.745227	.144305	.036117	.012438
	.5	.752090	.166489	.043471	.015214
	1.0	.758952	.188674	.050825	.017990
	1.5	.765815	.210859	.058179	.020766
	2.0	.772678	.233044	.065533	.023542
	3.0	.786403	.277413	.0802404	.029093
	4.0	.800128	.321782	.094948	.034645
.1	.5	.752471	.165926	.043250	.015127
	1.0	.759334	.188111	.050604	.017902
	1.5	.766196	.210295	.057958	.020678
	2.0	.773059	.232480	.065312	.023454
	3.0	.786784	.276850	.080020	.029006
.2	.5	.753615	.164236	.042590	.014865
	1.0	.760477	.186420	.049944	.017641
	1.5	.767340	.208605	.057298	.020416
	2.0	.774203	.230790	.064652	.023192
	3.0	.787928	.275159	.079359	.028744
.3	1.0	.762384	.183603	.048842	.017204
	1.5	.769246	.205788	.056196	.019980
	2.0	.776109	.227973	.063550	.022756
	3.0	.789834	.272342	.078258	.028308
.4	2.0	.778778	.224029	.062008	.022145
	3.0	.792503	.268398	.076716	.027696

TABLE 2.5.7 b

$\lvert\lambda_3\rvert$	λ_4	$P\left(\hat{C}pm > c \mid n=5, a_1=\frac{4}{3}, \lvert\lambda_3\rvert, \lambda_4\right)$			
		.5	1.0	1.5	2.0
.0	.0	.787493	.121041	.022341	.005983
	.5	.786477	.142525	.027854	.007625
	1.0	.785460	.164008	.033366	.009266
	1.5	.784444	.185492	.038879	.010908
	2.0	.783427	.206976	.044391	.012550
	3.0	.781395	.249944	.055416	.015833
	4.0	.779362	.292911	.066441	.019116
.1	.5	.786965	.142004	.027691	.007574
	1.0	.785949	.163488	.033204	.009215
	1.5	.784933	.184972	.038716	.010857
	2.0	.783916	.206455	.044229	.012498
	3.0	.781884	.249423	.055254	.015781
.2	.5	.788432	.140443	.027204	.007420
	1.0	.787416	.161926	.032716	.009062
	1.5	.786400	.183410	.038229	.010703
	2.0	.785383	.204894	.043741	.012345
	3.0	.783351	.247862	.054766	.015628
.3	1.0	.789861	.159324	.031904	.008806
	1.5	.788844	.180808	.037417	.010447
	2.0	.787828	.202292	.042929	.012089
	3.0	.785795	.245259	.053954	.015372
.4	2.0	.791250	.198648	.041792	.011730
	3.0	.789218	.241616	.052817	.015013

TABLE 2.5.7 c

$\lvert\lambda_3\rvert$	λ_4	$P\left(\hat{C}pm>c \mid n=6,\ a_1=\frac{2\sqrt{5}}{3},\ \lvert\lambda_3\rvert,\lambda_4\right)$			
		.5	1.0	1.5	2.0
.0	.0	.820076	.101831	.013924	.002905
	.5	.812916	.122215	.017950	.003849
	1.0	.805757	.142598	.021977	.004793
	1.5	.798597	.162981	.026003	.005737
	2.0	.791437	.183365	.030029	.006682
	3.0	.777118	.224131	.038082	.008570
	4.0	.762799	.264898	.046135	.010459
.1	.5	.813420	.121739	.017833	.003820
	1.0	.806261	.142122	.021860	.004764
	1.5	.799101	.162505	.025886	.005708
	2.0	.791941	.182888	.029912	.006652
	3.0	.777622	.223655	.037965	.008541
.2	.5	.814932	.120310	.017482	.003732
	1.0	.807772	.140693	.021508	.004676
	1.5	.800612	.161077	.025534	.005620
	2.0	.793453	.181460	.029561	.006565
	3.0	.779133	.222226	.037613	.008453
.3	1.0	.810291	.138312	.020922	.004530
	1.5	.803131	.158696	.024949	.005474
	2.0	.795972	.179079	.028975	.006418
	3.0	.781653	.219845	.037028	.008307
.4	2.0	.799499	.175746	.028155	.006214
	3.0	.785179	.216512	.036207	.008102

TABLE 2.5.7 d

$\lvert\lambda_3\rvert$	λ_4	$P\left(\hat{C}pm > c \mid n=10, a_1=2, \lvert\lambda_3\rvert, \lambda_4\right)$			
		.5	1.0	1.5	2.0
.0	.0	.900368	.052653	.002223	.000172
	.5	.881284	.067690	.003235	.000263
	1.0	.862199	.082728	.004247	.000353
	1.5	.843115	.097765	.005260	.000444
	2.0	.824031	.112802	.006272	.000534
	3.0	.785863	.142877	.008296	.000715
	4.0	.747695	.172951	.010321	.000896
.1	.5	.881556	.067368	.003207	.000260
	1.0	.862472	.08241	.004219	.000350
	1.5	.843388	.097443	.005231	.000441
	2.0	.824304	.112480	.006243	.000531
	3.0	.786136	.142554	.008267	.000712
.2	.5	.882374	.066401	.003121	.000252
	1.0	.863290	.081439	.004133	.000342
	1.5	.844206	.096476	.005145	.000433
	2.0	.825122	.111513	.006157	.000523
	3.0	.786954	.141588	.008182	.000704
.3	1.0	.864653	.079828	.003990	.000328
	1.5	.845569	.094865	.005002	.000419
	2.0	.826485	.109902	.006014	.000509
	3.0	.788317	.139976	.008039	.000690
.4	2.0	.828393	.107646	.005814	.000490
	3.0	.790225	.137721	.007838	.000671

2.6 Example

The adjusted breaking strength of a perforation was identified by the customer as the key quality variable of a particular process. An initial study of the process suggested that adjusted breaking strength did not follow a normal distribution. The first fifty observations from production were used to assess the normality of the breaking strengths. Both the histogram and the normal probability plot (Figure 2.6.1) suggest that breaking strengths do not behave normally. The customer specifications and numerical results for the first fifty observations were as follows

N	50
USL	2.0
LSL	-2.0
Target	0
Mean	-0.135
Std. Deviation	0.83
Skewness	0.10
Kurtosis	0.99
Cpm	0.79

The process was monitored at regular intervals (every 24 hours) at which time $\bar{x}$, s and $\hat{C}pm$ were calculated for subgroups of size 5 and $\bar{x}$ and s charts are plotted (see Figure 2.6.2). The first subgroup of size five resulted in the following observations

n	5
x_1	0.63
x_2	-1.04
x_3	0.37
x_4	0.99
x_5	-0.48
$\bar{x}$	0.09
s	0.835
$\hat{C}pm$	0.71

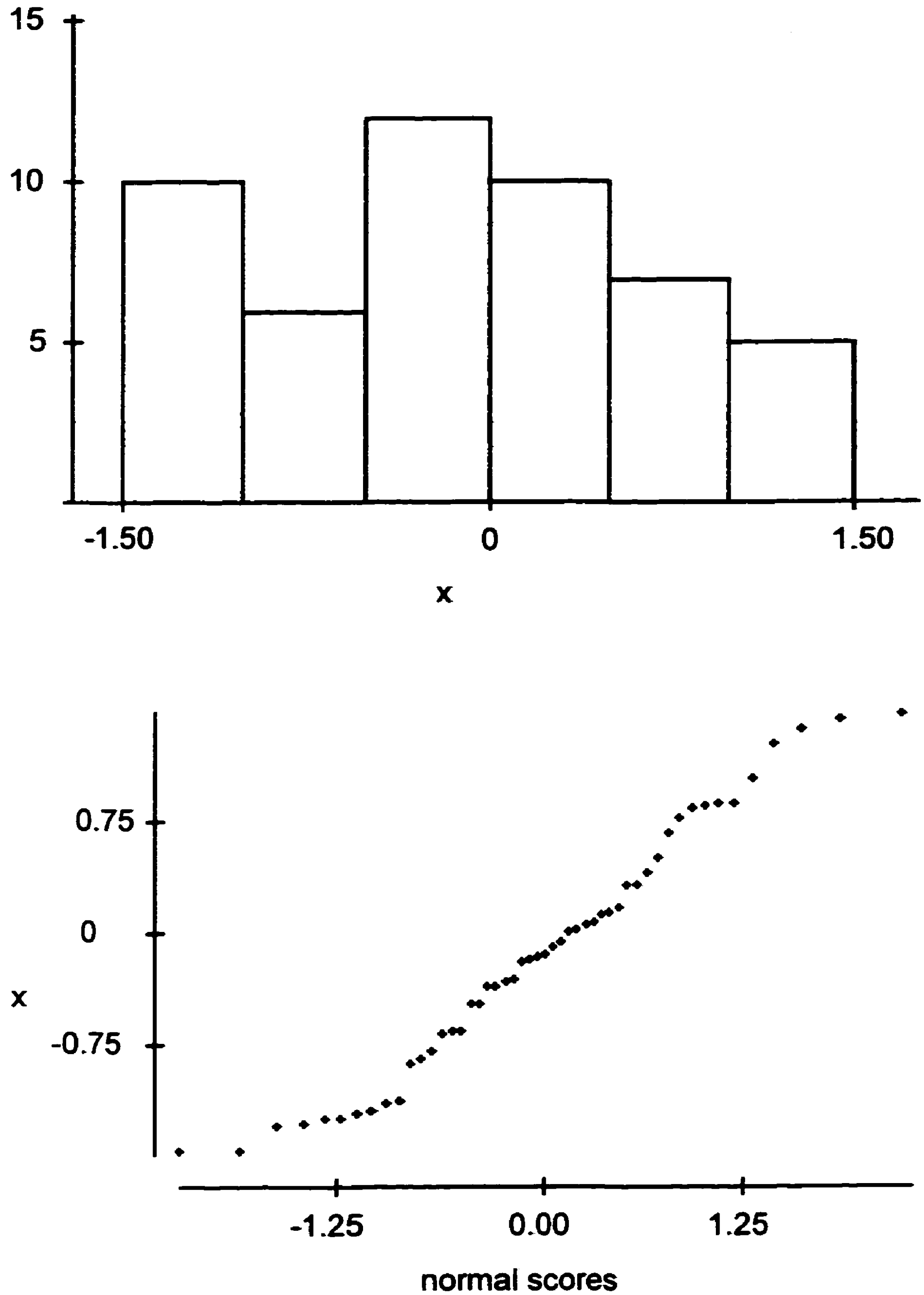


Figure 2.6.1. Histogram and Normal probability plot of Adjusted Breaking Strengths

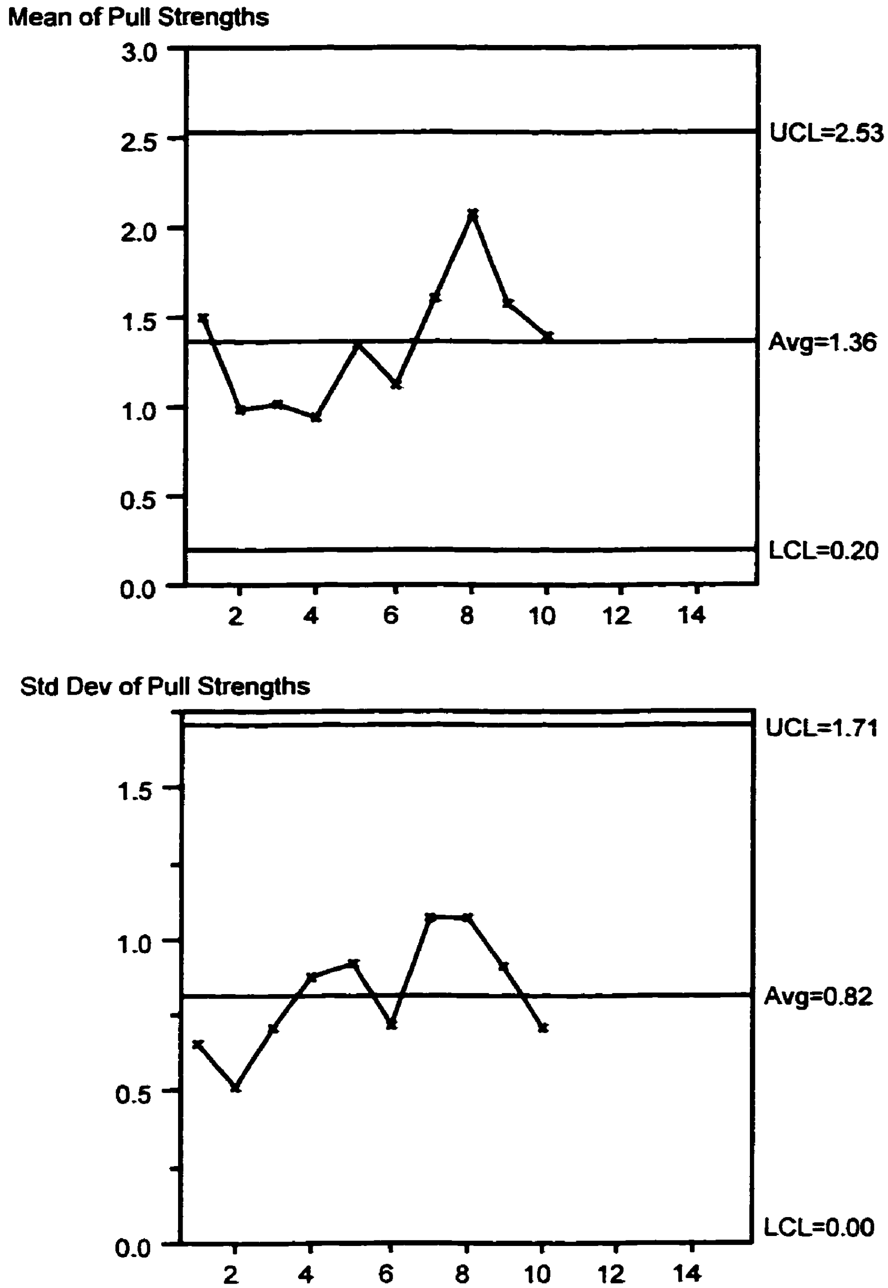


Figure 2.6.2 $\bar{x}$ and s chart of Adjusted Breaking Strengths

To investigate the process' Cpm (assuming that the observations behave similar to the first fifty), the upper and lower critical values were determined from Table 2.5.4a and Table 2.5.4b (for $n = 5$, $a_1 = \frac{4}{3}$, $\lambda_3 = 0.1$ and $\lambda_4 = 1.0$) to be $c_L = .2690$ and $c_U = 3.004$ ($\alpha = 0.0027$). Since $\hat{C}pm = 0.71$ fell inside the interval [.2690, 3.0042] the practitioner concluded that the process Cpm had not changed.

2.7 Relationship with Squared Error Loss Function

Kane (1986) noted that PCIs were receiving increased usage in process measurements and purchasing decisions especially in the automotive industry, and the indices Cp and Cpk were of interest. These indices are simple to manipulate, and are convenient because they are based on traditional specification limits. However, they are not related to the cost of failing to meet customers' desires. Taguchi emphasized the loss in a product's worth when one of its characteristics departs from the customers' ideal value T. To help account for this Hsiang and Taguchi (1985) introduced the index Cpm, which was also proposed independently by Chan, Cheng, and Spiring (1988b) and they related the index Cpm to the idea of squared error loss.

The index Cpm is defined in equation [1.3.1] is a function of the expected squared deviation from the target. The loss associated with a characteristic X missing its target is often assumed to be appropriately approximated by the symmetric squared error loss function

$$L(x) = w(x - T)^2$$

where w is a non-stochastic weight function. This implies that the loss is zero when the process is on target and the loss is positive for any deviation from target. In this case the expected loss is

$$E[L(X)]=w\left[\sigma^2+(\mu-T)^2\right] \qquad [2.7.1]$$

and the Cpm index can be expressed as

$$Cpm=\sqrt{w}\,\frac{USL-LSL}{6\sqrt{E[L(X)]}}. \qquad [2.7.2]$$

Equation [2.7.1] can be expressed in terms of Cpm

$$E[L(X)]=w\,\frac{[USL-LSL]^2}{36\,Cpm^2} \qquad [2.7.3]$$

Clearly when the expected loss increases as the value of Cpm becomes smaller and vice versa.

The relationship [2.7.2] is in terms of the expected loss of the product when the product is on target. This approach reduces the information and provides an interpretation of the index Cpm in terms of the percentage loss. This intuitive interpretation should increase the acceptance of this index by management. As a decision maker who may be interested in an upper limit on the loss from the process rather than just a point estimate of the loss, $\hat{L}(X)$. An unbiased estimator of E[L(X)] is

$$\hat{L}(X)=w\left[\hat{\sigma}^2+(\bar{X}-T)^2\right] \qquad [2.7.4]$$

Note that $\hat{L}(X)$ is a uniformly minimum variance unbiased estimator (UMVUE) of L(x) if X comes from $N(\mu, \sigma^2)$ since it is unbiased and is a function of jointly complete sufficient statistics. Hence

$$\hat{C}pm = \sqrt{w}\,\frac{USL - LSL}{6\sqrt{\hat{L}(X)}} \tag{2.7.5}$$

Notice that

$$\frac{\hat{L}(X)}{w} = \hat{\sigma}^2 + (\bar{X} - T)^2$$

$$\frac{n}{\sigma^2}\frac{\hat{L}(X)}{w} = \frac{n}{\sigma^2}\left[\hat{\sigma}^2 + (\bar{X} - T)^2\right]$$

which has a non-central chi square with n df and non-centrality parameter $\lambda = \frac{n(\mu - T)^2}{\sigma^2}$ if

X arises from $N(\mu, \sigma^2)$. The ratio

$$\frac{\hat{L}(X)}{E[L(X)]} = \frac{\left[\hat{\sigma}^2 + (\bar{X} - T)^2\right]\frac{n}{\sigma^2}}{\left[\sigma^2 + (\mu - T)^2\right]\frac{n}{\sigma^2}}$$

$$= \frac{\frac{n}{\sigma^2}\left[\hat{\sigma}^2 + (\bar{X} - T)^2\right]}{n + \lambda}$$

so that $$\frac{n + \lambda}{E[L(X)]}\hat{L}(X) \sim \chi^2_{n,\lambda}.$$

Then an upper (1 - α) 100% confidence limit for the loss function parameters, E[L(X)], is

$$P\left(\frac{n + \lambda}{E[L(X)]}\hat{L}(X) \geq \chi^2_{n,\lambda;1-\alpha}\right) = 1 - \alpha$$

$$\Rightarrow \qquad P\left(E[L(X)] \le \frac{(n+\lambda)}{\chi^2_{n,\lambda;1-\alpha}}\hat{L}(X)\right) = 1-\alpha \qquad [2.7.6]$$

where $\chi^2_{n,\lambda;1-\alpha}$ is the 100(1-α)th percentile of $\chi^2_{n,\lambda}$.

Therefore a (1 - α) 100% upper confidence limit for the loss function parameters, E[L(X)], is

$$\frac{(n+\lambda)}{\chi^2_{n,\lambda;1-\alpha}}\hat{L}(X) = \frac{(n+\lambda)}{\chi^2_{n,\lambda;1-\alpha}}\left[\hat{\sigma}^2 + (\overline{X}-T)^2\right]$$

Applying the classical Patnaik (1949) approximation by matching the first two moments of a scaled chi-square of the form $c\chi^2_\nu$, where the constants c and ν are determined by equating the means and variances of the two distributions, i.e., to solve the equations

$$c\nu = n\left(1+\frac{\lambda}{n}\right), \qquad 2c^2\nu = 2n\left(1+\frac{2\lambda}{n}\right)$$

$$\Rightarrow \qquad c = \frac{n+2\lambda}{n+\lambda}, \qquad \nu = \frac{(n+\lambda)^2}{n+2\lambda}.$$

So that

$$\chi^2_{n,\lambda} \approx c\chi^2_\nu$$

$$\frac{n+\lambda}{\chi^2_{n,\lambda}} \approx \frac{n+\lambda}{c\chi^2_\nu} = \frac{\nu}{\chi^2_\nu}$$

and results in an approximate upper (1 - α) 100% confidence limit for the loss function parameter, E[L(X)]

$$\frac{\hat{\nu}}{\chi^2_{\hat{\nu};1-\alpha}}\hat{L}(X) = \frac{\hat{\nu}}{\chi^2_{\hat{\nu};1-\alpha}}\left[\hat{\sigma}^2 + (\overline{X}-T)^2\right] \qquad [2.7.7]$$

where $\hat{\nu} = \frac{(n+\hat{\lambda})^2}{n+2\hat{\lambda}}$ and $\hat{\lambda} = \frac{n(\overline{X}-T)^2}{\hat{\sigma}^2}$.

2.8 Comments

We have attempted to indicate that robustness studies for those process capability indices whose magnitudes are translated into parts per million non-conforming are meaningless as the parameters are sensitive to departures from normality. Hence regardless of how robust the estimator maybe, its associated parameter is not stable and hence any robustness claims carry little meaning. Similarly, developing actual and approximate confidence intervals for these capability indices when the process characteristics arise from non-normal distributions is an academic pursuit with no application.

For those capability indices that attempt to assess the ability of the process to cluster around the target, the robustness of the estimator is a valid concern. We have examined the robustness of the traditional estimator which also has the smallest bias and mean square error, in the face of moderate departures from normality. From the examination we are able to make recommendations/adjustments to critical values associated with attempts to assess changes in the process capability. Similar alterations to the action limits associated with capability monitoring charts are possible.

The Cpm index, as well as its generalization Cpm*, can be estimated respectively using $\hat{C}pm$ and $\hat{C}pm^*$ for those cases where $\mu = T$. Both $\hat{C}pm$ and $\hat{C}pm^*$ have been shown to be biased estimators of Cpm and Cpm* respectively but are asymptotically unbiased.

The quantities, Cpm and E[L(X)], each have their own advantages and are familiar to quality practitioners. The expected loss does require the use of an explicit loss function such as Taguchi's modified loss function. However it is easily interpreted in terms of monetary loss , either to the practitioner and/or the society when the process characteristic misses the target.

Chapter 3

Alternative Measures of Process Capability

3.1 Introduction

The use of order statistics in industrial applications has been studied since late 1940's. In statistical quality control, usually small samples, say $n = 5$, are taken at intervals from a production process. For each sample, the means and the ranges are plotted on separate control charts to indicate whether the process is in- or out-of-control. The study of the range and the mean range as measures of dispersion under normality are outlined in David (1981) and Arnold, Balakrishnan and Nagaraja (1992). Efficiency and approximations of the mean range have been considered by Cox (1949), Patnaik (1950) and Cadwell (1953). The distribution of range in random samples has been discussed in depth by Hartley (1942), McKay and Pearson (1933) and McKay (1935).

Inferences regarding process measurements that do not appear to follow a normal distribution were earlier cautioned against. One reason for the caution being that $\hat{C}p$ (Chan, Cheng, and Spiring (1988c)), $\hat{C}pk$ and $\hat{C}pm$ (Section 2.5) have been shown to non-robust to departures from normality. For more dramatic departures from normality the problem does not lie so much in the non-robustness of the PCIs themselves. Regardless of the abilities of $\hat{C}p$, $\hat{C}pk$ and $\hat{C}pm$ to depict the true value of Cp, Cpk and Cpm respectively, if the measurements do not arise from a normal distribution poor inferences may be drawn. So all PCIs are not universally appropriate measures of the ability of a process to ensure 99.73% of the process measurements fall within the required tolerance limits. Another major difficulty in the ability of the PCIs to indicate process capability is their functional forms involve a function of the population standard deviation as mentioned earlier in Section 2.3. Hence in addition to other criticisms of the PCIs we now find that as a general measure of process capability, Cp, as well as other indices Cpk, and Cpm which incorporate σ into their computing algorithm, should be restricted to the family of normal distributions. An alternative measure of the actual process spread, which possesses the ability to provide consistent inferences over various distributions that the process measurements may assume, should be considered.

The applications of order statistics to assess the process capability have been considered in Chan, Cheng and Spiring (1988a) and Clements (1989). In this chapter, a new process capability index, Cpo, is proposed using order statistics. We will examine this index for various process distributions such as uniform, exponential and normal distributions. The sampling distribution of $\hat{C}po$, the estimate of Cpo, and its associated properties such as pdf, mean and variance when the sample size is small as well the bias and mean squared error when the sample size is large, are investigated. The right hand tail probabilities and critical values for small sample sizes are tabulated for reference. The sampling distribution of $\hat{C}po$ is distribution-free when the distribution of process measurements is uniform and exponential, and it works for any sample size and any value of r and s, for $r < s$.

3.2 The Density of $\hat{C}po$

Let X be the measurements of a process characteristic and a random sample of size n is drawn from X so as to measure the process capability. Let $Y_1 < Y_2 < \ldots < Y_n$ be the corresponding order statistics. We consider the spacing between the rth and the sth order statistics, that is

$$\hat{D} = Y_s - Y_r \qquad [3.2.1]$$

whose pdf can be shown to be

$$f_{\hat{D}}(w) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{-\infty}^{\infty} [F(y_r)]^{r-1} [F(y_r + w) - F(y_r)]^{s-r-1}$$

$$[1 - F(y_r + w)]^{n-s} f(y_r) f(y_r + w) dy_r, \quad 0 < w < \infty, \text{ for } r<s. \qquad [3.2.2]$$

Now, define

$$Cpo = \frac{USL - LSL}{D}$$

where D is the difference between the population quantiles, $\xi_{.00135}$ and $\xi_{.99865}$, such that the probability $P(X < \xi_\gamma) = \gamma$.

An estimator of Cpo is defined as $\hat{C}po = \dfrac{USL - LSL}{\hat{D}} = \dfrac{a'}{\hat{D}}$.

The distribution of $\hat{C}po$ can be found by making a transformation on $\hat{D}$ resulting in

$$g_{\hat{C}po}(y) = f_{\hat{D}}\left(\frac{a'}{y}\right)\left|\frac{d}{dy}\frac{a'}{y}\right| = f_{\hat{D}}\left(\frac{a'}{y}\right)\left|-\frac{a'}{y^2}\right|$$

$$= f_{\hat{D}}\left(\frac{a'}{y}\right)\frac{a'}{y^2}, \qquad 0 < y < \infty \qquad [3.2.3]$$

3.2.1 The Distribution of $\hat{C}po$ when X arises from Uniform Distribution

If $X_1, X_2, \ldots, X_n \sim U(0,1)$, then $\hat{D} \sim Be(s\text{-}r, n\text{-}s\text{+}r\text{-}1)$ (David (1981), Arnold, Balakrishnan and Nagaraja (1992)). Hence the pdf of $\hat{C}po$ has the form

$$g_{\hat{C}po}(y) = \frac{n!}{(s-r-1)!(n-s+r)!}\left(\frac{a'}{y}\right)^{s-r-1}\left(1-\frac{a'}{y}\right)^{n-s+r}\frac{a'}{y^2}$$

$$= \frac{1}{B(s-r, n-s+r+1)\,y}\left(\frac{a'}{y}\right)^{(s-r+1)-1}\left(1-\frac{a'}{y}\right)^{(n-s+r+1)-1}$$

$a' < y < \infty$, zero elsewhere. [3.2.1.1]

Figures 3.2.1.1a-c show the density of $\hat{C}po$ using equation [3.2.1.1] when a' = USL – LSL = .1, n = 5, and various values of r and s.

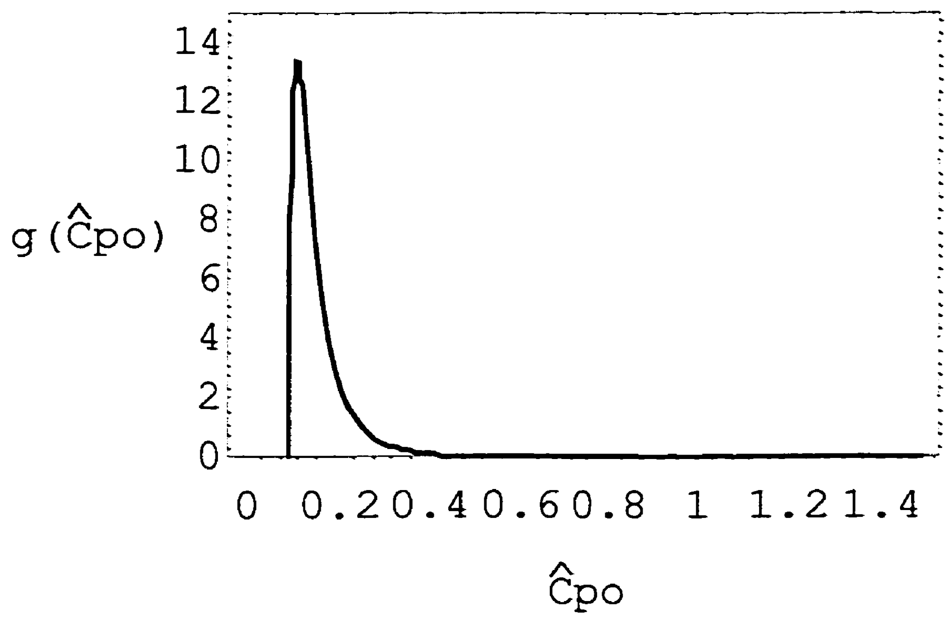


Figure 3.2.1.1a Density of $\hat{C}po$ with a' = .1, n = 5, r = 1 and s = 5.

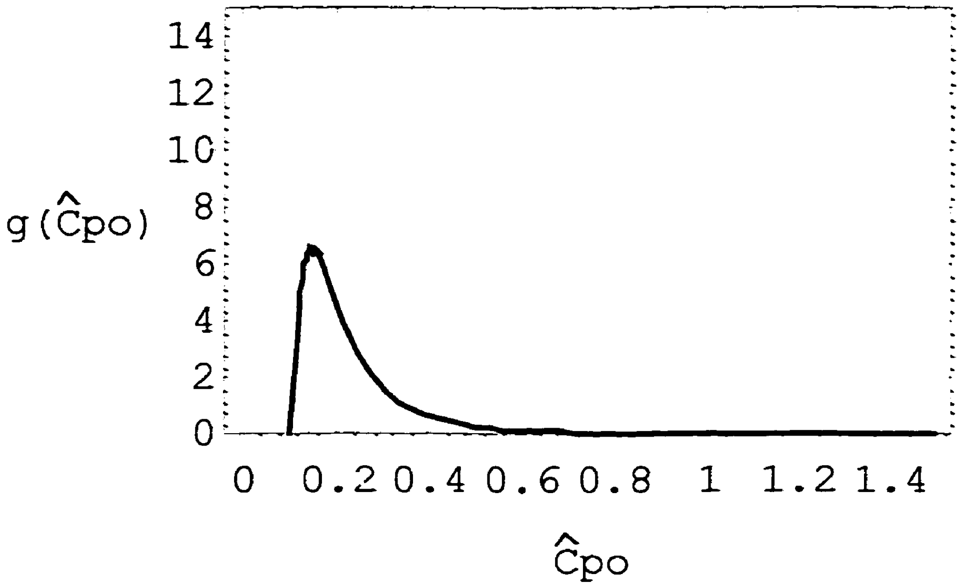


Figure 3.2.1.1b Density of Ĉpo with a' = .1, n = 5, r = 2 and s = 5.

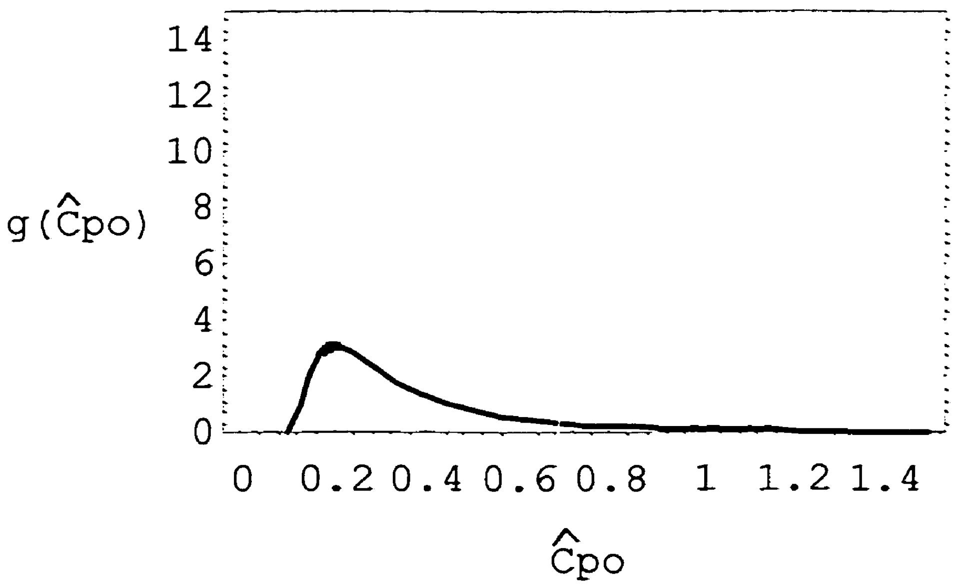


Figure 3.2.1.1c Density of Ĉpo with a' = .1, n = 5, r = 2 and s = 4.

Figures 3.2.1.2a, b show the density of $\hat{C}po$ using equation [3.2.1.1] when a' = USL – LSL = .1, n = 10, and different values of r and s.

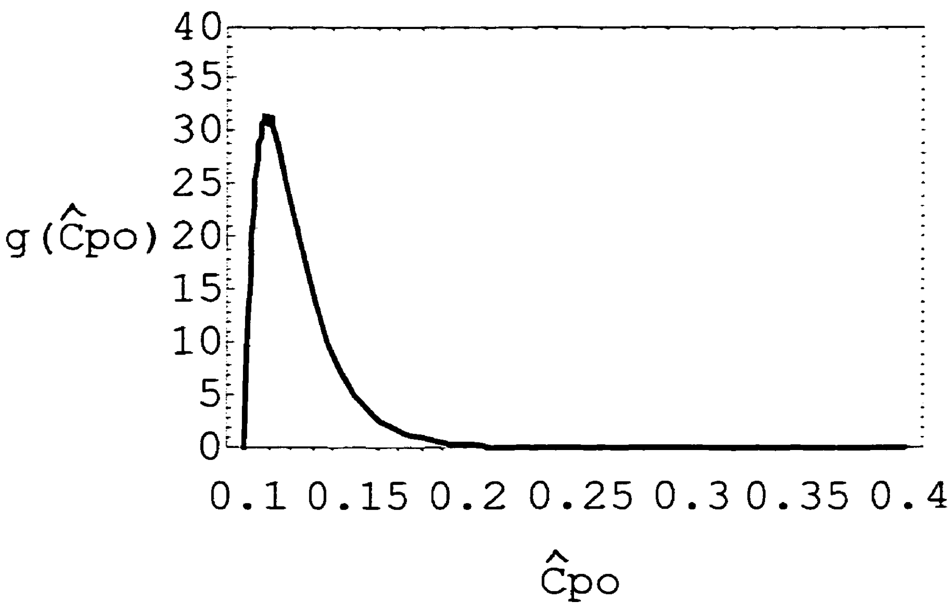


Figure 3.2.1.2a Density of $\hat{C}po$ with a' = .1, n = 10, r = 1 and s = 10.

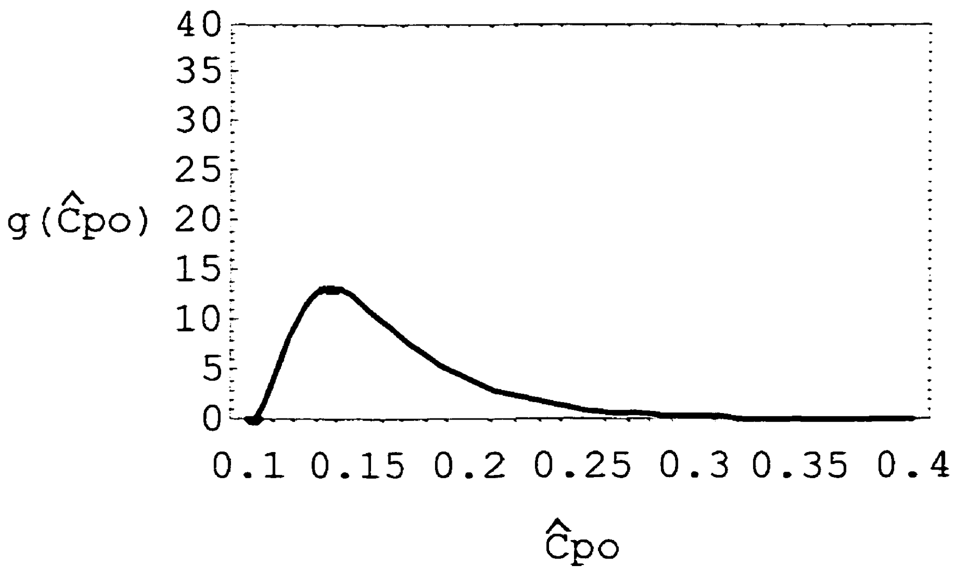


Figure 3.2.1.2b Density of $\hat{C}po$ with a' = .1, n = 10, r = 2 and s = 9.

Figures 3.2.1.3a, b show the density of $\hat{C}po$ using equation [3.2.1.1] when a' = USL − LSL = .1, n = 20, and different values of r and s.

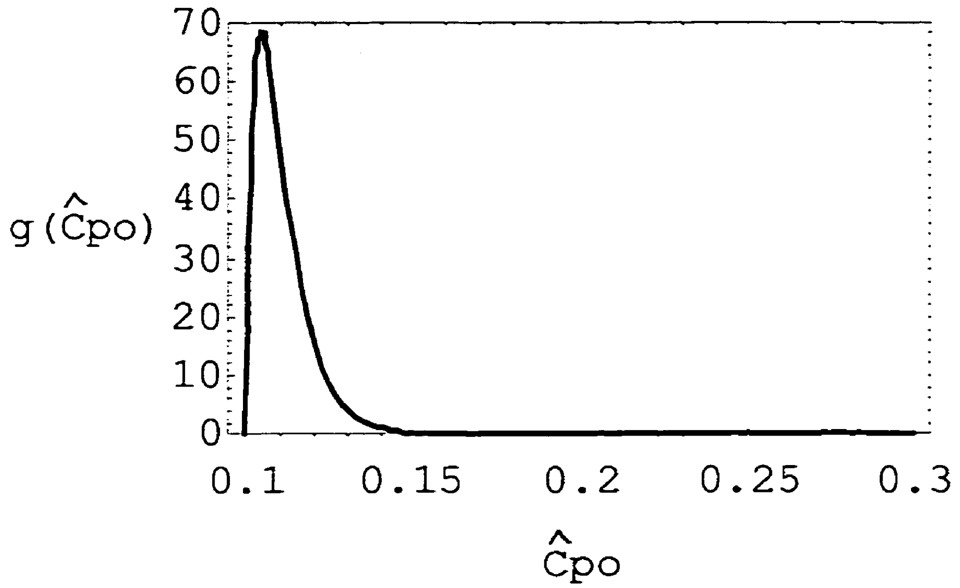


Figure 3.2.1.3a Density of $\hat{C}po$ with a' = .1, n = 20, r = 1 and s = 20.

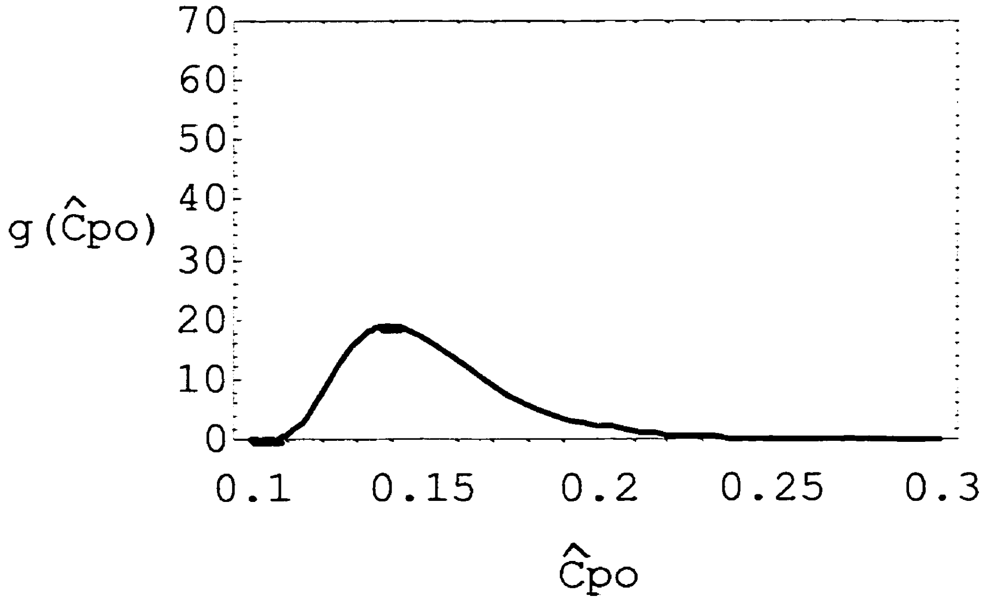


Figure 3.2.1.3b Density of $\hat{C}po$ with a' = .1, n = 20, r = 3 and s = 17.

From Figures 3.2.1.1a-c, Figures 3.2.1.2a-b and Figures 3.2.1.3a-b we can see that for a fixed sample size n, increasing the value of r or decreasing the value of s will skew the density of $\hat{C}po$ to the right. The magnitude of skewness is larger when decreasing s than increasing r meanwhile the magnitude of kurtosis will also be changed from leptokurtic to platykurtic.

Theorem 3.2.1:

The kth moment of $\hat{C}po$ when $X \sim U(0,1)$ is

$$E\left(\hat{C}po^{k}\right)=a'^{k}\frac{\Gamma(n+1)\Gamma(s-r-k)}{\Gamma(n-k+1)\Gamma(s-r)},\quad \text{for } s-r>k. \qquad [3.2.1.2]$$

Proof:

Consider the expectation of the kth moment of $\hat{C}po$ using equation [3.2.1.1] and substituting $u=\frac{a'}{y}$, then the expectation becomes

$$E\left(\hat{C}po^{k}\right)=\int_{a'}^{\infty}\frac{y^{k-1}}{B(s-r,n-s+r+1)}\left(\frac{a'}{y}\right)^{(s-r+1)-1}\left(1-\frac{a'}{y}\right)^{(n-s+r+1)-1}dy$$

$$=\int_{0}^{1}\frac{a'^{k}}{B(s-r,n-s+r+1)}u^{(s-r-k)-1}(1-u)^{(n-s+r+1)-1}\,du$$

$$=a'^{k}\frac{B(s-r-k,n-s+r+1)}{B(s-r,n-s+r+1)}$$

$$=a'^{k}\frac{\Gamma(n+1)\Gamma(s-r-k)}{\Gamma(n-k+1)\Gamma(s-r)},\quad \text{for } s-r>k.$$

Then the mean and variance are respectively

$$E(\hat{C}po)=a'\frac{\Gamma(s-r-1)\Gamma(n-s+r+1)}{\Gamma(n)}\frac{\Gamma(n+1)}{\Gamma(s-r)\Gamma(n-s+r+1)}$$

$$= \frac{a'n}{s-r-1}, \quad \text{for } s - r > 1, \qquad [3.2.1.3]$$

$$V(\hat{C}po) = \frac{a'^2 n(n-1)}{(s-r-1)(s-r-2)} - \frac{a'^2 n^2}{(s-r-1)^2}$$

$$= \frac{a'^2 n(n-s+r+1)}{(s-r-1)^2 (s-r-2)}, \quad \text{for } s - r > 2. \qquad [3.2.1.4]$$

Consider a special case of $\hat{C}po$ when n = 2, i.e., r = 1 and s = 2. The pdf in equation [3.2.1.1] becomes

$$g_{\hat{C}po}(y) = 2\left(1 - \frac{a'}{y}\right)\frac{a'}{y^2}, \quad a' < y < \infty, \text{ zero elsewhere.} \qquad [3.2.1.5]$$

It can be shown that equation [3.2.1.5] is equivalent to the density of $\hat{C}p$ when n = 2 except the constant a' is replaced by $\frac{a'}{3\sqrt{2}}$. From the definition of Cp,

$$Cp = \frac{\text{Allowable process spread}}{\text{Actual process spread}}$$

$$= \frac{a'}{6\sigma}.$$

The usual estimator of Cp is defined to be

$$\hat{C}p = \frac{a'}{6S}$$

$$= \frac{a'}{6\sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}}$$

$$= \frac{a'\sqrt{n-1}}{6\sqrt{\sum_{i=1}^{n}(X_i - \overline{X})^2}}.$$

For n = 2, the sample variance can be expressed as below:

$$S^2 = \frac{\sum_{i=1}^{2}(X_i - \overline{X})^2}{2-1}$$

$$= \left(X_1 - \frac{X_1 + X_2}{2}\right)^2 + \left(X_2 - \frac{X_1 + X_2}{2}\right)^2$$

$$= \frac{(X_1 - X_2)^2}{4} + \frac{(X_2 - X_1)^2}{4}$$

$$= \frac{(X_1 - X_2)^2}{2}$$

Let $\hat{D} = |X_1 - X_2| = Y_s - Y_r$, this is analogous to equation [3.2.1], then $S^2 = \frac{\hat{D}^2}{2}$, and

$S = \frac{\hat{D}}{\sqrt{2}}$. So

$$\hat{C}p = \frac{a'}{6S} = \frac{\sqrt{2}a'}{6\hat{D}} = \frac{a'}{3\sqrt{2}\,\hat{D}}.$$

The inverse transformation is

$$\hat{D} = \frac{a'}{3\sqrt{2}\,\hat{C}p}$$

with Jacobian

$$|J| = \left|\frac{d\hat{D}}{d\hat{C}p}\right|$$

$$= \left|-\frac{a'}{3\sqrt{2}\,\hat{C}p^2}\right| = \frac{a'}{3\sqrt{2}\,\hat{C}p^2}$$

Let $f_X(x)$ be the pdf of X_i, $i = 1, 2$; and $g_{\hat{D}}(x)$ be the pdf of $\hat{D}$, then the general form of the pdf associated with $\hat{C}p$ is

$$h_{\hat{C}p}(y) = g_{\hat{D}}\left(\frac{a'}{3\sqrt{2}\,y}\right)|\,J\,|$$

$$= g_{\hat{D}}\left(\frac{a'}{3\sqrt{2}\,y}\right)\frac{a'}{3\sqrt{2}\,y^2} \qquad [3.2.1.6]$$

Now, X has pdf $f_X(x) = 1,\ 0 < x < 1$. By a transformation it is easy to show that the pdf of $\hat{D} = |X_1 - X_2|$ is

$$g_{\hat{D}}(x) = 2(1 - x), \qquad 0 < x < 1.$$

Following equation [3.2.1.6], the pdf of $\hat{C}p$ under uniform distribution is

$$h_{\hat{C}p}(y) = 2\left(1 - \frac{a'}{3\sqrt{2}\,y}\right)\frac{a'}{3\sqrt{2}\,y^2}, \quad \frac{a'}{3\sqrt{2}} < y < \infty\text{, zero elsewhere.} \qquad [3.2.1.7]$$

Hence this is equivalent to equation [3.2.1.5] with a' replaced by $\frac{a'}{3\sqrt{2}}$. Note that both a' and $\frac{a'}{3\sqrt{2}}$ serve as a scale parameter respectively in the distribution of $\hat{C}po$ and $\hat{C}p$. All the moments do not exist in this special case when n = 2 for both $\hat{C}po$ and $\hat{C}p$ since the conditions in equations [3.2.1.3] and [3.2.1.4] are not satisfied.

Tables 3.2.1.1a and 3.2.1.1b show the lower (c_L) and upper (c_U) critical values of $\hat{C}po$ with a' = .1 and a' = .5 respectively, for various sample sizes, where $P(\hat{C}po < c_L) = .00135 = P(\hat{C}po < c_U)$.

Table 3.2.1.1a The critical values of $\hat{C}po$ with a' = .1.

n	c_L	c_U
2	.103815	148.0986
5	.101190	.758684
10	.100559	.256515

Table 3.2.1.1b The critical values of $\hat{C}po$ with a' = .5.

n	c_L	c_U
2	.519075	740.493
5	.505949	3.79342
10	.502795	1.28257
20	.501358	.789677

Tables 3.2.1.2a and 3.2.1.2b show the right hand tail probabilities of $\hat{C}po$ with a' = .1 and a' = .5 respectively.

Table 3.2.1.2a The right hand tail probabilities of $\hat{C}po$ with a' = .1.

	$P(\hat{C}po > c)$			
n c	.5	1.0	1.5	2.0
2	.360000	.190000	.128889	.097500
5	.006720	.000460	.000094	.000030
10	4.2×10^{-6}	9.1×10^{-9}	2.5×10^{-10}	1.9×10^{-11}

Table 3.2.1.2b The right hand tail probabilities of $\hat{C}po$ with a' = .5.

	$P(\hat{C}po > c)$		
n c	1.0	1.5	2.0
2	.750000	.555556	.437500
5	.187500	.045268	.015625
10	.010742	.000356	.000030
20	.000020	1.2×10^{-8}	5.5×10^{-11}

Let us examine the bias and mean squared error of $\hat{C}po$.

$$B(\hat{C}po) = E(\hat{C}po) - Cpo$$

$$= \frac{a'n}{s-r-1} - \frac{a'}{\gamma}$$

$$= a'\left[\frac{n}{s-r-1} - \frac{1}{\gamma}\right] \quad [3.2.1.8]$$

where γ is the probability contained between Y_r and Y_s.

$$MSE(\hat{C}po) = E\left[(\hat{C}po - Cpo)^2\right] = V(\hat{C}po) + \left[B(\hat{C}po)\right]^2$$

$$= \frac{a'^2 n(n-s+r+1)}{(s-r-1)^2(s-r-2)} + a'^2\left[\frac{n}{s-r-1} - \frac{1}{\gamma}\right]^2$$

$$= a'^2\left\{\frac{n(n-s+r+1)}{(s-r-1)^2(s-r-2)} + \left[\frac{n}{s-r-1} - \frac{1}{\gamma}\right]^2\right\} \quad [3.2.1.9]$$

Assuming an equal tailed out-of-control probability, s can be written as a function of n and r, i.e., $s = n - r + 1$. Hence the bias and mean squared error can be re-stated in terms of n and r as below.

$$B(\hat{C}po) = a'\left[\frac{n}{n-2r} - \frac{1}{\gamma}\right] \quad [3.2.1.10]$$

$$MSE(\hat{C}po) = a'^2\left\{\frac{n(2r)}{(n-2r)^2(n-2r-1)} + \left[\frac{n}{n-2r} - \frac{1}{\gamma}\right]^2\right\} \quad [3.2.1.11]$$

Tables 3.2.1.2a and 3.2.1.2b show the biases and mean squared errors of $\hat{C}po$ with $a' = .1$ and $a' = .5$ for various sample sizes.

Tables 3.2.1.3a Biases and mean squared errors of $\hat{C}po$ with a' = .1.

n	$B(\hat{C}po)$	$MSE(\hat{C}po)$
5	.066396	.009964
10	.024729	.001580
20	.010840	.000190

Tables 3.2.1.3b Biases and mean squared errors of $\hat{C}po$ with a' = .5.

n	$B(\hat{C}po)$	$MSE(\hat{C}po)$
5	.331980	.249099
10	.123646	.026449
20	.054202	.004753

For example, let n = 740, r = 1, i.e., s = 740, and $\gamma = .9973$, then

$$B(\hat{C}po) = a'\left[\frac{740}{738} - \frac{1}{.9973}\right] = a'[1.002710027 - 1.00270737] = 0.0000027173 \approx 0.$$

$$MSE(\hat{C}po) = a'^2\left\{\frac{740(2)}{(738)^2(737)} + \left[\frac{740}{738} - \frac{1}{.9973}\right]^2\right\}$$

$$= a'^2\left[\ 3.68707\times10^{-6} + 7.38426\times10^{-12}\ \right] \approx 0.$$

Consider another example, let n = 100000, r = 135, i.e., s = 99866, and $\gamma = .9973$, then

$$B(\hat{C}po) = a'\left[\frac{100000}{99730} - \frac{1}{.9973}\right] = 0$$

$$MSE(\hat{C}po) = a'^2\left\{\frac{100000(270)}{(99730)^2(99729)} + \left[\frac{100000}{99730} - \frac{1}{.9973}\right]^2\right\}$$

$$= a'^2\left[\ 2.72202\times10^{-8}\ \right] \approx 0.$$

Now, taking limits as n approaches infinity, then

$$\lim_{n\to\infty} B(\hat{C}po) = a' \lim_{n\to\infty} \left[\frac{n}{n-2r} - \frac{1}{\gamma}\right] \approx 0,$$

and
$$\lim_{n\to\infty} MSE(\hat{C}po) = a'^2 \lim_{n\to\infty} \left\{\frac{n(2r)}{(n-2r)^2(n-2r-1)} + \left[\frac{n}{n-2r} - \frac{1}{\gamma}\right]^2\right\} \approx 0.$$

This implies that $\hat{C}po$ is asymptotically unbiased and mean squared error consistent. These also imply that $\hat{C}po$ converges in probability to Cpo.

3.2.2 The Distribution of $\hat{C}po$ when X arises from Exponential Distribution

It follows directly from the transformation that $X = -\theta \ln Y \sim E\left(\frac{1}{\theta}\right)$, having the negative exponential distribution with parameter θ if Y is uniformly distributed between 0 and 1. If $X_1, X_2, \ldots, X_n$ is a random sample from a negative exponential distribution with the pdf

$$f_X(x) = \theta e^{-\theta x}, \qquad 0 < x < \infty$$

then $\hat{D}$ has the pdf of the form

$$f_{\hat{D}}(w) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^\infty \left[1 - e^{-\theta y_r}\right]^{r-1} \left[e^{-\theta y_r} - e^{\theta(y_r+w)}\right]^{s-r-1}$$
$$\left[e^{-\theta(y_r+w)}\right]^{n-s} \theta e^{-\theta y_r}\, \theta e^{-\theta(y_r+w)} dy_r$$

$$= \frac{n!\theta^2 e^{-\theta w(n-s+1)}\left(1-e^{-\theta w}\right)^{s-r-1}}{(r-1)!(s-r-1)!(n-s)!} \int_0^\infty \left[1-e^{-\theta y_r}\right]^{r-1} e^{-\theta y_r (n-r+1)} dy_r$$

making a substitution, $u = e^{-\theta y_r}$, then

$$f_{\hat{D}}(w)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}\theta^2 e^{-\theta w(n-s+1)}\left(1-e^{-\theta w}\right)^{s-r-1}\int_0^1 u^{n-r+1}(1-u)^{r-1}\frac{du}{\theta u}$$

$$=\frac{(n-r)!}{(s-r-1)!(n-s)!}\theta\left[e^{-\theta w}\right]^{n-s+1}\left[1-e^{-\theta w}\right]^{s-r-1}$$

$$=\frac{1}{B(n-s+1,s-r)}\theta\left[e^{-\theta w}\right]^{n-s+1}\left[1-e^{-\theta w}\right]^{s-r-1},\quad \text{for } 0<w<\infty. \qquad [3.2.2.1]$$

Or, it may be expressed in terms of the distribution function of X, $F_X(x)=1-e^{-\theta x}$

$$f_{\hat{D}}(w)=\frac{1}{B(n-s+1,s-r)}\theta[F_X(w)]^{s-r-1}[1-F_X(w)]^{n-s+1} \qquad [3.2.2.2]$$

Thus the pdf of $\hat{C}po$ utilizing equation [3.2.2] is

$$g_{\hat{C}po}(y)=\frac{1}{B(n-s+1,s-r)}\frac{a'\theta}{y^2}\left[F_X\left(\frac{a'}{y}\right)\right]^{s-r-1}\left[1-F_X\left(\frac{a'}{y}\right)\right]^{n-s+1},\quad 0<y<\infty. \qquad [3.2.2.3]$$

Figures 3.2.2.2a-c show various shapes of the pdf of $\hat{C}po$ using equation [3.2.2.3] when $a'\theta = .5$, $n=5$ with different values of r and s.

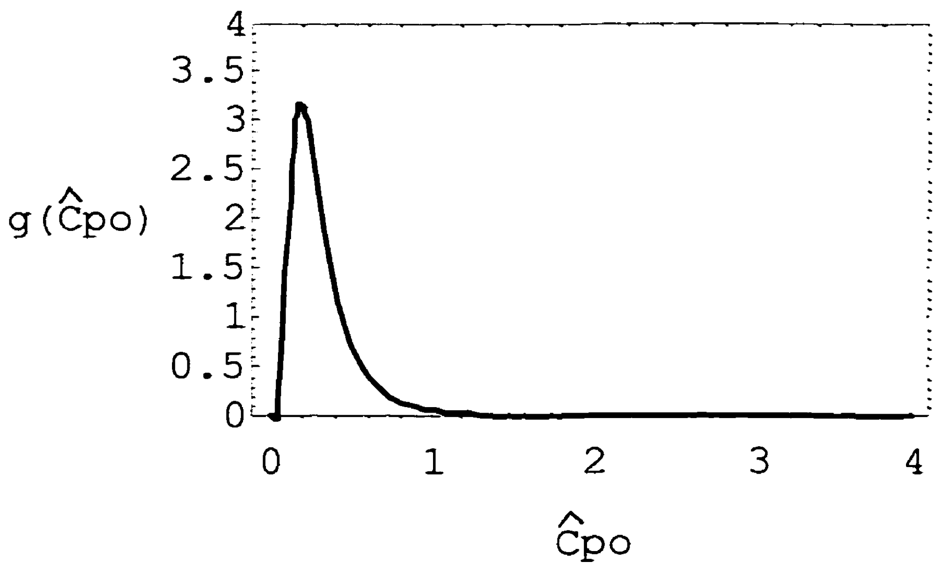


Figure 3.2.2.1a Density of $\hat{C}po$ with $a'\theta = .5$, $n=5$, $r=1$, and $s=5$.

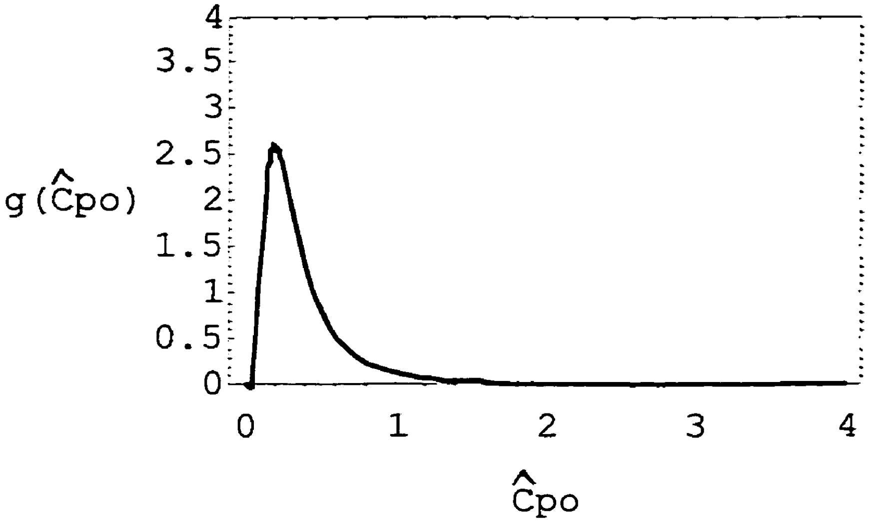


Figure 3.2.2.1b Density of Ĉpo with a'θ = .5, n = 5, r = 2, and s = 5.

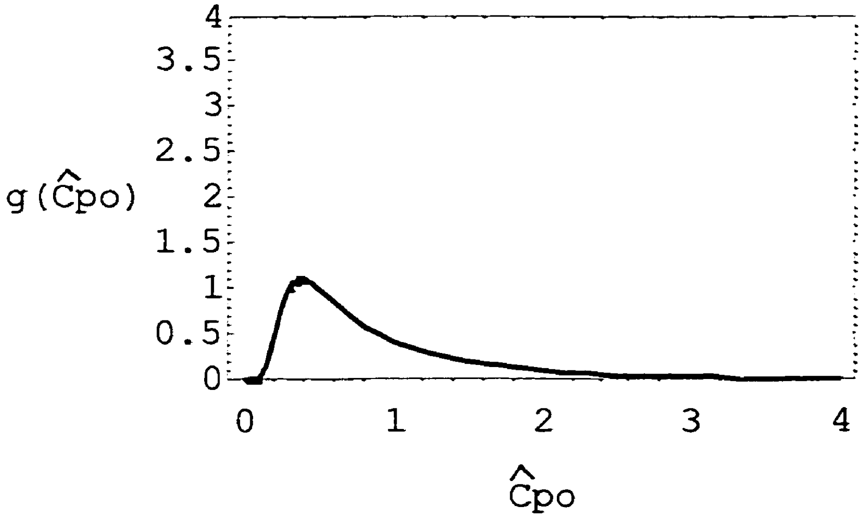


Figure 3.2.2.1c Density of Ĉpo with a'θ = .5, n = 5, r = 2, and s = 4.

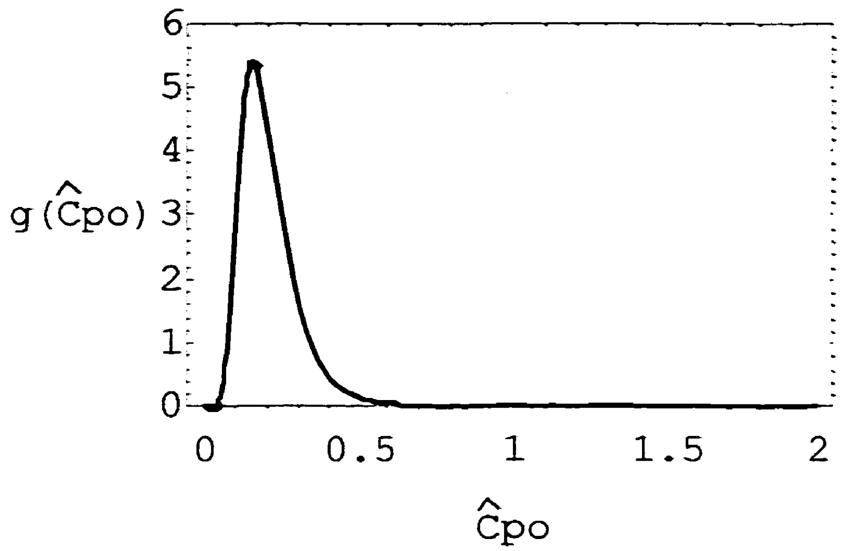


Figure 3.2.2.2a Density of Ĉpo with $a'\theta = .5$, $n = 10$, $r = 1$, and $s = 10$.

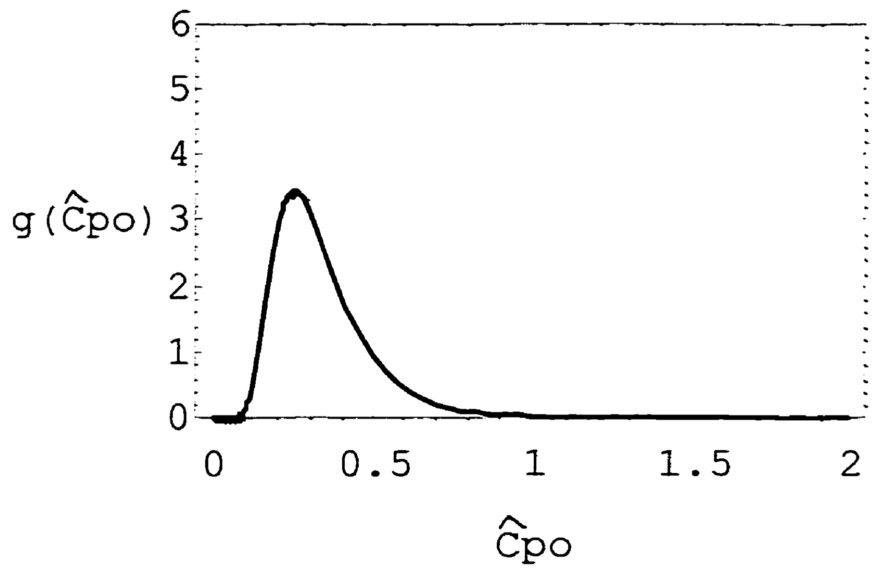


Figure 3.2.2.2b Density of Ĉpo with $a'\theta = .5$, $n = 10$, $r = 2$, and $s = 9$.

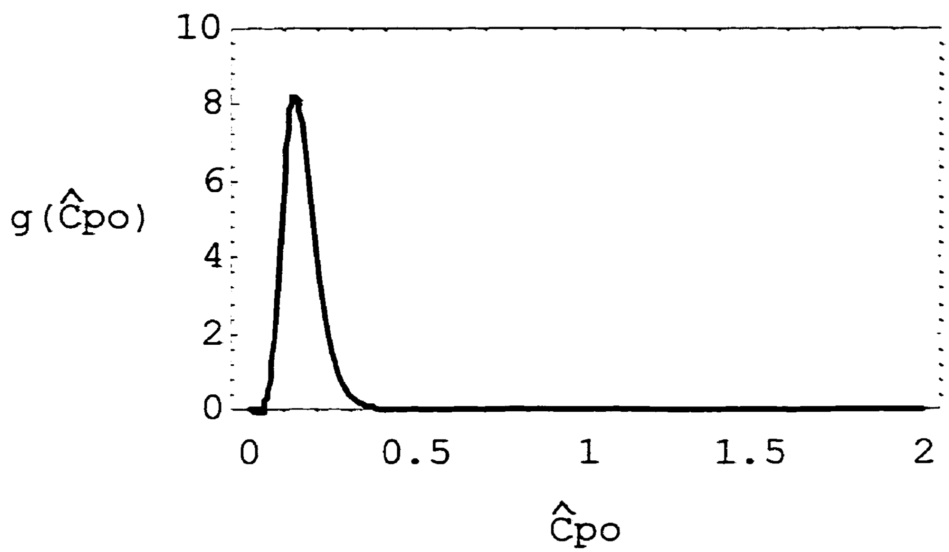


Figure 3.2.2.3a Density of Ĉpo with a'θ = .5, n = 20, r = 1, and s = 20.

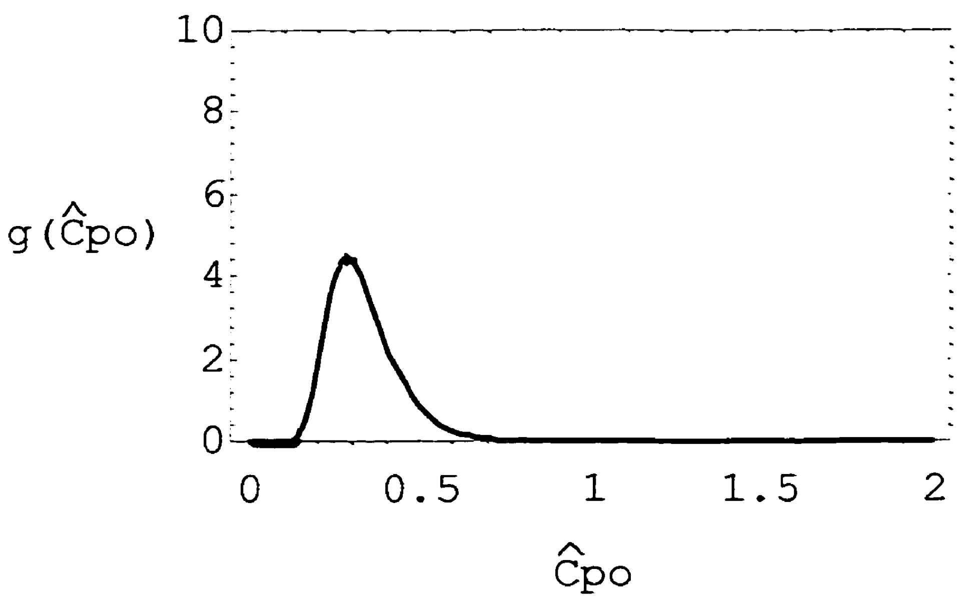


Figure 3.2.2.3b Density of Ĉpo with a'θ = .5, n = 20, r = 3, and s = 17.

From Figures 3.2.2.1a-c, Figures 3.2.2.2a-b and Figures 3.2.2.3a-b we can see that for a fixed sample size n, increasing the value of r will decrease skewness and increase kurtosis the density of $\hat{C}po$ simultaneously while decreasing the value of s will increase the kurtosis of density of $\hat{C}po$. Increasing the sample size n will cause the density curve flatter and shift the mode of the $\hat{C}po$ to the left.

Table 3.2.2.1 shows the lower (c_L) and upper (c_U) critical values of $\hat{C}po$ with $a'\theta = .5$ for various sample sizes, where $P(\hat{C}po < c_L) = .00135 = P(\hat{C}po < c_U)$ while Table 3.2.2.2 shows the right hand tail probabilities of $\hat{C}po$ with $a'\theta = .5$ for various sample sizes.

Table 3.2.2.1 The critical values of $\hat{C}po$ with $a'\theta = .5$.

n	c_L	c_U
2	.075670	370.1200
5	.062549	2.349615
10	.056789	.7648430
20	.052350	.4081427

Table 3.2.2.2 The right hand tail probabilities of $\hat{C}po$ with $a'\theta = .5$.

	$P(\hat{C}po > c)$			
n c	.5	1.0	1.5	2.0
2	.632121	.393469	.283469	.221199
5	.159661	.023969	.006457	.002394
10	.016114	.000226	.000012	1.27×10^{-6}
20	.000164	2.01×10^{-8}	3.96×10^{-11}	3.56×10^{-13}

The expectation and the variance of $\hat{C}po$ can be computed as follows :

$$E(\hat{C}po)=E(Y)$$

$$=\int_0^{\infty}\frac{1}{B(n-s+1,s-r)}\frac{a'\theta}{y}e^{-\frac{a'\theta}{y}(n-s+1)}\left(1-e^{-\frac{a'\theta}{y}}\right)^{s-r-1}dy\,,\ \text{for } s-r>1. \qquad [3.2.2.4]$$

Similarly, the second moment of $\hat{C}po$ is

$$E(\hat{C}po^2)=\int_0^{\infty}\frac{a'\theta}{B(n-s+1,s-r)}\,e^{-\frac{a'\theta}{y}(n-s+1)}\left(1-e^{-\frac{a'\theta}{y}}\right)^{s-r-1}dy\,,\ \text{for } s-r>2. \qquad [3.2.2.5]$$

Then, the variance is

$$V(\hat{C}po)=E(\hat{C}po^2)-\left[E(\hat{C}po)\right]^2 \qquad [3.2.2.6]$$

For equations [3.2.2.4] and [3.2.2.5] do not possess a closed form, numerical integration is used to evaluate the mean, variance, bias and mean squared error.

Consider a special case of $\hat{C}po$ when $n=2$, i.e., $r=1$ and $s=2$. Substituting into equation [3.2.2.3] results in

$$h_{\hat{C}p}(y)=\theta e^{-\frac{a'\theta}{y}}\frac{a'}{y^2}$$

$$=\frac{a'\theta}{y^2}e^{-\frac{a'\theta}{y}},\quad 0<y<\infty,\ \text{zero elsewhere} \qquad [3.2.2.7]$$

which is an inverted gamma distribution with shape parameter 1 and scale parameter $a'\theta$.

The pdf of an inverted gamma is

$$f(y)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\left(\frac{1}{y}\right)^{\alpha+1}e^{-\frac{\lambda}{y}},\qquad 0<y<\infty,$$

where α is the shape parameter and λ is the scale parameter, with

$$E(Y) = \frac{\lambda}{\alpha - 1}, \text{ for } \alpha > 1, \text{ and } V(Y) = \frac{\lambda^2}{(\alpha-1)^2(\alpha-2)}, \text{ for } \alpha > 2.$$

Following the development in Subsection 3.2.1, it can be shown that the pdf of $\hat{C}po$ is equivalent to the pdf of $\hat{C}p$ when n = 2 except the constant a' is replaced by $\frac{a'}{3\sqrt{2}}$.

Now, X has a negative exponential pdf with parameter θ. The distribution of $X_1 - X_2$, using the moment generating function (mgf) technique, is

$$\begin{aligned} M_{X_1-X_2}(t) &= E\left[e^{t(X_1-X_2)}\right] \\ &= E\left(e^{tX_1}\right)E\left(e^{-tX_2}\right) \\ &= \left(1-\frac{t}{\theta}\right)^{-1}\left(1+\frac{t}{\theta}\right)^{-1} \\ &= \left(1-\frac{t^2}{\theta^2}\right)^{-1} \end{aligned}$$

which is the mgf of a double exponential distribution with parameters $\alpha = 0$, and $\beta = \frac{1}{\theta}$.

Making use of the distribution function technique, the pdf of $\hat{D} = |X_1 - X_2|$ is

$$g_{\hat{D}}(x) = \theta\, e^{-\theta x}, \qquad 0 < x < \infty, \theta > 0$$

which is again a negative exponential distribution with parameter θ.

Analogous to equation [3.2.1.6], the pdf of $\hat{C}p$ under negative exponetial distribution is

$$h_{\hat{C}p}(y) = \frac{a'\theta}{3\sqrt{2}\,y^2}\, e^{-\frac{a'\theta}{3\sqrt{2}\,y}}, \quad 0 < y < \infty, \text{ zero elsewhere.} \qquad [3.2.2.8]$$

Hence both the distributions of $\hat{C}po$ and $\hat{C}p$ possess an inverted gamma when n = 2 with shape parameter 1, scale parameter $a'\theta$ (for $\hat{C}po$) and $\frac{a'\theta}{3\sqrt{2}}$ (for $\hat{C}p$). Thus, all the moments do not exist in this case since α = 1, and do not satisfy the conditions of equations [3.2.2.4] and [3.2.2.5].

Let us examine the bias and mean squared error of $\hat{C}po$ in the present.

$$B(\hat{C}po) = E(\hat{C}po) - Cpo$$

$$= \int_0^{\infty} \frac{a'\theta}{B(n-s+1, s-r)y} e^{-\frac{a'\theta}{y}(n-s+1)} \left(1-e^{-\frac{a'\theta}{y}}\right)^{s-r-1} dy - \frac{a'\theta}{\ln\left(\frac{1+\gamma}{1-\gamma}\right)} \qquad [3.2.2.9]$$

where γ is the probability contained between Y_r and Y_s,

$$MSE(\hat{C}po) = E\left[(\hat{C}po - Cpo)^2\right] = V(\hat{C}po) + \left[B(\hat{C}po)\right]^2. \qquad [3.2.2.10]$$

The following table shows the biases and mean squared errors of $\hat{C}po$ with $a'\theta$ = .5 for various sample sizes.

Table 3.2.2.3 Biases and mean squared errors of $\hat{C}po$ with $a'\theta$ = .5.

n	$B(\hat{C}po)$	$MSE(\hat{C}po)$
5	.264113	.137743
10	.136794	.028420
20	.082648	.009855

For example, let n = 740, r = 1, s = 740, γ = .9973, and $a'\theta$ = .5 then

$$B\left(\hat{C}po\right) = .0716295 - .07568533 = -.004055833$$

$$MSE\left(\hat{C}po\right) = 1.3662473\times10^{-4} + (-.004055833)^2 = 1.5307451\times10^{-4}$$

This seems to indicate numerically that $\hat{C}po$ is an asymptotically unbiased estimator and is mean squared error consistent. These also imply that $\hat{C}po$ converges in probability to Cpo.

3.2.3 The Distribution of $\hat{C}po$ when X arises from Normal Distribution

If $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$, then $Z_1, Z_2, \ldots, Z_n \sim N(0,1)$ by the transformation formula $Z = \dfrac{X-\mu}{\sigma}$. Then the pdf of $\hat{D}$ is

$$f_{\hat{D}}(w) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{-\infty}^{\infty}\left[\int_{-\infty}^{y_r} e^{-\frac{z^2}{2}}dz\right]^{r-1}\left[\int_{y_r}^{y_r+w} e^{-\frac{z^2}{2}}dz\right]^{s-r-1}$$
$$\left[\int_{y_r}^{\infty} e^{-\frac{z^2}{2}}dz\right]^{n-s} e^{-\frac{y_r^2}{2}} e^{-\frac{(y_r+w)^2}{2}} dy_r, \quad 0 < w < \infty$$

Notice that the probability of out-of-control condition in the usual practice is α = .0027 and assuming equal tailed probability then $\frac{\alpha}{2}$ = .00135. The distribution of $\hat{C}po$ can be affected by the sample size n because the distribution of $\hat{D}$ is based on the difference between the order statistics X_r and X_s. For, if $\frac{\alpha}{2}$ = .00135, then

$$\left(\frac{\alpha}{2}\right)n \le 1 \Rightarrow n \le \frac{2}{\alpha} = \frac{1}{.00135} = 740.7$$

When $n \leq 740$, the distribution of $\hat{D}$ is just the distribution of range. If we keep $\frac{\alpha}{2}$ = .00135 fixed and when $n > 740$, the distribution of $\hat{D}$ will depend on the distribution of quasi-range.

For a quality practitioner to monitor the quality characteristic in a process, usually smaller sample sizes are necessary. So we will consider small sample inferences of $\hat{C}po$. For $n \leq 740$, $\hat{D}$ is the range of the sample. Thus using equation [3.2.1] we can obtain the cumulative distribution function (cdf) of $\hat{C}po$.

$$G_{\hat{C}po}(y) = P(\hat{C}po \leq y)$$

$$= P\left(\hat{D} \leq \frac{a'}{y}\right)$$

$$= n\int_0^1 \left[F\left(z + \frac{a'}{y}\right) - F(z)\right]^{n-1} dF(z)$$

$$= \left[2F\left(\frac{a'}{2y}\right) - 1\right]^n + 2n \int_{\frac{a'}{2y}}^{\infty} \left[F(t) - F\left(t - \frac{a'}{y}\right)\right]^{n-1} f(t)dt \qquad [3.2.3.1]$$

Hartley (1942) has found the distribution of range, $\psi_n(w)$, for symmetric unimodal distribution as follows. Let $\psi_n(w)$ and $\Psi_n(w)$ be the pdf and cdf, respectively, of the range, W, and sample size n.

$$\psi_n(w) = n(n-1) \int_{-\infty}^{\infty} [F(x+w) - F(x)]^{n-2} f(x+w) f(x) dx$$

$$\Psi_n(w) = n \int_0^1 [F(x+w) - F(x)]^{n-1} dF(x)$$

And $\Psi_n(w)$ can be rewritten, for a symmetric variate with respect to $-\frac{w}{2}$, as

$$n\int_{-\infty}^{-\frac{w}{2}}[F(z+w)-F(z)]^{n-1}f(z)dz+n\int_{-\frac{w}{2}}^{\infty}[F(z+w)-F(z)]^{n-1}f(z)dz$$

$$=n\int_{-\frac{w}{2}}^{\infty}[F(-u)-F(-u-w)]^{n-1}f(-u-w)du+n\int_{-\frac{w}{2}}^{\infty}[F(z+w)-F(z)]^{n-1}f(z)dz,\ \text{where}\ u=-z-w$$

$$=n\int_{-\frac{w}{2}}^{\infty}[F(u+w)-F(u)]^{n-1}f(u+w)du+n\int_{-\frac{w}{2}}^{\infty}[F(z+w)-F(z)]^{n-1}f(z)dz$$

$$=n\int_{-\frac{w}{2}}^{\infty}[F(z+w)-F(z)]^{n-1}[f(z)+f(z+w)]dz$$

$$=-n\int_{-\frac{w}{2}}^{\infty}[F(z+w)-F(z)]^{n-1}[f(z)-f(z+w)]dz+2n\int_{-\frac{w}{2}}^{\infty}[F(z+w)-F(z)]^{n-1}f(z+w)dz$$

$$=\left[2F\left(\frac{w}{2}\right)-1\right]^{n}+2n\int_{\frac{w}{2}}^{\infty}[F(t)-F(t-w)]^{n-1}f(t)dt,\ \text{where}\ t=z+w.$$

The explicit pdf of $\hat{C}po$ can be obtained when $Z \sim N(0,1)$ and $n = 2$ and 3. For the pdf of $\hat{D}$ when $n = 2$ and 3 are respectively

$$f_{\hat{D}}(w)=\frac{1}{\sqrt{\pi}}e^{-\frac{w^2}{4}},\quad 0<w<\infty,$$

$$f_{\hat{D}}(w)=\frac{3\sqrt{2}}{\pi}e^{-\frac{w^2}{4}}\int_{0}^{\frac{w}{\sqrt{6}}}e^{-\frac{t^2}{2}}dt,\quad 0<w<\infty.$$

The corresponding pdf of $\hat{C}po$ when $n = 2$, for $0 < y < \infty$, is

$$g_{\hat{C}po}(y)=\frac{a'}{\sqrt{\pi}y^2}e^{-\frac{a'^2}{4y^2}}. \qquad [3.2.3.2]$$

It is equivalent to the distribution of $\hat{C}p$ when n = 2 if a' is replaced by $\frac{a'}{3\sqrt{2}\sigma}$.

Analogous to Subsections 3.2.1 and 3.2.2, let $X_1, X_2 \sim N(\mu, \sigma^2)$, with $-\infty < \mu < \infty$, $\sigma > 0$, be a random sample of size 2 and having pdf

$$f_X(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

Let $T = X_1 - X_2$, such that $T \sim N(0, 2\sigma^2)$ and has pdf

$$f_T(t)=\frac{1}{2\sigma\sqrt{\pi}}e^{-\frac{t^2}{4\sigma^2}}, \quad -\infty < t < \infty.$$

Now $\hat{D}=|T|$ and has cdf

$$G_{\hat{D}}(w)= P(|T|\leq w)$$

$$=P(-w \leq T \leq w)$$

$$=\int_{-w}^{w}\frac{1}{2\sigma\sqrt{\pi}}e^{-\frac{t^2}{4\sigma^2}}dt$$

Thus the pdf of $\hat{D}$ can be obtained through differentiating $G_{\hat{D}}(w)$ with respect to w and it is

$$g_{\hat{D}}(w)=\frac{d}{dw}G_{\hat{D}}(w)$$

$$=\frac{d}{dw}\int_{-w}^{w}\frac{1}{2\sigma\sqrt{\pi}}e^{-\frac{t^2}{4\sigma^2}}dt$$

$$=\frac{1}{2\sigma\sqrt{\pi}}e^{-\frac{w^2}{4\sigma^2}}-\frac{1}{2\sigma\sqrt{\pi}}e^{-\frac{(-w)^2}{4\sigma^2}}(-1)$$

$$=\frac{1}{\sigma\sqrt{\pi}}e^{-\frac{w^2}{4\sigma^2}},\quad 0<w<\infty.$$

Applying equation [3.2.1.6], the pdf of $\hat{C}p$ under normal distribution is

$$h_{\hat{C}p}(y)=\frac{1}{\sigma\sqrt{\pi}}e^{\frac{a'^2}{4(3\sqrt{2})^2\sigma^2y^2}}\frac{a'}{3\sqrt{2}\,y^2}$$

$$=\frac{a'}{3\sqrt{2\pi}\,\sigma\,y^2}e^{-\frac{a'^2}{72\sigma^2y^2}},0<y<\infty. \qquad [3.2.3.3]$$

This is equivalent to equation [3.2.3.2] when a' is replaced by $\frac{a'}{3\sqrt{2}\,\sigma}$. It can be shown that all the moments of $\hat{C}po$ and $\hat{C}p$ do not exist when $n = 2$. The expectation of $\hat{C}po$ is

$$E(\hat{C}po)=E(Y)$$

$$=\int_0^\infty \frac{a'}{\sqrt{\pi}\,\sigma y}e^{-\frac{a'^2}{4\sigma^2y^2}}dy$$

making a substitution $u=\frac{a'^2}{4\sigma^2y^2}\Rightarrow dy=-\frac{a'}{4\sigma}u^{-\frac{3}{2}}du$

$$E(\hat{C}po)=\int_0^\infty \frac{a'}{2\sigma\sqrt{\pi}}u^{-1}e^{-u}du$$

$$=\frac{a'}{2\sigma\sqrt{\pi}}\Gamma(0)$$

which is undefined.

Similarly, with the same substitution, the expectation of the second moment of $\hat{C}po$ is

$$E\left(\hat{C}po^2\right)=E\left(Y^2\right)=\int_0^\infty \frac{a'}{\sqrt{\pi}\sigma} e^{-\frac{a'^2}{4\sigma^2 y^2}} dy$$

$$=\int_0^\infty \frac{a^{*2}}{4\sigma^2\sqrt{\pi}} u^{-\frac{3}{2}} e^{-u} du$$

$$=\frac{a'^2}{4\sigma^2\sqrt{\pi}}\Gamma\left(-\frac{1}{2}\right)$$

which is again undefined. Neither do the higher moments. The expectations of $\hat{C}p$ follow directly.

The pdf of $\hat{C}po$ when $n = 3$ is

$$g_{\hat{C}po}(y)=\frac{3\sqrt{2}a'}{\pi y^2} e^{-\frac{a'^2}{4y^2}} \int_0^{\frac{a'}{\sqrt{6}y}} e^{-\frac{t^2}{2}} dt. \qquad [3.2.3.4]$$

The rth moment of $\hat{C}po$ is

$$E\left(\hat{C}po^r\right) = E\left(Y^r\right) = E\left(\frac{a'^r}{\hat{D}^r}\right)$$

$$= a'^r E\left(\hat{D}^{-r}\right)$$

$$= \frac{3\sqrt{2}\, a'^r}{\pi} \int_0^\infty w^{-r} e^{-\frac{w^2}{4}} \int_0^{\frac{w}{\sqrt{6}}} e^{-\frac{t^2}{2}} dt\, dw$$

Now, let $v = \frac{t}{w}$ and change order of integration, the expectation becomes

$$= \frac{3\sqrt{2}\, a'^r}{\pi} \int_0^\infty \int_0^{\frac{1}{\sqrt{6}}} w^{1-r} e^{-\frac{w^2}{4}\left(1+2v^2\right)} dv\, dw, \text{ for } r<2$$

$$= \frac{3\sqrt{2}\, 2^{1-r} a'^{r} \Gamma\left(1-\frac{r}{2}\right)}{\pi} \int_0^{\frac{1}{\sqrt{6}}} \frac{dv}{\left(1+2v^2\right)^{\frac{2-r}{2}}}, \text{ letting } v = \frac{\tan\theta}{\sqrt{2}}$$

$$= \frac{3\left(2^{1-r}\right) a'^{r} \Gamma\left(1-\frac{r}{2}\right)}{\pi} \int_0^{\frac{\pi}{6}} \sec^{r}\theta \; d\theta$$

When r = 1, the expectation of $\hat{C}po$ is

$$E\left(\hat{C}po\right) = \frac{3a'}{\sqrt{\pi}} \int_0^{\frac{\pi}{6}} \sec\theta \; d\theta$$

$$= \frac{3 \ln\sqrt{3}\; a'}{\sqrt{\pi}} = .92974 \; a'$$

The second and higher moments do not exist.

The bias of $\hat{C}po$ is

$$B\left(\hat{C}po\right) = a'[\, .92974 - \frac{1}{6}\,] = .763073 \; a'.$$

For $4 \le n \le 740$, numerical methods or approximation methods are needed to find the pdf of $\hat{C}po$ when the distribution is normal.

3.3 Comments

The pdfs of $\hat{C}po$ are determined for the cases if process characteristic is uniform distribution and exponential distribution. These pdfs can be used for any sample size and/or any position r and s can be. However the case for normal distribution is dependent upon the size of sample, n, and the probability of out-of-control, α. If $\left(\frac{\alpha}{2}\right)n \leq 1$ or $n \leq \frac{2}{\alpha}$, then the denominator of $\hat{C}po$ will depend on the distribution of range in equal tailed out-of-control condition. In other situations if $\left(\frac{\alpha}{2}\right)n \leq r$ or $n \leq \frac{2r}{\alpha}$, where $r > 1$, then the distribution of quasi-range is needed to be considered.

McKay (1935) suggested an approximation to the distribution of range if w is large such that $\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} f(x)dx = 1$, hence

$$\psi_n(w) = n(n-1)\int_{-\infty}^{\infty} f\left(t-\frac{w}{2}\right) f\left(t+\frac{w}{2}\right)\left[\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} f(x)dx\right]^{n-2} dt$$

$$= \frac{n(n-1)}{2\sqrt{\pi}} e^{-\frac{w^2}{4}}$$

McKay (1935) also suggested if w is small, then $\int_{t-\frac{w}{2}}^{t+\frac{w}{2}} f(x)dx = w f(t)$, hence

$$\psi_n(w) = n(n-1)w^{n-2}\int_{-\infty}^{\infty} f\left(t-\frac{w}{2}\right) f\left(t+\frac{w}{2}\right)[f(t)]^{n-2} dt$$

$$= \frac{\sqrt{n}\,(n-1)w^{n-2}e^{-\frac{w^2}{4}}}{(2\pi)^{\frac{n-1}{2}}}$$

Note that the distribution of $\hat{C}po$ is based on order statistics and is distribution-free. $\hat{C}po$ is not just an estimate of the process capability, its magnitude can also be compared to other process capability indices in terms of bias, mean squared error and relative efficiency. The relationships of Cpo to other process capability indices are as below:

$$Cpo = \frac{USL-LSL}{D}\frac{6\sigma}{6\sigma}$$
$$= \frac{6\sigma}{D}Cp.$$

$$Cpo = \frac{6\sigma}{D}\frac{1-k}{1-k}Cp$$

$$= \frac{6\sigma}{D(1-k)}Cpk*, \text{ for } 0<k<1.$$

$$Cpo = \frac{USL-LSL}{D}\frac{6\sigma\sqrt{1+p^2}}{6\sigma\sqrt{1+p^2}}$$
$$= \frac{6\sigma\sqrt{1+p^2}}{D}Cpm, \text{ for } p = \frac{|\mu-T|}{\sigma}.$$

$$Cpo = \frac{USL-LSL}{D}\frac{6\sigma\sqrt{1+wp^2}}{6\sigma\sqrt{1+wp^2}}$$
$$= \frac{6\sigma\sqrt{1+wp^2}}{D}Cpw, \text{ for w nonstochastic.}$$

Confidence intervals can be constructed and hypothesis testing can be elaborated if the sampling distribution $\hat{C}po$ is known.

Modifications can be made to $\hat{C}po$ so as to include asymmetric specification limits. A suggested form may look like

$$Cpo^* = \min\left(\frac{USL - T}{D_2}, \frac{T - LSL}{D_1} \right)$$

where $D_1 = \xi_{.5} - \xi_{.00135}$, and $D_2 = \xi_{.99865} - \xi_{.5}$.

Chapter 4

The Unifying Index Cpw

4.1 Introduction

Many authors have promoted the use of process capability indices Cp, Cpl, Cpu, Cpk and Cpm and have examined with different degrees of completeness their associated properties. In an attempt to simplify various process capability indices that have led to controversy (Nelson, 1992), Vännman and Kotz (1995a, 1995b), Spiring (1997), Vännman (1997) proposed families of indices that tie the various forms of measures together, while illustrating the statistical properties associated with each form. In this chapter, the Cpw index is defined and the probability density function of its estimate, $\hat{C}pw$, presented. Properties such as expectation, bias, mean squared error, probabilities and critical values on some selected weights (i.e. values of w) are further investigated.

4.2 The Probability Density Function of $\hat{C}pw$ and its Properties

A natural estimator of Cpw is

$$\hat{C}pw = \frac{USL - LSL}{6\sqrt{\hat{\sigma}^2 + w(\overline{X} - T)^2}}, \qquad [4.2.1]$$

while the the natural estimator of the generalized form of Cpw (equation [1.6.2]) is

$$\hat{C}pw* = \frac{\min[USL - T, T - LSL]}{3\sqrt{\hat{\sigma}^2 + w(\overline{X} - T)^2}} \qquad [4.2.2]$$

where $X \sim N(\mu, \sigma^2)$ are the measurements of a process characteristic, and w is a non-stochastic weight.

Let $\theta = \frac{n\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2}$ and it has a chi square distribution with n-1 degrees of freedom, and $\eta = \left[\frac{\overline{X}-T}{\frac{\sigma}{\sqrt{n}}}\right]^2 = \frac{n(\overline{X}-T)^2}{\sigma^2}$ and has a non-central chi square distribution with 1 df and the non-centrality parameter $\lambda = \frac{n(\mu - T)^2}{\sigma^2}$. Then

$$\hat{C}pw = \frac{USL-LSL}{6\frac{\sigma}{\sqrt{n}}\sqrt{\theta + w\eta}}$$

$$= \frac{a}{\sqrt{\theta + w\eta}} \qquad [4.2.3]$$

and

$$\hat{C}pw^* = \frac{\min[USL-T, T-LSL]}{3\frac{\sigma}{\sqrt{n}}\sqrt{\theta + w\eta}}$$

$$= \frac{a^*}{\sqrt{\theta + w\eta}} \qquad [4.2.4]$$

where $a = \frac{\sqrt{n}[USL-LSL]}{6\sigma} = \sqrt{n+\lambda w}\,Cpw$,

and $a^* = \frac{\sqrt{n}\min[USL-T, T-LSL]}{3\sigma} = \sqrt{n+\lambda w}\,Cpw^*$.

Consider the distribution function of $\hat{C}pw$, for $x > 0$

$$
\begin{aligned}
F_{\hat{C}pw}(x) &= P\left(\hat{C}pw \le x\right) \\
&= P\left(\frac{a}{\sqrt{\theta + w\eta}} \le x\right) \\
&= P\left(\frac{a^2}{\theta + w\eta} \le x^2\right) \\
&= P\left(\theta + w\eta \ge \frac{a^2}{x^2}\right) \\
&= 1 - P\left(\theta + w\eta \le \frac{a^2}{x^2}\right) \\
&= 1 - \int_0^{\infty} P\left(\theta \le \frac{a^2}{x^2} - w\eta \mid \eta = y\right) f_\eta(y)\,dy \\
&= 1 - \int_0^{\frac{a^2}{wx^2}} P\left(\theta \le \frac{a^2}{x^2} - wy\right) f_\eta(y)\,dy \qquad [4.2.5] \\
&= 1 - \sum_{j=0}^{\infty} P_j \int_0^{\frac{a^2}{wx^2}} P\left(\theta \le \frac{a^2}{x^2} - wy\right) f_{\eta_j}(y)\,dy
\end{aligned}
$$

where $P_j = \frac{e^{-\frac{\lambda}{2}}}{j!}\left(\frac{\lambda}{2}\right)^j$, the Poisson weights, and λ the non-centrality parameter

$$f_{\eta_j}(y) = \frac{y^{\frac{2j+1}{2}-1} e^{-\frac{y}{2}}}{\Gamma\left(\frac{2j+1}{2}\right) 2^{\frac{2j+1}{2}}},$$ $y > 0$, the chi square density with $2j+1$ df

$$f_\theta(t) = \frac{t^{\frac{n-1}{2}-1} e^{-\frac{t}{2}}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}},$$ $t > 0$, the chi square density with $n-1$ df

Differentiate $F_{\hat{C}pw}(x)$, i.e., equation [4.2.5], with respect to x we can get $f_{\hat{C}pw}(x)$ for $w > 0$ and $x > 0$.

$$f_{\hat{C}pw}(x) = \frac{d}{dx} F_{\hat{C}pw}(x)$$

$$= -P\left(\theta \le \frac{a^2}{x^2} - w\frac{a^2}{wx^2}\right) f_\eta\left(\frac{a^2}{wx^2}\right)\left[-\frac{2a^2}{wx^3}\right]$$

$$+ \int_0^{\frac{a^2}{wx^2}} f_\theta\left(\frac{a^2}{x^2} - wy\right)\frac{2a^2}{wx^3} f_\eta(y)dy \, .$$

The first term on the right hand side of the above equation is zero since θ is a nonnegative random variable. Letting $u = \frac{wx^2}{a^2} y$, then the pdf of $\hat{C}pw$ is

$$f_{\hat{C}pw}(x) = \int_0^1 f_\theta\left(\frac{a^2}{x^2} - \frac{a^2}{x^2}u\right)\frac{2a^4}{w^2x^5} f_\eta\left(\frac{a^2u}{wx^2}\right)du$$

$$= \int_0^1 f_\theta\left[\frac{a^2}{x^2}(1-u)\right]\frac{2a^4}{w^2x^5} f_\eta\left(\frac{a^2u}{wx^2}\right)du$$

$$= \sum_{j=0}^{\infty} P_j \int_0^1 f_\theta\left[\frac{a^2}{x^2}(1-u)\right]\frac{2a^4}{w^2x^5} f_{\eta_j}\left(\frac{a^2u}{wx^2}\right)du, \quad \text{for } x > 0.$$

$$= \sum_{j=0}^{\infty} P_j \int_0^1 \frac{a^{n+2j}\, u^{\frac{2j+1}{2}-1} (1-u)^{\frac{n-1}{2}-1} e^{-\frac{a^2}{2x^2}\left(1-u+\frac{u}{w}\right)}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{2j+1}{2}\right) 2^{\frac{n+2j}{2}-1}\, w^{\frac{2j+1}{2}+1}\, x^{n+2j+1}} du$$

$$= \sum_{j=0}^{\infty} P_j \frac{a^{n+2j}\, B\left(\frac{n-1}{2},\frac{2j+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{2j+1}{2}\right) 2^{\frac{n+2j}{2}-1}\, w^{\frac{2j+1}{2}+1}\, x^{n+2j+1}}$$

$$\left\{ e^{-\frac{a^2}{2x^2}} + {}_1F_1\left[\frac{2j+1}{2};\frac{n+2j}{2};\frac{a^2}{2x^2}\right] + {}_1F_1\left[\frac{2j+1}{2};\frac{n+2j}{2};-\frac{a^2}{2wx^2}\right]\right\}$$

$$= \sum_{j=0}^{\infty} P_j \frac{a^{n+2j}}{\Gamma\left(\frac{n+2j}{2}\right) 2^{\frac{n+2j}{2}-1}\, w^{\frac{2j+1}{2}+1}\, x^{n+2j+1}}$$

$$\left\{ e^{-\frac{a^2}{2x^2}} + {}_1F_1\left[\frac{2j+1}{2};\frac{n+2j}{2};\frac{a^2}{2x^2}\right] + {}_1F_1\left[\frac{2j+1}{2};\frac{n+2j}{2};-\frac{a^2}{2wx^2}\right]\right\}. \qquad [4.2.6]$$

When $w = 1$, the pdf of $\hat{C}pw$ becomes the pdf of $\hat{C}pm$ when $X \sim N(\mu,\sigma^2)$

$$f_{\hat{C}pw}(x) = \sum_{j=0}^{\infty} P_j \frac{a^{n+2j}\, e^{-\frac{a^2}{2x^2}}}{\Gamma\left(\frac{n+2j}{2}\right) 2^{\frac{n+2j}{2}-1}\, x^{n+2j+1}},\quad 0 < x < \infty. \qquad [4.2.7]$$

Further, if $\mu = T$, equivalently $\lambda = 0$, the pdf of $\hat{C}pw$ simplifies to

$$f_{\hat{C}pw}(x) = \frac{a^n\, e^{-\frac{a^2}{2x^2}}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}-1}\, x^{n+1}},\quad 0 < x < \infty, \qquad [4.2.8]$$

this is equivalent to the pdf of $\hat{C}p$ (Chan, Cheng and Spiring (1988b), (1988c)) when $X \sim N(\mu,\sigma^2)$.

Consider again equation [4.2.6], if $\mu = T$, then the pdf of $\hat{C}pw$ becomes

$$f_{\hat{C}pw}(x)=\frac{a^{n}\left\{e^{-\frac{a^{2}}{2x^{2}}}+{}_{1}F_{1}\left[\frac{1}{2};\frac{n}{2};\frac{a^{2}}{2x^{2}}\right]+{}_{1}F_{1}\left[\frac{1}{2};\frac{n}{2};-\frac{a^{2}}{2wx^{2}}\right]\right\}}{\Gamma\left(\frac{n}{2}\right)\ 2^{\frac{n}{2}-1}\ w^{\frac{3}{2}}\ x^{n+1}},\ w>0,\ 0<x<\infty. \quad [4.2.9]$$

Further, if $w = 1$, equation [4.2.9] becomes equation [4.2.8].

The pdf of $\hat{C}pw*$ can be obtained through the same procedure as in determining the pdf of $\hat{C}pw$. From equations [4.2.3] and [4.2.4] we can see that $\hat{C}pw$ and $\hat{C}pw*$ are having the same distribution and their distributions are differed by replacing the constant $a = \sqrt{n+\lambda w}\,Cpw$ (for Cpw) to $a* = \sqrt{n+\lambda w}\,Cpw*$ (for Cpw*) hence the pdf of $\hat{C}pw*$ and its special cases associated with different values of w are listed as follows:

$$f_{\hat{C}pw*}(x) = \sum_{j=0}^{\infty} P_j \frac{a^{*n+2j}}{\Gamma\left(\frac{n+2j}{2}\right)\ 2^{\frac{n+2j}{2}-1}\ w^{\frac{2j+1}{2}+1}\ x^{n+2j+1}}$$

$$\left\{e^{-\frac{a^{*2}}{2x^{2}}}+{}_{1}F_{1}\left[\frac{2j+1}{2};\frac{n+2j}{2};\frac{a^{*2}}{2x^{2}}\right]+{}_{1}F_{1}\left[\frac{2j+1}{2};\frac{n+2j}{2};-\frac{a^{*2}}{2wx^{2}}\right]\right\}$$

$$w>0,\ 0<x<\infty. \quad [4.2.10]$$

When $w = 1$, the pdf $\hat{C}pw*$ of becomes

$$f_{\hat{C}pw}(x)=\sum_{j=0}^{\infty} P_j \frac{a^{*n+2j}\ e^{-\frac{a^{*2}}{2x^{2}}}}{\Gamma\left(\frac{n+2j}{2}\right)2^{\frac{n+2j}{2}-1}\ x^{n+2j+1}},\ 0<x<\infty, \quad [4.2.11]$$

the pdf of $\hat{C}pm*$ when $X \sim N(\mu,\sigma^{2})$.

If $\mu = T$, equivalently $\lambda = 0$, the pdf of $\hat{C}pw^*$ simplifies to

$$f_{\hat{C}pw^*}(x) = \frac{a^{*n} e^{-\frac{a^{*2}}{2x^2}}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}-1} x^{n+1}}, \quad 0 < x < \infty. \qquad [4.2.12]$$

Consider again equation [4.2.10], if $\mu = T$, then the pdf of $\hat{C}pw$ becomes

$$f_{\hat{C}pw^*}(x) = \frac{a^{*n}\left\{e^{-\frac{a^{*2}}{2x^2}} + {}_1F_1\left[\frac{1}{2};\frac{n}{2};\frac{a^{*2}}{2x^2}\right] + {}_1F_1\left[\frac{1}{2};\frac{n}{2};-\frac{a^{*2}}{2wx^2}\right]\right\}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}-1} w^{\frac{3}{2}} x^{n+1}},$$

$$w > 0, \ 0 < x < \infty. \qquad [4.2.13]$$

Further, if $w = 1$, then equation [4.2.13] reduces to equation [4.2.12].

The statistical properties associated with $\hat{C}pw$ are analytically intractable for the general case. However, when the target value and the process population mean are identical, i.e., $\mu = T$, it is possible to derive some properties with different values of w.

Theorem 4.2.1:

The rth moment of $\hat{C}pw$ when $X \sim N(\mu, \sigma^2)$ is

$$E\left(\hat{C}pw^r\right) = \left(\frac{a}{\sqrt{2}}\right)^r \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{r}{2}, \frac{2j+1}{2}; \frac{n-1}{2}; 1-w\right].$$

for $r = 0, 1, 2, \ldots, k$. [4.2.14]

Proof:

For $$E\left(\hat{C}pw^r\right)=E\left[a^r(\theta+w\eta)^{-\frac{r}{2}}\right]$$

$$= a^r\, E\left[\sum_{j=0}^{\infty} P_j\left(\theta+w\eta_j\right)^{-\frac{r}{2}}\right]$$

$$= a^r \sum_{j=0}^{\infty} P_j\, E\left[\left(\theta+w\eta_j\right)^{-\frac{r}{2}}\right]$$

Notice that $\gamma=\theta+\eta_j \sim \chi^2_{n+2j}$ and $\delta_j=\dfrac{\eta_j}{\gamma}=\dfrac{\eta_j}{\theta+\eta_j}=\dfrac{1}{1+\dfrac{\theta}{\eta_j}} \sim \text{Beta}\left(\dfrac{2j+1}{2},\dfrac{n-1}{2}\right)$ are independent. Now, consider

$$\left(\theta+w\eta_j\right)^{-\frac{r}{2}}=\left(\gamma-\eta_j+w\eta_j\right)^{-\frac{r}{2}}$$

$$=\left[\gamma+(w-1)\eta_j\right]^{-\frac{r}{2}}$$

$$=\gamma^{-\frac{r}{2}}\left[1+(w-1)\frac{\eta_j}{\gamma}\right]^{-\frac{r}{2}}$$

$$=\gamma^{-\frac{r}{2}}\left[1+(w-1)\delta_j\right]^{-\frac{r}{2}}$$

and, $$E\left(\gamma^{-\frac{r}{2}}\right)=2^{-\frac{r}{2}}\frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} \qquad [4.2.15]$$

$$E\left\{\left[1+(w-1)\delta_j\right]^{-\frac{r}{2}}\right\}=\int_0^1 \frac{1}{B\left(\frac{2j+1}{2},\frac{n-1}{2}\right)}\left[1+(w-1)x\right]^{-\frac{r}{2}} x^{\frac{2j+1}{2}-1}\left(1-x\right)^{\frac{n-1}{2}-1} dx$$

$$={}_2F_1\left[\frac{r}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right] \qquad [4.2.16]$$

Therefore $$E\left(\hat{C}pw^r\right)=\left(\frac{a}{\sqrt{2}}\right)^r \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}\,{}_2F_1\left[\frac{r}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right],$$

for $r = 0, 1, 2, \ldots, k$.

The mean and variance are respectively

$$E\left(\hat{C}pw\right)=\frac{a}{\sqrt{2}}\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}\,{}_2F_1\left[\frac{1}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right]$$

$$=\sqrt{\frac{n+\lambda w}{2}}\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}\,{}_2F_1\left[\frac{1}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right]Cpw \qquad [4.2.17]$$

$$V\left(\hat{C}pw\right)=\frac{a^2}{2}\left\{\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n+2j-2}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}{}_2F_1\left[1,\frac{2j+1}{2};\frac{n+2j}{2};1-w\right]\right.$$

$$\left.-\left[\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}{}_2F_1\left[\frac{1}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right]\right]^2\right\}$$

$$=\frac{n+\lambda w}{2}\left\{\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n+2j-2}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}{}_2F_1\left[1,\frac{2j+1}{2};\frac{n+2j}{2};1-w\right]\right.$$

$$\left.-\left[\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}{}_2F_1\left[\frac{1}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right]\right]^2\right\}Cpw^2. \quad [4.2.18]$$

If w = 0, equation [4.2.16] becomes

$$E\left\{\left[1-\delta_j\right]^{-\frac{r}{2}}\right\}=\int_0^1\frac{1}{B\left(\frac{2j+1}{2},\frac{n-1}{2}\right)}[1-x]^{-\frac{r}{2}}x^{\frac{2j+1}{2}-1}\left(1-x\right)^{\frac{n-1}{2}-1}dx$$

$$=\frac{\Gamma\left(\frac{n-r-1}{2}\right)\Gamma\left(\frac{n+2j}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+2j-r}{2}\right)}$$

then $$E\left(\hat{C}pw^r\right)=\left(\sqrt{\frac{n+\lambda w}{2}}Cpw\right)^r\sum_{j=0}^{\infty}P_j\frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}. \quad [4.2.19]$$

Further, if $\mu = T$, i.e., $\lambda = 0$, then equation [4.2.15] becomes

$$E\left(\gamma^{-\frac{r}{2}}\right) = 2^{-\frac{r}{2}} \frac{\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

and hence,
$$E\left(\hat{C}pw^r\right) = \left(\sqrt{\frac{n}{2}}\,Cpw\right)^r \frac{\Gamma\left(\frac{n-r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}. \qquad [4.2.20]$$

Thus, the mean and variance of $\hat{C}pw$ when $w = 0$ and $\mu = T$ are respectively

$$E\left(\hat{C}pw\right) = \sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} Cpw \qquad [4.2.21]$$

$$V\left(\hat{C}pw\right) = \frac{n}{2}\left\{\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - \left[\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2\right\} Cpw^2. \qquad [4.2.22]$$

If $w = 1$, equation [4.2.16] becomes

$$E\left\{\left[\ 1-(1-1)\delta_j\ \right]^{-\frac{r}{2}}\right\} = 1$$

then
$$E\left(\hat{C}pw^r\right) = \left(\sqrt{\frac{n+\lambda}{2}}\,Cpw\right)^r \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}. \qquad [4.2.23]$$

Further, if $\mu = T$, i.e., $\lambda = 0$, then

$$E\left(\hat{C}pw^r\right) = \left(\sqrt{\frac{n}{2}}Cpw\right)^r \frac{\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \qquad [4.2.24]$$

Thus, the mean and variance of $\hat{C}pw$ when $w = 1$ and $\mu = T$ are respectively

$$E\left(\hat{C}pw\right) = \sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} Cpw \qquad [4.2.25]$$

$$V\left(\hat{C}pw\right) = \frac{n}{2}\left\{\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - \left[\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2\right\} Cpw^2. \qquad [4.2.26]$$

Replacing $a = \sqrt{n+\lambda w}\,Cpw$ by $a^* = \sqrt{n+\lambda w}\,Cpw^*$ in equations [4.2.14], and [4.2.17] through [4.2.26], we can obtain the moments of $\hat{C}pw^*$ and its special cases for different values of w.

The rth moment of $\hat{C}pw^*$ $X \sim N\left(\mu, \sigma^2\right)$ is

$$E\left(\hat{C}pw^{*r}\right) = \left(\frac{a^*}{\sqrt{2}}\right)^r \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{r}{2}, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right],$$

for $r = 0, 1, 2, \ldots, k$. [4.2.27]

The mean and variance are respectively

$$E\left(\hat{C}pw^*\right) = \sqrt{\frac{n+\lambda w}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right] Cpw^* \qquad [4.2.28]$$

$$V(\hat{C}pw*) = \frac{n+\lambda w}{2}\left\{ \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-2}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[1, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right] - \left[\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right] \right]^2 \right\} Cpw*^2 \quad [4.2.29]$$

If $w = 0$, then
$$E(\hat{C}pw*^r) = \left(\sqrt{\frac{n+\lambda w}{2}} Cpw*\right)^r \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n-r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}. \quad [4.2.30]$$

Further, if $\mu = T$, i.e., $\lambda = 0$, then

$$E(\hat{C}pw*^r) = \left(\sqrt{\frac{n}{2}} Cpw*\right)^r \frac{\Gamma\left(\frac{n-r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}. \quad [4.2.31]$$

Thus, the mean and variance of $\hat{C}pw*$ when $w = 0$ and $\mu = T$ are respectively

$$E(\hat{C}pw*) = \sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} Cpw* \quad [4.2.32]$$

$$V(\hat{C}pw*) = \frac{n}{2}\left\{ \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - \left[\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2 \right\} Cpw*^2 \quad [4.2.33]$$

If $w = 1$, then $$E\left(\hat{C}pw^{*r}\right) = \left(\sqrt{\frac{n+\lambda}{2}}\,Cpw^*\right)^r \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)}. \qquad [4.2.34]$$

Further, if $\mu = T$, i.e., $\lambda = 0$, then

$$E\left(\hat{C}pw^{*r}\right) = \left(\sqrt{\frac{n}{2}}\,Cpw^*\right)^r \frac{\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \qquad [4.2.35]$$

Thus, the mean and variance of $\hat{C}pw^*$ when $w = 1$ and $\mu = T$ are respectively

$$E\left(\hat{C}pw^*\right) = \sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} Cpw^* \qquad [4.2.36]$$

$$V\left(\hat{C}pw^*\right) = \frac{n}{2}\left\{\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - \left[\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2\right\} Cpw^{*2}. \qquad [4.2.37]$$

Now, let us examine the biases and mean squared errors of $\hat{C}pw$ and $\hat{C}pw^*$. Let $B\left(\hat{\theta}\right)$ and $MSE\left(\hat{\theta}\right)$ be the bias and mean squared error respectively of an estimator $\hat{\theta}$, recall equations [4.2.17], [4.2.18] [4.2.28] and [4.2.29], then

$B(\hat{C}pw)$

$= E(\hat{C}pw) - Cpw$

$$= \sqrt{\frac{n+\lambda w}{2}}\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right] Cpw - Cpw$$

$$= \left[\sqrt{\frac{n+\lambda w}{2}}\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right] - 1\right] Cpw \qquad [4.2.38]$$

$B(\hat{C}pw^*)$

$= E(\hat{C}pw^*) - Cpw^*$

$$= \left[\sqrt{\frac{n+\lambda w}{2}}\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2},\frac{2j+1}{2};\frac{n+2j}{2};1-w\right] - 1\right] Cpw^* \qquad [4.2.39]$$

$$\mathrm{MSE}\left(\hat{C}pw\right)$$

$$= E\left[\left(\hat{C}pw - Cpw\right)^2\right]$$

$$= V\left(\hat{C}pw\right) + \left[B\left(\hat{C}pw\right)\right]^2$$

$$= \frac{n+\lambda w}{2}\left\{\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-2}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[1, \frac{2j+1}{2}; \frac{n-1}{2}; 1-w\right]\right.$$

$$\left. - \left[\sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-2}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right]\right]^2 \right\}$$

$$+ \left[\sqrt{\frac{n+\lambda w}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right] - 1\right]^2 Cpw^2$$

$$= \left\{ \frac{n+\lambda w}{2} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-2}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[1, \frac{2j+1}{2}; \frac{n-1}{2}; 1-w\right]\right.$$

$$\left. -2\sqrt{\frac{n+\lambda w}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right] + 1 \right\} Cpw^2 \qquad [4.2.40]$$

$\mathrm{MSE}\left(\hat{C}pw*\right)$

$$= E\left[\left(\hat{C}pw* - Cpw*\right)^2\right]$$

$$= V\left(\hat{C}pw*\right) + \left[B\left(\hat{C}pw*\right)\right]^2$$

$$= \left\{ \frac{n+\lambda w}{2} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-2}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[1, \frac{2j+1}{2}; \frac{n-1}{2}; 1-w\right] \right.$$

$$\left. -2\sqrt{\frac{n+\lambda w}{2}} \sum_{j=0}^{\infty} P_j \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{2j+1}{2}; \frac{n+2j}{2}; 1-w\right] + 1 \right\} Cpw*^2. \quad [4.2.41]$$

When $\mu = T$, i.e., $\lambda = 0$, then

$$B\left(\hat{C}pw\right) = \left[\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; \frac{n}{2}; 1-w\right] - 1 \right] Cpw \quad [4.2.42]$$

$$B\left(\hat{C}pw*\right) = \left[\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; \frac{n}{2}; 1-w\right] - 1 \right] Cpw* \quad [4.2.43]$$

$$MSE(\hat{C}pw) = \left\{ \frac{n}{2}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left[1,\frac{1}{2};\frac{n-1}{2};1-w\right] \right.$$

$$\left. -2\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left[\frac{1}{2},\frac{1}{2};\frac{n}{2};1-w\right] + 1 \right\} Cpw^2 \qquad [4.2.44]$$

$$MSE(\hat{C}pw*) = \left\{ \frac{n}{2}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left[1,\frac{1}{2};\frac{n-1}{2};1-w\right] \right.$$

$$\left. -2\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left[\frac{1}{2},\frac{1}{2};\frac{n}{2};1-w\right] + 1 \right\} Cpw*^2. \qquad [4.2.45]$$

If $w = 0$, then equations [4.2.42], [4.2.43], [4.2.44] and [4.2.45] become

$$B(\hat{C}pw) = \left[\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 1 \right] Cpw$$

$$B(\hat{C}pw*) = \left[\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 1 \right] Cpw*$$

$$MSE(\hat{C}pw) = \left\{ \frac{n}{2}\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 2\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} + 1 \right\} Cpw^2$$

$$MSE(\hat{C}pw^*) = \left\{ \frac{n}{2}\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 2\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} + 1 \right\} Cpw^{*2}$$

Taking limits as $n \to \infty$, it can be shown that

$$\lim_{n\to\infty} \frac{n}{2}\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = 1 \text{ and } \lim_{n\to\infty} \sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = 1,$$

for $\lim_{n\to\infty} n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = 1$. See Abramowitz and Stegun (1965).

Thus both the biases of $\hat{C}pw$ and $\hat{C}pw^*$ become zero, i.e.,

$$B(\hat{C}pw) = 0, \text{ and } B(\hat{C}pw^*) = 0.$$

These imply that both $\hat{C}pw$ and $\hat{C}pw^*$ are asymptotically unbiased. Also, the mean squared errors of $\hat{C}pw$ and $\hat{C}pw^*$ are zero, i.e.,

$$MSE(\hat{C}pw) = 0, \text{ and } MSE(\hat{C}pw^*) = 0.$$

These imply that when $\mu = T$ and $w = 0$ both $\hat{C}pw$ and $\hat{C}pw^*$ are mean squared error consistent. These also imply that both $\hat{C}pw$ and $\hat{C}pw^*$ converge in probability to Cpw and Cpw*, respectively.

If $w = 1$, then equations [4.2.42], [4.2.43], [4.2.44] and [4.2.45] become

$$B(\hat{C}pw) = \left[\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 1 \right] Cpw$$

$$B(\hat{C}pw*) = \left[\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 1 \right] Cpw*$$

$$MSE(\hat{C}pw) = \left\{ \frac{n}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 2\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} + 1 \right\} Cpw^2$$

$$MSE(\hat{C}pw*) = \left\{ \frac{n}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 2\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} + 1 \right\} Cpw*^2$$

Taking limits as $n \to \infty$, it can be shown that

$$\lim_{n\to\infty} \frac{n}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = 1 \text{ and } \lim_{n\to\infty} \sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = 1.$$

Thus both the biases of $\hat{C}pw$ and $\hat{C}pw*$ become zero, i.e.,

$$B(\hat{C}pw) = 0, \text{ and } B(\hat{C}pw*) = 0.$$

These imply that both $\hat{C}pw$ and $\hat{C}pw*$ are asymptotically unbiased. Also, the mean squared errors of $\hat{C}pw$ and $\hat{C}pw*$ are zero, i.e.,

$$MSE(\hat{C}pw) = 0, \text{ and } MSE(\hat{C}pw*) = 0.$$

These imply that when $\mu = T$ and $w = 1$ both $\hat{C}pw$ and $\hat{C}pw^*$ are mean squared error consistent. These also imply that both $\hat{C}pw$ and $\hat{C}pw^*$ converge in probability to Cpw and Cpw*, respectively.

4.3 Confidence Intervals for Cpw and its Relationship to Squared Error Loss

Similar to Section 2.7 we define the loss function

$$L(x) = w(x-T)^2$$

with expectation

$$E[L(X)] = w\sigma^2 + w(\mu - T)^2 \quad [4.3.1]$$

$$= (w-1)\sigma^2 + \sigma^2 + w(\mu - T)^2.$$

Define the expected weighted loss of X when X is not on target as

$$E[L_W(X)] = \sigma^2 + w(\mu - T)^2 \quad [4.3.2]$$

$$= E[L(X)] + (1-w)\sigma^2$$

$$= E\left[\frac{n-w}{n-1}\hat{\sigma}^2 + w(\overline{X} - T)^2\right]$$

$$= E[\hat{L}_W(X)]$$

where $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n}$, and $\hat{L}_W(X) = \frac{n-w}{n-1}\hat{\sigma}^2 + w(\overline{X} - T)^2$.

Note that $\hat{L}_W(X) = \frac{n-w}{n-1}\hat{\sigma}^2 + w(\overline{X}-T)^2$ is an unbiased estimator of $E[L_W(X)]$ and is a function of jointly complete sufficient statistics if $X \sim N(\mu, \sigma^2)$, hence it is a uniformly minimum variance unbiased estimator (UMVUE) for $E[L_W(X)]$. Hence the Cpw index of equation [1.6.1] can be written in terms of $E[L_W(X)]$

$$Cpw = \frac{USL-LSL}{6\sqrt{E[L_W(X)]}} \qquad [4.3.3]$$

$$E[L_W(X)] = \frac{[USL-LSL]^2}{36\,Cpw^2} \qquad [4.3.4]$$

thus, an estimator of Cpw is

$$\hat{C}pw = \frac{USL-LSL}{6\sqrt{\hat{L}_W(X)}} \qquad [4.3.5]$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, then it follows that

$$(\overline{X}-T)^2 \sim \frac{\sigma^2}{n}\chi^2_{1,\lambda} \quad \Rightarrow \quad w(\overline{X}-T)^2 \sim \frac{w\sigma^2}{n}\chi^2_{1,\lambda}$$

and

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{n}\chi^2_{n-1} \quad \Rightarrow \quad \frac{n-w}{n-1}\hat{\sigma}^2 \sim \frac{n-w}{n-1}\frac{\sigma^2}{n}\chi^2_{n-1}$$

and $\overline{X}$, $\hat{\sigma}^2$ are independent, where λ is the non-centrality parameter. Analogous to Spiring (1997) and define

$$Q^2_{n,\lambda} = \frac{n}{\sigma^2}\left[\frac{n-w}{n-1}\hat{\sigma}^2 + w(\overline{X}-T)^2\right]$$

$Q^2_{n,\lambda}$ becomes a linear combination of two independent chi square random variables

$$\frac{n-w}{n-1}\chi^2_{n-1} + w\chi^2_{1,\lambda}.$$

Denoting $Q^2_{n,\lambda}(x)$ as the cumulative distribution function (cdf) associated with $Q^2_{n,\lambda}$, Press (1966) showed that the $Q^2_{n,\lambda}(x)$ can be expressed as a mixture of central chi square distribution with general form

$$Q^2_{n,\lambda}(x) = \sum_{j=0}^{\infty} d_j\,\chi^2_{n+2j}(x)$$

with the d_j's being the weights such that $\sum_{j=0}^{\infty} d_j = 1$, where the d_j's are the functions of the degrees of freedom (i.e., n - 1 and 1), the non-centrality parameter, and the weight function. The functional form of the d_j's are given in Press (1966), which for the general $Q^2_{n,\lambda}(x)$, are as follows:

$$d_0 = \left(\frac{w(n-1)}{n-w}\right)^{-\frac{1}{2}} \exp\left(-\frac{\lambda}{2}\right)$$

$$\begin{aligned} d_i = \sum_{j=0}^{i}\sum_{k=0}^{j} & \frac{\exp\left(-\frac{\lambda}{2}\right)\left(\frac{\lambda}{2}\right)^{j-k}}{(j-k)!}\left(\frac{w(n-1)}{n-w}\right)^{-\frac{1}{2}}\left(1-\frac{n-w}{w(n-1)}\right)^{i-j} \\ & \times\frac{\Gamma\left(i-j+\frac{1}{2}\right)}{\Gamma(i-j+1)\Gamma\left(\frac{1}{2}\right)}\left(\frac{w(n-1)}{n-w}\right)^{k-j}\left(1-\frac{n-w}{w(n-1)}\right)^{k}\binom{j-1}{k} \end{aligned}$$

$$= \sum_{j=0}^{i}\sum_{k=0}^{j}\frac{\exp\left(-\frac{\lambda}{2}\right)\left(\frac{\lambda}{2}\right)^{j-k}\binom{j-1}{k}}{(j-k)!}\frac{\Gamma\left(i-j+\frac{1}{2}\right)}{\Gamma(i-j+1)\Gamma\left(\frac{1}{2}\right)}\left(\frac{w(n-1)}{n-w}\right)^{k-j-\frac{1}{2}}\left(\frac{n(w-1)}{w(n-1)}\right)^{i-j+k}$$

$i = 1, 2, 3, \dots, \infty.$

Thus a $(1 - \alpha)$ 100% confidence interval for Cpw can be constructed as follows:

$$P\left[Q^2_{n,\lambda;\frac{\alpha}{2}} \le \frac{n}{\sigma^2}\left[w(\overline{X}-T)^2+\frac{n-w}{n-1}\hat{\sigma}^2\right] \le Q^2_{n,\lambda;1-\frac{\alpha}{2}}\right]=1-\alpha$$

$$P\left[\sqrt{Q^2_{n,\lambda;\frac{\alpha}{2}}} \le \sqrt{\frac{n}{\sigma^2}\left[w(\overline{X}-T)^2+\frac{n-w}{n-1}\hat{\sigma}^2\right]} \le \sqrt{Q^2_{n,\lambda;1-\frac{\alpha}{2}}}\right]=1-\alpha$$

$$P\left[\frac{USL-LSL}{6\sigma\sqrt{Q^2_{n,\lambda;1-\frac{\alpha}{2}}}} \le \frac{USL-LSL}{6\sigma\sqrt{\frac{n}{\sigma^2}\left[w(\overline{X}-T)^2+\frac{n-w}{n-1}\hat{\sigma}^2\right]}} \le \frac{USL-LSL}{6\sigma\sqrt{Q^2_{n,\lambda;\frac{\alpha}{2}}}}\right]=1-\alpha$$

for $$Cpw=\frac{USL-LSL}{6\sigma\sqrt{1+\frac{\lambda w}{n}}}$$

$\Rightarrow$ $$\sqrt{1+\frac{\lambda w}{n}}\,Cpw=\frac{USL-LSL}{6\sigma},$$

then $$P\left[\sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;1-\frac{\alpha}{2}}}}\,Cpw \le \hat{C}pw \le \sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;\frac{\alpha}{2}}}}\,Cpw\right]=1-\alpha$$

$$P\left[\sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;1-\frac{\alpha}{2}}}}\,\hat{C}pw \le Cpw \le \sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;\frac{\alpha}{2}}}}\,\hat{C}pw\right]=1-\alpha$$

where $P\left[\; Q^2_{n,\lambda} > Q^2_{n,\lambda;\alpha}\right]=\alpha$.

Therefore a (1 - α) 100% confidence interval for Cpw is

$$\left(\sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;1-\frac{\alpha}{2}}}}\,\hat{C}pw,\ \sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;\frac{\alpha}{2}}}}\,\hat{C}pw \right). \qquad [4.3.6]$$

A (1 - α) 100% confidence interval for Cpw* can be constructed similarly and it is

$$\left(\sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;1-\frac{\alpha}{2}}}}\,\hat{C}pw*,\ \sqrt{\frac{1+\frac{\lambda w}{n}}{Q^2_{n,\lambda;\frac{\alpha}{2}}}}\,\hat{C}pw* \right). \qquad [4.3.7]$$

Analogous to Section 2.7, we are going to find an upper confidence limit for the loss function parameter, $E[L_W(X)]$. Consider the ratio

$$\frac{\hat{L}_W(X)}{E[L_W(X)]} = \frac{\left[\frac{n-w}{n-1}\hat{\sigma}^2 + w(\overline{X}-T)^2\right]}{\sigma^2 + w(\mu-T)^2} \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}}$$

$$= \frac{Q^2_{n,\lambda}}{n+\lambda w}$$

Hence, $\frac{n+\lambda w}{E[L_W(X)]}\hat{L}_W(X) \sim Q^2_{n,\lambda}$ and an upper (1 - α) 100% confidence limit for the loss function parameter, $E[L_W(X)]$ can be derived

$$P\left[\ Q^2_{n,\lambda} \geq Q^2_{n,\lambda;1-\alpha}\right] = 1-\alpha$$

$$P\left[\ \frac{n+\lambda w}{E[L_W(X)]}\hat{L}_W(X) \geq Q^2_{n,\lambda;1-\alpha}\right] = 1-\alpha$$

$$P\left[\; E[L_W(X)] \le \frac{n+\lambda w}{Q^2_{n,\lambda;1-\alpha}} \hat{L}_W(X) \right] = 1-\alpha \qquad [4.3.8]$$

Therefore, $\frac{n+\lambda w}{Q^2_{n,\lambda;1-\alpha}} \hat{L}_W(X)$ is an upper (1 - α) 100% confidence limit for $E[L_W(X)]$.

4.4 Effects of Non-Normality on $\hat{C}pw$

In order to examine the effects of non-normality of $\hat{C}pw$, we follow the approach in Section 2.5. The estimator for $\hat{C}pw$ is equation [4.2.1]. Now let $X_1, X_2, \ldots, X_n$ be a random sample from $X \sim N(0, 1)$ and define

$$U = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n} + w(\overline{X}-T)^2 = \frac{S_2}{n} + w\left(\frac{S_1}{n} - T\right)^2$$

$$Z = (\overline{X}-T)^2 = \left(\frac{S_1}{n} - T\right)^2$$

where $S_1 = \sum_{i=1}^{n} X_i$ and $S_2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 = (n-1)S^2$

and results in $\begin{cases} u = \dfrac{s_2}{n} + w\left(\dfrac{s_1}{n} - T\right)^2 \\ z = \left(\dfrac{s_1}{n} - T\right)^2 \end{cases}$

with inverse transformation $\begin{cases} s_1 = n\left(\pm\sqrt{z} + T\right) \\ s_2 = n(u - wz) \end{cases}$

and the Jacobian is $\mathbf{J} = \left|\frac{\partial(s_1, s_2)}{\partial(u,z)}\right| = \begin{vmatrix} 0 & n\left(\pm\dfrac{1}{2\sqrt{z}}\right) \\ n & -n \end{vmatrix} = \mp\frac{n^2}{2\sqrt{z}}$.

If $T = 0$, then the inverse transformation becomes $\begin{cases} s_1 = \pm n\sqrt{z} \\ s_2 = n(u - wz) \end{cases}$ and the Jacobian of the inverse transformation remains the same. Following Gayen (1949) equation [2.1], we have:

$$W_1(n-1)=\frac{\exp\left\{-\frac{n}{2}[u+z(1-w)]\right\}[n(u-wz)]^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}\sqrt{n}}=W_2(n-1)=W(n-1)$$

and the Hermite polynomials

$$\begin{aligned} H_1(x)&=x \\ H_2(x)&=x^2-1 \\ H_3(x)&=x^3-3x \\ H_4(x)&=x^4-6x^2+3 \end{aligned} \quad\Rightarrow\quad \begin{aligned} H_1(-\sqrt{z})&=-\sqrt{z} \\ H_2(-\sqrt{z})&=z^2-1 \\ H_3(-\sqrt{z})&=(-\sqrt{z})^3+3\sqrt{z} \\ H_4(-\sqrt{z})&=z^2-6z+3 \end{aligned} \quad\text{and}\quad \begin{aligned} H_1(\sqrt{z})&=\sqrt{z} \\ H_2(\sqrt{z})&=z^2-1 \\ H_3(\sqrt{z})&=(\sqrt{z})^3-3\sqrt{z} \\ H_4(\sqrt{z})&=z^2-6z+3 \end{aligned}$$

Note that $H_1(-\sqrt{z})+H_1(\sqrt{z})=0$

$$H_3(-\sqrt{z})+H_3(\sqrt{z})=0$$

$$H_2(-\sqrt{z})+H_2(\sqrt{z})=2(z-1)$$

$$H_4(-\sqrt{z})+H_4(\sqrt{z})=2(z^2-6z+3).$$

The joint density of U and Z is

$$
\begin{aligned}
h(u,z)=W_1(n-1)\Bigg\{1+\frac{n\lambda_3}{6}\Big[H_3\left(-\sqrt{z}\right)+3(u-wz)H_1\left(-\sqrt{z}\right)\Big] \\
+\frac{n\lambda_4}{24}\left[H_4\left(-\sqrt{z}\right)+6(u-wz)H_2\left(-\sqrt{z}\right)+\frac{3(n-1)}{n+1}(u-wz)^2\right] \\
+\frac{n\lambda_3^2}{72}\Bigg[\; n\left(-\sqrt{z}\right)^6-3(2n+3)\left(-\sqrt{z}\right)^4+9(n+4)\left(-\sqrt{z}\right)^2-15 \\
+6(u-wz)\left(n\left(-\sqrt{z}\right)^4-3(n+3)\left(-\sqrt{z}\right)^2+6\right) \\
+\frac{9}{(n+1)}(u-wz)^2\left(n(n+1)\left(-\sqrt{z}\right)^2-3(n-1)\right) \\
+\frac{6n(n-2)}{(n+3)(n+1)}(u-wz)^3\;\Bigg]\Bigg\}\left|\frac{n^2}{2\sqrt{z}}\right| \\
+W_2(n-1)\Bigg\{1+\frac{n\lambda_3}{6}\Big[H_3\left(\sqrt{z}\right)+3(u-wz)H_1\left(\sqrt{z}\right)\Big] \\
+\frac{n\lambda_4}{24}\left[H_4\left(\sqrt{z}\right)+6(u-wz)H_2\left(\sqrt{z}\right)+\frac{3(n-1)}{n+1}(u-wz)^2\right] \\
+\frac{n\lambda_3^2}{72}\Bigg[\; n\left(\sqrt{z}\right)^6-3(2n+3)\left(\sqrt{z}\right)^4+9(n+4)\left(\sqrt{z}\right)^2-15 \\
+6(u-wz)\left(n\left(\sqrt{z}\right)^4-3(n+3)\left(\sqrt{z}\right)^2+6\right) \\
+\frac{9}{(n+1)}(u-wz)^2\left(n(n+1)\left(\sqrt{z}\right)^2-3(n-1)\right) \\
+\frac{6n(n-2)}{(n+3)(n+1)}(u-wz)^3\;\Bigg]\Bigg\}\left|-\frac{n^2}{2\sqrt{z}}\right|
\end{aligned}
$$

$$h(u,z)=\frac{\exp\left\{-\frac{n}{2}[u+z(1-w)]\right\}n^{\frac{n}{2}}(u-wz)^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\frac{n}{2}}\sqrt{n}}\frac{1}{2\sqrt{z}}\left\{1\right.$$

$$+\frac{n\lambda_4}{24}\left[\left(z^2-6z+3\right)+6(u-wz)+\frac{3(n-1)}{(n+1)}(u-wz)^2\right]$$

$$+\frac{n\lambda_3^2}{72}\left[nz^3-3(2n+3)z^2+9(n+4)z-15+6(u-wz)\left[nz^2-3(n+3)z+6\right]\right.$$

$$\left.\left.+\frac{9}{(n+1)}(u-wz)^2[n(n+1)z-3(n-1)]+\frac{6n(n-2)}{(n+3)(n+1)}(u-wz)^3\right]\right\}$$

$$0<z<\frac{u}{w},\ 0<u<\infty,\ \text{zero elsewhere.} \qquad [4.4.1]$$

In order to find the marginal pdf, $h_1(u)$, of U we need to integrate z between the limits 0 and $\frac{u}{w}$. By making a substitution $t=\frac{u-wz}{u}$ which makes the integration easier. Now, let $t=\frac{u-wz}{u}$, then $z=\frac{u}{w}(1-t)$, $dz=-\frac{u}{w}dt$. Thus

$$h(u,t)=h(u,z)\left|-\frac{u}{w}\right|, \quad \text{for } 0<t<1,\ 0<u<\infty$$

$$=h\left(u,\frac{u}{w}(1-t)\right)\frac{u}{w}$$

$$h(u,t)=\frac{\exp\left(-\frac{n}{2w}u\right)u^{\frac{n}{2}-1}\exp\left[-\frac{nu(1-w)}{2w}t\right]t^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}\right)^{\frac{n}{2}}w^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}\left\{1\right.$$

$$+\frac{n\lambda_4}{24}\left[\frac{u^2}{w^2}(1-t)^2-6\frac{u}{w}(1-t)+3+6\frac{u^2}{w^2}t(1-t)-6ut+3\frac{(n-1)}{n+1}u^2t^2\right]$$

$$+\frac{n\lambda_3^2}{72}\left[n\frac{u^3}{w^3}(1-t)^3-3(2n+3)\frac{u^2}{w}(1-t)^2+9(n+4)\frac{u}{w}(1-t)-15\right.$$

$$+6n\frac{u^3}{w^2}t(1-t)^2-18(n+3)\frac{u^2}{w}t(1-t)+36ut$$

$$\left.\left.+9n\frac{u^3}{w}t^2(1-t)-27\frac{(n-1)}{n+1}u^2t^2+6\frac{n(n-2)}{(n+3)(n+1)}u^3t^3\right]\right\}$$

Hence the marginal pdf of U is

$$h_1(u)=\int_0^1 h(u,t)dt$$

$$h_1(u)=\frac{\exp\left(-\frac{n}{2w}u\right)u^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}\right)^{\frac{n}{2}}}\left\{B\left(\frac{n-1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]\right.$$

$$+\frac{n\lambda_4}{24}\left[\frac{u^2}{w^2}B\left(\frac{n-1}{2},\frac{5}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{5}{2};-\frac{nu(1-w)}{2w}\right]\right.$$

$$-6\frac{u}{w}B\left(\frac{n-1}{2},\frac{3}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{3}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+3B\left(\frac{n-1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+6\frac{u^2}{w^2}B\left(\frac{n+1}{2},\frac{3}{2}\right){}_1F_1\left[\frac{n+1}{2};\frac{3}{2};-\frac{nu(1-w)}{2w}\right]$$

$$-6uB\left(\frac{n+1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n+1}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+3\frac{(n-1)}{n+1}u^2 B\left(\frac{n+3}{2},\frac{1}{2}\right)\,{}_1F_1\left[\frac{n+3}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]\Bigg]$$

$$+\frac{n\lambda_3^2}{72}\Bigg[\;n\frac{u^3}{w^3}B\left(\frac{n-1}{2},\frac{7}{2}\right)\,{}_1F_1\left[\frac{n-1}{2};\frac{7}{2};-\frac{nu(1-w)}{2w}\right]$$

$$-3(2n+3)\frac{u^2}{w}B\left(\frac{n-1}{2},\frac{5}{2}\right)\,{}_1F_1\left[\frac{n-1}{2};\frac{5}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+9(n+4)\frac{u}{w}B\left(\frac{n-1}{2},\frac{3}{2}\right)\,{}_1F_1\left[\frac{n-1}{2};\frac{3}{2};-\frac{nu(1-w)}{2w}\right]$$

$$-15B\left(\frac{n-1}{2},\frac{1}{2}\right)\,{}_1F_1\left[\frac{n-1}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+6n\frac{u^3}{w^2}B\left(\frac{n+1}{2},\frac{5}{2}\right)\,{}_1F_1\left[\frac{n+1}{2};\frac{5}{2};-\frac{nu(1-w)}{2w}\right]$$

$$-18(n+3)\frac{u^2}{w}B\left(\frac{n+1}{2},\frac{3}{2}\right)\,{}_1F_1\left[\frac{n+1}{2};\frac{3}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+36uB\left(\frac{n+1}{2},\frac{1}{2}\right)\,{}_1F_1\left[\frac{n+1}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+9n\frac{u^3}{w}B\left(\frac{n+3}{2},\frac{3}{2}\right)\,{}_1F_1\left[\frac{n+3}{2};\frac{3}{2};-\frac{nu(1-w)}{2w}\right]$$

$$-27\frac{(n-1)}{n+1}u^2 B\left(\frac{n+3}{2},\frac{1}{2}\right)\,{}_1F_1\left[\frac{n+3}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]$$

$$+6\frac{n(n-2)}{(n+3)(n+1)}u^3 B\left(\frac{n+5}{2},\frac{1}{2}\right)\,{}_1F_1\left[\frac{n+5}{2};\frac{1}{2};-\frac{nu(1-w)}{2w}\right]\Bigg]\Bigg\}$$

$0 < u < \infty$, zero elsewhere. [4.4.2]

For $\hat{C}pw=\frac{a}{\sqrt{U}}=V$, then $v=\frac{a}{\sqrt{u}} \Rightarrow u=\frac{a^2}{v^2}$ with Jacobian of the inverse transformation, $J=\left|\frac{du}{dv}\right|=-\frac{2a^2}{v^3}$. So that the pdf of $\hat{C}pw$ is

$$g_{\hat{C}pw}(v)=h_1(u)|J|=h_1\left(\frac{a^2}{v^2}\right)\left|-\frac{2a^2}{v^3}\right|$$

$$=\frac{\exp\left(-\frac{n}{2w}\frac{a^2}{v^2}\right)\left(\frac{a^2}{v^2}\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}\right)^{\frac{n}{2}}}\left\{B\left(\frac{n-1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]\right.$$

$$+\frac{n\lambda_4}{24}\left[\frac{a^2}{w^2v^2}B\left(\frac{n-1}{2},\frac{5}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{5}{2};-\frac{na^2(1-w)}{2wv^2}\right]\right.$$

$$-6\frac{a^2}{wv^2}B\left(\frac{n-1}{2},\frac{3}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{3}{2};-\frac{na^2(1-w)}{2wv^2}\right]$$

$$+3B\left(\frac{n-1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]$$

$$+6\frac{a^4}{w^2v^4}B\left(\frac{n+1}{2},\frac{3}{2}\right){}_1F_1\left[\frac{n+1}{2};\frac{3}{2};-\frac{na^2(1-w)}{2wv^2}\right]$$

$$-6\frac{a^2}{v^2}B\left(\frac{n+1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n+1}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]$$

$$\left.+3\frac{(n-1)}{n+1}\frac{a^4}{v^4}B\left(\frac{n+3}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n+3}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]\right]$$

$$
\begin{aligned}
&+\frac{n\lambda_3^2}{72}\Bigg[\; n\frac{a^6}{w^3v^6}B\left(\frac{n-1}{2},\frac{7}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{7}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad -3(2n+3)\frac{a^4}{wv^4}B\left(\frac{n-1}{2},\frac{5}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{5}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad +9(n+4)\frac{a^2}{wv^2}B\left(\frac{n-1}{2},\frac{3}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{3}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad -15B\left(\frac{n-1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n-1}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad +6n\frac{a^6}{w^2v^6}B\left(\frac{n+1}{2},\frac{5}{2}\right){}_1F_1\left[\frac{n+1}{2};\frac{5}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad -18(n+3)\frac{a^4}{wv^4}B\left(\frac{n+1}{2},\frac{3}{2}\right){}_1F_1\left[\frac{n+1}{2};\frac{3}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad +36\frac{a^2}{v^2}B\left(\frac{n+1}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n+1}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad +9n\frac{a^6}{wv^6}B\left(\frac{n+3}{2},\frac{3}{2}\right){}_1F_1\left[\frac{n+3}{2};\frac{3}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad -27\frac{(n-1)}{n+1}\frac{a^4}{v^4}B\left(\frac{n+3}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n+3}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]\\
&\quad +6\frac{n(n-2)}{(n+3)(n+1)}\frac{a^6}{v^6}B\left(\frac{n+5}{2},\frac{1}{2}\right){}_1F_1\left[\frac{n+5}{2};\frac{1}{2};-\frac{na^2(1-w)}{2wv^2}\right]\Bigg]\Bigg\}
\end{aligned}
$$

where $a=\dfrac{\sqrt{n}\left[USL-LSL\right]}{6\sigma}=\sqrt{n+\lambda w}\,Cpw$,

$0<v<\infty$, zero elsewhere. [4.4.3]

4.5 Comments

The Cpw index, as well as its generalization Cpw*, can be estimated respectively using $\hat{C}pw$ and $\hat{C}pw^*$ for those cases where $\mu = T$. Both $\hat{C}pw$ and $\hat{C}pw^*$ have been shown to be biased estimators of Cpw and Cpw* respectively. Asymptotically the biases and mean squared errors associated with both $\hat{C}pw$ and $\hat{C}pw^*$ are zero implying that both $\hat{C}pw$ and $\hat{C}pw^*$ are asymptotically unbiased estimators and both converge in probability to their respective parameters, Cpw and Cpw*, for w = 0 and 1. It can be further shown, for w non-negative, that

$$\lim_{n\to\infty} {}_2F_1\left[1,\frac{1}{2};\frac{n}{2};1-w\right]=1 \quad \text{and} \quad \lim_{n\to\infty} {}_2F_1\left[\frac{1}{2},\frac{1}{2};\frac{n}{2};1-w\right]=1$$

where

$${}_2F_1[b,c;d;z]=1+\frac{bc}{1d}z+\frac{b(b+1)c(c+1)}{1\cdot 2d(d+1)}z^2+\ldots+\frac{b(b+1)\ldots(b+s-1)c(c+1)\ldots(c+s-1)}{s!\,d(d+1)\ldots(d+s-1)}z^s+\ldots$$

b, c and z are real numbers.

Assuming $\mu = T$, hence equations [4.2.42] through [4.2.45] can be shown equal to zero as n approaches infinity. These imply that both $\hat{C}pw$ and $\hat{C}pw^*$ are asymptotically unbiased and are consistent estimators of Cpw and Cpw* respectively for non-negative values of w.

The quantities, Cpw and $E[L_W(X)]$, each have their own advantages and are familiar to quality practitioners. The expected weighted loss does require the use of an explicit loss function such as Taguchi's modified loss function attached with an appropriate weight function, w. Allowing the weight function (Spiring (1997)) to assume different forms allows Cpw to be analogous to the different types of loss and utility functions available and used by practitioners. Using different weight functions allows one to customize the capability index to the process of interest, thereby allowing different shaped loss functions to be used for various processes. However it is easily interpreted in terms of monetary loss, either to the practitioner and/or the society when the process characteristic misses the target.

We have developed the statistical properties of $\hat{C}pw$ and $\hat{C}pw^*$ where the process characteristic is normal distribution and when the normal distribution is distorted with non zero values of skewness and kurtosis. We have also developed alternative techniques to obtain the confidence intervals for Cpw as well as Cpw*. Similarly to what we have done in Section 2.7 we have examined the relationship of Cpw to the squared error loss function.

Chapter 5

The Inverted Beta Loss Function and its Applications

5.1 Introduction

The applications of loss functions in quality assurance settings have been increasingly studied with the recognized importance of off-targetness by both customers and manufacturers. In this chapter, a general class of loss functions based on the inversion of a family of probability density functions is examined. Applications in industry including reliability and quality assurance, process monitoring and control using economic loss are used to demonstrate applications of this general class of loss functions. Mathematical derivations are included as theorems in this chapter. Some properties arised from the IBLF are discussed in Section 5.7.

5.2 The Inverted Beta Loss Function

The maximum value of a standard beta pdf with parameters $\alpha > 0$, $\beta > 0$ having functional form

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \qquad 0 < x < 1,\ \alpha > 0, \beta > 0,$$

can be found as follows:

$$\ln f(x) = -\ln B(\alpha,\beta) + (\alpha-1)\ln x + (\beta-1)\ln(1-x)$$

$$\frac{\partial \ln f(x)}{\partial x} = \frac{\alpha-1}{x} - \frac{\beta-1}{1-x} = 0$$

Solving for x yields

$$x = \frac{\alpha-1}{\alpha+\beta-2}$$

and represents the value of x where the beta pdf is a maximum. With the existence of unique maximum we must have $\alpha > 1$, $\beta > 1$ and $\alpha < \beta$ since

$$\frac{\partial^2 \ln f(x)}{\partial x^2} = -\frac{\alpha-1}{x^2} - \frac{\beta-1}{(1-x)^2}\text{, and evaluate at } x = \frac{\alpha-1}{\alpha+\beta-2}$$

$$\frac{\partial^2 \ln f(x)}{\partial x^2} = -\frac{(\alpha+\beta-2)^3}{(\alpha-1)(\beta-1)} < 0 \Rightarrow \text{ maximum if both } \alpha > 1 \text{ and } \beta > 1.$$

Following the development for general inverted probability loss function outlined in Spiring and Yeung (1998), let T denote the target of the process, and define T = $\frac{\alpha-1}{\alpha+\beta-2}$ to be fixed. Using the unique maximum conditions associated with the beta distribution a linear relationship can be established between α and β through T. The relationship can be written as

$$\alpha = \frac{T}{1-T}\beta + \frac{1-2T}{1-T} \qquad \text{and} \qquad \alpha - 1 = \frac{T(\beta-1)}{1-T}.$$

Letting $\pi(x, T)$ denote a function of the form of a beta probability density function

$$\pi(x,T) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1,$$

with $$m = \sup_x \pi(x,T) = \frac{1}{B(\alpha,\beta)} T^{\alpha-1}(1-T)^{\beta-1},$$

where m denotes the supremum of $\pi(x,T)$. Then analogous to Spiring and Yeung (1998), the loss inversion ratio becomes

$$\frac{\pi(x,T)}{m} = \left(\frac{x}{T}\right)^{\alpha-1}\left(\frac{1-x}{1-T}\right)^{\beta-1}$$

$$= \left(\frac{x}{T}\right)^{\alpha-1}\left(\frac{1-x}{1-T}\right)^{\frac{(1-T)(\alpha-1)}{T}}$$

$$= \left[\frac{x}{T}\left(\frac{1-x}{1-T}\right)^{\frac{1-T}{T}}\right]^{(\alpha-1)}$$

$$= C\left\{x(1-x)^{\frac{1-T}{T}}\right\}^{(\alpha-1)} \qquad [5.2.1]$$

where $C = \left[T(1-T)^{\frac{1-T}{T}}\right]^{1-\alpha}$.

The loss function associated with inverting the beta pdf is referred to as the Inverted Beta Loss Function which, for a K, the maximum loss, is :

$$L(x,T) = K\left[1-\frac{\pi(x,T)}{m}\right]$$

$$= K\left\{1-\left[\frac{x}{T}\left(\frac{1-x}{1-T}\right)^{\frac{1-T}{T}}\right]^{(\alpha-1)}\right\} \qquad [5.2.2]$$

$$= K\left\{1-C\left[x(1-x)^{\frac{1-T}{T}}\right]^{(\alpha-1)}\right\}$$

Figures 5.2.1 a, b and c are IBLF's with K = 1

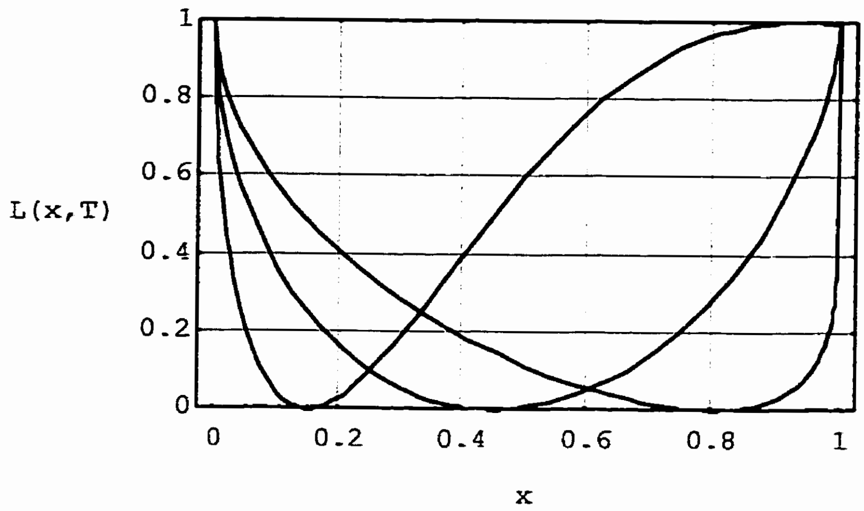


Figure 5.2.1a L (x, T = .15, α = 1.5, β = 3.83), L (x, T = .45, α = 1.5, β = 1.61) and L (x, T = .80, α = 1.5, β = 1.125)

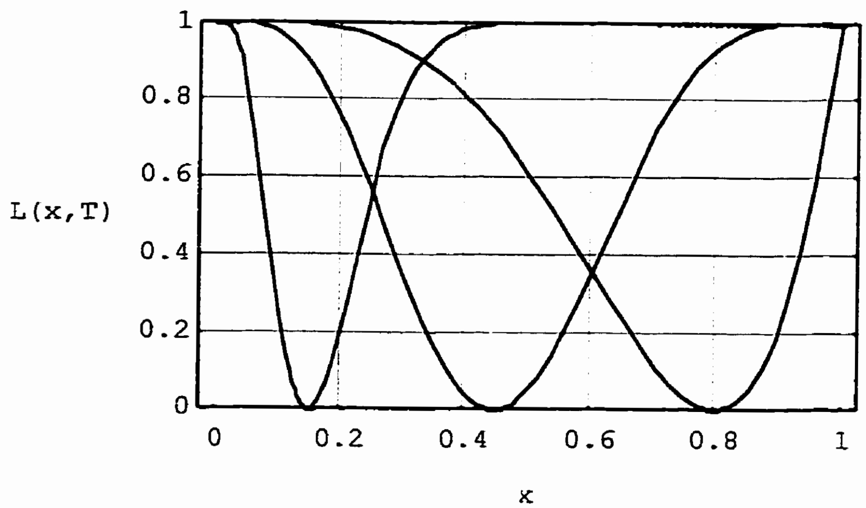


Figure 5.2.1b L (x, T = .15, α = 5, β = 23.67), L (x, T = .45, α =5, β =5.89) and L (x, T = .80, α = 5, β = 2.00)

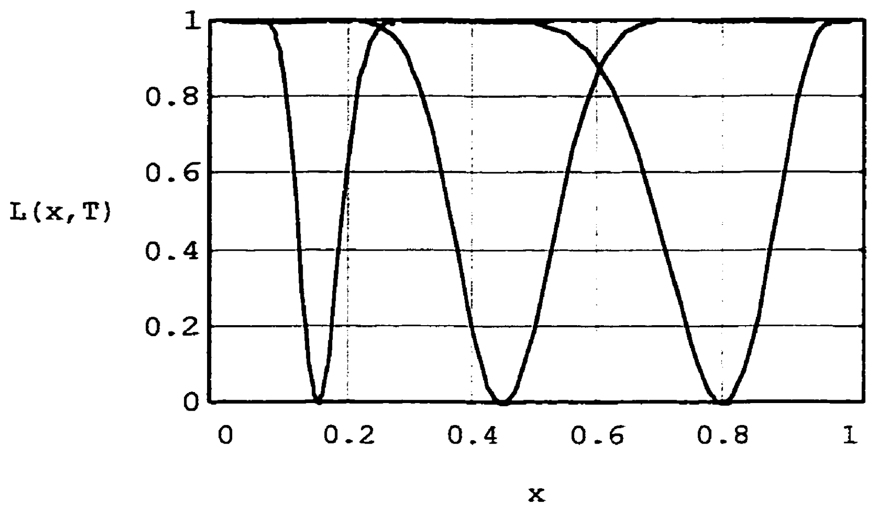


Figure 5.2.1c $L(x, T=.15, \alpha=20, \beta=108.67)$, $L(x, T=.45, \alpha=20, \beta=24.22)$ and $L(x, T=.80, \alpha=20, \beta=5.75)$

5.3 The Risk Function

The loss function is employed to describe the precise situation that the loss incurs when process measurements depart from the target. In decision theory the Risk Function provides the average loss associated with the process given the loss function and some assumed distribution for the process measurements. It measures, in monetary units, the average loss to customers or society when the target is missed. The IBLF can be so chosen to fit the practitioners' need and its associated risk function can be evaluated easily.

In particular if the process characteristic X has a beta distribution with parameters $\alpha_R > 0$ and $\beta_R > 0$, then the expected loss or risk is given by

$$E[L(X,T)] = \int_0^1 K\left\{1 - C\left[x(1-x)^{\frac{1-T}{T}}\right]^{(\alpha-1)}\right\}\frac{1}{B(\alpha_R,\beta_R)}x^{\alpha_R-1}(1-x)^{\beta_R-1}dx$$

$$= K\left(1-\frac{C}{B(\alpha_R,\beta_R)}\int_0^1 x^{(\alpha+\alpha_R-1)-1}(1-x)^{\frac{1-T}{T}(\alpha-1)+\beta_R-1}dx\right)$$

$$= K\left\{1-C\frac{B(\alpha+\alpha_R-1,\frac{1-T}{T}(\alpha-1)+\beta_R)}{B(\alpha_R,\beta_R)}\right\}. \qquad [5.3.1]$$

The conjugate distribution for this IBLF gives a closed form for the beta distribution. An application of this risk function will be illustrated through an example from the printing industry. However there are other distributions which are suitable for this IBLF. (See Section 5.7 for details.)

5.4 Choosing the Appropriate IBLF

In those cases where only the "primary loss information" is specified, i.e. T and K, the general form of the IBLF is

$$L(x,T) = K\left[1-\frac{\pi(x,T)}{m}\right],\quad 0 < x < 1$$

where K is the maximum loss, $\pi(x,T)$ a standard beta pdf with parameters α and β, $T = \frac{\alpha-1}{\alpha+\beta-2}$ is the target of the process, m is the supremum of $\pi(x,T)$.

The associated risk function is equation [5.3.1] when assuming $X \sim Be(\alpha_R,\beta_R)$. The shape of this loss function can be controlled through the selection of α and/or β as long as both $\alpha > 1$ and $\beta > 1$. Since α, β and T are related as follows :

$$\alpha - 1 = \frac{T(\beta-1)}{1-T} \quad \text{or} \quad \beta - 1 = \frac{(1-T)(\alpha-1)}{T} \qquad [5.4.1]$$

and assuming T to be fixed, there are many combinations of α and β which can be used to create various shapes for the IBLF. Three such combinations of α and β are illustrated in Figure 5.4.1.

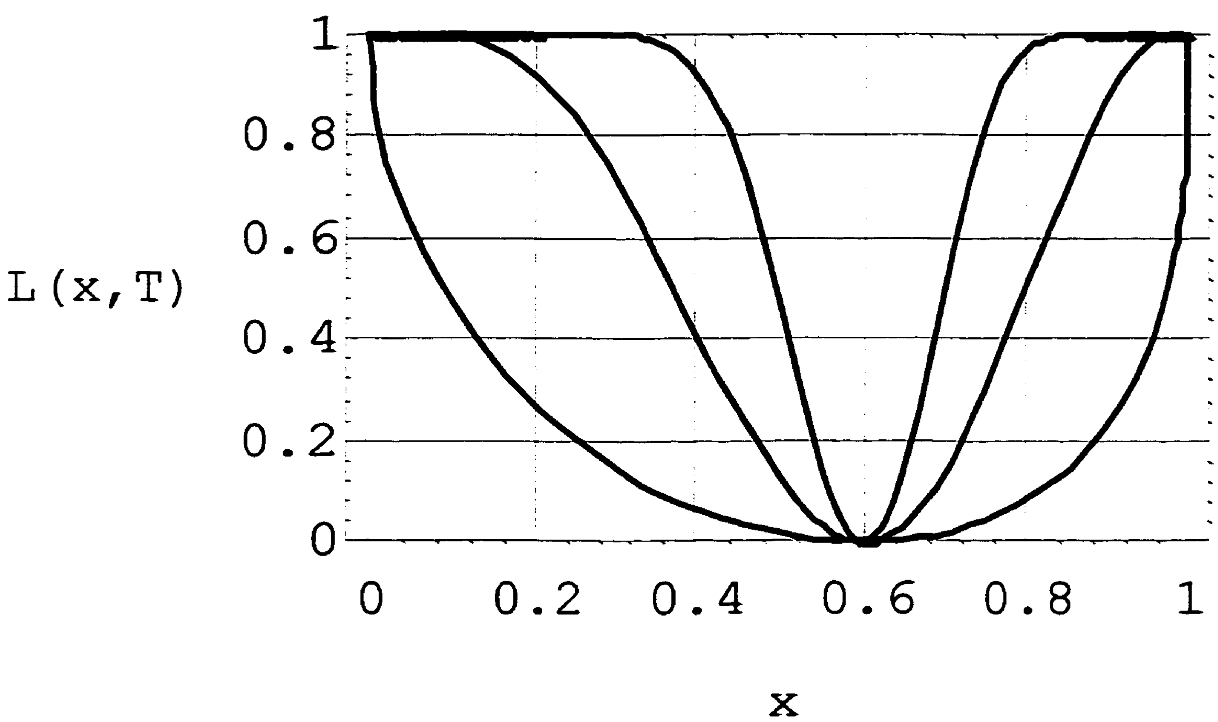


Figure 5.4.1 L_1 (x, T = .60, α = 1.5, β = 1.33), L_2 (x, T = .60, α =5, β =3.67) and L_3 (x, T = .60, α = 20, β = 13.67)

When T = 1/2, from equation [5.4.1] it is easy to verify that $\alpha = \beta$. Assuming T = 1/2 and $\alpha = \beta > 1$, the resulting IBLF is symmetric around T with the maximum loss reached at similar distances from T in both directions. Figure 5.4.2 illustrates three of the many symmetric forms the IBLF may take on.

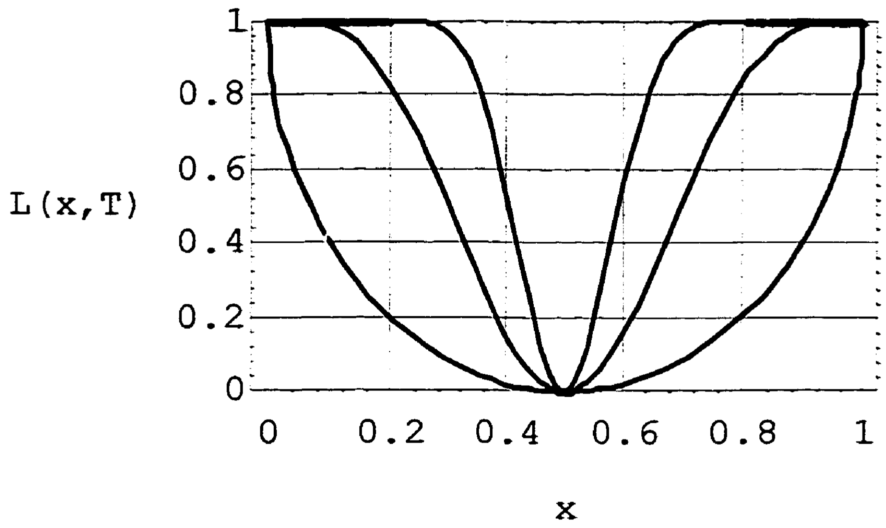


Figure 5.4.2 L_1 (x, T = .50, α = 1.5, β = 1.5), L_2 (x, T = .50, α =1.5,β =1) and L_3 (x, T = .50, α = 1.5, β = 1.5)

When T $\neq$ 1/2, the IBLF will be asymmetric and have many potential shapes. Figure 5.4.3 illustrates three IBLFs for the case where T < 1/2, while Figure 5.4.4 illustrates three IBLFs for the case where T > 1/2.

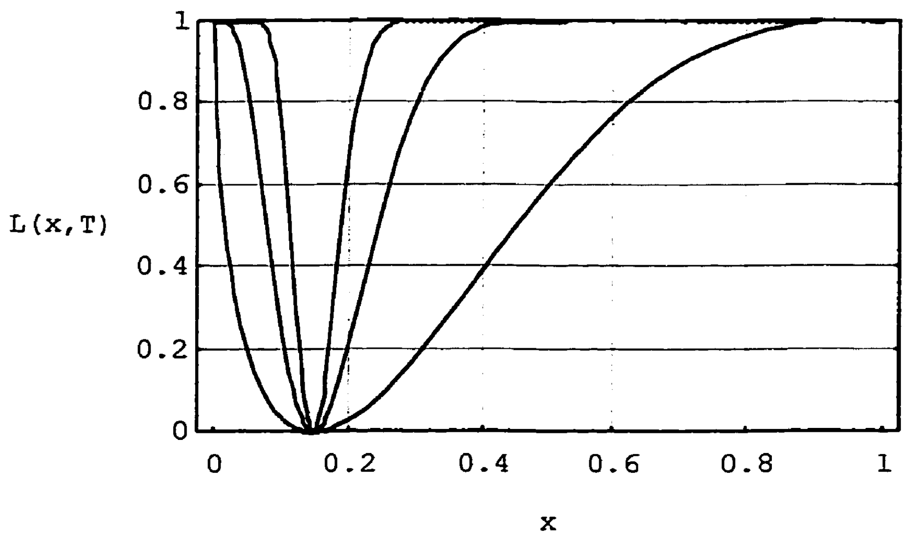


Figure 5.4.3 Various Asymmetric IBLFs with T < 1/2

Outer : T = .15, K = 1, α = 1.5, β = 3.833
Middle : T = .15, K = 1, α = 5, β = 23.6667
Inner : T = .15, K = 1, α = 20, β = 108.6667

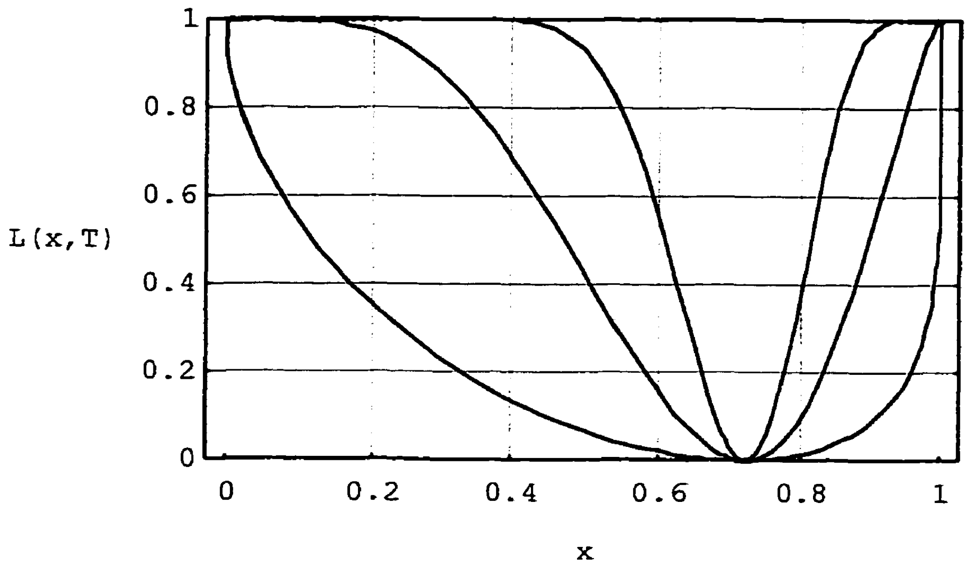


Figure 5.4.4 Various Asymmetric IBLFs with $T > 1/2$
Outer: $T = .72, K = 1, \alpha = 1.5, \beta = 1.1944$
Middle: $T = .72, K = 1, \alpha = 5, \beta = 2.5556$
Inner: $T = .72, K = 1, \alpha = 20, \beta = 8.3889$

From the above figures we see small values of α "open up the arms" of the loss function around T, while larger α values "tighten the arms around T". Small α values result in smaller economic losses for slightly off target processes, while larger values of α assign a more severe penalty (loss) for similar departures from the target.

In those cases where T, K and an auxiliary piece of information about the loss are known (e.g., $[x_1, L_1]$, where L_1 represents the loss at x_1), the value of α is such that

$$L_1(x_1, T) = K\left[1 - \frac{\pi(x_1, T)}{m}\right] \qquad [5.4.2]$$

while continuing to satisfy the conditions outlined in equation [5.4.1]. α can be solved using equation [5.4.2] using $K_i = K$ for $i = 1,2$. The associated risk function will be the same as that described in equation [5.3.1]. It can apply whenever $x_1 < T$ or $x_1 > T$.

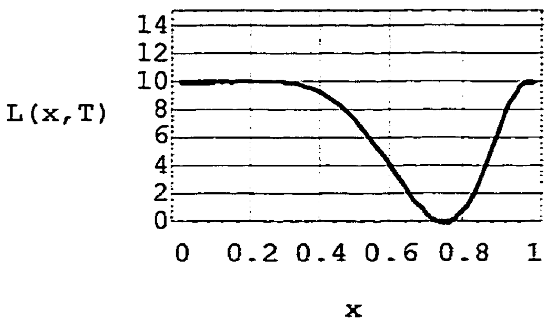


Figure 5.4.5 a
$T = .75, K = 10, L[.60, .75] = 4,$
$\alpha = 8.6844, \beta = 3.5615$

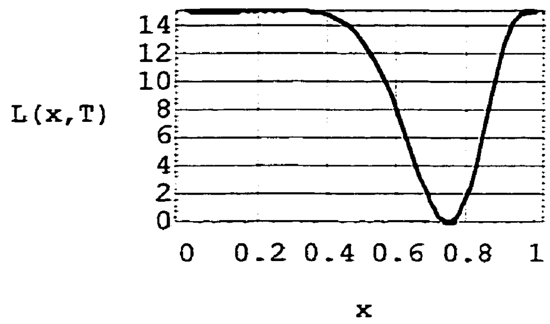


Figure 5.4.5 b
$T = .75, K = 15, L[.85, .75] = 6,$
$\alpha = 12.3235, \beta = 4.7745$

When T, K and two auxiliary pieces of information (e.g., $[x_1, L_1]$ and $[x_2, L_2]$) are known and such that $x_1 < T$ and $x_2 > T$, we need to solve for α_1 and α_2 such that

$$L_1(x_1, T) = K\left[1 - \frac{\pi_1(x_1, T)}{m_1}\right] \quad \text{and} \quad L_2(x_2, T) = K\left[1 - \frac{\pi_2(x_2, T)}{m_2}\right].$$

It is easy to show the solutions are of the form

$$\alpha_i = \frac{\ln\left[1 - \dfrac{L_i(x_i, T)}{K_i}\right]}{\ln\left[\dfrac{x_i}{T}\left(\dfrac{1 - x_i}{1 - T}\right)^{\frac{1-T}{T}}\right]} + 1\,, \qquad i = 1, 2. \qquad [5.4.3]$$

where L_1 and L_2 represent the loss associated with the values of the process characteristic x_1 and x_2 respectively. Combining the resulting curves provides practitioners with a versatile loss function of the form

$$L(x, T) = \begin{cases} K\left(1 - \dfrac{\pi_1(x,T)}{m_1}\right) & \text{if } 0 < x < T \\ K\left(1 - \dfrac{\pi_2(x,T)}{m_2}\right) & \text{if } T < x < 1 \end{cases} \qquad [5.4.4]$$

allowing both sides of the target to have a maximum loss of K and shape based on $\pi_1(x,T)$ [which has parameters α_1 and T] and $\pi_2(x,T)$ [which has parameters α_2 and T]. Figure 5.4.6 illustrates the combined loss function based on two different Beta pdfs.

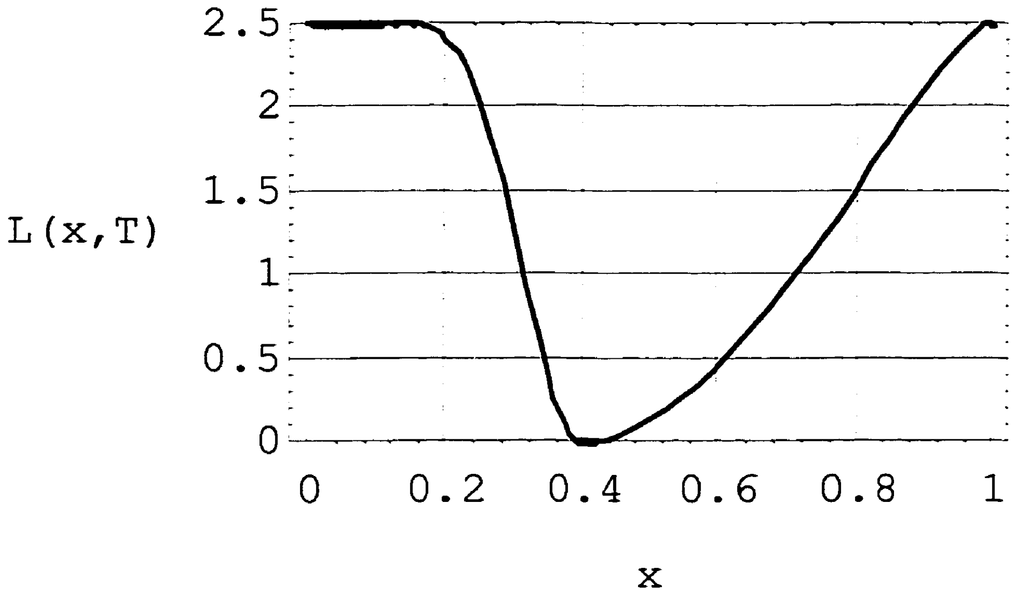


Figure 5.4.6 $\quad T = .4,\ K = 2.5, L_1(.3, .4) = 1.25, L_2(.75, .4) = 1.2$

Theorem 5.4.1:

The associated risk function of the IBLF, assuming $X \sim Be(\alpha_R, \beta_R)$, is

$$E[L(X,T)] = K\{1 - C_1 B_T(l_1 + \alpha_R, n_1 + \beta_R)/B(\alpha_R, \beta_R) -$$

$$C_2 [B(l_2 + \alpha_R, n_2 + \beta_R) - B_T(l_2 + \alpha_R, n_2 + \beta_R)]/B(\alpha_R, \beta_R)\}$$

[5.4.5]

where $C_i = \left[T(1-T)^{\frac{1-T}{T}} \right]^{1-\alpha_i}$, $l_i = \alpha_i - 1$, $n_i = \frac{1-T}{T}(\alpha_i - 1)$, $i = 1, 2.$

Proof:

Let X have a standard beta distribution with $\alpha_R > 0$, and $\beta_R > 0$, then

$$E[L(X,T)] = \int_0^1 L(x,T)\, f(x)\, dx$$

$$= \int_0^T K\left\{1 - C_1\left[x^{l_1}(1-x)^{n_1}\right]\right\} f(x)\, dx + \int_T^1 K\left\{1 - C_2\left[x^{l_2}(1-x)^{n_2}\right]\right\} f(x)\, dx$$

$$= K\left[\int_0^T f(x)\, dx - C_1 \int_0^T x^{l_1}(1-x)^{n_1} \frac{x^{\alpha_R - 1}(1-x)^{\beta_R - 1}}{B(\alpha_R, \beta_R)} dx\right] +$$

$$K\left[\int_T^1 f(x)\, dx - C_2 \left[\int_T^1 x^{l_2}(1-x)^{n_2} \frac{x^{\alpha_R - 1}(1-x)^{\beta_R - 1}}{B(\alpha_R, \beta_R)} dx\right]\right.$$

$$= K\left\{ 1 - C_1 \int_0^T \frac{x^{l_1 + \alpha_R - 1}(1-x)^{n_1 + \beta_R - 1}}{B(\alpha_R, \beta_R)} dx - C_2 \int_T^1 \frac{x^{l_2 + \alpha_R - 1}(1-x)^{n_2 + \beta_R - 1}}{B(\alpha_R, \beta_R)} dx \right\}$$

$$= K\left\{1 - C_1 \frac{B_T(l_1 + \alpha_R, n_1 + \beta_R)}{B(\alpha_R, \beta_R)} - C_2 \frac{B(l_2 + \alpha_R, n_2 + \beta_R) - B_T(l_2 + \alpha_R, n_2 + \beta_R)}{B(\alpha_R, \beta_R)}\right\}.$$

For those situations where the maximum is different on either side of the target the IBLF can be combined as follows :

$$L(x, T) = \begin{cases} K_1\left(1 - \dfrac{\pi_1(x,T)}{m_1}\right) & \text{if } 0 < x < T \\ K_2\left(1 - \dfrac{\pi_2(x,T)}{m_2}\right) & \text{if } T < x < 1 \end{cases} \qquad [5.4.6]$$

allowing either side of the target to have maximum losses of K_1 and K_2 respectively and shape based on $\pi_1(x,T)$ and $\pi_2(x,T)$. See Figure 5.4.7 for an example.

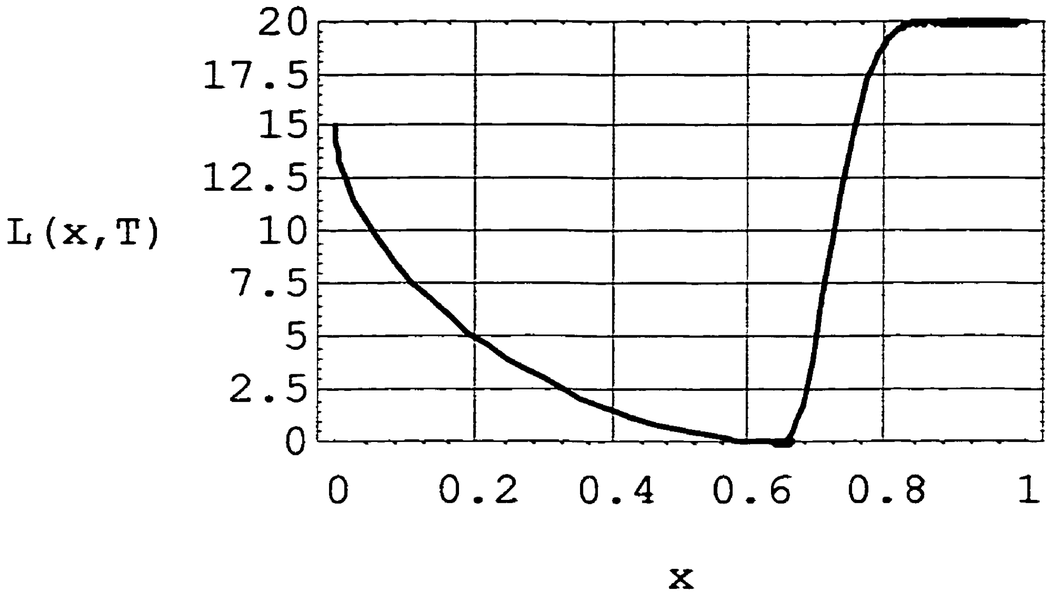


Figure 5.4.7 $T = .65$, $K_1 = 15$, $L_1(.2, .65) = 5$, $K_2 = 20$, $L_2(.7, .65) = 5$

Theorem 5.4.2:

The corresponding risk function of the IBLF, assuming $X \sim Be(\alpha_R, \beta_R)$, is

$$\begin{aligned} E[L(X,T)] = & K_1 \{ I_T(\alpha_R, \beta_R) - C_1 B_T(l_1 + \alpha_R, n_1 + \beta_R)/ B(\alpha_R, \beta_R)\} \\ & + K_2 \{[1 - I_T(\alpha_R, \beta_R)] - \\ & C_2 [B(l_2 + \alpha_R, n_2 + \beta_R) - B_T(l_2 + \alpha_R, n_2 + \beta_R)]/ B(\alpha_R, \beta_R)\}. \end{aligned}$$

[5.4.7]

Proof:

Let X have a standard beta distribution with $\alpha_R > 0$, and $\beta_R > 0$, then

$$E[L(X,T)] = \int_0^1 L(x,T)\, f(x)\, dx$$

$$= \int_0^T K_1 \left\{1 - C_1\left[x^{l_1}(1-x)^{n_1}\right]\right\} f(x)\, dx + \int_T^1 K_2 \left\{1 - C_2\left[x^{l_2}(1-x)^{n_2}\right]\right\} f(x)\, dx$$

$$= K_1 \left[\int_0^T f(x)\, dx - C_1 \int_0^T x^{l_1}(1-x)^{n_1} \frac{x^{\alpha_R - 1}(1-x)^{\beta_R - 1}}{B(\alpha_R, \beta_R)} dx\right] +$$

$$K_2 \left[\int_T^1 f(x)\, dx - C_2 \int_T^1 x^{l_2}(1-x)^{n_2} \frac{x^{\alpha_R - 1}(1-x)^{\beta_R - 1}}{B(\alpha_R, \beta_R)} dx\right]$$

$$= K_1 \left[I_T(\alpha_R, \beta_R) - C_1 \frac{B_T(l_1 + \alpha_R, n_R + \beta_R)}{B(\alpha_R, \beta_R)}\right] +$$

$$K_2 \left\{ \frac{B(\alpha_R, \beta_R) - B_T(\alpha_R, \beta_R)}{B(\alpha_R, \beta_R)} - C_2 \left[\frac{B(l_2 + \alpha_R, n_2 + \beta_R) - B_T(l_2 + \alpha_R, n_2 + \beta_R)}{B(\alpha_R, \beta_R)}\right]\right\}$$

$$= K_1 \left[I_T(\alpha_R, \beta_R) - C_1 \frac{B_T(l_1 + \alpha_R, n_R + \beta_R)}{B(\alpha_R, \beta_R)}\right] +$$

$$K_2 \left\{ \left[1 - I_T(\alpha_R, \beta_R)\right] - C_2 \left[\frac{B(l_2 + \alpha_R, n_R + \beta_R) - B_T(l_2 + \alpha_R, n_2 + \beta_R)}{B(\alpha_R, \beta_R)}\right]\right\}.$$

Notice that equation [5.4.7] reduces to equation [5.4.5] when $K_1 = K_2 = K$.

5.5 Properties of the IBLF

1. The shape of the IBLF is scale invariant under linear transformation.

   If the IBLF is based on a generalized beta distribution (i.e., f(x)) with unique maximum conditions, then a transformation of the form y = a + bx results in an IBLF with similar shape but different scale and/or target. Assuming f(x) to be a standard beta pdf with unique maximum conditions, then the transformation y = a + bx results in $y_{max} = a + bx_{max}$. As a result the ratio of the pdf to its unique mode is independent of scale. It follows that 1 minus this ratio is also independent of scale, and hence the IBLF is said to be scale invariant under linear transformations. To illustrate, Figures 5.5.1 a and 5.5.1 b contain the IBLF associated with the standard Beta pdf and K = 10 (i.e., L(x, T = .65)) and the IBLF associated with the transformation y = 20 + 20x again with K = 10 (i.e., IBLF L(y, T = 33)).

Theorem 5.5.1:

The shape of the IBLF is scale invariant under linear transformation.

Proof:

If Y has a generalized beta distribution with parameters $\alpha > 0$, $\beta > 0$ and ranging from p to q (with $p \leq q$), then we can transform it to a standard beta distribution that possesses the same loss function as X.

The pdf of Y is

$$g(y) = \frac{1}{B(\alpha,\beta)(q-p)}\left(\frac{y-p}{q-p}\right)^{\alpha-1}\left(\frac{q-y}{q-p}\right)^{\beta-1}, \quad p < y < q.$$

Let $Y = (q - p)X + p$ and then $X = \dfrac{Y-p}{q-p}$ with $|\,\boldsymbol{J}\,| = \left|\dfrac{dy}{dx}\right| = (q - p)$.

Similarly, the mode of Y can be obtained through differentiation and found to be

$$y = \frac{(q-p)(\alpha-1)}{\alpha+\beta-2} + p$$

and let it equal to T', the target with respect to Y.

Note that, 1) $\frac{T'-p}{q-p}=\frac{\alpha-1}{\alpha+\beta-2}=T$, the target value in X,

2) $\beta-1=\frac{1-T}{T}(\alpha-1)$, and

3) $\left\{\left(\frac{T'-p}{q-p}\right)\left(\frac{q-T'}{q-p}\right)^{\frac{1-T}{T}}\right\}^{1-\alpha}=\left\{T(1-T)^{\frac{1-T}{T}}\right\}^{1-\alpha}=C$

So, $\pi(y,T')=\frac{1}{B(\alpha,\beta)(q-p)}\left(\frac{y-p}{q-p}\right)^{\alpha-1}\left(\frac{q-y}{q-p}\right)^{\beta-1}$ and correspondingly m', the supremum of $\pi(y,T')$, is

$$m'=\frac{1}{B(\alpha,\beta)(q-p)}\left(\frac{T'-p}{q-p}\right)^{\alpha-1}\left(\frac{q-T'}{q-p}\right)^{\beta-1}=\frac{1}{B(\alpha,\beta)(q-p)}C^{-1}.$$

Hence the loss inversion ratio

$$\frac{\pi(y,T')}{m'}=C\left(\frac{y-p}{q-p}\right)^{\alpha-1}\left(\frac{q-y}{q-p}\right)^{\frac{1-T}{T}(\alpha-1)}$$

$$=C\left\{x(1-x)^{\frac{1-T}{T}}\right\}^{\alpha-1}$$, which is same as [5.2.1]

$$=\frac{\pi(x,T)}{m}$$

$$\therefore\ L(x,T)=L(y,T').$$

Figures 5.5.1a and 5.5.1b are IBLFs with K = 10.

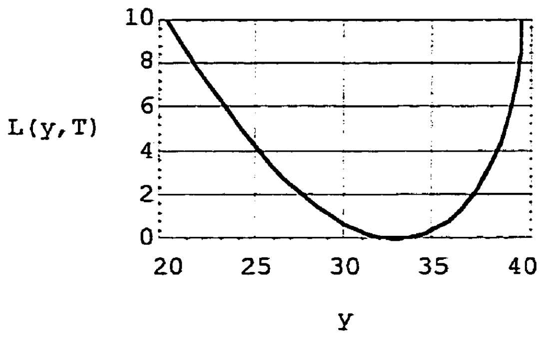


Figure 5.5.1 a
$L(y, T' = 33, \alpha = 2, \beta = 1.54)$

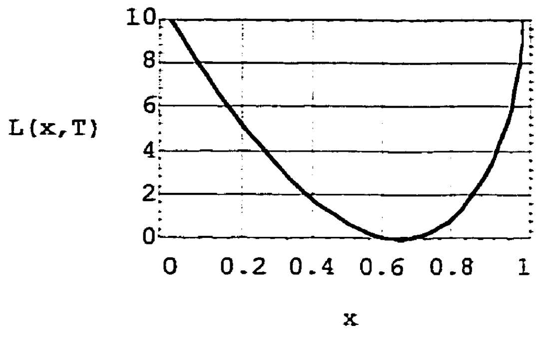


Figure 5.5.1 b
$L(x, T = .65, \alpha = 2, \beta = 1.54)$

2. Theorem 5.5.2

The risk function is scale invariant under linear transformation.

Proof:

It follows from Theorem 5.5.1, the pdf of X is

$$f(x) = g(y) \mid \mathbf{J} \mid = g[x(q - p) + p]\ (q - p)$$

$$= \frac{1}{B(\alpha,\beta)(q-p)}\ x^{\alpha-1}\ (1-x)^{\beta-1}\ (q - p)$$

$$= \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1.$$

Then the risk function of Y is

$$E[L(Y, T')] = \int_p^q L(y, T')\ g(y)\ dy = \int_p^q L(x,T)\ g(y)\ dy$$

$$= \int_0^1 L(x,T)\ g[x(q - p) + p]\ (q - p)\ dx$$

$$= \int_0^1 L(x,T)\ f(x)\ dx$$

$$= E[L(X,T)]$$

3. The risk function associated with the IBLF has a closed form for all distributions with finite raw moments.

The general form of the risk function for the IBLF is $C \cdot E[X^{l}(1-X)^{n}]$ and can be evaluated for all cases where the moments exist.

In general, the IBLF has a better evaluating form through the raw moments of the process characteristic X. For

$$E[L(X,T)] = \int_{-\infty}^{\infty} K\left(1 - \frac{\pi(x,T)}{m} f(x)dx\right)$$

$$= K - K \int_{-\infty}^{\infty} \frac{\pi(x,T)}{m} f(x)dx$$

$$= K\left[1 - \int_{-\infty}^{\infty} \frac{\pi(x,T)}{m} f(x)dx\right]$$

$$= K\left[1 - C\int_{-\infty}^{\infty} x^{l}(1-x)^{n} f(x)dx\right]$$

$$= K\left[1 - C\sum_{i=0}^{\infty} {}_{n}C_{i}(-1)^{i}\left(\mu'_{l+i}\right)\right]$$

where $C = \left[T(1-T)^{\frac{1-T}{T}}\right]^{1-\alpha}$, and

${}_{n}C_{i}$ is the general binomial coefficient and μ'_{r} is the rth raw moment.

If n is a positive integer then the expectation above has finite number of terms, otherwise it has infinite number of terms.

Notice that this risk function can be evaluated for any process characteristic distribution including normal, gamma, Weibull distributions ... etc. as long as all the raw moments exist.

As another example to show the ease of evaluating IBLF. Let's consider the IBLF if $X \sim U(\alpha_R, \beta_R)$, where $0 < \alpha_R < \beta_R$, then the associated risk function is :

$$E[L(X,T)] = \int_{\alpha_R}^{\beta_R} K\left\{1 - Cx^{\alpha-1}(1-x)^{\frac{1-T}{T}(\alpha-1)}\right\}\frac{1}{\beta_R - \alpha_R}dx$$

$$= K\left\{1 - \frac{C}{\beta_R - \alpha_R}\int_{\alpha_R}^{\beta_R} x^{\alpha-1}(1-x)^{\frac{1-T}{T}(\alpha-1)}dx\right\}$$

let $l = \alpha - 1$, $n = \dfrac{1-T}{T}(\alpha - 1)$, be both positive integers,

and let $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$

$$E[L(X,T)] = K\left\{1 - \frac{2C}{\beta_R - \alpha_R}\int_{\arcsin\sqrt{\alpha_R}}^{\arcsin\sqrt{\beta_R}} \sin^{2l+1}\theta\cos^{2n+1}\theta\, d\theta\right\}$$

$$= K\left\{1 - \frac{2C\sin^{2l+1}\theta}{(\beta_R - \alpha_R)(2n+2l+2)}\left[\cos^{2n}\theta + \sum_{k=1}^{n}\frac{2^k n(n-1)\ldots(n-k+1)\cos^{2n-2k}\theta}{(2n+2l)(2n+2l-2)\ldots(2n+2l-2k+2)}\right]_{\arcsin\sqrt{\alpha_R}}^{\arcsin\sqrt{\beta_R}}\right\}$$

$$= K\left\{1 - \frac{C\sin^{2l+1}\theta}{(\beta_R - \alpha_R)(n+l+1)}\left[\cos^{2n}\theta + \sum_{k=1}^{n}\frac{n(n-1)\ldots(n-k+1)\cos^{2n-2k}\theta}{(n+l)(n+l-1)\ldots(n+l-k+1)}\right]_{\arcsin\sqrt{\alpha_R}}^{\arcsin\sqrt{\beta_R}}\right\}.$$

5.6 An Application of the IBLF

A lottery ticket manufacturer produced tickets that were to be distributed through a vending machine. The tickets were to be folded and stacked in columns within the vending machine and dispensed one at a time through a dispensing slot. After inserting sufficient funds, a ticket would be exposed and the purchaser required to tear the ticket from the dispenser. The vending machine operators identified the critical characteristic in this process as the force required to remove the ticket from the dispenser. This force was directly related to the "pull strength" associated with perforations made on the tickets during manufacturing.

The vending machine operators found that when the pull strength associated with the perforation was above 60 pounds per square inch (psi), tickets would not necessarily break along the perforations, leaving portions of the ticket inside the vending machine. It was also found that in those case where pull strength of the perforation was less than 40 psi, the vending machine tended to supply more than one ticket at a time. This resulted in the vending machine jamming as the next ticket would not feed properly through the mechanism.

In the case of a vending machine jam the company felt the cost to restore the machine to working condition was $0.10 per ticket. If the pull strength went beyond 60 psi, the vending machine was unable to break the perforation cleanly and the loss per ticket was also considered to be $0.10. The manufacturer agreed to compensate the vending machine company on a sliding scale that accurately reflected the costs associated with off-target perforations. Both parties agreed that $0.10 per ticket fairly depicted the costs associated with a complete failure of the perforation and that this occurred when the perforation strengths were outside 40-60 psi interval. In addition they both agreed that the scale must include a \$0.05 per ticket penalty to the manufacturer if the pull strength were 45 psi or 57.5 psi.

Using this information above, an IBLF was ultimately used to reflect the compensatory package for perforations that were off target. In addition the manufacturer was interested in determining their risk exposure under normal operating conditions. Using equation [5.4.3], for $T = 55$, $K = .1$, $x_1 = 45$, $L_1(x_1, T) = .05, \alpha_1 = 1.9464$ and $x_2 = 57.5$, $L_2(x_2, T) = .05, \alpha_2 = 10.0138$, results in the IBLF illustrated in Figure 5.6.1. The original and transformed pull strength data are listed in Tables 5.6.1 and 5.6.2. The pull strength data appears to follow a Be[α_R =2.0994, β_R =2.3184] (verified by the Chi-square Goodness-of-fit test (p-value = .0987)) whose expected loss (evaluated using equation [5.4.5]) is \$0.028.

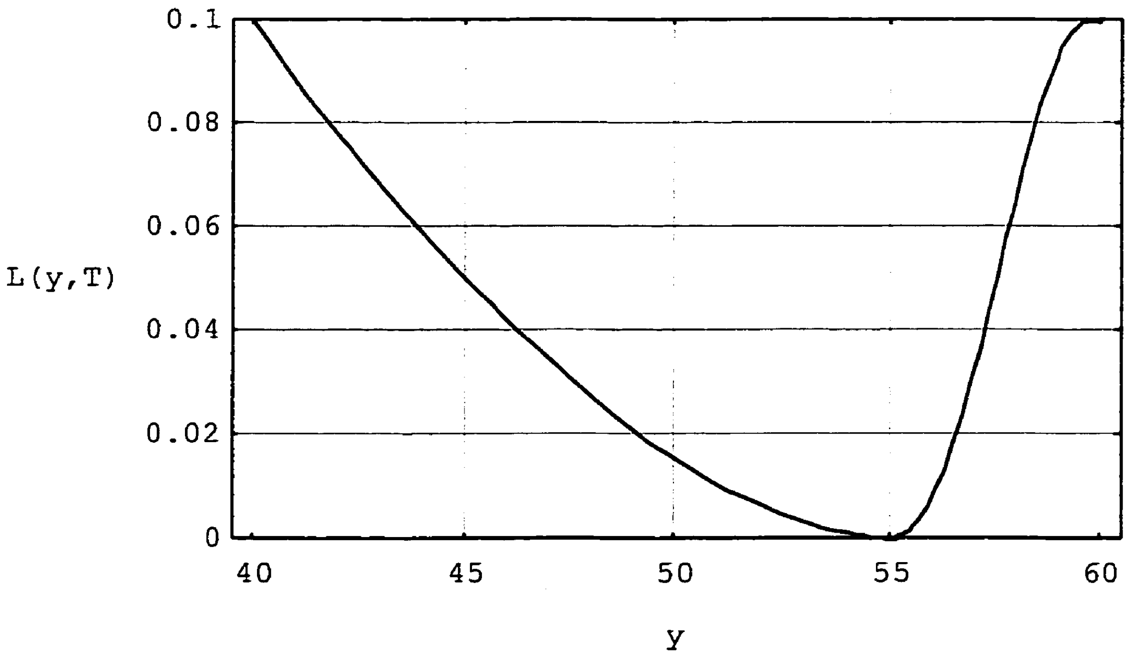


Figure 5.6.1
$L_1(x_1 = 45, T=55, K=.10) = .05, \alpha_1 = 1.9464$ and
$L_2(x_2 = 57.5, T=55, K=.10) = .05, \alpha_2 = 10.0138$

Table 5.6.1 Pull Strength data

40.6	47.6	49.5	52.8	45.0	51.6
48.5	58.3	46.5	53,5	48.9	58.3
42.0	41.0	47.3	47.5	47.0	54.5
47.7	44.8	47.5	47.0	41.7	54.8
42.6	56.5	52.4	55.9	42.2	52.6
50.5	49.7	48.6	58.6	43.7	53.7
47.6	55.0	45.0	54.6	55.0	
46.6	51.8	51.0	46.2	53.8	
56.9	48.6	47.6	44.0	49.4	
53.7	44.2	52.0	44.5	48.1	

Table 5.6.2 The transformed Pull Strength data

.03	.38	.475	.64	.25	.58
.425	.915	.325	.675	.445	.915
.1	.05	.365	.375	.35	.725
.385	.24	.375	.35	.085	.74
.13	.825	.62	.795	.11	.63
.525	.485	.43	.93	.185	.685
.38	.75	.25	.73	.75	
.33	.59	.55	.31	.69	
.845	.43	.38	.20	.47	
.685	.21	.60	.225	.405	

5.7 Comments and Recommendations

The shape of IBLF is easy to construct as the various choices of α for a fixed T allows the IBLF to be tailored to the practitioners' need. In general the relationship that exists among T, α and β suggests that for a fixed T, as α increases, β will increase.

Alternatively, keeping α fixed and increasing T, β will decrease. It can be seen that when T = 1/2, this may not necessary be that $\alpha = \beta$ and possessing symmetric shape of loss function. We can make use of equations [5.4.4] or [5.4.6] to adjust the loss function properly and having asymmetric shape. Figure 5.7.1 illustrates two IBLFs with T = 1/2 but $\alpha \neq \beta$ and various values of K.

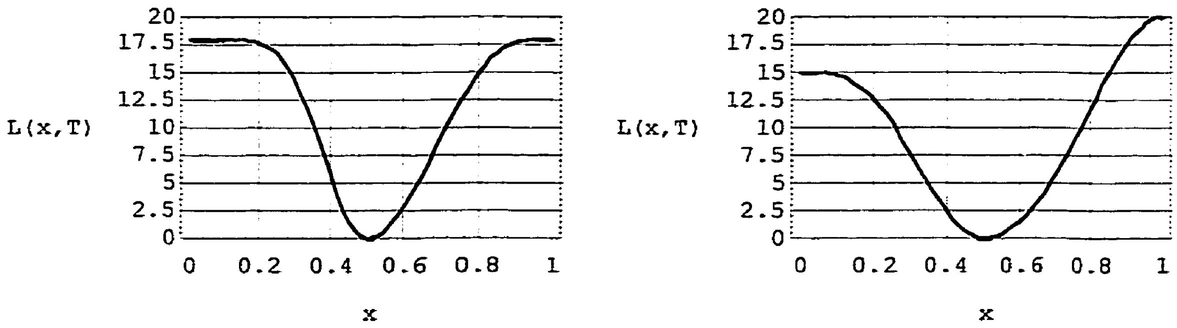


Figure 5.7.1

$L_1(x, K_1 = 18, \alpha_1 = 10)$, $L_2(x, K_2 = 18, \alpha_2 = 5)$ and
$L_1(x, K_1 = 15, \alpha_1 = 5)$, $L_2(x, K_2 = 20, \alpha_2 = 3)$

The general form of the IBLF admits a closed form risk function for those distributions having finite moments. The expectation of L(x, T) can be obtained easily even if the loss function is a combination of two different loss functions due to the nice form this IBLF possesses. In particular, if the beta distribution is employed as the conjugate distribution, equations [5.4.5] and [5.4.7] are the solutions. If the process measurements follow a distribution of the form of a normal, gamma, Weibull, ... etc., the risk function can still be found.

Under linear transformation the IBLF and its risk function are the same as those IBLFs associated with the standard beta pdf. This follows directly from the result that a generalized beta distribution can always be transformed to a standard beta distribution.

In general, the scale invariant nature of the IBLF and its associated risk function under linear transformation holds for any distribution having a unique maximum. These can be shown as follows:

Let $X \sim N(\mu,\sigma^2)$, its pdf has a unique maximum at $x = \mu$ and let it be T.

Then $\pi(x,T) = \dfrac{\exp\left[-\dfrac{(x-T)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\,\sigma}$ and its maximum is $m = \dfrac{1}{\sqrt{2\pi}\,\sigma}$. Hence

$$\frac{\pi(x,T)}{m} = \exp\left[-\frac{(x-T)^2}{2\sigma^2}\right].$$

Now, let $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$, where a and b are constants, its pdf g(y) has a unique maximum at $y = a\mu + b$ and let it be T'. Then $\pi(y,T') = \dfrac{\exp\left[-\dfrac{(y-T')^2}{2a^2\sigma^2}\right]}{\sqrt{2\pi}\,a\sigma}$ and its maximum is $m' = \dfrac{1}{\sqrt{2\pi}\,a\sigma}$. Hence the loss inversion ratio, after simplification, is

$$\frac{\pi(y,T')}{m'} = \exp\left[-\left(\frac{y-T'}{a}\right)^2 \frac{1}{2\sigma^2}\right], \text{ for } \frac{y-T'}{a} = x - T$$

$$= \exp\left[-\frac{(x-T)^2}{2\sigma^2}\right]$$

$$= \frac{\pi(x,T)}{m}.$$

Following this approach and if $X \sim G(\alpha,\beta)$ with pdf $f(x) = \dfrac{x^{\alpha-1}\exp[-\frac{x}{\beta}]}{\Gamma(\alpha)\beta^{\alpha}}$ with a unique maximum at $x = \beta(\alpha-1) = T$, $\alpha > 1$. Then

$$\pi(x,T) = \frac{x^{\alpha-1}\exp[-\frac{x(\alpha-1)}{T}]}{\Gamma(\alpha)\left(\frac{T}{\alpha-1}\right)^{\alpha}}$$

and

$$m = \frac{\left(\frac{T}{e}\right)^{\alpha-1}}{\Gamma(\alpha)\left(\frac{T}{\alpha-1}\right)^{\alpha}}$$

with loss inversion ratio

$$\frac{\pi(x,T)}{m} = \left(\frac{xe}{T}\right)^{\alpha-1}\exp[-\frac{x(\alpha-1)}{T}].$$

Then, if $Y = aX + b$, Y will possess a three-parameter Gamma distribution with location parameter b, scale parameter $a\beta$ and shape parameter α, with the pdf

$$g(y) = \frac{\left(\frac{y-b}{a}\right)^{\alpha-1}\exp\left[-\frac{y-b}{a\beta}\right]}{\Gamma(\alpha)\beta^{\alpha}a}$$

which has a unique maximum at $y = a\beta(\alpha-1) + b = T'$. Thus

$$\pi(y,\ T') = \frac{\left(\frac{y-b}{a}\right)^{\alpha-1}\exp\left[-\frac{(y-b)(\alpha-1)}{T'-b}\right]}{\Gamma(\alpha)\left[\frac{T'-b}{a(\alpha-1)}\right]^{\alpha}a},$$

and $$m' = \frac{[\beta(\alpha-1)]^{\alpha-1}\exp[-(\alpha-1)]}{\Gamma(\alpha)\left[\frac{T'-b}{a(\alpha-1)}\right]^{\alpha}} = \frac{\left[\frac{\beta(\alpha-1)}{e}\right]^{\alpha-1}}{\Gamma(\alpha)\left[\frac{T'-b}{a(\alpha-1)}\right]^{\alpha}},$$

hence the loss inversion ratio, after simplification, is

$$\frac{\pi(y,T')}{m'} = \left[\frac{(y-b)e}{a\beta(\alpha-1)}\right]^{\alpha-1}\exp\left[-\frac{(y-b)(\alpha-1)}{T'-b}\right], \text{ for } \frac{y-b}{a}=x,\ \frac{T'-b}{a}=T=\beta(\alpha\text{-}1)$$

$$= \left(\frac{xe}{T}\right)^{\alpha-1}\exp[-\frac{x(\alpha-1)}{T}]$$

$$= \frac{\pi(x,T)}{m}.$$

Similarly, if X ~ W(c,d) having the pdf $f(x) = cdx^{d-1}\exp[-cx^{d}]$ and X has a unique maximum at $x = \left(\frac{cd}{d-1}\right)^{\frac{1}{d}} = T,\ d > 1$. Then

$$\pi(x,T) = (d\text{-}1)T^{d}x^{d-1}\exp\left[-\frac{d-1}{d}(xT)^{d}\right]$$

and $$m = cdT^{\frac{d-1}{d}}\exp\left[-\frac{c^{2}d}{d-1}\right] = cdT^{\frac{d-1}{d}}\exp\left[-cT^{d}\right]$$

with loss inversion ratio

$$\frac{\pi(x,T)}{m} = \left(\frac{x}{T}\right)^{d-1}\exp\left\{-\left[\frac{d-1}{d}(xT)^{d} - cT^{d}\right]\right\}.$$

For Y = aX + b, Y will have a three-parameter Weibull distribution with location parameter b, scale parameter a and shape parameter c, and its pdf is

$$g(y) = cd\left(\frac{y-b}{a}\right)^{d-1}\exp\left[-c\left(\frac{y-b}{a}\right)^{d}\right]$$

g(y) has a unique maximum at $y = a\left(\frac{cd}{d-1}\right)^{\frac{1}{d}} + b$. Its loss inversion ratio is

$$\frac{\pi(y,T')}{m'} = \left[\frac{y-b}{a}\left(\frac{d-1}{cd}\right)^{\frac{1}{d}}\right]^{d-1} \exp\left\{-\left\{\frac{d-1}{d}\left[\frac{(T'-b)(y-b)}{a^2}\right]^d - \frac{c^2 d}{d-1}\right\}\right\}$$

$$= \left(\frac{x}{T}\right)^{d-1} \exp\left\{-\left[\frac{d-1}{d}(xT)^d - cT^d\right]\right\}, \text{ for } \frac{y-b}{a} = x,\ \left(\frac{T'-b}{a}\right)^d = \frac{cd}{d-1} = T^d$$

$$= \frac{\pi(x,T)}{m}.$$

Hence the loss function and risk function follow.

There are some limitations of this IBLF when the unique maximum conditions do not hold. For example, taking $\alpha = 1$ and $\beta = 1$ with any target value T, the loss will be zero over the range (0, 1) when standard beta distribution is concerned. It is unrealistic to have zero loss between the two specification limits.

To conclude, the applications of this proposed inverted beta loss function is not limited to industry as they can be used in any application where reflecting economic or monetary loss to the company or to the society is of interest.

Chapter 6

Some Properties of Inverted Probability Loss Function

6.1 Introduction

Loss functions have been studied for several decades and have been widely used for various purposes such as business decision making, quality assurance and reliability settings. Taguchi (1986) used a quadratic loss function to motivate and illustrate losses to society associated with departures from the target in industrial applications. Spiring (1993) modified this loss function approach using an inverted normal probability density function which provided a reasonable assessment of loss. Spiring and Yeung (1998) developed a class of loss functions based on inverting various pdfs including gamma, Tukey's Symmetric Lambda and Laplace distributions which not only provided the traditional properties of loss functions but also emphasized the asymmetric loss cases.

Loss functions are used to quantify losses associated with deviation from a desired target value in both decision theory applications and quality assurance settings. In decision theory, loss is generally defined as a nonnegative function of the deviation of an estimator from the parameter value to be estimated. In quality assurance settings, loss functions are used to reflect the economic loss associated with variation about, and deviations from, the process target or the target value of a process characteristic.

In decision theory, traditional loss functions usually take forms such as quadratic loss, absolute loss, step loss and the generalized loss (which includes the quadratic and absolute losses as special cases) and possess nice properties such as boundedness, invariance under linear transformation, and closed under sampling in prior-posterior analysis, ... etc. In this chapter, we investigate several statistical properties associated with the family of Inverted Probability Loss Functions. We have found IPLFs do have all these nice properties that the traditional loss functions possess, and in addition IPLFs are more flexible in expressing the economic consequences associated with deviation from target as long as the selected probability density function possessing a unique maximum.

6.2 Basis of Inverted Probability Loss Function

The proposed general class of loss functions is based on the inversion of common probability density functions. This family of loss functions satisfies the criteria that the loss must be non-negative, is zero worth at the target value, is monotonically increasing as the process drifts from either side of target, and attains a quantifiable maximum near the lower and/or upper specification limits of the process. In this section we are going to develop the basis of this family of loss functions.

Let f(x) be a probability density function possessing a unique maximum at x. Let T = x be the value of this unique maximum, where T is the target value. Let $\pi(x,T) =$ f(x), which is in terms of x and T, use in creating the economic loss function for the process of interest, then $m = \sup_x f(x) = f(T)$. So that the IPLF takes the form

$$L(x,T)=K\left[1-\frac{\pi(x,T)}{m}\right] \qquad [6.2.1]$$

where K is the maximum loss incurred when the target is not attained and $\frac{\pi(x,T)}{m}$ is the Loss Inversion Ratio.

It can be seen from the structure of equation [6.2.1] that $\pi(x,T)$ is the form of a pdf in terms of x and T, m is the supremum (or the maximum) of $\pi(x,T)$, the ratio $\frac{\pi(x,T)}{m}$, named the loss inversion ratio which has no unit and has a minimum value of zero when x takes on values far from the target value T, and a maximum value of one when x is exactly on target, i.e., $0 \leq \frac{\pi(x,T)}{m} \leq 1$. One property of the LIR with respect to 1 is the true percentage of the x values that are missed with respect to the target, and hence this percentage loss, $1 - \frac{\pi(x,T)}{m}$, is the penalty it pays subject to the maximum loss amount incurred in the process. The concept of percent defect (percentage of products that lie within specification limits but deviate from target) has been widely used as a measure of the quality level of process characteristics of manufactured products. Usually the percentage of defective products in shipped goods is small. There are a number of properties this LIR possesses, in this section we are going to develop and examine these properties, and summarized in the following theorems.

Theorem 6.2.1:

The rth expectation of LIR is bounded between 0 and 1, i.e., $0 \leq E\left\{\left[\frac{\pi(x,T)}{m}\right]^r\right\} \leq 1$.

Proof:

For $0 \leq \frac{\pi(x,T)}{m} \leq 1$, which implies $0 \leq \left[\frac{\pi(x,T)}{m}\right]^r \leq 1$, then multiply the inequality by $f_R(x)$, the density of the process characteristic, and integrate over the space of X. Thus $0 \leq E\left\{\left[\frac{\pi(x,T)}{m}\right]^r\right\} \leq 1$.

Theorem 6.2.2:

The variance of LIR is

$$V\left[\frac{\pi(X,T)}{m}\right] = E\left\{\left[\frac{\pi(X,T)}{m}\right]^2\right\} - \left\{E\left[\frac{\pi(X,T)}{m}\right]\right\}^2. \qquad [6.2.2]$$

Theorem 6.2.3:

The variance of LIR is bounded between 0 and 1, i.e., $0 \leq V\left[\frac{\pi(X,T)}{m}\right] \leq 1$.

Proof:

From equation [6.2.2], we know that $E\left\{\left[\frac{\pi(X,T)}{m}\right]^2\right\} - \left\{E\left[\frac{\pi(X,T)}{m}\right]\right\}^2 \geq 0$ and, from Theorem 6.2.1 that the maximum value of the second moment of LIR is 1 and the minimum value of square of the first expectation is 0. Therefore $0 \leq V\left[\frac{\pi(X,T)}{m}\right] \leq 1$.

6.3 Properties of IPLF

In manufacturing, loss functions usually comprehend the economic consequences associated with deviations from target regardless of how small the deviation is. Since different processes have different sets of economic consequences, a flexible approach to developing loss functions is desirable. Taguchi's modified quadratic loss, the INLF (Spiring (1993)), and Sun, Laramee, and Ramberg's (1996) refinement are flexible, but do not cover the spectrum of potential loss functions.

The IPLF defined in equation [6.2.1] can reflect losses arising from processes with observations not on target. It also has properties inherent to the structure of the function. They can be expressed as follows:

1. The general form of the Risk Function for IPLFs is

$$E[L(X,T)] = E\left\{K\left[1-\frac{\pi(X,T)}{m}\right]\right\} = K\left\{1-E\left[\frac{\pi(X,T)}{m}\right]\right\}. \qquad [6.3.1]$$

It can be evaluated either directly taking expectation of $\pi(x,T)$

$$E[\pi(X,T)] = \int_{-\infty}^{\infty} \pi(x,T)\, f_R(x)\, dx \qquad [6.3.2]$$

or through the use of expectation of LIR

$$E\left[\frac{\pi(X,T)}{m}\right] = \int_{-\infty}^{\infty} \frac{\pi(x,T)}{m} f_R(x)\, dx \qquad [6.3.3]$$

where $f_R(x)$ is the distribution of the process characteristic, or a conjugate distribution.

Similarly, the higher moments can be obtained

$$E\left[[\pi(X,T)]^r\right] = \int_{-\infty}^{\infty} [\pi(x,T)]^r f_R(x)\, dx, \qquad r=0,1,2,\ldots,k \qquad [6.3.4]$$

or

$$E\left[\left[\frac{\pi(X,T)}{m}\right]^r\right] = \int_{-\infty}^{\infty} \left[\frac{\pi(x,T)}{m}\right]^r f_R(x)\, dx, \qquad r=0,1,2,\ldots,k \qquad [6.3.5]$$

provided that the expectations exist.

Theorem 6.3.1:

The IPLF is bounded between 0 and K, i.e., $0 \le E[L(X,T)] \le K$, where K is the maximum loss incurred when the target is missed.

Proof:

For
$$0 \le \frac{\pi(x,T)}{m} \le 1$$

$$0 \le 1 - \frac{\pi(x,T)}{m} \le 1$$

$$0 \le K\left[1 - \frac{\pi(x,T)}{m}\right] \le K$$

$$0 \le \int_{-\infty}^{\infty} K\left[1 - \frac{\pi(x,T)}{m}\right] f_R(x)dx \le \int_{-\infty}^{\infty} K\, f_R(x)dx$$

$$0 \le E[L(X,T)] \le K.$$

2. The variance of L(x, T) arises from using $f_R(x)$ as the process characteristic distribution is given by

$$V[L(X,T)] = K^2\left[E\left\{\left[\frac{\pi(X,T)}{m}\right]^2\right\} - \left\{E\left[\frac{\pi(X,T)}{m}\right]\right\}^2\right] \qquad [6.3.6]$$

Theorem 6.3.2:

The variance of an IPLF is K^2 times the variance of LIR. Hence it is bounded between 0 and K^2, i.e., $0 \le V[L(X,T)] \le K^2$.

Proof:

$$V[L(X,T)] = E\left\{[L(X,T)]^2\right\} - \left\{E[L(X,T)]\right\}^2$$

$$= E\left\{K^2\left[1-\frac{\pi(X,T)}{m}\right]^2\right\} - K^2\left\{1-E\left[\frac{\pi(X,T)}{m}\right]\right\}^2$$

$$=K^2\left[1-2E\left[\frac{\pi(X,T)}{m}\right]+E\left\{\left[\frac{\pi(X,T)}{m}\right]^2\right\}-1+2E\left[\frac{\pi(X,T)}{m}\right]-\left\{E\left[\frac{\pi(X,T)}{m}\right]\right\}^2\right]$$

$$= K^2\left[E\left\{\left[\frac{\pi(X,T)}{m}\right]^2\right\} - \left\{E\left[\frac{\pi(X,T)}{m}\right]\right\}^2\right].$$

For $0 \le V[L(X,T)] \le K^2$, follows from Theorem 6.2.3.

3. The Loss Inversion Ratio is scale invariant under linear transformation.
4. The shape of IPLF is scale invariant under linear transformation.
5. The Loss Function is scale invariant under linear transformation.
6. The Risk Function is scale invariant under linear transformation.

Theorem 6.3.3:

Let $\pi(x, T)$ be a continuous pdf denoted by $f(x)$, having a unique maximum at $x = T$. Then under any linear transformation $Y = a + bX$, $b \neq 0$, the following are scale invariant :

(1) the Loss Inversion Ratio, i.e., $\frac{\pi(y,T')}{m'} = \frac{\pi(x,T)}{m}$;

(2) the shape of IPLF;

(3) the IPLF;

(4) the Risk Function.

Proof :

Given that $\pi(x, T)$ is a continuous pdf denoted by f(x), then the linear transformation $Y = a + bX$ has pdf

$$g(y) = f(x)|J|$$

$$= f\left(\frac{y-a}{b}\right)\left|\frac{dx}{dy}\right|$$

$$= f\left(\frac{y-a}{b}\right)\frac{1}{b}, \qquad -\infty < y < \infty$$

$$\Rightarrow \qquad b\,g(y) = f\left(\frac{y-a}{b}\right) = f(x)$$

and

$$\sup_{x \in X} f(x) = b \sup_{y \in Y} g(y)$$

$$= b \sup_{y \in Y} f\left(\frac{y-a}{b}\right)\frac{1}{b}$$

$$= \sup_{x \in X} f(x)$$

$$\Rightarrow \qquad T' = a + bT, \quad \text{and} \quad bm' = m.$$

Let $g(y) = \pi(y, T')$, then its LIR is

$$\frac{\pi(y,T')}{m'} = \frac{g(y)}{\frac{m}{b}} = \frac{b\,g(y)}{m} = \frac{f(x)}{m} = \frac{\pi(x,T)}{m}.$$

This proves (1).

It follows from (1) that $L(y,T') = K\left[1 - \frac{\pi(y,T')}{m'}\right] = K\left[1 - \frac{\pi(x,T)}{m}\right]$, hence (2) and (3) follow directly.

From (3), it follows

$$E[L(Y,T')] = E\left\{K\left[1-\frac{\pi(Y,T')}{m'}\right]\right\}$$

$$= K\left\{1-E\left[\frac{\pi(Y,T')}{m'}\right]\right\}$$

$$= K\left\{1-E\left[\frac{\pi(X,T)}{m}\right]\right\}$$

$$= E[L(X,T)]$$

this completes the proof.

6.4 The Selection of IPLFs

Different choices of IPLFs can reveal different levels of penalties for similar deviations from a target. Similarly, different process characteristics (conjugate distributions) with suitable choice of IPLF can succintly reflect the correct loss incurred by practitioners and hence to society. Investigation of several IPLFs with appropriate conjugate distribution are examined. How to evaluate the quality level of products shipped to consumers is the problem of concern. We introduce a monetory evaluation of the quality of products, assuming that the tolerances are correct and the process measurements are in-control. In the subsequent subsections, selected loss functions with plausible conjugate distributions/statistical distributions associated with the process measurements are studied and compared in order to provide more information for the practitioners' selection.

6.4.1 The Inverted Normal Loss Function

Let $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$,which has a maximum at $x = \mu$ and let it be T.

Hence
$$\pi(x,T)=\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-T)^2}{2\sigma^2}\right]$$

and
$$m=\sup_{x\in X} f(x)=\frac{1}{\sqrt{2\pi}\sigma}$$

with LIR
$$\frac{\pi(x,T)}{m}=\exp\left[-\frac{(x-T)^2}{2\sigma^2}\right] \qquad [6.4.1.1]$$

and hence the IPLF is
$$L(x,T)=K\left\{1-\exp\left[-\frac{(x-T)^2}{2\sigma^2}\right]\right\}. \qquad [6.4.1.2]$$

It can be seen from equation [6.4.1.1] that the larger the σ^2, the smaller the LIR and hence the larger the INLF. The Risk Function associated with INLF, assuming a conjugate distribution of $N(\mu_R,\sigma_R^2)$, using equations [6.3.1] and [6.3.3] is

$$E[L(X,T)]=\int_{-\infty}^{\infty} K\left\{1-\exp\left[-\frac{(x-T)^2}{2\sigma^2}\right]\right\}\frac{1}{\sqrt{2\pi}\,\sigma_R}\exp\left[-\frac{(x-\mu_R)^2}{2\sigma_R^2}\right]dx$$

$$=K\left\{1-\int_{-\infty}^{\infty}\frac{\exp\left\{-\frac{1}{2}\left[\frac{(x-T)^2}{\sigma^2}+\frac{(x-\mu_R)^2}{\sigma_R^2}\right]\right\}}{\sqrt{2\pi}\,\sigma_R}dx\right\}$$

$$=K\left\{1-\frac{\sigma}{\sqrt{\sigma^2+\sigma_R^2}}\exp\left[-\frac{(\mu_R-T)^2}{2(\sigma^2+\sigma_R^2)}\right]\right\}. \qquad [6.4.1.3]$$

Theorem 6.4.1.1 :

The rth moment of LIR associated with the INLF when the conjugate distribution is $N\left(\mu_R, \sigma_R^2\right)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}=\frac{\sigma}{\sqrt{\sigma^2+r\sigma_R^2}}\exp\left[-\frac{r(\mu_R-T)^2}{2\left(\sigma^2+r\sigma_R^2\right)}\right],\ r=0,1,2,\ldots,k \qquad [6.4.1.4]$$

provided the expectations exist.

Proof :

For the rth moment of LIR of INLF with respect to a normal conjugate distribution is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}=E\left\{\exp\left[-r\frac{(x-T)^2}{2\sigma^2}\right]\right\}$$

$$=\int_{-\infty}^{\infty}\exp\left[-r\frac{(x-T)^2}{2\sigma^2}\right]\frac{1}{\sqrt{2\pi}\,\sigma_R}\exp\left[-\frac{(x-\mu_R)^2}{2\sigma_R^2}\right]dx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma_R}\exp\left\{-\frac{1}{2}\left[r\frac{(x-T)^2}{\sigma^2}+\frac{(x-\mu_R)^2}{\sigma_R^2}\right]\right\}dx$$

Now, to complete the square in the exponent and it is

$$r\frac{(x-T)^2}{\sigma^2}+\frac{(x-\mu_R)^2}{\sigma_R^2}=\frac{\sigma^2+r\sigma_R^2}{\sigma^2\sigma_R^2}\left\{\left(x-\frac{rT\sigma_R^2+\mu_R\sigma^2}{\sigma^2+\sigma_R^2}\right)^2+\frac{r\sigma^2\sigma_R^2(\mu_R-T)^2}{\left(\sigma^2+r\sigma_R^2\right)^2}\right\}.$$

Then the expectation becomes

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}$$

$$=\int_{-\infty}^{\infty}\frac{\frac{\sigma\sigma_R}{\sqrt{\sigma^2+r\sigma_R^2}}}{\sqrt{2\pi}\sigma_R\frac{\sigma\sigma_R}{\sqrt{\sigma^2+r\sigma_R^2}}}\exp\left\{-\frac{1}{2\frac{\sigma^2\sigma_R^2}{\sigma^2+r\sigma_R^2}}\left[\left(x-\frac{rT\sigma_R^2+\mu_R\sigma^2}{\sigma^2+r\sigma_R^2}\right)^2+\frac{r\sigma^2\sigma_R^2(\mu_R-T)^2}{\left(\sigma^2+r\sigma_R^2\right)^2}\right]\right\}dx$$

$$=\frac{\sigma}{\sqrt{\sigma^2+r\sigma_R^2}}\exp\left[-\frac{r(\mu_R-T)^2}{2\left(\sigma^2+r\sigma_R^2\right)}\right].$$

Using equation [6.3.6] and Theorem 6.4.1.1, the variation of INLF arises from using the conjugate distribution of $N\left(\mu_R,\sigma_R^2\right)$, is

$$V[L(X,T)]=K^2\left\{\frac{\sigma}{\sqrt{\sigma^2+2\sigma_R^2}}\exp\left[-\frac{(\mu_R-T)^2}{\sigma^2+2\sigma_R^2}\right]-\frac{\sigma^2}{\sigma^2+\sigma_R^2}\exp\left[-\frac{(\mu_R-T)^2}{\sigma^2+\sigma_R^2}\right]\right\}. \quad [6.4.1.5]$$

Theorem 6.4.1.2:

The rth moment of LIR associated with the INLF when the process characteristic distribution is $U(\alpha_R,\beta_R)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}=\sqrt{\frac{\pi}{2r}}\frac{\sigma}{(\beta_R-\alpha_R)}\left[\operatorname{erf}\left(\sqrt{\frac{r}{2}}\frac{(T-\alpha_R)}{\sigma}\right)-\operatorname{erf}\left(\sqrt{\frac{r}{2}}\frac{(\beta_R-T)}{\sigma}\right)\right], r=1, 2, \ldots, k.$$

$$=1 \qquad\qquad r=0. \quad [6.4.1.6]$$

provided the expectations exist.

Thus, the mean and variance of INLF for the uniform process characteristic distribution are respectively

$$E[L(X,T)]=K\left\{1-\sqrt{\frac{\pi}{2}}\frac{\sigma}{(\beta_R-\alpha_R)}\left[\operatorname{erf}\left(\frac{T-\alpha_R}{\sqrt{2}\sigma}\right)-\operatorname{erf}\left(\frac{\beta_R-T}{\sqrt{2}\sigma}\right)\right]\right\} \qquad [6.4.1.7]$$

$$V[L(X,T)]=\frac{K^2\sqrt{\pi}\sigma}{2(\beta_R-\alpha_R)}\left\{\left[\operatorname{erf}\left(\frac{T-\alpha_R}{\sigma}\right)-\operatorname{erf}\left(\frac{\beta_R-T}{\sigma}\right)\right]\right.$$

$$\left.-\frac{\sqrt{\pi}\sigma}{(\beta_R-\alpha_R)}\left[\operatorname{erf}\left(\frac{T-\alpha_R}{\sqrt{2}\sigma}\right)-\operatorname{erf}\left(\frac{\beta_R-T}{\sqrt{2}\sigma}\right)\right]^2\right\}. \qquad [6.4.1.8]$$

Theorem 6.4.1.3 :

The rth moment of LIR associated with the INLF when the process characteristic distribution is $G(\alpha_R, \beta_R)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}$$

$$=\frac{2\left(\frac{\sigma}{\sqrt{2r}\beta_R}\right)^{\alpha_R+1}\Gamma\left(\frac{1}{2}\right)\exp\left[-\frac{rT^2}{2\sigma^2}\right]}{\sigma^2\,\Gamma\left(\frac{\alpha_R}{2}\right)\Gamma\left(\frac{\alpha_R+1}{2}\right)}\left\{\sqrt{\frac{r}{2}}\beta_R\sigma\Gamma\left(\frac{\alpha_R}{2}\right){}_1F_1\left[\frac{\alpha_R}{2},\frac{1}{2},\left(\frac{\sigma^2-rT}{\sqrt{2r}\beta_R\sigma^2}\right)^2\right]\right.$$

$$\left.+\left(r\beta_R T-\sigma^2\right)\Gamma\left(\frac{\alpha_R+1}{2}\right){}_1F_1\left[\frac{\alpha_R+1}{2},\frac{3}{2},\left(\frac{\sigma^2-rT}{\sqrt{2r}\beta_R\sigma^2}\right)^2\right]\right\}, r=0, 1, 2, \ldots, k. \qquad [6.4.1.9]$$

provided the expectations exist.

The mean and variance of INLF for the gamma distribution are respectively

$$E\left[\frac{\pi(X,T)}{m}\right]$$

$$= K\left\{1 - \frac{2\left(\dfrac{\sigma}{\sqrt{2}\beta_R}\right)^{\alpha_R+1}\Gamma\left(\dfrac{1}{2}\right)\exp\left[-\dfrac{T^2}{2\sigma^2}\right]}{\sigma^2\,\Gamma\left(\dfrac{\alpha_R}{2}\right)\Gamma\left(\dfrac{\alpha_R+1}{2}\right)}\left\{\frac{\beta_R\sigma}{\sqrt{2}}\Gamma\left(\frac{\alpha_R}{2}\right){}_1F_1\left[\frac{\alpha_R}{2},\frac{1}{2},\left(\frac{\sigma^2-T}{\sqrt{2}\beta_R\sigma^2}\right)^2\right]\right.\right.$$

$$\left.\left.+\left(\beta_R T-\sigma^2\right)\Gamma\left(\frac{\alpha_R+1}{2}\right){}_1F_1\left[\frac{\alpha_R+1}{2},\frac{3}{2},\left(\frac{\sigma^2-T}{\sqrt{2}\beta_R\sigma^2}\right)^2\right]\right\}\right\}$$

[6.4.1.10]

$$
\begin{aligned}
&V[L(X,T)] \\
&= 2\Gamma\left(\frac{1}{2}\right)K^2 \Bigg(\frac{\left(\frac{\sigma}{2\beta_R}\right)^{\alpha_R+1} \exp\left[-\frac{T^2}{\sigma^2}\right]}{\sigma^2\,\Gamma\left(\frac{\alpha_R}{2}\right)\Gamma\left(\frac{\alpha_R+1}{2}\right)} \Bigg\{ \beta_R\sigma\,\Gamma\left(\frac{\alpha_R}{2}\right){}_1F_1\left[\frac{\alpha_R}{2},\frac{1}{2},\left(\frac{\sigma^2-2T}{2\beta_R\sigma^2}\right)^2\right] \\
&\qquad + \left(2\beta_R T-\sigma^2\right)\Gamma\left(\frac{\alpha_R+1}{2}\right){}_1F_1\left[\frac{\alpha_R+1}{2},\frac{3}{2},\left(\frac{\sigma^2-2T}{2\beta_R\sigma^2}\right)^2\right] \Bigg\} \\
&\quad -2\Gamma\left(\frac{1}{2}\right) \Bigg[\frac{\left(\frac{\sigma}{\sqrt{2}\beta_R}\right)^{\alpha_R+1} \exp\left[-\frac{T^2}{2\sigma^2}\right]}{\sigma^2\,\Gamma\left(\frac{\alpha_R}{2}\right)\Gamma\left(\frac{\alpha_R+1}{2}\right)} \Bigg\{ \frac{\beta_R\sigma}{\sqrt{2}}\Gamma\left(\frac{\alpha_R}{2}\right){}_1F_1\left[\frac{\alpha_R}{2},\frac{1}{2},\left(\frac{\sigma^2-T}{\sqrt{2}\beta_R\sigma^2}\right)^2\right] \\
&\qquad + \left(\beta_R T-\sigma^2\right)\Gamma\left(\frac{\alpha_R+1}{2}\right){}_1F_1\left[\frac{\alpha_R+1}{2},\frac{3}{2},\left(\frac{\sigma^2-T}{\sqrt{2}\beta_R\sigma^2}\right)^2\right] \Bigg\} \Bigg]^2 \Bigg). \qquad [6.4.1.11]
\end{aligned}
$$

For the expectation and variance of INLF when the process characteristic distribution is $Be(\alpha_R, \beta_R)$, we need to expand the INLF into a series and then perform term by term integration to approximate these expected values.

The normal pdf provides the basis for a variety of IPLFs, all with the familiar inverted symmetric bell shape. By varying T and σ^2, quality practitioners can customize a loss function in order to accurately depict losses associated with process departures from the target. The risk function associated with INLF can be evaluated for most distributions such as uniform, gamma and beta that the process measurements may follow. However, difficulties may arise in determining the functional form of the risk function for some distributions, for examples, Weibull ($W(a_R, b_R)$, with $b_R > 2$) and lognormal. These associated risk functions will result in a complex number. Hence, the conjugate distribution for the INLF appears to be the normal distribution.

6.4.2 The Inverted Gamma Loss Function

Letting $\pi(x,T)$ take the functional form of the gamma distribution allows us to expand the class of IPLFs. The shape of the corresponding loss function will be different from the INLF as $\pi(x,T)$ will now be asymmetric around the target. The gamma distribution will form the basis for a group of loss functions that can be used to represent processes with continuous asymmetric loss. The IGLF is developed as follows:

Let $f(x) = \dfrac{x^{\alpha-1}\exp\left(-\dfrac{x}{\beta}\right)}{\Gamma(\alpha)\beta^{\alpha}}$ with a unique maximum at $x = \beta(\alpha-1)$, $\alpha > 1$, and let it be T. Hence

$$\pi(x,T)=\frac{x^{\alpha-1}\exp\left[-\frac{x(\alpha-1)}{T}\right]}{\Gamma(\alpha)\left(\frac{T}{\alpha-1}\right)^{\alpha}}$$

and $$m=\sup_{x\in X} f(x)=\frac{\left(\frac{T}{e}\right)^{\alpha-1}}{\Gamma(\alpha)\left(\frac{T}{\alpha-1}\right)^{\alpha}}$$

with LIR $$\frac{\pi(x,T)}{m}=\left[\frac{e}{T}x\exp\left(-\frac{x}{T}\right)\right]^{\alpha-1}, \qquad [6.4.2.1]$$

thus, the IGLF is $$L(x,T)=K\left\{1-\left[\frac{e}{T}x\exp\left(-\frac{x}{T}\right)\right]^{\alpha-1}\right\}. \qquad [6.4.2.2]$$

It can be seen from equation [6.4.2.1] that if α increases, the LIR will decrease, thus the IGLF in equation [6.4.2.2] will increase.

Theorem 6.4.2.1:

The rth moment of LIR associated with the IGLF, when the conjugate distribution is $G(\alpha_R,\beta_R)$, is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}=\frac{\exp[r(\alpha-1)]\,\Gamma[\alpha_R+r(\alpha-1)]\beta_R^{r(\alpha-1)}T^{\alpha_R}}{\Gamma(\alpha_R)[r\beta_R(\alpha-1)+T]^{\alpha_R+r(\alpha-1)}},\quad r=0,1,2,\dots,k. \qquad [6.4.2.3]$$

provided the expectations exist.

Proof:

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}=\int_0^\infty\left[\frac{e}{T}x\exp\left(-\frac{x}{T}\right)\right]^{r(\alpha-1)}\frac{x^{\alpha_R-1}\exp\left(-\frac{x}{\beta_R}\right)}{\Gamma(\alpha_R)\beta_R^{\alpha_R}}dx$$

$$= \frac{\exp[r(\alpha-1)]}{\Gamma(\alpha_R)\beta_R^{\alpha_R} T^{r(\alpha-1)}} \int_0^\infty x^{[\alpha_R + r(\alpha-1)]-1} \exp\left[-\frac{x}{\frac{\beta_R T}{r\beta_R(\alpha-1)+T}}\right] dx$$

$$= \frac{\exp[r(\alpha-1)]\,\Gamma[\alpha_R + r(\alpha-1)]\beta_R^{r(\alpha-1)} T^{\alpha_R}}{\Gamma(\alpha_R)[r\beta_R(\alpha-1)+T]^{\alpha_R + r(\alpha-1)}}, \quad r=0,1,2,\ldots,k,$$

provided the expectations exist.

The mean and variance of IGLF for the gamma distribution are respectively

$$E[L(X,T)] = K\left(1 - \frac{\exp(\alpha-1)\Gamma[\alpha_R+\alpha-1]\beta_R^{\alpha-1} T^{\alpha_R}}{\Gamma(\alpha_R)[\beta_R(\alpha-1)+T]^{\alpha_R+\alpha-1}}\right) \qquad [6.4.2.4]$$

$$V[L(X,T)] = \frac{K^2 \exp[2(\alpha-1)]\beta_R^{2(\alpha-1)} T^{\alpha_R}}{\Gamma(\alpha_R)} \cdot$$

$$\left\{\frac{\Gamma[\alpha_R + 2(\alpha-1)]}{[2\beta_R(\alpha-1)+T]^{\alpha_R+2(\alpha-1)}} - \frac{T^{\alpha_R}[\Gamma(\alpha_R+\alpha-1)]^2}{\Gamma(\alpha_R)[\beta_R(\alpha-1)+T]^{2(\alpha_R+\alpha-1)}}\right\} \qquad [6.4.2.5]$$

Theorem 6.4.2.2:

The rth moment of LIR associated with the IGLF, when the process characteristic distribution is $U(\alpha_R, \beta_R)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}$$

$$= \left[\frac{e}{r(\alpha-1)}\right]^{r(\alpha-1)} \frac{T}{r(\alpha-1)(\beta_R-\alpha_R)}\left\{\Gamma\left[r(\alpha-1)+1, 0, \frac{r\beta_R(\alpha-1)}{T}\right] - \Gamma\left[r(\alpha-1)+1, 0, \frac{r\alpha_R(\alpha-1)}{T}\right]\right\}$$

$$r = 1, 2, \ldots, k. \qquad [6.4.2.6]$$

provided the expectations exist. The expectation equals to 1 if $r = 0$.

The mean and variance of IGLF for the uniform distribution are respectively

$$E[L(X,T)]=K\left(1-\left(\frac{e}{\alpha-1}\right)^{(\alpha-1)}\frac{T}{(\alpha-1)(\beta_R-\alpha_R)}\left\{\Gamma\left[\alpha,0,\frac{\beta_R(\alpha-1)}{T}\right]-\Gamma\left[\alpha,0,\frac{\alpha_R(\alpha-1)}{T}\right]\right\}\right)$$

[6.4.2.7]

V[L(X, T)]

$$=K^2\left(\left[\frac{e}{2(\alpha-1)}\right]^{2(\alpha-1)}\frac{T}{2(\alpha-1)(\beta_R-\alpha_R)}\left\{\Gamma\left[2\alpha-1,0,\frac{2\beta_R(\alpha-1)}{T}\right]-\Gamma\left[2\alpha-1,0,\frac{2\alpha_R(\alpha-1)}{T}\right]\right\}\right.$$

$$\left.-\left(\frac{e}{\alpha-1}\right)^{2(\alpha-1)}\left[\frac{T}{(\alpha-1)(\beta_R-\alpha_R)}\right]^2\left\{\Gamma\left[\alpha,0,\frac{\beta_R(\alpha-1)}{T}\right]-\Gamma\left[\alpha,0,\frac{\alpha_R(\alpha-1)}{T}\right]\right\}^2\right).$$

[6.4.2.8]

Theorem 6.4.2.3:

The rth moment of LIR associated with the IGLF when the process characteristic distribution is $N\left(\mu_R, \sigma_R^2\right)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\} = \frac{\left(\frac{\sqrt{2}\, e\sigma_R}{T}\right)^{r(\alpha-1)} \exp\left(-\frac{\mu_R^2}{2\sigma_R^2}\right)}{\sqrt{2\pi}}$$

$$\left\{ \frac{1}{\sqrt{2}}\Gamma\left[\frac{r(\alpha-1)+1}{2}\right] {}_1F_1\left[\frac{r(\alpha-1)+1}{2}, \frac{1}{2}, \frac{\left[T\mu_R - r(\alpha-1)\sigma_R^2\right]^2}{2T^2\sigma_R^2}\right]\left[1+(-1)^{r(\alpha-1)}\right] \right.$$

$$\left. + \left(\frac{r\sigma_R}{T}(1-\alpha) + \frac{\mu_R}{\sigma_R}\right)\Gamma\left[\frac{r(\alpha-1)+2}{2}\right] {}_1F_1\left[\frac{r(\alpha-1)+2}{2}, \frac{3}{2}, \frac{\left[T\mu_R - r(\alpha-1)\sigma_R^2\right]^2}{2T^2\sigma_R^2}\right]\left[1-(-1)^{r(\alpha-1)}\right] \right\}$$

$$r = 0, 1, 2, \ldots, k. \quad [6.4.2.9]$$

provided the expectations exist.

The mean and variance of IGLF for the normal distribution are respectively

$$E[L(X,T)] = K\left(1 - \frac{\left(\frac{\sqrt{2}\,e\sigma_R}{T}\right)^{\alpha-1} \exp\left(-\frac{\mu_R^2}{2\sigma_R^2}\right)}{\sqrt{2\pi}} \right.$$

$$\left\{ \frac{1}{\sqrt{2}}\Gamma\left[\frac{\alpha}{2}\right] {}_1F_1\left[\frac{\alpha}{2},\frac{1}{2},\frac{\left[T\mu_R-(\alpha-1)\sigma_R^2\right]^2}{2T^2\sigma_R^2}\right]\left[1+(-1)^{(\alpha-1)}\right] \right.$$

$$\left. + \left(\frac{\sigma_R}{T}(1-\alpha)+\frac{\mu_R}{\sigma_R}\right)\Gamma\left[\frac{\alpha+1}{2}\right] {}_1F_1\left[\frac{\alpha+1}{2},\frac{3}{2},\frac{\left[T\mu_R-(\alpha-1)\sigma_R^2\right]^2}{2T^2\sigma_R^2}\right]\left[1-(-1)^{(\alpha-1)}\right] \right\} \quad [6.4.2.10]$$

V[L(X, T)]

$$= \frac{K^2\left(\frac{2e^2\sigma_R^2}{T}\right)^{\alpha-1}\exp\left(-\frac{\mu_R^2}{2\sigma_R^2}\right)}{\sqrt{2\pi}}\left(\sqrt{2}\Gamma\left(\frac{2\alpha-1}{2}\right) {}_1F_1\left[\frac{2\alpha-1}{2},\frac{1}{2},\frac{\left[T\mu_R-(\alpha-1)\sigma_R^2\right]^2}{2T^2\sigma_R^2}\right] \right.$$

$$- \frac{\exp\left(-\frac{\mu_R^2}{2\sigma_R^2}\right)}{\sqrt{2\pi}}\left\{ \frac{1}{\sqrt{2}}\Gamma\left[\frac{\alpha}{2}\right] {}_1F_1\left[\frac{\alpha}{2},\frac{1}{2},\frac{\left[T\mu_R-(\alpha-1)\sigma_R^2\right]^2}{2T^2\sigma_R^2}\right]\left[1+(-1)^{(\alpha-1)}\right] \right.$$

$$\left.\left. + \left(\frac{\sigma_R}{T}(1-\alpha)+\frac{\mu_R}{\sigma_R}\right)\Gamma\left[\frac{\alpha+1}{2}\right] {}_1F_1\left[\frac{\alpha+1}{2},\frac{3}{2},\frac{\left[T\mu_R-(\alpha-1)\sigma_R^2\right]^2}{2T^2\sigma_R^2}\right]\left[1-(-1)^{(\alpha-1)}\right] \right\}^2 \right)$$

[6.4.2.11]

Note that equations [6.4.2.10] and [6.4.2.11] may result in complex numbers if α is not a positive integer.

Theorem 6.4.2.4:

The rth moment of LIR associated with the IGLF when the process characteristic distribution is $Be(\alpha_R,\beta_R)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}=\frac{\Gamma(\alpha_R+\beta_R)\Gamma[r(\alpha-1)+\alpha_R]}{\Gamma(\alpha_R)\Gamma[r(\alpha-1)+\alpha_R+\beta_R]}\,{}_1F_1\left[r(\alpha-1)+\alpha_R, r(\alpha-1)+\alpha_R+\beta_R, \frac{r(\alpha-1)}{T}\right]$$

$r = 0, 1, 2, \dots, k.$ [6.4.2.12]

The mean and variance of the IGLF for the beta distribution are respectively

$$E[L(X,T)]=K\left(1-\frac{\Gamma(\alpha_R+\beta_R)\Gamma[\alpha+\alpha_R-1]}{\Gamma(\alpha_R)\Gamma[\alpha+\alpha_R+\beta_R-1]}\,{}_1F_1\left[\alpha+\alpha_R-1, \alpha+\alpha_R+\beta_R-1, \frac{\alpha-1}{T}\right]\right)$$

[6.4.2.13]

$$V[L(X,T)]$$

$$= K^2\left(\frac{\Gamma(\alpha_R+\beta_R)\Gamma[2\alpha+\alpha_R-2]}{\Gamma(\alpha_R)\Gamma[2\alpha+\alpha_R+\beta_R-2]}\,{}_1F_1\left[2\alpha+\alpha_R-2, 2\alpha+\alpha_R+\beta_R-2, \frac{2(\alpha-1)}{T}\right]\right.$$

$$\left.-\left(\frac{\Gamma(\alpha_R+\beta_R)\Gamma[\alpha+\alpha_R-1]}{\Gamma(\alpha_R)\Gamma[\alpha+\alpha_R+\beta_R-1]}\,{}_1F_1\left[\alpha+\alpha_R-1, \alpha+\alpha_R+\beta_R-1, \frac{\alpha-1}{T}\right]\right)^2\right).$$

[6.4.2.14]

The gamma pdf provides the basis for another range of IPLFs, all with inverted asymmetric shape having the right arm of the IGLF open up wider on the right hand side of the target. By varying T and α, quality practitioners can customize a loss function to depict losses associated with process departures from the target. However, care should be taken when practitioners fit an asymmetric loss using IGLF. If the target of the process is near the upper specification limit, IGLF may not depict losses adequately in this situation.

The risk function associated with the IGLF can be evaluated for distributions such as uniform, normal and beta, that the process measurements may follow. As mentioned earlier in Section 6.4.1, the Weibull and lognormal distributions may enhance difficulties in determining the functional form of the risk function. Thus, the conjugate distribution for the IGLF appears to be the gamma distribution.

6.4.3 The Inverted Beta Loss Function (IBLF)

The derivation of IBLF and its associated Risk Function have been discussed in Chapter 5, so that only the expectation of it's LIR is shown here. Recall equation [5.2.1] that when α increases, LIR will decrease and thus decreases the IBLF.

Theorem 6.4.3.1:

The rth moment of LIR associated with the IBLF when the conjugate distribution is $Be(\alpha_R, \beta_R)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\} = C^r \frac{B\left(r(\alpha-1)+\alpha_R, r(\alpha-1)\frac{1-T}{T}+\beta_R\right)}{B(\alpha_R,\beta_R)} \qquad [6.4.3.1]$$

where $C = \left[T(1-T)^{\frac{1-T}{T}}\right]^{1-\alpha}$, $r = 0, 1, 2, \dots, k$.

The mean and variance of the IBLF for the beta distribution are respectively

$$E[L(X,T)]=K\left[1-C\frac{B\left(\alpha+\alpha_R-1,(\alpha-1)\frac{1-T}{T}+\beta_R\right)}{B(\alpha_R,\beta_R)}\right] \qquad [6.4.3.2]$$

$$V[L(X,T)]=K^2\frac{C^2}{B(\alpha_R,\beta_R)}\left\{B\left(\alpha+\alpha_R-2,2(\alpha-1)\frac{1-T}{T}+\beta_R\right)\right.$$

$$\left.-\frac{\left[B\left(\alpha+\alpha_R-1,(\alpha-1)\frac{1-T}{T}+\beta_R\right)\right]^2}{B(\alpha_R,\beta_R)}\right\}. \qquad [6.4.3.3]$$

Theorem 6.4.3.2:

The rth moment of LIR associated with the IBLF when the process characteristic distribution is $U(\alpha_R,\beta_R)$ is

$$E\left\{\left[\frac{\pi(X,T)}{m}\right]^r\right\}=\frac{C^r}{(\beta_R-\alpha_R)(rl+1)}\left\{\beta_R^{rl+1}{}_2F_1[rl+1,-rn,rl+2,\beta_R]\right.$$

$$\left.-\alpha_R^{rl+1}{}_2F_1[rl+1,-rn,rl+2,\alpha_R]\right\} \qquad [6.4.3.4]$$

The expectations of LIR of IBLF when the process characteristic distribution is $N(\mu_R,\sigma_R^2)$ or $G(\alpha_R,\beta_R)$ can be evaluated by

$$C\cdot E[X^l(1\text{-}X)^n]=C\sum_{i=0}^{\infty}{}_nC_i(-1)^i(\mu'_{l+i}) \qquad [6.4.3.5]$$

where $C=\left[T(1-T)^{\frac{1-T}{T}}\right]^{1-\alpha}$, and $l=\alpha-1$, $n=\frac{1-T}{T}(\alpha-1)$.

If n is a positive integer then the expectation above has finite number of terms, otherwise it has infinite number of terms.

The beta pdf provides the basis for a variety of IPLFs, all with various inverted symmetric and asymmetric shapes. By varying T and α, quality practitioners can customize a loss function in order to accurately depict losses associated with process departures from the target. The risk function associated with IBLF can be evaluated for distributions such as uniform, normal and gamma, that the process measurements may follow. Difficulties may arise in determining the functional form of the risk function as mentioned in Sections 6.4.1 and 6.4.2. However, numerical value of the risk function associated with IBLF can be obtained using computing packages such as Mathematica. Hence, the conjugate distribution for the IBLF is the beta distribution.

6.5 Comparison of IPLFs

The IPLFs that we have considered include the INLF, IGLF, IBLF and their associated properties for the uniform, normal, gamma and beta distributions. Let us compare the performance of these IPLFs under homogeneous situations by fixing the target at some T with the same variation on their open-upward "arms" and hence setting the same mean and variance for all the process characteristic distributions.

Because the IPLFs are scale invariant, we can fix the target value of T between 0 and 1 to reduce calculations. The values of T equal to .1, .5 and .8 allow us to compare the various IPLFs. Two sets of homogeneous process distributions were chosen for consideration, one with small variation and the other with large variation.

The distributions associated with parameters of IPLFs selected with fixed T and same variation on "arms" are shown in Table 6.5.1 while the process characteristic distributions associated with parameters with small and large variations are shown respectively in Table 6.5.2a and Table 6.5.2b.

TABLE 6.5.1 Fixed T with same variation on "arms"

T Distribution	.1	.5	.8
Normal N(T, σ^2)	N(.1, .0107)	N(.5, .0500)	N(.8, .0255)
Gamma G(α,β)	G(2.5427, .0648)	G(6.8541, .08541)	G(27.0510,.0307)
Beta Be(α,β)	Be(2,10)	Be(2, 2)	Be(2,5)

Table 6.5.2a Homogeneous process characteristic distributions

with $\mu_R = \frac{1}{2}, \sigma_R^2 = \frac{1}{12}$

Distribution	Sample space
Uniform U(0, 1)	$0 < x < 1$
Normal N($\frac{1}{2}, \frac{1}{12}$)	$-\infty < x < \infty$
Gamma G(3, $\frac{1}{6}$)	$0 < x < \infty$
Beta Be(6, 2)	$0 < x < 1; -1 < y < 1$*
Beta Be(2, 6)	$0 < x < 1; 0 < y < 2$*

* $X = \frac{Y-p}{q-p} \sim$ Beta (α, β)

Table 6.5.2b Homogeneous process characteristic distributions with $\mu_R=12, \sigma_R^2=24$

Distribution	Sample space
Uniform U(12- $6\sqrt{2}$, 12+$6\sqrt{2}$)	$12-6\sqrt{2} < x < 12+6\sqrt{2}$
Normal N(10, 12)	$-\infty < x < \infty$
Gamma G(6,2)	$0 < x < \infty$
Beta Be(3.2, 4.8)	$0 < x < 1; 0 < y < 30$*
Beta Be(4.64,1.69)	$0 < x < 1; -10 < y < 20$*

* $X = \dfrac{Y-p}{q-p} \sim \text{Beta}(\alpha, \beta)$

To compare the risk function is equivalent to comparing the expected value of LIR. If we are going to select the smallest risk function it is equivalent to choosing the largest expected LIR. Tables 6.5.3a to 6.5.5b show the expectations of LIR, with different T values, associated with different process characteristic distributions.

Table 6.5.3a Expectation of LIR with T = .1, σ^2= .0107

$\mu_R=.5, \sigma_R^2=.0107$	N(.1, .0107)	G(2.5427, .0648)	Be(2,10)
Uniform U(0, 1)	.215912	.212866#	.234652
Normal N(.5, .0107)	.143951	.026858#	.003701
Gamma G(3, .1666)	.167626	.195636	.218479
Beta Be(6, 2)	.477809	.488535	.531418
Beta Be(2, 6)	.002230	.007775	.007963

Approximated value using G(2, .1)

Table 6.5.3b Expectation of LIR with T = .1, σ^2= .0107

$\mu_R=12, \sigma_R^2=24$	N(.1, .0107)	G(2.5427, .0648)	Be(2,10)
Uniform U(12- $6\sqrt{2}$, 12+$6\sqrt{2}$)	.131552	.000000#	.013827
Normal N(12,24)	.772373	.001725#	.001463
Gamma G(6,2)	.000000	.000000	.000000
Beta Be(3.2, 4.8)	.175546	.214813	.243123
Beta Be(4.64, 1.68)	.006008	.013854	.015170

Approximated value using G(2, .1)

From Tables 6.5.3a and 6.5.3b the target value T = .1 is near the lower specification limit, the INLF depicts the loss quite satisfactory if the process measurements are uniform or normal, irrespective of large or small variations. However, it will overstate the true loss when the process measurements are skewed to the left such as Beta(6, 2) in Table 6.5.3a, G(6, 2) and Be(4.64, 1.68) in Table 6.5.3b. The IGLF does a little better among these cases especially where the process measurements are gamma. The IBLF appears best in all situations when the process variation is small.

Table 6.5.4a Expectation of LIR with T = .5, σ^2 = .05

μ_R=.5, σ_R^2=.0107	N(.5, .05)	G(6.8541, .0854)	Be(2,2)
Uniform U(0, 1)	.546292	.499985	.666667
Normal N(.5, .0107)	.612372	.602064#	.690305
Gamma G(3, .1666)	.632013	.571549	.666667
Beta Be(6, 2)	.519218	.364509	.666667
Beta Be(2, 6)	.519218	.561889	.666667

Approximated value using G(6, .1)

Table 6.5.4b Expectation of LIR with T = .5, σ^2 = .05

μ_R=12, σ_R^2=24	N(.5, .05)	G(6.8541, .0854)	Be(2,2)
Uniform U(12- 6$\sqrt{2}$, 12+6$\sqrt{2}$)	.000000	.000000#	.039284
Normal N(12,24)	.407509	.000000#	.000000
Gamma G(6,2)	.000011	.000019	.000013
Beta Be(3.2, 4.8)	.743910	.661656	.853333
Beta Be(4.64, 1.68)	.537525	.575051	.675556

Approximated value using G(6, .1)

Tables 6.5.4a and 6.5.4b show the expectation of the LIR of the IPLFs when the target value (i.e., T = .5) is at the middle of the process. The IBLF depicts loss extremely well where the process variation is small or large. The INLF reflects loss much better than the other two IPLFs whenever the process variation is large. The IGLF is the worst in these situations.

Table 6.5.5a Expectation of LIR with T = .8, σ^2 = .0255

$\mu_R=.5, \sigma_R^2=.0107$	N(.8, .0255)	G(27.051, .0307)	Be(2,5)
Uniform U(0, 1)	.358219	.337034	.664604
Normal N(.5, .0107)	.320186	.292612#	.651712
Gamma G(3, .1666)	.250092	.223922	.623138
Beta Be(6, 2)	.037514	.022892	.420013
Beta Be(2, 6)	.741539	.722601	.926726

Approximated value using G(27, .0308)

Table 6.5.5b Expectation of LIR with T = .8, σ^2 = .0255

$\mu_R=12, \sigma_R^2=24$	N(.8, .0255)	G(27.051, .0307)	Be(2,5)
Uniform U(12- $6\sqrt{2}$, 12+$6\sqrt{2}$)	.087345	.000000#	.039162
Normal N(12,24)	.549375	.000000#	.000000
Gamma G(6,2)	.000030	.000035	.000025
Beta Be(3.2, 4.8)	.157804	.121304	.633012
Beta Be(4.64, 1.68)	.696812	.675903	.908127

Approximated value using G(27, .0308)

From Tables 6.5.5a and 6.5.5b, it can be seen that the IBLF is the best when the target value (T = .8) is near the upper specification limit and process variation is small. The IBLF ranked second after the INLF when the process variation is large. The IGLF is the worst among all these situations.

Based on the tabulated values above (Tables 6.5.3a to 6.5.5b), we have shown numerically that the overall performance of IBLF is the best among the three IPLFs considered. Even though the variance of the process is large or small, the IBLF prescribes the loss consistently regardless of whether the target value is located near the lower or upper specification limits. The performance of INLF is good only when the target value is near the middle and when the variation of the process characteristic distribution is small. The performance of IGLF is good only when the target value is near the lower specification limit irrespective of whether the variation of the process is large or small.

6.6 Comments

The increasing use of loss functions in quality assurance has created a demand for realistic and representative loss functions. The family of inverted probability loss functions provides practitioners with a wide variety of loss functions that can be used to accurately depict process loss. We have examined the general properties of the Inverted Probability Loss Function physically and statistically. Different cases of IPLF with their plausible process characteristic distributions are compared. The risk function associated with IPLF is dependent upon the distribution of the process characteristic. Hence, the process characteristic distribution is important in selecting the best IPLF to reflect their true loss.

From the evaluation above we can confirm that the conjugate distributions for the IPLFs based on "closed under sampling" and "make calculation easy" are respectively normal distribution for the INLF, gamma distribution for the IGLF and beta distribution for the IBLF. Also, we have suggested some other possible process characteristic distributions for use in evaluating the associated risks when the process measurements do not follow exactly their corresponding conjugate distributions.

The properties of IPLF captioned in this chapter are limited to only one piece of primary loss information, i.e., the target T, the maximum loss K and $[x_1, L_1]$ where L_1 represents the loss at x_1. We can examine the properties of a composite inverted probability loss function with two pieces of loss information, i.e., T, K, $[x_1, L_1]$ and $[x_2, L_2]$, or more generally with T, K_1, $[x_1, L_1]$, K_2, and $[x_2, L_2]$. The IPLF has the following form

$$L(x,T)=\begin{cases} K_1\left(1-\dfrac{\pi_1(x,T)}{m_1}\right) & \text{if } x < T, \\ K_2\left(1-\dfrac{\pi_2(x,T)}{m_2}\right) & \text{if } x > T, \end{cases}$$

allowing either side of the target to have maximum losses of K_1 and K_2 respectively and shape based on $\pi_1(x,T)$ and $\pi_2(x,T)$, where m_1 is the supremum of $\pi_1(x,T)$ and m_2 is the supremum of $\pi_2(x,T)$.

Chapter 7

Conclusion

7.1 The Process Capability Indices (PCIs)

The process capability indices Cpm and Cpw are techniques which can be used to evaluate the ability of a process to attain a preset target value T and to fall within required specification limits concurrently. While Cpm* and Cpw* are the generalizations of Cpm and Cpw respectively so as including asymmetric specification limits. However, Cpw* is a unified approach which draws together a particular class of process capability indices (Spiring (1997)) and allows one to examine the statistical properties associated with estimators of the various indices using a similar perspective. Varying different weight functions allows one to customize the capability index to the process of interest and again to customize the index to the appropriate loss functions being used for various process depicting losses due to variation from T.

Those PCIs which incorporate σ into their computing algorithm and whose magnitudes are translated into parts per million non-conforming are meaningless in the face of departures from normality. Regardless of how robust the estimator maybe, its associated parameter is not stable and hence any robustness claim carry little meaning. In the face of normality distorted with non-zero values of skewness and kurtosis, adjustments to critical values associated with attempts to assess changes in the process capability. All PCIs involving σ in their computation may contain much less than 99.73% of the process measurements if the process distribution is not normal. The process capability index Cpo is proposed as an alternative for assessing process capability when the underlying distribution of the process measurements is uniform and exponential. The estimate of Cpo is distribution-free and can guarantee a coverage of at least 99.73% of the process measurements irrespective to the target value. Determination of the numerical value of $\hat{C}po$ becomes a matter of finding a 99.73% probability between two order statistics, i.e., $P\left(Y_r < \hat{D} < Y_s\right) = .9973$, and using the width of this interval ($Y_s - Y_r$, where $r < s$) as a measure of the actual process spread. The bias and mean squared error of $\hat{C}po$ has been shown convergent even though the rate of convergent is slower than that of $\hat{C}pm$, $\hat{C}pw$ as well $\hat{C}pm^*$ and $\hat{C}pw^*$.

7.2 Inverted Probability Loss Functions

Over the last few years, loss functions have becoming increasingly important in quality assurance settings for quantifying economic losses associated with variation about, and deviation from, a desired target value or the target value of a product characteristic. In Chapter 7, we have developed some properties of loss functions with primary loss information based on inversions of probability density functions and make numerical comparison of some selected IPLFs. With an auxilliary piece of loss information we have found that the performance of an inverted beta loss function is better than the inverted normal loss function and the inverted gamma loss function irrespective to whether the target value is near the lower or upper specification limits or near the center of the process distribution. An industrial application of IBLF as well as its properties are included in

Chapter 6. Further study of IPLF can be extended to loss functions with two pieces of loss information allowing two maximum losses on either side of the target value.

The family of inverted probability loss functions provides practitioners with a wide variety of loss functions that can be used to quality assurance settings. It also provides flexibility for practitioners to accurately depict process loss and succintly reflect monetary loss to the company or to the society of interest.

7.3 Future Research Direction

There are several possible extensions in some of the work done in previous chapters which may lead to significant results. The Cpm index, as mentioned in Section 2.3, can be used to measure process capability when the measurements are normally distributed. The index Cpm is expressed as a number which summarizes the process variation and the process drift from its target as the actual process spread relative to the allowable spread. A measure can be constructed to quantify how the process measurements are clustering around the target. Analogous to Section 2.5, the robustness of the unifying index, Cpw, to departures from normality is a possible extension of Chapter 4. Allowing different weights (Spiring(1997)) in Cpw we can compare the performance of the PCIs under normality distorted with non-zero values of skewness and kurtosis.

The bivariate beta distribution has the following pdf

$$f(x_1,x_2) = \frac{\Gamma(\alpha_1+\alpha_2+\beta)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} (1-x_1-x_2)^{\beta-1},$$

$$x_1>0,\ x_2>0, x_1+x_2<1,\quad \alpha_1,\alpha_2,\beta>0,\ \text{zero elsewhere.}$$

Then X_1 and X_2 are said to possess a bivariate beta distribution. Following a similar approach of IBLF in Chapter 5, an inverted bivariate beta loss function can be examined. It may extend to inverted multivariate beta loss function. Similarly, inverted bivariate gamma loss function and inverted multivariate gamma loss function can also be studied.

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