THE UNIVERSITY OF MANITOBA

CONTRIBUTIONS TO THE THEORY OF HYPERSPACES

by

John N. Ginsburg

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS AND ASTRONOMY

Winnipeg, Manitoba

October 1975

CONTRIBUTIONS TO THE THEORY OF HYPERSPACES

by

John N. Ginsburg

A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

TABLE OF CONTENTS

ABSTRACT

One of the most natural and most interesting objects associated with a topological space X is its space of closed subsets 2^X. Of the various topologies with which 2^X may be endowed, the one that concerns us here is the so-called finite topology introduced by Vietoris in [48]. We shall refer to the space of closed subsets of X, endowed with the finite topology, as the hyperspace of X. Hyperspaces have been studied by several authors from several points of view. 2^X has been studied in the context of set-valued mappings, fixed-point theorems, and selections. This approach is illustrated in the collected papers in [46]. The study of 2^X when X is a continuum or metric continuum has occupied the interest of many topologists. (A few examples of this are [8], [31], [50]). The comprehensive work of E. Michael [36] is the standard reference for the fundamental properties of 2^X. In [36], Michael describes various topologies and uniformities on spaces of subsets and examines such basic topics as separation axioms, countability, compactness, continuous functions, connectedness and selections. Further basic properties of 2^X are examined in [32], where one may find a treatment of such topics as set-valued mappings and decomposition spaces. The relation of 2^X to lattices and Brouwerian algebras, and the role of 2^X as a topological semilattice are also elucidated in [32].

In this work, we are primarily concerned with properties related to compactness in 2^X. Such properties are of great interest and have received considerable attention. One of the earliest and most elegant results on hyperspaces is the fundamental compactness theorem, established by Vietoris, asserting that 2^X is compact when X is. This result is basic in the study of several compactness-related properties of 2^X. Important progress in the study of compactness-related properties of 2^X has recently been made by J. Keesling, who, in a series of papers, [27], [28], [29], [30], obtained many significant results, including the fascinating result that normality and compactness are equivalent in hyperspaces [28]. Keesling's results have motivated much of the present work.

The first chapter is devoted to a study of pseudocompact and countably compact spaces, the emphasis being on powers and products.

In Chapter 2, we apply the results of the first chapter in examining the countable compactness and pseudocompactness of 2^X.

In the third chapter, our attention is focused on the Stone-Cech compactification of 2^X, and particularly on the validity of the relation $\beta(2^X) = 2^{\beta X}$. The results of Chapter 2 provide us with a fairly large class of spaces for which this relation is valid.

The role of $2^{\beta X}$ as a compactification of 2^X is further examined in Chapter 4, where we describe the G_δ-closure of 2^X

in $2^{\beta X}$. This description enables us to obtain information on the realcompactness of 2^X.

In the final chapter, in a somewhat different, though not unrelated vein, we examine some of the cardinal invariants of 2^X, including weight, character and cellularity.

ACKNOWLEDGEMENTS

I am deeply indebted to my advisor, Professor R. G. Woods, whose guidance and inspiration have been invaluable in preparing this thesis. The Topology Seminar at the University of Manitoba, headed by Professor Woods, has been a valuable source of motivation and experience, and has been a pleasure to take part in. I express my sincere thanks to Professor M. Rayburn and to S. Broverman for the time and assistance they contributed to the many discussions we had during my stay at the University of Manitoba. I am especially grateful to Professor Victor Saks of the University of Costa Rica, who collaborated with the author on much of the material of Chapter 1 during a recent visit to Canada. This interaction has greatly benefited the author. I thank the National Research Council of Canada for its financial assistance throughout my graduate studies. Finally, I thank Geri for her patience and understanding throughout the preparation of this thesis.

Chapter 0

PRELIMINARIES

Our topological terminology and notation are well established, and follow the standard texts on point-set topology. For background material on rings of continuous functions and compactifications, we refer the reader to the Gillman and Jerison text [15]. Following [15], the ring of continuous real-valued functions on a topological space X is denoted by $C(X)$, and its subring of bounded members by $C^*(X)$. For $f \varepsilon C(X)$, the set $\{x \varepsilon X: f(x) = 0\}$ is called the zero-set of f, and is denoted by $Z(f)$. A cozero-set is the complement of a zero-set. The set of all zero-sets of functions in $C(X)$ is denoted by $Z(X)$. The Stone-Cech compactification of a completely regular, Hausdorff space X is denoted by βX. It is characterized as the compactification of X to which all bounded continuous real-valued functions on X may be continuously extended. The set (and discrete space) of positive integers is denoted by $\underset{\sim}{N}$. The points of $\beta\underset{\sim}{N}-\underset{\sim}{N}$ are the free ultrafilters on $\underset{\sim}{N}$, and they play a dual role in this thesis, as points of the space $\beta\underset{\sim}{N}$, and as ultrafilters. The space of real numbers is denoted by $\underset{\sim}{R}$, and the cardinality of a set S is denoted by $|S|$.

The notions from set theory that we shall employ are standard. An ordinal is thought of as the set of its predecessors, and a cardinal as an initial ordinal. The symbol ω_α is used to denote the α'th infinite cardinal. For a discussion of the cardinal invariants discussed in this thesis, we refer the reader to [4] and [25].

In the present work, the main object of our study is the space of closed subsets of a topological space. We now recall the definition of the space of closed sets, and state several basic facts concerning this space.

Let X be a topological space. Let 2^X denote the set of all non-empty closed subsets of X. For a subset A of X, we let $2^A = \{F \in 2^X: F \subseteq A\}$. We generate a topology on 2^X by taking all sets of the form 2^G and all sets of the form $2^X - 2^{X-G}$, for G open in X, as a sub-basis. This topology on 2^X is known as the finite topology, and 2^X, endowed with this topology, is called the <u>hyperspace of X</u>.

Our basic references for the fundamental properties of 2^X are [32] and [36].

Following [32], we make the following notational convention. For subsets $A_0, A_1, \ldots, A_n$ of X, we let $B(A_0; A_1, \ldots, A_n) = 2^{A_0} \cap \bigcap_{i=1}^{n} (2^X - 2^{X-A_i}) = \{F \in 2^X: F \subseteq A_0 \text{ and } F \cap A_i \neq \phi \text{ for all } i = 1,2\ldots,n\}$. Using this notation we see that the sets $B(G_0; G_1, \ldots, G_n)$ where $G_0, G_1, \ldots, G_n$ are open and $\bigcup_{i=1}^{n} G_i \subseteq G_0$, form a basis for the open subsets of 2^X.

We now state several basic facts about hyperspaces which we will need in the course of our discussion.

0.1. If X is T_1, the singletons of 2^X form a subspace homeomorphic to X. ([36])

0.2. For each positive integer n, we set $\mathcal{F}_n(X) = \{F \in 2^X: |F| \leq n\}$, and we set $\mathcal{F}(X) = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n(X)$. If X is T_1, then $\mathcal{F}(X)$ is dense in 2^X. If X is Hausdorff, then $\mathcal{F}_n(X)$ is closed in 2^X for each $n \in \mathbb{N}$. (See 2.4 in [36].)

0.3. The operation of set-theoretic union, $(A,B) \to A \cup B$, is a continuous map from $2^X \times 2^X$ into 2^X. (See page 166 of [32].)

0.4 2^X is compact Hausdorff if, and only if, X is compact Hausdorff. (See 4.9 in [36].)

0.5. 2^X is completely regular (and Hausdorff) if, and only if, X is normal and Hausdorff. (See 4.9 in [36].)

0.6. If X is normal and Hausdorff, the natural mapping $i: 2^X \to 2^{\beta X}$ defined by $i(F) = cl_{\beta X}F$ is an embedding of 2^X onto a dense subspace of $2^{\beta X}$. (See [27].)

0.7. Let f be a bounded, real-valued function on X. We define real-valued functions f^s and f^i on 2^X by $f^s(F) = \sup\{f(x): x \in F\}$ and $f^i(F) = \inf\{f(x): x \in F\}$. Then if f is continuous, so are f^i and f^s. We have for $f \geq 0$, $Z(f^s) = 2^{Z(f)}$ and, if X is countably compact, $Z(f^i) = B(X: Z(f))$. Identifying X with the singletons in 2^X, we see that, for a T_1 space X, X is C^*-embedded in 2^X. (See 4.7 and 4.8 in [36].)

0.8. If X is normal and T_1, the sets of the form $B(X; Z_0) \cup 2^{Z_1} \cup \ldots \cup 2^{Z_n}$, where $Z_0, Z_1, \ldots, Z_n \in Z(X)$, form a base for the closed sets in 2^X. This can be verified in a straightforward manner.

From Chapter 2 on, we will assume that all spaces under consideration are T_1, and this assumption will be used without explicit mention in some cases. These spaces are not consistently assumed to satisfy separation axioms other than T_1. Higher separation axioms do enter in certain of our results and arguments in a significant and essential way, and in such situations we are explicit as to what separation axioms are assumed. But we repeat that the assumption that all topological spaces discussed are T_1 is tacit from Chapter 2 on. One further word on separation axioms: the term completely regular, even when unmodified, implies Hausdorff throughout this thesis.

Theorems are referred to by number. "Theorem 2.6 of Chapter 1" indicates the sixth theorem in the second section of Chapter 1. When the number of the chapter is not indicated, it is to be understood that the reference is to the present chapter.

Chapter 1

COUNTABLY COMPACT AND PSEUDOCOMPACT SPACES

1. In this chapter we are concerned with certain aspects of the theory of countable compactness and pseudocompactness. Several of the ideas and results of this chapter will subsequently be applied to the countable compactness and pseudocompactness of hyperspaces; however, the main interest and significance of these results lie in their contribution to the general theory of countable compactness and pseudocompactness. The material presented in this chapter is part of joint work by the author and Victor Saks, whose contribution the author gratefully acknowledges. This work appears in [20].

We characterize spaces all of whose powers are countably compact, and obtain partial results on the corresponding question for pseudocompactness. The basic tool in this work is A. R. Bernstein's concept of $\mathcal{D}$-compactness ([1]). The maximal $\mathcal{D}$-compact extension of a completely regular space is constructed. Additional product theorems for pseudocompact spaces are proved, imposing conditions closely related to $\mathcal{D}$-compactness on the factors, which imply the pseudocompactness of the product. In the last section of the chapter, we prove several theorems which provide new examples of non-trivial pseudocompact spaces. In particular, we exhibit a homogeneous space, all of whose powers are pseudocompact, in which no discrete countable set has a cluster point.

2. Countably Compact Powers. Let us recall several definitions of compactness-like conditions which depend on the behaviour of countable sets.

Let X be a topological space.

X is said to be countably compact, if every countably infinite subset of X has a cluster point.

A subset A of X is relatively countably compact in X, if every countably infinite subset of A has a cluster point in X.

X is sequentially compact, if every sequence in X has a convergent subsequence.

X is called strongly ω_0-compact, if every infinite subset of X meets some compact subset of X in an infinite set.

Finally, we call X ω_0-bounded, if every countable subset of X is contained in a compact subset of X.

Our first result characterizes those spaces X such that every power of X is countably compact. The main tool in this investigation is Bernstein's concept of $\mathcal{D}$-compactness. In [1] the concept was introduced, and some of the basic theory of $\mathcal{D}$-compact spaces was developed. We now give his definition of $\mathcal{D}$-compactness, and quote the major results in [1], including a proof of his result that $\mathcal{D}$-compactness is a productive property.

2.1 Definition. Let $\mathcal{D}$ be a free ultrafilter on $\underset{\sim}{N}$. Let X be a topological space, and let $(x_n\colon n \in \underset{\sim}{N})$ be a sequence in X. A point $z \in X$ is said to be a <u>$\mathcal{D}$-limit point</u> of the sequence $(x_n\colon n \in \underset{\sim}{N})$ if, for every neighbourhood W of z, $\{n\colon x_n \in W\} \in \mathcal{D}$. We shall express this by writing $z \in \mathcal{D}\text{-}\lim\limits_{n\to\infty} x_n$. In Hausdorff spaces, $\mathcal{D}$-limit points, when they exist, are unique, in which case we write $z = \mathcal{D}\text{-}\lim\limits_{n\to\infty} x_n$. A space X is said to be <u>$\mathcal{D}$-compact</u> if every sequence in X has a $\mathcal{D}$-limit point.

Observe that a $\mathcal{D}$-limit point of a sequence of distinct points $(x_n\colon n \in \underset{\sim}{N})$ is, in particular, a cluster point of the set $\{x_n\colon n \in \underset{\sim}{N}\}$. Therefore, a $\mathcal{D}$-compact space is countably compact.

2.2 Lemma. <u>Let</u> $\{x_n\colon n \in \underset{\sim}{N}\} \subseteq X$ <u>and let</u> $z \in X$ <u>be a cluster point of</u> $\{x_n\colon n \in \underset{\sim}{N}\}$. <u>Then there exists</u> $\mathcal{D}$ <u>in</u> $\beta N - \underset{\sim}{N}$ <u>such that</u> $z \in \mathcal{D}\text{-}\lim\limits_{n\to\infty} x_n$.

Proof. Let $G(z)$ denote the family of all neighbourhoods of z in X. For $W \in G(z)$, let $s(W) = \{n\colon x_n \in W\}$. The family $F = \{s(W) - \{k\}\colon W \in G(z),\ k \in N\}$ has the finite intersection property, and so there is an ultrafilter $\mathcal{D}$ on $\underset{\sim}{N}$ such that $F \subseteq \mathcal{D}$. Obviously $\mathcal{D}$ is free and $z \in \mathcal{D}\text{-}\lim\limits_{n\to\infty} x_n$.

2.3 Lemma. <u>Let</u> $f\colon X \to Y$ <u>be a continuous map. Let</u> $(x_n\colon n \in \underset{\sim}{N})$ <u>be a sequence in X, and let</u> $z \in X$ <u>such that</u> $z \in \mathcal{D}\text{-}\lim\limits_{n\to\infty} x_n$. <u>Then</u> $f(z) \in \mathcal{D}\text{-}\lim\limits_{n\to\infty} f(x_n)$.

Proof. For every neighbourhood W of $f(z)$ in Y, $f^{-1}(W)$ is a neighbourhood of z in X. Since $\{n\colon x_n \in f^{-1}(W)\} = \{n\colon f(x_n) \in W\}$, the result follows.

2.4 Theorem (Bernstein). _$\mathcal{D}$-compactness is closed hereditary and productive. A completely regular space is ω_0-bounded if, and only if, it is $\mathcal{D}$-compact for every $\mathcal{D}$ in_ $\beta\mathbb{N}-\mathbb{N}$.

Proof. Obviously a closed subset of a $\mathcal{D}$-compact space is $\mathcal{D}$-compact. We will prove the statement concerning products, and refer the reader to Theorems 3.4 and 3.5 of [1] for the last statement.

Thus, let $\{X_\alpha : \alpha \in I\}$ be a family of $\mathcal{D}$-compact spaces, and let $X = \prod_{\alpha\in I} X_\alpha$. We will show that X is $\mathcal{D}$-compact. Let $(x^{(n)} : n \in \mathbb{N})$ be a sequence in X. Then, for each α in I, $(x_\alpha^{(n)} : n \in \mathbb{N})$ has a $\mathcal{D}$-limit point z_α in X_α. This defines a point $z = (z_\alpha)_{\alpha\in I}$ in X. We claim that $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x^{(n)}$. For, let W be any neighbourhood of z in X. There is a finite subset F of I, and open sets W_α in X_α, for each α in F, such that

$$z \in \prod_{\alpha\in F} W_\alpha \times \prod_{\alpha\notin F} X_\alpha \subseteq W.$$

But $\{n : x^{(n)} \in W\} \supseteq \bigcap_{\alpha\in F} \{n : x_\alpha^{(n)} \in W_\alpha\}$, and therefore $\{n : x^{(n)} \in W\} \in \mathcal{D}$. This proves that $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x^{(n)}$ and thus X is $\mathcal{D}$-compact.

2.5 Corollary. _Any product of $\mathcal{D}$-compact spaces is countably compact_.

Proof. Immediate.

We are now in a position to characterize spaces all of whose powers are countably compact.

2.6 Theorem. Let X be a topological space. The following statements are equivalent:

(i) Every power of X is countably compact;

(ii) X^{2^c} is countably compact;

(iii) $X^{|X|^{\omega_0}}$ is countably compact;

(iv) There exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that X is $\mathcal{D}$-compact.

Proof. (i) => (ii). This is trivial.

(ii) => (iv). We show that if (iv) fails, so does (ii). Thus, suppose X is not $\mathcal{D}$-compact for any $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$. Then, for each $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$, there is a sequence $(x_n^{(\mathcal{D})}: n \in \mathbb{N})$ in X which has no $\mathcal{D}$-limit point in X. Define a sequence $(y^{(n)}: n \in \mathbb{N})$ in $X^{\beta\mathbb{N}-\mathbb{N}}$ as follows: $y_{\mathcal{D}}^{(n)} = x_n^{(\mathcal{D})}$.

For the sake of contradiction, assume (ii) holds. Then $X^{\beta\mathbb{N}-\mathbb{N}}$ is countably compact, and therefore the sequence $(y^{(n)}: n \in \mathbb{N})$ has a cluster point z in $X^{\beta\mathbb{N}-\mathbb{N}}$. By Lemma 2.2, there exists $\mathcal{E}$ in $\beta\mathbb{N}-\mathbb{N}$ such that $z \in \mathcal{E}\text{-}\lim_{n\to\infty} y^{(n)}$. But this implies, by Lemma 2.3, that

$$\Pi_{\mathcal{E}}(z) \in \mathcal{E}\text{-}\lim_{n\to\infty} \Pi_{\mathcal{E}}(y^{(n)}) = \mathcal{E}\text{-}\lim_{n\to\infty} x_n^{(\mathcal{E})}.$$

But this is ridiculous, since $(x_n^{(\mathcal{E})}: n \in \mathbb{N})$ has no $\mathcal{E}$-limit point. Thus (ii) must also fail.

(iv) => (i). This follows immediately from 2.5.

(i) => (iii). This is trivial.

(iii) => (iv). Let Σ be the set of all sequences in X. We write $\sigma \in \Sigma$ as $\sigma = (x_n^{(\sigma)} : n \in \mathbb{N})$. Now $|\Sigma| = |X|^{\omega_0}$, and so (iii) implies that X^Σ is countably compact. Define a sequence $(z^{(n)} : n \in \mathbb{N})$ in X^Σ as follows: $z_\sigma^{(m)} = x_m^{(\sigma)}$. Let $p \in X^\Sigma$ be a cluster point of $(z^{(n)} : n \in \mathbb{N})$. By 2.2, there exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that $p \in \mathcal{D}\text{-}\lim_{n\to\infty} z^{(n)}$. We claim that, for this $\mathcal{D}$, X is $\mathcal{D}$-compact. For, if $\sigma = (x_n^{(\sigma)} : n \in \mathbb{N})$ is any sequence in X, Lemma 2.3 implies that

$$\Pi_\sigma(p) \in \mathcal{D}\text{-}\lim_{n\to\infty} \Pi_\sigma(z^{(n)}) = \mathcal{D}\text{-}\lim_{n\to\infty} x_n^{(\sigma)}.$$

Thus every sequence in X has a $\mathcal{D}$-limit point, and so X is $\mathcal{D}$-compact.

2.7 Remark. In [45], Scarborough and Stone have shown that, if $X = \prod_{\alpha\in I} X_\alpha$, then X is countably compact if, and only if, every subproduct of 2^{2^c} factors is countably compact. Thus the conditions (ii) and (iii) in Theorem 2.6 may be regarded as an improvement of their result in the case where all the factors are the same.

2.8 Corollary. If $|X| \leq c$, then X^c is countably compact if, and only if, there exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that X is $\mathcal{D}$-compact.

In [44] the following theorem is proved.

2.9 Theorem. (Saks-Stephenson). <u>The product of not more than ω_1 strongly ω_0-compact spaces is countably compact.</u>

Assuming the continuum hypothesis [CH], we obtain the following corollary, which gives natural examples of $\mathcal{D}$-compact spaces.

2.10 Corollary. [CH]. <u>If</u> $|X| \leq c$, <u>and if X is strongly</u> ω_0<u>-compact, then there exists</u> $\mathcal{D}$ <u>in</u> $\beta\tilde{N}-\tilde{N}$ <u>such that X is</u> $\mathcal{D}$<u>-compact. In particular, every countably compact k-space of cardinality</u> $\leq c$ <u>is</u> $\mathcal{D}$<u>-compact for some</u> $\mathcal{D}$ <u>in</u> $\beta\tilde{N}-\tilde{N}$.

Proof. The first assertion is obvious from 2.8 and 2.9, while the second is a special case, by Theorem 1.2 in [39].

2.11 Remark. Since every sequentially compact space is strongly ω_0-compact, the conclusion of 2.10 holds for sequentially compact spaces, of cardinal $\leq c$. This special case of 2.10 also follows directly from Theorem 5.8 in [45], together with our Theorem 2.6.

For non-trivial examples of the spaces hypothesized in 2.10, the reader is referred to [10].

We now turn to another aspect of $\mathcal{D}$-compactness. It follows from the corollary to Theorem 1 in [23], that every completely regular space has a maximal $\mathcal{D}$-compact extension. That is, for every completely regular space X, there is a completely regular $\mathcal{D}$-compact space $\mathcal{D}(X)$ containing X as a dense subspace, such that every continuous map of X into any (completely regular) $\mathcal{D}$-compact space extends continuously to $\mathcal{D}(X)$. From the final section of [23], it follows that, in fact, we may take $X \subseteq \mathcal{D}(X) \subseteq \beta X$ where $\mathcal{D}(X)$ is the intersection of all $\mathcal{D}$-compact subspaces of βX

containing X.

We now show how $\mathcal{D}(X)$ is built up from X. The construction is an exact analogue of Example 4 in [1]. In this example, Bernstein is constructing a $\mathcal{D}$-compact space which is not ω_0-bounded. His construction, when slightly modified, gives the maximal $\mathcal{D}$-compact extension of an arbitrary completely regular space. R. G. Woods independently characterized $\mathcal{D}(X)$ by the same method as given here, in [52].

Let X be a completely regular space. We first construct a transfinite sequence $(X_\alpha : \alpha < \omega_1)$ of subspaces of βX containing X.

Let $X_0 = X$. Assume we have constructed the spaces X_α, for $\alpha < \beta$ such that

(i) $\alpha_1 \leq \alpha_2 < \beta \Rightarrow X_{\alpha_1} \subseteq X_{\alpha_2} \subseteq \beta X$

(ii) $\alpha_1 < \alpha_2 < \beta \Rightarrow$ every sequence in X_{α_1} has a $\mathcal{D}$-limit point in X_{α_2}.

We now construct X_β. Let Σ_β be the set of all sequences in $\bigcup_{\alpha<\beta} X_\alpha$. For each $\sigma \in \Sigma_\beta$, let x_σ be a $\mathcal{D}$-limit point of σ in βX. Finally, let $X_\beta = (\bigcup_{\alpha<\beta} X_\alpha) \cup \{x_\sigma : \sigma \in \Sigma_\beta\}$. This completes the induction step, and gives a sequence $(X_\alpha : \alpha < \omega_1)$ satisfying (i) and (ii) for all $\alpha_1 < \alpha_2 < \omega_1$.

2.12 Theorem. $\mathcal{D}(X) = \bigcup_{\alpha<\omega_1} X_\alpha$.

Proof. Obviously $X \subseteq \bigcup_{\alpha<\omega_1} X_\alpha \subseteq \beta X$. If $(x_n : n \in \underset{\sim}{N})$ is a sequence in $\bigcup_{\alpha<\omega_1} X_\alpha$, it lies entirely within one X_β, and thus has a $\mathcal{D}$-limit point in $X_{\beta+1}$. Therefore $\bigcup_{\alpha<\omega_1} X_\alpha$ is $\mathcal{D}$-compact. A straightforward induction shows that any $\mathcal{D}$-compact subspace of βX containing X must contain every X_α, that is, must contain $\bigcup_{\alpha<\omega_1} X_\alpha$. From the result we have quoted from the Herrlich and van der Slot paper [23], it now follows that $\mathcal{D}(X) = \bigcup_{\alpha<\omega_1} X_\alpha$.

2.13 Corollary. $|\mathcal{D}(X)| \leq |X|^{\omega_0}$.

Proof. This is obvious from the construction described above.

Information on the role of $\mathcal{D}$-compactness as an extension property of topological spaces can be found in [52].

3. Powers and Products of Pseudocompact Spaces. Recall that a space X is pseudocompact if every continuous real-valued function on X is bounded. There is an obvious modification of $\mathcal{D}$-compactness which is suited to the study of pseudocompactness in completely regular spaces. This is because, as Glicksberg observed in [21], a completely regular space X is pseudocompact if, and only if, every sequence of non-empty open subsets of X has a cluster point. (A cluster point of a sequence of sets is a point such that each of its neighbourhoods meets infinitely many sets in the sequence.)

In fact, as Glicksberg shows, it is necessary and sufficient that every sequence of pairwise disjoint, non-empty open sets have a cluster point. This condition in general, (that is, for non-completely regular spaces) is stronger than pseudocompactness. (See [45].)

3.1 Definition. Let $\mathcal{D}$ be a free ultrafilter on $\underset{\sim}{N}$. Let $(S_n : n \in \underset{\sim}{N})$ be a sequence of subsets of a topological space X. A point $p \in X$ is called a $\mathcal{D}$-limit point of the sequence $(S_n : n \in \underset{\sim}{N})$ if, for every neighbourhood W of p, $\{n : S_n \cap W \neq \phi\} \in \mathcal{D}$. A space X is called $\mathcal{D}$-pseudocompact if every sequence of non-empty open subsets of X has a $\mathcal{D}$-limit point.

Making use of arguments similar to those in 2.4, we can readily establish the following facts.

3.2 Theorem. Every $\mathcal{D}$-pseudocompact space is pseudocompact.

3.3 Theorem. Every product of $\mathcal{D}$-pseudocompact spaces is $\mathcal{D}$-pseudocompact.

A corollary of these two theorems is that every power of a $\mathcal{D}$-pseudocompact space is pseudocompact. Now, it follows from Theorem 4 of [21], that any product of pseudocompact, locally compact spaces is pseudocompact, and that any product of pseudocompact, first countable space is pseudocompact. Since there is no reason, in general, to expect such products to be $\mathcal{D}$-pseudocompact, one cannot hope for a result analogous to Theorem 2.6 for pseudocompact powers of completely regular spaces. This can be seen in another way. In [21], Glicksberg shows that a product of completely regular spaces $\prod_{\alpha\in I} X_\alpha$ is pseudocompact if, and only if, every countable subproduct is pseudocompact. Now, for a sequence of sets $\sigma = (S_n : n \in \underset{\sim}{N})$ in X, let $L(\sigma) = \{\mathcal{D} \in \beta\underset{\sim}{N}-\underset{\sim}{N} : \sigma$ has a $\mathcal{D}$-limit point$\}$. Let Σ be the set of sequences (of points) in X, and let Σ_G be the set of all sequences of non-empty open subsets of X. The proof of 2.6 really shows that every power of X is countably compact if, and only if, for <u>every</u> subset T of Σ, $\bigcap_{\sigma\in T} L(\sigma) \neq \phi$. Since every power of a completely regular space X is pseudocompact if, and only if, X^{ω_0} is pseudocompact, we can, in a similar way, conclude that every power of X is pseudocompact if, and only if, for every <u>countable</u> subset T of Σ_G, $\bigcap_{\sigma\in T} L(\sigma) \neq \phi$.

3.4 Example. <u>A completely regular space, all of whose powers are pseudocompact, which is not $\mathcal{D}$-pseudocompact for any $\mathcal{D}$ in $\beta\underset{\sim}{N}-\underset{\sim}{N}$.</u>

For each p in $\beta\underset{\sim}{N}-\underset{\sim}{N}$, let $X_p = \beta\underset{\sim}{N}-\{p\}$. Let $X = \prod_{p\in\beta\underset{\sim}{N}-\underset{\sim}{N}} X_p$. Since every power of X is a product of locally compact, pseudocompact spaces, every power of X is pseudocompact. But the factor $X_{\mathcal{D}}$ of X is not $\mathcal{D}$-pseudocompact, and so X is not $\mathcal{D}$-pseudocompact for any $\mathcal{D}$ in $\beta\underset{\sim}{N}-\underset{\sim}{N}$.

3.5 If a space X has a dense subset D such that every sequence in D has an accumulation point in X, then obviously X is pseudocompact. Many of the familiar examples of pseudocompact spaces have this property, and this criterion for pseudocompactness has been used profitably in many instances. We refer the reader to [3] and [13] for excellent examples of this.

With this in mind, another natural application of $\mathcal{D}$-compactness to the study of pseudocompactness arises. Let us consider spaces X which have a dense subset A such that every sequence in A has a $\mathcal{D}$-limit point in X. Calling such spaces <u>densely-$\mathcal{D}$-compact</u>, we can establish the following theorem.

3.6 Theorem. <u>Every product of densely-$\mathcal{D}$-compact spaces is densely-$\mathcal{D}$-compact. Every densely-$\mathcal{D}$-compact space is $\mathcal{D}$-pseudocompact.</u>

Proof. The first assertion follows in a straightforward manner, using an argument similar to that in 2.4. To prove the second statement, let X be densely-$\mathcal{D}$-compact. Let A be a dense subset of X such that every sequence in A has a $\mathcal{D}$-limit point in X. Now, let $(G_n : n \in \underset{\sim}{N})$ be any sequence of non-empty open sets in X. For each n, there exists a point $a_n \in G_n \cap A$. Let $p \in X$ be a $\mathcal{D}$-limit point of the sequence $(a_n : n \in \underset{\sim}{N})$. Then, clearly p is a $\mathcal{D}$-limit point of the sequence $(G_n : n \in \underset{\sim}{N})$. Therefore, X is $\mathcal{D}$-pseudocompact.

4. Examples of Pseudocompact Spaces. In this section we prove several theorems which provide new examples of non-trivial pseudocompact spaces.

Let us first recall the notion of type in $\beta\mathbb{N}-\mathbb{N}$. The equivalence relation ~ defined on $\beta\mathbb{N}-\mathbb{N}$ by $x \sim y$ if there exists a homeomorphism of $\beta\mathbb{N}$ onto itself taking x to y, decomposes $\beta\mathbb{N}-\mathbb{N}$ into equivalence classes called types. For $p \in \beta\mathbb{N}-\mathbb{N}$, $T(p)$ denotes the type of p. Recall that, for any $p \in \beta\mathbb{N}-\mathbb{N}$, $T(p)$ is dense in $\beta\mathbb{N}-\mathbb{N}$. (See 6S in [15], and [12].) Note also that every type is a homogeneous space.

We are indebted to Z. Frolik for communicating the following lemma.

4.1 Lemma. (Frolik). Let T be a type of $\beta\mathbb{N}-\mathbb{N}$. Then no countable discrete subset of T has a cluster point in T.

Proof. Suppose the statement is false. We shall derive a contradiction. Thus, let $(x_n : n \in \mathbb{N})$ be a discrete subset of a type T which has a cluster point in T, say x. Find pairwise disjoint, infinite subsets $\{A_n : n \in \mathbb{N}\}$ of N such that $\mathbb{N} = \bigcup_{n\in\mathbb{N}} A_n$ and $x_n \in cl_{\beta\mathbb{N}} A_n$ for each n.

Now, for each n, x_n and x are of the same type, so we can find, for each n, a homeomorphism $f_n : \beta\mathbb{N} \to \beta\mathbb{N}$ such that $f_n(x_n) = x$. Let g_n denote the restriction of f_n to A_n. Define

$F: \mathbb{N} \to \mathbb{N}$ by $F = \bigcup_{n \in N} g_n$. Let F^β denote the Stone extension of F to $\beta\mathbb{N}$. Continuity implies that $F^\beta(x_n) = x$ for each n, and therefore implies that $F^\beta(\mathrm{cl}_{\beta\mathbb{N}}\{x_n : n \in \mathbb{N}\}) = \{x\}$. Thus $F^\beta(x) = x$.

We now appeal to a result of Katetov, in [26], which implies that the fixed points of F^β are precisely the points in the $\beta\mathbb{N}$-closure of the set of fixed points of F. (For a detailed proof, see Lemma 9.1 in [6].). Thus, letting $U = \{p \in \beta\mathbb{N} : F^\beta(p) = p\}$, we have $U = \mathrm{cl}_{\beta\mathbb{N}}(U \cap \mathbb{N})$. In particular, U is open in $\beta\mathbb{N}$. Since $x \in U$, and since x is a cluster point of $\{x_n : n \in \mathbb{N}\}$, there is an integer k such that $x_k \in U$. For such an integer k, we then have to conclude that $x_k = F^\beta(x_k) = x$.

But this is ridiculous, since x is a cluster point of $\{x_n : n \in \mathbb{N}\}$ and $\{x_n : n \in \mathbb{N}\}$ is discrete.

As was remarked in 3.5, the pseudocompactness of many familiar spaces can be deduced by the presence of a relatively countably compact dense subspace. One of the first examples of a pseudocompact space which has no dense countably compact subspace appears in [35]. The following Theorem 4.2, together with Lemma 4.1, shows there are pseudocompact spaces in which no countable discrete set has a cluster point. Assuming the continuum hypothesis, in 4.3 below, we exhibit a pseudocompact space in which no countable set has a cluster point. It follows that, in all of these examples, there is no dense relatively countably compact subspace. In Theorem 4.5 we show that these spaces have all of their powers pseudocompact. These results show that pseudocompact spaces can be as far from countably compact as is imaginable.

We will show that, if q is a non-P-point of $\beta\mathbb{N}-\mathbb{N}$, then $T(q)$ is pseudocompact. (For the definition and basic properties of P-points, see 4K and 4L of [15].) We use the fact that, if q is a non-P-point of $\beta\mathbb{N}-\mathbb{N}$, there exists a partition $\{B_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into infinite sets, such that for each $A \in q$, we have $\{n : A \cap B_n \text{ is infinite}\}$ is infinite. This can be shown directly, as in Lemma 9.14 of [6].

Our original theorem on the pseudocompactness of types held for a more restricted class of types. We are grateful to W. W. Comfort for pointing out that our construction works for all non-P-point types.

4.2 Theorem. _If q is a non-P-point of $\beta\mathbb{N}-\mathbb{N}$, then $T(q)$ is pseudocompact._

Proof. By the result of Glicksberg's quoted earlier, it is sufficient to prove that every sequence of pairwise disjoint, non-empty open subsets of $T(q)$ has a cluster point in $T(q)$.

Thus, let $(G_n : n \in \mathbb{N})$ be a sequence of pairwise disjoint, non-empty open subsets of $T(q)$. For each n, there is an infinite subset A_n of $\mathbb{N}$ such that $(\mathrm{cl}_{\beta\mathbb{N}} A_n) \cap T(q) \subseteq G_n$. We claim that, for $n \neq m$, $A_n \cap A_m$ is finite. For, if $A_m \cap A_n$ was infinite, then $cl_{\beta\underset{\sim}{N}} A_m \cap cl_{\beta\underset{\sim}{N}} A_n$ would be an open subset of $\beta\underset{\sim}{N}$ that meets $\beta\underset{\sim}{N}-\underset{\sim}{N}$. The density of $T(q)$ would imply that $(cl_{\beta\underset{\sim}{N}} A_m) \cap (cl_{\beta\underset{\sim}{N}} A_n) \cap T(q) \neq \phi$, which contradicts the disjointness of G_n and G_m. Thus $n \neq m$ implies that $A_n \cap A_m$ is finite. For each n, let $A_n' = A_n - \bigcup_{i<n} A_i$. Then $\{A_n' : n \in \underset{\sim}{N}\}$ is a family of pairwise disjoint infinite subsets of $\underset{\sim}{N}$ such that $(cl_{\beta\underset{\sim}{N}} A_n') \cap T(q) \subseteq G_n$ for each n. Let $C_1 = A_1' \cup (N - \bigcup_{n \in N} A_n')$ and let $C_n = A_n'$ for $n > 1$. To find a cluster point of $(G_n : n \in \underset{\sim}{N})$ it clearly suffices to find a cluster point of the sequence $((cl_{\beta\underset{\sim}{N}} C_n) \cap T(q) : n \in \underset{\sim}{N})$.

We have thus reduced the task of showing $T(q)$ is pseudocompact to the following: We must show that, for every partition of $\underset{\sim}{N}$ into infinite sets $\{A_n : n \in \underset{\sim}{N}\}$, the sequence $((cl_{\beta\underset{\sim}{N}} A_n) \cap T(q) : n \in \underset{\sim}{N})$ has a cluster point in $T(q)$. To this end, let $\underset{\sim}{N} = \bigcup_{n \in \underset{\sim}{N}} A_n$ be such a partition. Since q is not a P-point of $\beta\underset{\sim}{N}-\underset{\sim}{N}$, there is a partition $\underset{\sim}{N} = \bigcup_{n \in \underset{\sim}{N}} B_n$ of $\underset{\sim}{N}$ into infinite sets, such that, for each $A \in q$, $\{n : A \cap B_n \text{ is infinite}\}$ is infinite. Let $f: \underset{\sim}{N} \to \underset{\sim}{N}$ be a bijection taking B_n onto A_n for every n. Let f^β denote its Stone-extension to $\beta\underset{\sim}{N}$, and let $p = f^\beta(q)$. Then $p \in T(q)$. We claim that p is a cluster point of the sequence $((cl_{\beta\underset{\sim}{N}} A_n) \cap T(q) : n \in \underset{\sim}{N})$. To prove this, let $(cl_{\beta\underset{\sim}{N}} A) \cap T(q)$ be any basic neighbourhood of p in $T(q)$. Then $A \in p$, and so $f^{-1}(A) \in q$. The set $\{n: f^{-1}(A) \cap B_n$ is infinite$\}$ is infinite. Since f is a bijection, for infinitely many n, $A \cap A_n$ is an infinite set. But for any such n, $cl_{\beta\mathbb{N}} A \cap cl_{\beta\mathbb{N}} A_n$ is an open subset of $\beta\mathbb{N}$ that meets $\beta\mathbb{N}-\mathbb{N}$. Since $T(q)$ is dense in $\beta\mathbb{N}-\mathbb{N}$, for any such n, $[(cl_{\beta\mathbb{N}} A) \cap T(q)] \cap [(cl_{\beta\mathbb{N}} A_n) \cap T(q)] = (cl_{\beta\mathbb{N}} A) \cap (cl_{\beta\mathbb{N}} A_n) \cap T(q) \neq \phi$. Thus every neighbourhood of p in $T(q)$ meets infinitely many of the sets $(cl_{\beta\mathbb{N}} A_n) \cap T(q)$. That is, p is a cluster point of $((cl_{\beta\mathbb{N}} A_n) \cap T(q)\colon n \in \mathbb{N})$. As we observed at the beginning of the proof, this enables us to conclude that $T(q)$ is pseudocompact.

4.3 Remark. In [42], assuming the continuum hypothesis, M. E. Rudin shows there exists a non-P-point q in $\beta\mathbb{N}-\mathbb{N}$ such that q is not in the closure of any countable subset of $\beta\mathbb{N}-\mathbb{N}$. By our Theorem 4.2, for such q, $T(q)$ is a pseudocompact space in which no countable subset has a limit point. The assumption that q is not a P-point in 4.2 is essential, since a pseudocompact P-space is finite, and therefore no P-point type is pseudocompact.

The following theorem shows that the non-P-point types are not only pseudocompact, they are, in fact $\mathcal{D}$-pseudocompact. We thus see that $\mathcal{D}$-pseudocompactness arises in very natural and fundamental spaces. As the reader will observe, our proof of the $\mathcal{D}$-pseudocompactness of types exploits the "homogeneity" of $\beta\mathbb{N}$.

4.4 Theorem. <u>Let q be a non-P-point of $\beta\mathbb{N}-\mathbb{N}$. Then there exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that $T(q)$ is $\mathcal{D}$-pseudocompact.</u>

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be an infinite collection of pairwise disjoint infinite subsets of N with $N - \bigcup_{n \in N} A_n$ infinite. Since $T(q)$ is pseudocompact, the sequence $((cl_{\beta\mathbb{N}} A_n) \cap T(q) : n \in \mathbb{N})$ has a cluster point $p \in T(q)$. A proof completely analogous to Lemma 2.2 shows there exists a free ultrafilter $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that p is a $\mathcal{D}$-limit point of $((cl_{\beta\mathbb{N}} A_n) \cap T(q) : n \in \mathbb{N})$. We will show that for this $\mathcal{D}$, $T(q)$ is $\mathcal{D}$-pseudocompact.

Thus, let $(G_n : n \in \mathbb{N})$ be any sequence of non-empty open subsets of $T(q)$. For each n, find an infinite subset B_n of $\mathbb{N}$ such that $(cl_{\beta\mathbb{N}} B_n) \cap T(q) \subseteq G_n$. Using the Disjoint Refinement Lemma 7.5 of [6], we can find a pairwise disjoint sequence $(C_n : n \in \mathbb{N})$ of infinite subsets of $\mathbb{N}$, such that $\mathbb{N} - \bigcup_{n \in \mathbb{N}} C_n$ is infinite and $C_n \subseteq B_n$ for each n. It clearly suffices to show that $((cl_{\beta\mathbb{N}} C_n) \cap T(q) : n \in \mathbb{N})$ has a $\mathcal{D}$-limit point in $T(q)$, for such a point will be a $\mathcal{D}$-limit point of $(G_n : n \in \mathbb{N})$. Now, let $f : \mathbb{N} \to \mathbb{N}$ be a bijection taking A_n onto C_n for every n. Let $r = f^{\beta}(p)$. Then $r \in T(q)$. It follows easily that r is a $\mathcal{D}$-limit point of $((cl_{\beta\mathbb{N}} C_n) \cap T(q) : n \in \mathbb{N})$ in $T(q)$. Thus every sequence of open sets in $T(q)$ has a $\mathcal{D}$-limit point. That is, $T(q)$ is $\mathcal{D}$-pseudocompact.

4.5 Theorem. If q is a non-P-point of $\beta\underset{\sim}{N}-\underset{\sim}{N}$, then every power of $T(q)$ is pseudocompact.

Proof. This follows immediately from 4.4, 3.2, and 3.3.

4.6 Remark. Theorem 4.4 and Lemma 4.1 show that, for a non-P-point q of $\beta\underset{\sim}{N}-\underset{\sim}{N}$, $T(q)$ is an example of a $\mathcal{D}$-pseudocompact space which is not densely E-compact for any E in $\beta\underset{\sim}{N}-\underset{\sim}{N}$. Thus the converse of the second statement in 3.6 is false.

Our last result in this section is very special in nature. It exhibits certain pseudocompact subspaces between X and βX.

4.7 Recall that a completely regular space X is said to be extremally disconnected if every open subset of X has open closure, and basically disconnected if every cozero-set in X has open closure. For the elementary properties of these spaces, the reader is referred to 1H and 6M in [15].

4.8 Theorem. Let X be basically disconnected and locally compact, and let D be a dense subset of $\beta X-X$. Then $X \cup D$ is pseudocompact.

Proof. We will assume not and reach a contradiction. If $X \cup D$ is not pseudocompact, there is an unbounded function f in $C(X \cup D)$. Let $g = (f^2+1)^{-1}$. Then $g \in C(X \cup D)$, $0 < g \leq 1$ and $\inf_{t \in X \cup D} g(t) = 0$. Let g^β denote the Stone extension of g to $\beta(X \cup D) = \beta X$. Since X is dense in $X \cup D$, for every n there is a point x_n in X such that $g(x_n) < \frac{1}{n}$. For each n, let $G_n = \{x \in X: g(x) < \frac{1}{n}\}$. Since X is locally compact, we can find, for each n, a cozero-set H_n containing x_n, whose closure is compact and is contained in G_n. Let $W = \bigcup_{n \in N} H_n$. Then W is a cozero-set in X. Now $cl_X W$ is not compact, since the function g on $cl_X W$ does not attain its infimum on $cl_X W$. Thus $cl_{\beta X} W - X \neq \phi$. But, since X is basically disconnected, $cl_{\beta X} W = cl_{\beta X}(cl_X W)$ is open in βX. Since it meets $\beta X - X$, it must meet D, since D is dense in $\beta X - X$. Let $p \in (cl_{\beta X} W) \cap D$. Since g is strictly positive on $X \cup D$, there is an integer m such that $g(p) > \frac{1}{m}$. Let $V = \{x \in \beta X: g^{\beta}(x) > \frac{1}{m}\}$. Then $V - \bigcup_{i=1}^{m} cl_X H_i$ is a neighbourhood of p in βX which is disjoint from W. This is ridiculous, since $p \in cl_{\beta X} W$. This contradiction proves that $X \cup D$ is pseudocompact.

4.9 Remark. The conclusion of Theorem 4.8 holds whenever X is locally compact and realcompact. This result was established by Fine and Gillman in Theorem 3.1 of [9]. A special case of 4.8, for extremally disconnected, locally compact spaces, combined with Theorem 3.1 in [51], yields the Fine - Gillman result as a corollary.

Chapter 2

COUNTABLE COMPACTNESS AND PSEUDOCOMPACTNESS IN HYPERSPACES

1. We are concerned here with the countable compactness and pseudocompactness of 2^X. The first significant results concerning this theme are found in [27], where it is shown that 2^X is ω_0-bounded when X is normal and ω_0-bounded, and where it is pointed out that 2^X need not be countably compact or pseudocompact when X is. Although we are unable to characterize those spaces whose hyperspaces are countably compact (or pseudocompact), we obtain substantial generalizations of the results in [27] mentioned above. The concepts and theorems of Chapter 1 are applied in establishing the following results. X is $\mathcal{D}$-compact if, and only if, 2^X is $\mathcal{D}$-compact. X is $\mathcal{D}$-pseudocompact if, and only if, 2^X is $\mathcal{D}$-pseudocompact. If all powers of X are countably compact, then 2^X is countably compact. If 2^X is countably compact, then all finite powers of X are countably compact. If X is completely regular and 2^X is pseudocompact, then all finite powers of X are pseudocompact. We give an example of a completely regular space Y, all of whose finite powers are countably compact, such that 2^Y is not pseudocompact.

We assume, from now on, that all spaces considered are T_1 spaces.

2. Some Theorems on the Countable Compactness and Pseudocompactness of 2^X. Our first result compares the $\mathcal{D}$-compactness of 2^X with that of X.

2.1 Theorem. Let $\mathcal{D}$ be a free ultrafilter on $\underset{\sim}{N}$. Then X is $\mathcal{D}$-compact if, and only if, 2^X is $\mathcal{D}$-compact.

Proof. Suppose 2^X is $\mathcal{D}$-compact. Let $(x_n: n \in \underset{\sim}{N})$ be any sequence in X. The sequence $(\{x_n\}: n \in \underset{\sim}{N})$ in 2^X has a $\mathcal{D}$-limit point F in 2^X. Let p be any point of F. If G is a neighbourhood of p in X, then, since $F \cap G \neq \phi$, $B(X; G)$ is a neighbourhood of F in 2^X. Since F is a $\mathcal{D}$-limit point of $(\{x_n\}: n \in \underset{\sim}{N})$, we have $\{n: \{x_n\} \in B(X; G)\} \in \mathcal{D}$. But $\{n: \{x_n\} \in B(X; G)\} = \{n: x_n \in G\}$. Thus, for every neighbourhood G of p in X, $\{n: x_n \in G\} \in \mathcal{D}$, and so p is a $\mathcal{D}$-limit point of $(x_n: n \in \underset{\sim}{N})$ in X. This shows X is $\mathcal{D}$-compact.

For the converse, suppose X is $\mathcal{D}$-compact. We show 2^X is $\mathcal{D}$-compact. Thus, let $(F_n: n \in \underset{\sim}{N})$ be a sequence in 2^X. Let $L = \{p \in X: p \text{ is a } \mathcal{D}\text{-limit point of the sequence } (F_n: n \in \underset{\sim}{N})\}$. Clearly L is a non-empty, closed subset of X. That is, $L \in 2^X$. We claim that L is a $\mathcal{D}$-limit point of the sequence $(F_n: n \in \underset{\sim}{N})$ in 2^X. To see this, let $\mathcal{W} = B(G_0;G_1,\ldots,G_T)$ be a basic neighbourhood of L in 2^X. We must show that $\{n: F_n \in \mathcal{W}\} \in \mathcal{D}$. Now let $N_0 = \{n \in \underset{\sim}{N}: F_n \subseteq G_0\}$, and for $i \in \{1,2,\ldots,T\}$ let $N_i = \{n \in \underset{\sim}{N}: F_n \cap G_i \neq \phi\}$. Clearly $\{n \in \underset{\sim}{N}: F_n \in \mathcal{W}\} = \bigcap_{i=0}^{T} N_i$. Thus, to show that $\{n \in \underset{\sim}{N}: F_n \in \mathcal{W}\} \in \mathcal{D}$, we need to prove that

$N_i \in \mathcal{D}$ for each $i \in \{0,1,\ldots,T\}$. Now, since $L \in \mathcal{U}$, we have $L \cap G_i \neq \phi$. Let $p \in L \cap G_i$. Then p is a $\mathcal{D}$-limit point of the sequence $(F_n: n \in \underset{\sim}{N})$ and G_i is a neighbourhood of p, so $\{n: G_i \cap F_n \neq \phi\} = N_i \in \mathcal{D}$. Thus $N_i \in \mathcal{D}$ for $i = 1,2,\ldots,T$. Finally, we show $N_0 \in \mathcal{D}$. For the sake of contradiction, assume $N_0 \notin \mathcal{D}$. Then $\underset{\sim}{N}-N_0 \in \mathcal{D}$. For each $n \in \underset{\sim}{N}-N_0$, choose a point $x_n \in F_n - G_0$. For each $n \in N_0$, choose a point x_n arbitrarily from F_n. The sequence $(x_n: n \in \underset{\sim}{N})$, so obtained, has a $\mathcal{D}$-limit point a, by the $\mathcal{D}$-compactness of X. Clearly, a is a $\mathcal{D}$-limit point of the sequence $(F_n: n \in \underset{\sim}{N})$, and so $a \in L$. But $L \in \mathcal{U}$, so that $L \subseteq G_0$. Therefore, $a \in G_0$. Since a is a $\mathcal{D}$-limit point of the sequence $(x_n: n \in \underset{\sim}{N})$, we have $\{n: x_n \in G_0\} \in \mathcal{D}$. But this last set is disjoint from $\underset{\sim}{N}-N_0$, which also lies in $\mathcal{D}$. This is a contradiction. Therefore, $N_0 \in \mathcal{D}$, and L is a $\mathcal{D}$-limit point of the sequence $(F_n: n \in \underset{\sim}{N})$ in 2^X. Thus 2^X is $\mathcal{D}$-compact.

From 2.1, we can obtain, as a corollary, the following theorem due to J. Keesling, [27].

2.2 Corollary. Let X be a normal space. Then X is ω_0-bounded if, and only if, 2^X is ω_0-bounded.

Proof. If X is normal, then, by 4.9.5 of [36], 2^X is completely regular. By Theorem 2.4 of Chapter 1, 2^X is ω_0-bounded if, and only if, it is $\mathcal{D}$-compact for every free ultrafilter $\mathcal{D}$ on $\underset{\sim}{N}$. By 2.1, this happens exactly when X is $\mathcal{D}$-compact for all free ultrafilters $\mathcal{D}$ on $\underset{\sim}{N}$, which is equivalent to X being ω_0-bounded.

Theorem 2.1 also allows us to establish the following relation between the countable compactness of 2^X and that of powers of X.

2.3 Corollary. Let X be a Hausdorff space. If all powers of X are countably compact, then 2^X is countably compact. If 2^X is countably compact, then all finite powers of X are countably compact.

Proof. If all powers of X are countably compact, then by 2.6 of Chapter 1, there is a free ultrafilter $\mathcal{D}$ on $\underset{\sim}{N}$ such that X is $\mathcal{D}$-compact. By 2.1 above, 2^X is also $\mathcal{D}$-compact, and so, in particular, is countably compact.

Suppose 2^X is countably compact. For each $n \in \underset{\sim}{N}$, let $F_n(X) = \{F \in 2^X\colon |F| \leq n\}$. By 2.4 of [36], $F_n(X)$ is a closed subspace of 2^X for each $n \in \underset{\sim}{N}$. For each n, define the map $S_n\colon X^n \to F_n(X)$ by $S_n(x_1,x_2,\ldots,x_n) = \{x_1,x_2,\ldots,x_n\}$. Then, for each n, S_n is a continuous, closed, finite-to-one map from X^n onto $F_n(X)$, [14]. As countable compactness is closed hereditary and preserved under perfect pre-images, the countable compactness of 2^X implies that of X^n for each $n \in \underset{\sim}{N}$.

We next turn to pseudocompactness. The next result is an analogy to 2.1.

2.4 Theorem. Let $\mathcal{D}$ be a free ultrafilter on $\mathbb{N}$. Then X is $\mathcal{D}$-pseudocompact if, and only if, 2^X is $\mathcal{D}$-pseudocompact.

Proof. Suppose 2^X is $\mathcal{D}$-pseudocompact. We show that X is $\mathcal{D}$-pseudocompact. Thus, let $(G_n : n \in \mathbb{N})$ be a sequence of non-empty open subsets of X. Then $(2^{G_n} : n \in \mathbb{N})$ is a sequence of non-empty open subsets of 2^X. As 2^X is $\mathcal{D}$-pseudocompact, this sequence has a $\mathcal{D}$-limit point $F \in 2^X$. Choose any point $p \in F$. We show that p is a $\mathcal{D}$-limit point, in X, of the sequence $(G_n : n \in \mathbb{N})$. For, let W be any neighbourhood of p in X. Then, since $F \cap W \neq \phi$, $2^X - 2^{X-W}$ is a neighbourhood of F in 2^X. Since F is a $\mathcal{D}$-limit point of the sequence $(2^{G_n} : n \in \mathbb{N})$, $\{n : 2^{G_n} \cap (2^X - 2^{X-W}) \neq \phi\} \in \mathcal{D}$. But $2^{G_n} \cap (2^X - 2^{X-W}) \neq \phi$ if, and only if, $G_n \cap W \neq \phi$. Thus $\{n : G_n \cap W \neq \phi\} \in \mathcal{D}$, and so p is a $\mathcal{D}$-limit point of the sequence $(G_n : n \in \mathbb{N})$. Therefore X is $\mathcal{D}$-pseudocompact.

Conversely, suppose X is $\mathcal{D}$-pseudocompact. Since the sets $B(G_0, G_1, \ldots, G_T)$, with $G_0, G_1, \ldots, G_T$ open in X and $\bigcup_{i=1}^{T} G_i \subseteq G_0$, form a basis for the topology on 2^X, to show that 2^X is $\mathcal{D}$-pseudocompact we need only show that sequences of such open sets have $\mathcal{D}$-limit points. Thus, suppose we are given a sequence $(\mathcal{G}_n : n \in \mathbb{N})$ of non-empty basic open sets $\mathcal{G}_n$ in 2^X. Write $\mathcal{G}_n$ as $B(G_{0,n}, G_{1,n}, \ldots, G_{T_n,n})$, with $G_{i,n}$ open in X and $\bigcup_{i=1}^{T_n} G_{i,n} \subseteq G_{0,n}$. Let $L = \{p \in X : p$ is a $\mathcal{D}$-limit point of the sequence $(G_{0,n}: n \in \underset{\sim}{N})$. Then L is a non-empty, closed subset of X. That is, $L \in 2^X$. We claim that L is a $\mathcal{D}$-limit point, in 2^X, of the sequence $(\mathcal{G}_n: n \in \underset{\sim}{N})$. Now the sets of the form 2^G and B(X; G) form a sub-basis for 2^X. Since filters are closed under finite intersection, to show that L is a $\mathcal{D}$-limit point of $(\mathcal{G}_n: n \in \underset{\sim}{N})$, it is enough to establish the following two statements:

(i) If G is open in X and $L \in 2^G$, then $\{n \in \underset{\sim}{N}: 2^G \cap \mathcal{G}_n \neq \phi\} \in \mathcal{D}$.

(ii) If G is open in X and $L \in B(X; G)$, then

$\{n \in \underset{\sim}{N}: B(X; G) \cap \mathcal{G}_n \neq \phi\} \in \mathcal{D}$.

Let us first establish (i). Note that $2^G \cap \mathcal{G}_n \neq \phi$ if, and only if, $G \cap G_{i,n} \neq \phi$ for all $i = 1,2,\ldots,T_n$. Let $S = \{n \in \underset{\sim}{N}: 2^G \cap \mathcal{G}_n \neq \phi\}$ and let $T = \underset{\sim}{N}-S$. For the sake of contradiction, suppose $S \notin \mathcal{D}$. Then $T \in \mathcal{D}$. For each $n \in T$, find an integer $i_n \in \{1,2,\ldots,T_n\}$ such that $G \cap G_{i_n,n} = \phi$. Define a sequence $(H_n: n \in \underset{\sim}{N})$ of non-empty open subsets of X as follows. For $n \in T$, $H_n = G_{i_n,n}$, and for $n \in S$, $H_n = G_{1,n}$. Now, since X is $\mathcal{D}$-pseudocompact, the sequence $(H_n: n \in \underset{\sim}{N})$ has a $\mathcal{D}$-limit point $a \in X$. Clearly $a \in L$. Since $L \in 2^G$, we have $L \subseteq G$, and so $a \in G$. Since a is a $\mathcal{D}$-limit point of $(H_n: n \in \underset{\sim}{N})$, $\{n \in \underset{\sim}{N}: G \cap H_n \neq \phi\} \in \mathcal{D}$. But this latter set is disjoint from T, and $T \in \mathcal{D}$. This is a contradiction. Therefore, $S \in \mathcal{D}$, establishing (i). To establish (ii), suppose G is open in X and $L \in B(X; G)$. Observe that $B(X; G) \cap \mathcal{G}_n \neq \phi$ if, and only if, $G \cap G_{0,n} \neq \phi$. Let $M = \{n \in \underset{\sim}{N}: B(X; G) \cap \mathcal{G}_n \neq \phi\}$ $= \{n \in \underset{\sim}{N}: G \cap G_{0,n} \neq \phi\}$. Now, since $L \in B(X; G)$, $L \cap G \neq \phi$. Let $p \in L \cap G$. Then p is a $\mathcal{D}$-limit point of the sequence $(G_{0,n}: n \in \underset{\sim}{N})$, and G is a neighbourhood of p. Therefore, $\{n \in \underset{\sim}{N}: G \cap G_{0,n} \neq \phi\} \in \mathcal{D}$. That is, $M \in \mathcal{D}$, establishing (ii).

We have thus shown that L is a $\mathcal{D}$-limit point of $(\mathcal{G}_n: n \in \underset{\sim}{N})$. Therefore, 2^X is $\mathcal{D}$-pseudocompact, as desired.

Even having established 2.4, we cannot conclude that the pseudocompactness of all powers of X implies the pseudocompactness of 2^X, at least not by an argument analogous to the one used in 2.3. The problem here is that $\mathcal{D}$-pseudocompactness is not a necessary condition for pseudocompact powers. (See 3.4 in Chapter 1.) We can, however, establish a pseudocompact counterpart to the second assertion in 2.3. Let us call a space X <u>$\mathcal{G}$-pseudocompact</u> if every sequence of non-empty open subsets of X has a cluster point in X. (That is, a point in X, each of whose neighbourhoods meets infinitely many sets in the sequence.) These spaces have also been called feebly compact in the literature. (See [45].) As was mentioned in the first chapter, in the class of completely regular spaces, $\mathcal{G}$-pseudocompactness and pseudocompactness coincide. In general, $\mathcal{G}$-pseudocompactness implies pseudocompactness.

2.5 Theorem. Let X be regular. If 2^X is G-pseudocompact, then all finite powers of X are G-pseudocompact.

Proof. Assume 2^X is G-pseudocompact. Firstly, X is G-pseudocompact. For, if $(G_n : n \in \underset{\sim}{N})$ is a sequence of non-empty open subsets of X, the sequence $(2^{G_n} : n \in \underset{\sim}{N})$ has a limit point L in 2^X. Choosing any point $p \in L$, it is easy to see that p is a limit point of $(G_n : n \in \underset{\sim}{N})$. Thus every sequence of non-empty open subsets of X has a limit point in X. That is, X is G-pseudocompact.

Next, we show that $X \times X$ is G-pseudocompact, for which it suffices to show that every sequence $(U_n \times V_n : n \in \underset{\sim}{N})$ where U_n, V_n are non-empty, open subsets of X, has a limit point in $X \times X$. We will assume not, and we will derive a contradiction. So assume $(U_n \times V_n : n \in \underset{\sim}{N})$ has no limit point in $X \times X$. Now X is G-pseudocompact, as has already been established, so the sequence $(U_n : n \in \underset{\sim}{N})$ has a limit point $p \in X$. Since $(U_n \times V_n : n \in \underset{\sim}{N})$ has no limit point in $X \times X$, in particular, (p,p) is not a limit point of $(U_n \times V_n : n \in \underset{\sim}{N})$. Therefore, there is a neighbourhood W of p in X such that $\{n \in \underset{\sim}{N} : (W \times W) \cap (U_n \times V_n) \neq \phi\}$ is finite. Let $S = \{n \in \underset{\sim}{N} : (W \times W) \cap (U_n \times V_n) \neq \phi\}$. By regularity, find a neighbourhood W_1 of p in X such that $cl_X W_1 \subseteq W$. Let $T = \{n \in \underset{\sim}{N} : W_1 \cap U_n \neq \phi\}$. Since p is a limit point of $(U_n : n \in \underset{\sim}{N})$,

T is infinite. Let $N_1 = T-S$. Then N_1 is infinite. Consider the sequence $((W_1 \cap U_n) \times V_n \colon n \in N_1)$. Being a refinement of a subsequence of $(U_n \times V_n \colon n \in \underset{\sim}{N})$, the sequence $((W_1 \cap U_n) \times V_n \colon n \in N_1)$ also has no limit point in $X \times X$. Let $A = \mathrm{cl}_X W_1$, and let $B = \mathrm{cl}_X(\bigcup_{n\in N_1} V_n)$. Then A and B are disjoint regular-closed subsets of X and $\bigcup_{n\in N_1} [(W_1 \cap U_n) \times V_n] \subseteq A \times B$. Now, since A and B are disjoint closed sets, $A \cup B$ is homeomorphic to $A + B$, the free union of A and B. By 5a., page 166 of [32], 2^{A+B} is homeomorphic to $2^A \times 2^B$. Now G-pseudocompactness is evidently inherited by regular-closed subsets. As 2^X is G-pseudocompact, so is $2^{A\cup B}$, and so, by the above remarks, so is $2^A \times 2^B$. It follows easily that $A \times B$ is G-pseudocompact. But $((W_1 \cap U_n) \times V_n \colon n \in \underset{\sim}{N})$ has no limit point in $X \times X$, which is a contradiction. Thus $X \times X$ is G-pseudocompact.

One can now prove by induction on n, that X^n is G-pseudocompact for all $n \in \underset{\sim}{N}$. The essential idea in going from X^n to X^{n+1} is the same as going from X^2 to X^3, but the details are more cumbersome. Accordingly, we will show how to deduce the G-pseudocompactness of X^3 from that of X^2 (and that of 2^X, of course), and leave the induction as a straightforward extension of this step.

Thus, from the G-pseudocompactness of 2^X and $X \times X$, we are to deduce the G-pseudocompactness of $X \times X \times X$. We assume that

$X \times X \times X$ is not G-pseudocompact, and we will reach a contradiction. So, let $(A_n \times B_n \times C_n : n \in \underset{\sim}{N})$ be an open sequence in X^3 which has no limit point. Now X^2 is G-pseudocompact, so the sequence $(A_n \times B_n : n \in \underset{\sim}{N})$ has a limit point (a,b) in $X \times X$. Neither (a,b,a) nor (a,b,b) is a limit point of $(A_n \times B_n \times C_n : n \in \underset{\sim}{N})$ in X^3. Thus we can find neighbourhoods G and H of a and b respectively, such that the two sets

$$M_1 = \{n \in \underset{\sim}{N} : (G \times H \times G) \cap (A_n \times B_n \times C_n) \neq \phi\} \text{ and}$$

$$M_2 = \{n \in \underset{\sim}{N} : (G \times H \times H) \cap (A_n \times B_n \times C_n) \neq \phi\} \text{ are finite.}$$

Find neighbourhoods G_1 and H_1 of a and b respectively such that $cl_X G_1 \subseteq G$ and $cl_X H_1 \subseteq H$. Let $M_3 = \{n \in \underset{\sim}{N} : (G_1 \times H_1) \cap (A_n \times B_n) \neq \phi\}$. Since (a,b) is a limit point of $(A_n \times B_n : n \in \underset{\sim}{N})$, M_3 is infinite. Now let $N_1 = M_3 - (M_1 \cup M_2)$. Then N_1 is infinite. Let $A_n' = G_1 \cap A_n$, and let $B_n' = H_1 \cap B_n$. The sequence $(A_n' \times B_n' \times C_n : n \in N_1)$, being a refinement of a subsequence of $(A_n \times B_n \times C_n : n \in \underset{\sim}{N})$, also has no limit point in X^3. But X^2 is G-pseudocompact, so the sequence $(B_n' \times C_n : n \in N_1)$ has a cluster point (c,d) in X^2. Neither (c,c,d) nor (d,c,d) is a cluster point of $(A_n' \times B_n' \times C_n : n \in N_1)$. So we find neighbourhoods U and V of c and d respectively, such that the two sets

$$L_1 = \{n \in N_1 : (U \times U \times V) \cap (A_n' \times B_n' \times C_n) \neq \phi\} \text{ and}$$

$$L_2 = \{n \in N_1 : (V \times U \times V) \cap (A_n' \times B_n' \times C_n) \neq \phi\} \text{ are finite.}$$

Find neighbourhoods U_1 and V_1 of c and d respectively, such that $cl_X U_1 \subseteq U$ and $cl_X V_1 \subseteq V$. Now, let

$$L_3 = \{n \in N_1 : (U_1 \times V_1) \cap (B_n' \times C_n) \neq \phi\}.$$

Since (c,d) is a limit point of $(B_n' \times C_n : n \in N_1)$, the set $N_2 = L_3 - (L_1 \cup L_2)$ is infinite. For $n \in N_2$, set $A_n'' = A_n'$, $B_n'' = U_1 \cap B_n'$, $C_n'' = V_1 \cap C_n$. The sequence $(A_n'' \times B_n'' \times C_n'' : n \in N_2)$ has no limit point in X^3. Let $A = cl_X(\bigcup_{n\in N_2} A_n'')$, $B = cl_X(\bigcup_{n\in N_2} B_n'')$, $C = cl_X(\bigcup_{n\in N_2} C_n'')$. Then A,B,C are pairwise disjoint regular-closed subsets of X, and $\bigcup_{n\in N_2} (A_n'' \times B_n'' \times C_n'') \subseteq A \times B \times C$. By the same argument used earlier, $2^A \times 2^B \times 2^C$ is homeomorphic to $2^{A\cup B\cup C}$, which, as a regular-closed subspace of 2^X, inherits G-pseudocompactness. Thus $A \times B \times C$ is G-pseudocompact, which contradicts the fact that $(A_n'' \times B_n'' \times C_n'' : n \in N_2)$ has no limit point. This contradiction proves that X^3 is G-pseudocompact.

As was mentioned above, a completely regular space X is G-pseudocompact if, and only if, it is pseudocompact. Although 2^X is completely regular only when X is normal, these concepts remain equivalent for 2^X when X is completely regular, as we now show.

2.6 Proposition. Let X be completely regular. Then 2^X is G-pseudocompact if, and only if, it is pseudocompact.

Proof. $\mathcal{G}$-pseudocompactness always implies pseudocompactness. We need only show that if 2^X is not $\mathcal{G}$-pseudocompact, then 2^X is not pseudocompact. If 2^X is not $\mathcal{G}$-pseudocompact, there is a sequence $\mathcal{G}_n = B(G_{0,n};G_{1,n},\ldots,G_{T_n,n})$ of non-empty basic open subsets of 2^X, which has no limit point in 2^X. For each n and each $i \in \{1,2,\ldots,T_n\}$, choose a point $p_{n,i} \in G_{i,n}$. Let $F_n = \{p_{n,i}: i = 1,2,\ldots,T_n\}$. Now $F_n \subseteq G_{0,n}$, so, by complete regularity, we can find, for each n, a continuous, real-valued function f_n on X such that $f_n(x) = 1$ for each $x \in F_n$, and $f_n(x) = 0$ for each $x \in X - G_{0,n}$, and such that $0 \leq f_n \leq 1$. Given n and $i \in \{1,2,\ldots,T_n\}$, by complete regularity, we can find a continuous, real-valued function $g_{n,i}$ on X such that $0 \leq g_{n,i} \leq 1$, $g_{n,i}(p_{n,i}) = 1$, and $g_{n,i}(x) = 0$ for each $x \in X - G_{i,n}$. Now, for each n, define f_n^- on 2^X by $f_n^-(F) = \inf_{x\in F} f_n(x)$. For each n and each $i \in \{1,2,\ldots,T_n\}$, define $g_{n,i}^+$ on 2^X by $g_{n,i}^+(F) = \sup_{x\in F} g_{n,i}(x)$. By 4.7 of [36], the functions f_n^- and $g_{n,i}^+$ are all continuous, real-valued functions on 2^X. Now, for each n, let $G_n = f_n^-\cdot g_{n,i}^+\cdot\ldots\cdot g_{n,T_n}^+$. Then G_n is continuous and $G_n(F_n) = 1$, and $G_n(F) = 0$ for each $F \in 2^X - \mathcal{G}_n$. Since the sequence $(\mathcal{G}_n: n \in \underset{\sim}{N})$ has no limit point, the function $\sum_{n\in N} n\, G_n$ is continuous on 2^X, and is clearly unbounded. Thus 2^X is not pseudocompact.

2.7 Corollary. Let X be completely regular. If 2^X is pseudocompact, then all finite powers of X are pseudocompact.

Proof. This follows immediately from 2.6 and 2.7.

3. An Example. In [12], Z. Frolik constructs, for each positive integer n, a space X, such that X^n is countably compact, but X^{n+1} is not pseudocompact. In [27], J. Keesling shows that the hyperspaces of these spaces are not pseudocompact. This conclusion also follows from 2.7. Also in [12], Frolik constructs a space Y, all of whose finite powers are countably compact, such that Y^{ω_0} is not pseudocompact. We will see below that 2^Y is not pseudocompact, thus providing a counterexample to the converse of 2.7 and to the converse of the last statement in 2.3.

3.1 Example. A completely regular space Y, all of whose finite powers are countably compact, such that 2^Y is not pseudocompact.

Frolik constructs a sequence X_i, for $i \in \underset{\sim}{N}$, of subspaces of $\beta\underset{\sim}{N}-\underset{\sim}{N}$, such that $\prod_{k\in\underset{\sim}{N}} \underset{\sim}{N} \cup X_k$ is not pseudocompact, while every finite subproduct is countably compact. In his example, $\bigcap_{i\in\underset{\sim}{N}} X_i = \phi$. The desired space Y is the free union of the spaces $\underset{\sim}{N} \cup X_i$, together with a point at infinity, whose neighbourhoods are complements of finitely many of the spaces $\underset{\sim}{N} \cup X_i$. To avoid ambiguity, let us replace $\underset{\sim}{N} \cup X_i$ by $Y_i = (\underset{\sim}{N} \cup X_i) \times \{i\}$. The space Y is then $(\bigcup_{i \in \underset{\sim}{N}} Y_i) \cup \{\infty\}$, with the topology described above. We will show that 2^Y is not pseudocompact. We will in fact produce an open-closed subspace of 2^Y homeomorphic to $\underset{\sim}{N}$. For each n, we let $F_n = \{(n,1), (n,2), \ldots, (n,n)\}$. Since each point of each copy of $\underset{\sim}{N}$ is isolated in Y, it follows that, for every n, F_n is an isolated point of-2^Y. Thus $D = \{F_n : n \in \underset{\sim}{N}\}$ is a discrete, open subspace of 2^Y, and our proof will be complete if we show D is closed in 2^Y. Let $A \in 2^Y$. We show that A is not a cluster point of D.

Case 1. $A \cap [\bigcup_{k \in \underset{\sim}{N}} \underset{\sim}{N} \times \{k\}] \neq \phi$. In this case, let $(n,k) \in A$. Now (n,k) is isolated in Y, so $B(Y; \{(n,k)\})$ is a neighbourhood of A in 2^Y. At most one F_i is in $B(Y; \{(n,k)\})$. Therefore A is not a cluster point of D.

Case 2. There is an integer i such that $A \cap Y_i = \phi$. In this case, 2^{Y-Y_i} is a neighbourhood of A in 2^Y meeting D in a finite set. Thus A is not a cluster point of D.

Case 3. For some integer i, $|A \cap Y_i| > 1$. In this case, A meets two disjoint open subsets G_1 and G_2 of Y_i. Since each F_n contains at most one element from each Y_i, $B(Y; G_1, G_2)$ is a neighbourhood of A in 2^Y that is disjoint from D. So again, A is not a cluster point of D.

Case 4. In light of the first three cases, we may now assume that $A = \{(x_n,n): n \in \underset{\sim}{N}\} \cup \{\infty\}$, where, for each n, $x_n \in X_n$. Now, since $\bigcap_{n\in \underset{\sim}{N}} X_n = \phi$, we can find integers n and m such that $x_n \neq x_m$. Find disjoint open sets U and V in $\beta\underset{\sim}{N}$ such that $x_n \in U$ and $x_m \in V$. Now set $U_1 = [U \cap (\underset{\sim}{N} \cup X_n)] \times \{n\}$, and $V_1 = [V \cap (\underset{\sim}{N} \cup X_m)] \times \{m\}$. Then U_1 and V_1 are open in Y, and $(x_n,n) \in U_1$, $(x_m,m) \in V_1$. Thus $B(Y; U_1,V_1)$ is a neighbourhood of A in 2^Y. Since $B(Y; U_1,V_1)$ is clearly disjoint from D, A is not a cluster point of D.

Cases 1 to 4 combine to show that D is closed in 2^Y, completing the proof.

3.2 Remark. In light of the results of 2.3 and 2.7, and Example 3.1, it is natural to ask whether there is any relation between the pseudocompactness (countable compactness) of X^{ω_0} and that of 2^X. It would also be interesting to characterize those spaces X whose hyperspaces are countably compact (pseudocompact). The author has been unable to resolve these questions, and leaves them open to the reader. Natural examples of $\mathcal{D}$-compact and $\mathcal{D}$-pseudocompact spaces can be found in Chapter 1. These spaces provide non-trivial examples of pseudo-compact and countably compact hyperspaces.

Chapter 3

THE STONE-CECH COMPACTIFICATION OF 2^X

1. In this chapter our attention is focused on the Stone-Cech compactification of the space of closed sets. Since 2^X is completely regular only when X is normal, (see 0.5 in Chapter 0), we must confine our attention to hyperspaces of normal spaces. As noted in 0.6 of Chapter 0, if X is normal the mapping $i: 2^X \to 2^{\beta X}$ defined by $i(F) = cl_{\beta X}F$ is an embedding of 2^X onto a dense subspace of $2^{\beta X}$. In this way we can regard $2^{\beta X}$ as a compactification of 2^X.

When given a "natural" compactification αY of a completely regular space Y, one of the most obvious questions one may ask about αY is whether it coincides with the Stone-Cech compactification of Y. Investigations of this sort have led to many interesting results. One of the most natural ways to form a compactification of a product of completely regular spaces is to take the product of the corresponding Stone-Cech compactifications. Glicksberg's elegant results in [21] show that this natural compactification of the product coincides with the Stone-Cech compactification of the product exactly when the product is pseudocompact. (Assuming all factors are infinite.) The simplest compactification of a locally compact space is its one-point compactification. Spaces whose one-point compactifications coincide with their Stone-Cech compactifications are characterized in 6J of [15]. Another example of this type of investigation arises in the study of topological groups. In [49], Weil proves that each totally bounded group G is a dense subgroup of a compact group, and that this compactification is unique up to a topological isomorphism fixing G pointwise. This compactification of G is known as its Weil completion, and is denoted by $\bar{G}$. Groups G for which $\bar{G}$ can be identified as the Stone-Cech compactification of G are characterized by Comfort and Ross in [7].

A similar situation presents itself for hyperspaces. $2^{\beta X}$ is a natural compactification of 2^X (for normal X). When can we identify $2^{\beta X}$ as the Stone-Cech compactification of 2^X? It is to this question that our efforts are directed in this chapter. J. Keesling has stated in [30] that $\beta(2^X) = 2^{\beta X}$ implies 2^X is pseudocompact. We give a proof of this result and obtain a partial converse, namely, if $2^X \times 2^X$ is pseudocompact, then $\beta(2^X) = 2^{\beta X}$. We also obtain two other characterizations of the relation $\beta(2^X) = 2^{\beta X}$. Using the results of Chapter 1 and Chapter 2, we obtain a fairly large class of spaces for which the relation $\beta(2^X) = 2^{\beta X}$ is valid.

Throughout this chapter we assume that X is normal, (and T_1), and when we speak of $2^{\beta X}$ as a compactification of 2^X, we are identifying 2^X with the subspace $i(2^X)$ of $2^{\beta X}$ as described above.

2. <u>A Necessary Condition for $\beta(2^X) = 2^{\beta X}$.</u> Our first result gives a necessary condition for $\beta(2^X) = 2^{\beta X}$. This theorem was also established independently by J. Keesling, who announced it without proof in [30].

2.1 Theorem. <u>If $\beta(2^X) = 2^{\beta X}$, then 2^X is pseudocompact.</u>

Proof. Before proceeding with the argument, let us examine the meaning of the equality $\beta(2^X) = 2^{\beta X}$. Let $i: 2^X \to 2^{\beta X}$ be the canonical embedding of 2^X onto a dense subspace of $2^{\beta X}$. Now to say that $2^{\beta X}$ is $\beta(2^X)$ is exactly the statement that $i(2^X)$ is C^*-embedded in $2^{\beta X}$. So assume that $\beta(2^X) = 2^{\beta X}$. We will first show that X must be pseudocompact. For the sake of contradiction, suppose X is not pseudocompact. Then there is a sequence $(G_n : n \in \underset{\sim}{N})$ of non-empty open subsets of X with the property that $cl_X G_{n+1} \subseteq G_n$ for all n, snd such that $\bigcap_{n\in\underset{\sim}{N}} G_n = \phi$ (see 9.13 in [15]). We may assume that

$cl_X G_{n+1} \subsetneq G_n$ for each n. For each n, we set $F_n = cl_X G_{n+1}$, and we define the following sequence of open sets in 2^X. We let $\mathcal{G}_n = B(G_n : X - F_{n+1}) = 2^{G_n} \cap (2^X - 2^{F_{n+1}})$ for each n. Then $F_n \in \mathcal{G}_n$ for each n. We claim that $(\mathcal{G}_n : n \in \underset{\sim}{N})$ is locally finite. For, let $A \in 2^X$. Let $p \in A$. Since $\bigcap_{n\in\underset{\sim}{N}} F_n = \phi$, there is an integer k such that $p \notin F_k$. But then $B(X; X - F_k)$ is a neighbourhood of A in 2^X meeting only finitely many $\mathcal{G}_n$.

Therefore $(\mathcal{G}_n: n \in \underset{\sim}{N})$ is locally finite. Now, let $\mathcal{D} = \{F_n: n \in \underset{\sim}{N}\}$. Since $\mathcal{G}_n \cap \mathcal{D} = \{F_n\}$, $\mathcal{D}$ is a discrete subset of 2^X. So we may find a sequence $(\mathcal{H}_n: n \in \underset{\sim}{N})$ of pairwise disjoint open subsets of 2^X such that $F_n \in \mathcal{H}_n$ for each n. Let $\mathcal{U}_n = \mathcal{H}_n \cap \mathcal{G}_n$ for each n. Then $(\mathcal{U}_n: n \in \underset{\sim}{N})$ is a locally finite sequence of pairwise disjoint open sets in 2^X, and $F_n \in \mathcal{U}_n$ for each n. It follows that $\mathcal{D}$ is C^*-embedded in 2^X. Applying the homeomorphism i, we see that $\{i(F_n)\ n \in \underset{\sim}{N}\}$ is discrete and C^*-embedded in $i(2^X)$. Since $\beta(2^X) = 2^{\beta X}$, $i(2^X)$ is C^*-embedded in $2^{\beta X}$. Thus, $\{i(F_n): n \in \underset{\sim}{N}\}$ is C^*-embedded in $2^{\beta X}$. But since the F_n's descend, it is easy to see that, in $2^{\beta X}$, $\lim\limits_{n\to\infty} i(F_n) = \bigcap\limits_{n\in \underset{\sim}{N}} i(F_n)$. Letting $L = \bigcap\limits_{n\in \underset{\sim}{N}} i(F_n)$, we see that no function on $\{i(F_n): n \in \underset{\sim}{N}\}$ which is 0 for infinitely many $i(F_n)$ and 1 for infinitely many $i(F_n)$ can be extended continuously to L. But this is a contradiction, as $\{i(F_n): n \in \underset{\sim}{N}\}$ is C^*-embedded in $2^{\beta X}$. So we see that indeed X is pseudocompact. Now we show that 2^X is pseudocompact. To do this, we use the following familiar characterization of pseudocompactness: A completely regular space Y is pseudocompact if, and only if, every non-empty G_δ set in βY meets Y. In light of this characterization, to show 2^X is pseudocompact, we need only show that every non-empty G_δ in $2^{\beta X}$ intersects $i(2^X)$. Thus let $\mathcal{S}$ be a non-empty G_δ in $2^{\beta X}$. Find a sequence $(\mathcal{G}_n: n \in \underset{\sim}{N})$ of open subsets of $2^{\beta X}$ such that $\mathcal{S} = \bigcap\limits_{n\in \underset{\sim}{N}} \mathcal{G}_n$. Let $A \in \mathcal{S}$. Then, for each n, we can find open sets $G_{n,0}, G_{n,1}, \ldots, G_{n,K_n}$ in βX so that $A \in B(G_{n,0}; G_{n,1}, \ldots, G_{n,K_n}) \subseteq \mathcal{G}_n$. For each n, let $H_{n,0}$ be open in βX such that $A \subseteq H_{n,0} \subseteq cl_{\beta X} H_{n,0} \subseteq G_{n,0}$. For all n, and for all $j \in \{1,2,\ldots,K_n\}$, let $H_{n,j} = G_{n,j} \cap H_{n,0}$. Then

$A \in \bigcap_{n\in\mathbb{N}} B(H_{n,0}; H_{n,1}, \ldots, H_{n,K_n}) \subseteq \bigcap_{n\in\mathbb{N}} B(G_{n,0}; G_{n,1}, \ldots, G_{n,K_n}) \subseteq \mathcal{S}$.

Let $H = \bigcap_{n\in\mathbb{N}} H_{n,0}$, and let $G = \bigcap_{n\in\mathbb{N}} G_{n,0}$. Obviously $cl_{\beta X} H \subseteq G$.

Now, for each n and $j \in \{1,2,\ldots,K_n\}$, $H_{n,j} \cap H$ is a non-empty G_δ in βX. By the pseudocompactness of X, we can find, for each n and each $j \in \{1,2,\ldots,K_n\}$, a point $x_{n,j}$ in $H_{n,j} \cap H \cap X$. Let $B = cl_{\beta X}\{x_{n,j}: n \in \mathbb{N}, j \in \{1,2,\ldots,K_n\}\}$. Then $B \in i(2^X)$ and clearly $B \in \bigcap_{n\in\mathbb{N}} B(G_{n,0}; G_{n,1}, \ldots, G_{n,K_n}) \subseteq \mathcal{S}$. Therefore every non-empty G_δ in $2^{\beta X}$ meets $i(2^X)$. Thus 2^X is pseudocompact.

2.2 Theorem. (i) Let A denote the subalgebra of $C^*(2^X)$ generated by $\{f^s: f \in C^*(X)\}$. Then $\beta(2^X) = 2^{\beta X}$ if, and only if, A is uniformly dense in $C^*(2^X)$. (Here $C^*(2^X)$ is provided with the usual sup-norm topology.)

(ii) Let 2^X be pseudocompact. Then $\beta(2^X) = 2^{\beta X}$ if, and only if, every zero set in 2^X is a countable intersection of basic zero sets of the form $B(X;Z_0) \cup 2^{Z_1} \cup \ldots \cup 2^{Z_n}$, for $Z_0, \ldots, Z_n \in Z(X)$.

Proof. (i) Recall from Chapter 0 that f^s is defined on 2^X by $f^s(F) = \sup\{f(x): x \in F\}$. We first observe that each f^s can be extended continuously to $2^{\beta X}$. This is immediate, since, given $f \in C^*(X)$, we can extend f continuously to $f^\beta \in C^*(\beta X)$, and then $(f^\beta)^s$ is a continuous real-valued function on $2^{\beta X}$ whose restriction to 2^X (recall we are identifying 2^X as the subspace $i(2^X)$ of $2^{\beta X}$) is clearly f^s. It follows that every function in A can be extended continuously to $2^{\beta X}$, and thus that every function in the uniform closure of A in $C^*(2^X)$ may be so extended. (See Prop. 5 of [22].) So if A is uniformly dense in $C^*(2^X)$, every function in $C^*(2^X)$ extends continuously to $2^{\beta X}$. That is, $\beta(2^X) = 2^{\beta X}$. Conversely, if $\beta(2^X) = 2^{\beta X}$, then $C^*(2^X)$ and $C^*(2^{\beta X})$ are uniformly isomorphic, under the map $g \to g^\beta$, where g^β represents the Stone extension to $2^{\beta X}$ of $g \in C^*(2^X)$. In light of the preceding remarks we see that $(f^s)^\beta = (f^\beta)^s$, for all $f \in C^*(X)$. And so the converse in question becomes equivalent to the following assertion: If X is compact, the subalgebra of $C^*(2^X)$ generated by the functions $\{f^s: f \in C^*(X)\}$ is uniformly dense in $C^*(2^X)$. This assertion follows from the Stone-Weierstrass theorem, since $\{f^s: f \in C^*(X)\}$ contains the constant functions on 2^X, and separates points and closed sets, by 0.7 and 0.8 of Chapter 0.

(ii) We first show that, if X is compact, every zero-set in 2^X is a countable intersection of basic sets of the form $B(X;Z_0) \cup 2^{Z_1} \cup \ldots \cup 2^{Z_n}$, where $Z_0, Z_1, \ldots, Z_n \in Z(X)$. So, assume that X is compact, and let W be a zero-set in 2^X. Find open sets $(G_n : n \in \mathbb{N})$ in 2^X so that $W = \bigcap_{n\in\mathbb{N}} G_n$. Now, by 0.8 in Chapter 0, every closed set in 2^X is an intersection of certain of the basic sets in question. In particular this is true of W. So write $W = \bigcap_{i\in I} A_i$, where each A_i is a basic set of the form $B(X;Z_0) \cup 2^{Z_1} \cup \dots \cup 2^{Z_n}$. For each n, $W \subseteq G_n$, and so, by the compactness of 2^X, there is, for each n, a finite subset I_n of I such that $W \subseteq \bigcap_{i\in I_n} A_i \subseteq G_n$. Letting $J = \bigcup_{n\in\mathbb{N}} I_n$, we see that J is countable and $W = \bigcap_{i\in J} A_i$. Thus W is a countable intersection as required.

Now, if $\beta(2^X) = 2^{\beta X}$, every zero-set in $i(2^X)$ is the restriction of a zero-set in $2^{\beta X}$. Since βX is compact, the representation in (ii) holds for zero-sets in $2^{\beta X}$, as proved above, and so, by restricting to $i(2^X)$ and applying the homeomorphism i^{-1}, the corresponding representation for zero-sets in 2^X is also seen to be valid. There is one minor detail to be checked here, namely, that if Z is a zero-set in βX, the $i^{-1}(B(\beta X;Z) \cap i(2^X))$ $= B(X; Z \cap X)$. This can be verified as follows. Since 2^X is pseudocompact, so is X, by 2.7 in Chapter 2. Since X is normal, X is countably compact, and every closed subset of X is countably compact and C^*-embedded in X. So, if $A \in 2^X$, $\mathrm{cl}_{\beta X} A = \beta A$. If Z is a zero-set in βX such that $(\mathrm{cl}_{\beta X} A) \cap Z \neq \phi$, then

$(\mathrm{cl}_{\beta X}A) \cap Z$ is a non-empty zero-set in βA. Since A is pseudocompact, indeed countably compact, it follows that $(\mathrm{cl}_{\beta X}A) \cap Z$ meets A. That is, $Z \cap A = (Z \cap X) \cap A \neq \phi$. Therefore, $i^{-1}(B(\beta X;Z) \cap i(2^X)) = B(X; Z \cap X)$.

Now, for the converse of (ii). We assume that 2^X is pseudocompact, and that every zero-set in 2^X has the indicated representation. Observe that, since the functions f^s and f^i on 2^X extend continuously to $2^{\beta X}$, namely to $(f^\beta)^s$ and $(f^\beta)^i$, every basic zero-set $2^{Z_1} \cup \ldots \cup 2^{Z_n} \cup B(X,Z_{n+1})$ in 2^X is the restriction of a zero-set in $2^{\beta X}$. This is clear from 0.7 in Chapter 0. Now since a countable intersection of zero-sets is again a zero-set, if every zero-set in 2^X has a representation as in (ii), we conclude that every zero-set in 2^X is the restriction of a zero-set in $2^{\beta X}$. Together with the pseudocompactness of 2^X, this enables us to conclude, by 4.4 of [2], that 2^X is C^*-embedded in $2^{\beta X}$, that is, $\beta(2^X) = 2^{\beta X}$.

3. <u>A Partial Converse to 2.1</u>. It is our aim in this section to establish a partial converse to 2.1. Now to find sufficient conditions for $\beta(2^X) = 2^{\beta X}$, is to find conditions which imply that 2^X is C^*-embedded in $2^{\beta X}$. In this approach, we are asking when will $2^{\beta X}$ have the properties, as a compactification of 2^X, that characterize $\beta(2^X)$ as a compactification of 2^X? Without question, this is the most obvious and most direct means of approaching the condition $\beta(2^X) = 2^{\beta X}$. However, the approach

taken here is to reverse the roles of $\beta(2^X)$ and $2^{\beta X}$. We take the point of view that $2^{\beta X}$ has the nice properties, and that the relation $\beta(2^X) = 2^{\beta X}$ says that certain structure on X imposes these properties on $\beta(2^X)$. That is, we propose to describe $2^{\beta X}$ as a compact extension of 2^X, and try to determine when $\beta(2^X)$ has the same description. The key to this approach is that 2^X and $2^{\beta X}$ are topological join semi-lattices. $2^{\beta X}$ is a compact join semi-lattice containing 2^X as a dense sub-join semi-lattice, and continuous join homomorphisms $2^X \to \underset{\sim}{R}$ extend continuously to $2^{\beta X}$. We thus describe $2^{\beta X}$ as a compact, algebraic extension of 2^X, and determine conditions when $\beta(2^X)$ enjoys this algebraic structure.

We first recall the definition of a topological join semi-lattice.

3.1 Definition: A <u>join semi-lattice</u> is a set Y, equipped with a binary operation $\vee$ that satisfies the following identities:

(i) $x \vee x = x$

(ii) $x \vee y = y \vee x$

(iii) $(x \vee y) \vee z = x \vee (y \vee z)$.

The element $x \vee y$ is called the join of x and y. We shall use the term $\vee$-semi-lattice as an abbreviation for join semi-lattice.

If Y is a $\vee$-semi-lattice and we define, for $a,b \in Y$, $a \leq b$ to mean that $a \vee b = b$, then $\leq$ is a partial ordering on Y relative to which every pair of elements has a least upper bound. Indeed, in this ordering, $\sup\{a,b\} = a \vee b$. Whenever we speak of order in a semi-lattice it is always this natural order to which we refer. The supremum of a subset A, if it exists, is also called its join, and is denoted by $\vee A$. We say a $\vee$-semi-lattice is <u>$\vee$-complete</u>, if every non-empty subset has a supremum. A subset S of a $\vee$-semi-lattice Y is called a <u>sub-$\vee$-semi-lattice</u> of Y if $a,b \in S$ imply $a \vee b \in S$. A mapping $f\colon Y_1 \to Y_2$ between $\vee$-semi-lattices Y_1 and Y_2 is a <u>$\vee$-homomorphism</u> if $f(a \vee b) = f(a) \vee f(b)$ for all $a,b \in Y_1$.

A <u>topological $\vee$-semi-lattice</u> is a $\vee$-semi-lattice Y equipped with a topology such that the $\vee$-operation is continuous as a mapping from $Y \times Y$ to Y, where $Y \times Y$ carries the product topology. When speaking of topological $\vee$-semi-lattices, terms like compact $\vee$-semi-lattice, continuous $\vee$-homomorphism, carry their obvious meaning.

3.2 Proposition. <u>2^X and $2^{\beta X}$ are topological $\vee$-semi-lattices. $2^{\beta X}$ is a compact $\vee$-semi-lattice, and the natural mapping $i\colon 2^X \to 2^{\beta X}$ is a topological isomorphism of 2^X onto a dense $\vee$-complete sub-$\vee$-semi-lattice of $2^{\beta X}$. Every continuous $\vee$-homomorphism from 2^X to $\underset{\sim}{R}$ extends to a continuous $\vee$-homomorphism of $2^{\beta X}$ into $\underset{\sim}{R}$</u>.

Proof. The first statement follows from statement 0.3 in Chapter 0, the $\vee$-operation being the usual set-theoretic union. We know that the map i is a homeomorphism, and clearly it preserves the operation. Therefore i is a topological isomorphism of 2^X into a dense sub-$\vee$-semi-lattice of $2^{\beta X}$. The natural ordering involved in these semi-lattices is just set-theoretic inclusion. Thus each semi-lattice is $\vee$-complete. Now $i(2^X)$ is a $\vee$-complete sub-$\vee$-semi-lattice of $2^{\beta X}$, since for $\{A_j : j \in I\} \subseteq 2^X$, $\bigvee_{j\in I} i(A_j) = \bigvee_{j\in I} cl_{\beta X} A_j$ in $2^{\beta X}$, is just $cl_{\beta X}(\bigcup_{j\in I} cl_{\beta X} A_j)$ which equals $cl_{\beta X}(cl_X(\bigcup_{j\in I} A_j))$ which lies in $i(2^X)$.

To prove the last assertion, let g be a continuous $\vee$-homomorphism from 2^X into $\underset{\sim}{R}$. ($\underset{\sim}{R}$, of course, carries the natural $\vee$ defined by $a \vee b = \max\{a,b\}$.) We claim that g preserves all suprema. For, let $\{A_i : i \in I\} \subseteq 2^X$. We show that $g(\bigvee_{i\in I} A_i) = g(cl_X \bigcup_{i\in I} A_i)$ coincides with $\bigvee_{i\in I} g(A_i)$. Clearly, since g preserves order, $\bigvee_{i\in I} g(A_i) \leq g(\bigvee_{i\in I} A_i)$. For the sake of contradiction, assume $\bigvee_{i\in I} g(A_i) < g(\bigvee_{i\in I} A_i)$. By the continuity of g, we can find a basic neighbourhood $B(G_0;G_1,\ldots,G_n)$ of $\bigvee_{i\in I} A_i = cl_X \bigcup_{i\in I} A_i$ in 2^X such that $T \in B(G_0;G_1,\ldots,G_n)$ implies that $g(T) > \bigvee_{i\in I} g(A_i)$. Now $cl_X \bigcup_{i\in I} A_i \in B(G_0;G_1,\ldots,G_n)$ so we can find, for each $j = 1,2,\ldots,n$ an index $i_j \in I$ so that $A_{i_j} \cap G_j \neq \phi$. Letting $P = \bigcup_{j=1}^{n} A_{i_j}$, we see that $P \in B(G_0;G_1,\ldots,G_n)$, and thus that $g(P) > \bigvee_{i\in I} g(A_i)$. But $P = \bigcup_{j=1}^{n} A_{i_j}$, and g preserves finite joins, so that $g(P) = g(\bigvee_{j=1}^{n} A_{i_j}) = \bigvee_{j=1}^{n} g(A_{i_j}) \leq \bigvee_{i\in I} g(A_i)$. This is a contradiction. Thus g preserves all joins, and this establishes our claim.

Now, let $f = g|\{\{x\}: x \in X\}$. Then $f \in C(X)$ and f is bounded above. We claim that $g = f^s$. To verify this, note that if $T \in 2^X$, $g(T) = g(\bigvee_{x\in T} \{x\}) = \bigvee_{x\in T} g(\{x\}) = \bigvee_{x\in T} f(x) = f^s(T)$.

(We have thus shown that every continuous join homomorphism $2^X \to \underset{\sim}{R}$ has the form f^s; it is easy to see conversely, that if f is a member of C(X) that is bounded above, then f^s is a continuous $\vee$-homomorphism $2^X \to \underset{\sim}{R}$.) The continuous extension of g to $2^{\beta X}$ whose existence is asserted in the proposition is simply $(f^\beta)^s$.

We thus see, by this proposition, that comparing $\beta(2^X)$ with $2^{\beta X}$ involves casting $\beta(2^X)$ in the role of an algebraic extension of 2^X with the properties in 3.2. We shall see, in the following theorem, that pseudocompactness enables us to impose the required algebraic structure on $\beta(2^X)$.

3.3 Theorem. <u>Let $2^X \times 2^X$ be pseudocompact. Then there is a $\vee$-semi-lattice structure on $\beta(2^X)$ relative to which $\beta(2^X)$ is a compact $\vee$-semi-lattice, and 2^X is a dense, sub-$\vee$-semi-lattice.</u>

Proof. Let u denote the continuous join operation on 2^X. Being a continuous map from $2^X \times 2^X$ into 2^X, u has a Stone extension u^β from $\beta(2^X \times 2^X)$ into $\beta(2^X)$. But $2^X \times 2^X$ is pseudocompact, so by Glicksberg's Theorem 1 of [21], $\beta(2^X \times 2^X) = \beta(2^X) \times \beta(2^X)$. Thus u has a continuous extension $u^\beta\colon \beta(2^X) \times \beta(2^X) \to \beta(2^X)$. The map u^β defines a continuous operation on $\beta(2^X)$. It is readily verified that u^β defines a $\vee$-semi-lattice structure on $\beta(2^X)$. In order to check that the three identities in 3.1 hold for the operation u^β, one observes that, in each case, the identities are valid on a dense set, and since the identities are continuous functions of their variables, they hold everywhere. The remaining assertions in 3.3 are immediate. When the occasion arises, we will denote $u^\beta(s,t)$ by $s \vee t$, for $s,t \in \beta(2^X)$.

At this point we may pose the following question. Do 3.2 and 3.3 give enough information so that, with the assumptions in 3.3, we are able to conclude that $\beta(2^X)$ and $2^{\beta X}$ are identical compactifications of 2^X? This is indeed the case, but to prove it we will need several facts about topological $\vee$-semi-lattices. The first two facts are standard and easily proved results about compact $\vee$-semi-lattices, while the last two results are much more technical. We will content ourselves here with stating these results without proofs, giving appropriate references in each case.

3.4 Let K be a compact $\vee$-semi-lattice. Let $(x_\alpha : \alpha \varepsilon D)$ be an increasing net in K. Then $\bigvee_{\alpha \varepsilon D} x_\alpha$ exists, and $\bigvee_{\alpha \varepsilon D} x_\alpha = \lim_{\alpha \varepsilon D} x_\alpha$. If $(x_\alpha : \alpha \varepsilon D)$ is a decreasing net bounded below, then $\bigwedge_{\alpha \varepsilon D} x_\alpha$ exists and $\bigwedge_{\alpha \varepsilon D} x_\alpha = \lim_{\alpha \varepsilon D} x_\alpha$. (See [33], [47].)

3.5 A compact $\vee$-semi-lattice is $\vee$-complete. ([33], [47]).

3.6 Let f be a $\vee$-homomorphism from a compact $\vee$-semi-lattice S onto a compact $\vee$-semi-lattive T. If f preserves the suprema of increasing nets and the infima of decreasing nets, then f is continuous. ([33].)

3.7 Let K be a compact $\vee$-semi-lattice. Then the continuous $\vee$-homomorphisms of K onto metrizable compact $\vee$-semi-lattices separate the points of K. (See page 49 of [24].)

Now it is easy to see that, in 2^X, the join of a subset $\mathcal{S} \subseteq 2^X$ is the limit of the net of its finite sub-joins, the net being directed by the finite subsets of $\mathcal{S}$. This fact is tacitly proved in 3.2 in showing that continuous $\vee$-homomorphisms from 2^X into $\underset{\sim}{R}$ preserve suprema. From this fact, we see that 2^X is a $\vee$-complete subset of any compact $\vee$-semi-lattice K in which

2^X is a sub-$\vee$-semi-lattice. Indeed, if $S \subseteq 2^X$, then, by the above remarks, the join of S in 2^X is the limit of the increasing net of its finite sub-joins. These finite sub-joins in 2^X are the same as the corresponding joins in K. But by 3.4, this net converges to the join of S in K. Thus the join of S in K coincides with the join of S in 2^X. That is, 2^X is a $\vee$-complete subset of K. In particular, this statement is valid, in the case that $2^X \times 2^X$ is pseudocompact, for the compact $\vee$-semi-lattice $K = \beta(2^X)$.

We now have the necessary tools to establish the main result of this chapter.

3.8 Theorem. <u>Let $2^X \times 2^X$ be pseudocompact. Then $\beta(2^X) = 2^{\beta X}$.</u>

Proof. By statement 0.7 in Chapter 0, X, considered as the singletons in 2^X, is C^*-embedded in 2^X. Thus the closure of this copy of X in $\beta(2^X)$ is a copy of βX. So, there is an embedding $h\colon \beta X \to \beta(2^X)$ such that $h(x) = \{x\}$ for each $x \in X$. Since $\beta(2^X)$ is the largest compactification of 2^X, there is a quotient map $Q\colon \beta(2^X) \to 2^{\beta X}$, whose restriction to 2^X is the identity on 2^X. We endow $\beta(2^X)$ with the $\vee$-semi-lattice structure described in 3.3. Since Q is a $\vee$-homomorphism on a dense subset, by continuity, Q is a $\vee$-homomorphism. Since the supremum of a set is the limit of the suprema of its finite subsets, by continuity, Q preserves all suprema. Now, we define a mapping

$F: 2^{\beta X} \to \beta(2^X)$ by $F(S) = \bigvee_{p\varepsilon S} h(p)$. Note that $F|\beta X = h$, that is, the restriction of F to the singletons in $2^{\beta X}$ coincides with h. We claim that F preserves joins. For, let $\{A_i: i \varepsilon I\} \subseteq 2^{\beta X}$. Then $\bigvee_{i\varepsilon I} F(A_i) = \bigvee_{p\varepsilon \bigcup_{i\varepsilon I} A_i} h(p)$, and $F(\bigvee_{i\varepsilon I} A_i) = F(\mathrm{cl}_{\beta X} \bigcup_{i\varepsilon I} A_i)$

$= \bigvee_{p\varepsilon \mathrm{cl}_{\beta X} \bigcup_{i\varepsilon I} A_i} h(p)$. Clearly $\bigvee_{i\varepsilon I} F(A_i) \leq F(\bigvee_{i\varepsilon I} A_i)$. To prove the reverse inequality, let $a = \bigvee_{i\varepsilon I} F(A_i) = \bigvee_{p\varepsilon \bigcup_{i\varepsilon I} A_i} h(p)$. Now the set $\{t \varepsilon \beta(2^X): t \leq a\}$ is closed in $\beta(2^X)$. So, by the continuity of h, $\{p \varepsilon \beta X: h(p) \leq a\}$ is closed in βX. Since this latter set evidently contains $\bigcup_{i\varepsilon I} A_i$, it therefore contains $\mathrm{cl}_{\beta X} \bigcup_{i\varepsilon I} A_i$. So, for all $p \varepsilon \mathrm{cl}_{\beta X} \bigcup_{i\varepsilon I} A_i$, $h(p) \leq a$. Thus $F(\bigvee_{i\varepsilon I} A_i) = F(\mathrm{cl}_{\beta x} \bigcup_{i\varepsilon I} A_i)$

$= \bigvee_{p\varepsilon \mathrm{cl}_{\beta X} \bigcup_{i\varepsilon I} A_i} h(p) \leq a = \bigvee_{i\varepsilon I} F(A_i)$. Combining the two inequalities, we conclude that F preserves joins. Now consider the map $Q \circ F: 2^{\beta X} \to 2^{\beta X}$. By the continuity of Q and h, and the fact that $Q \circ h$ is the identity on $\{\{p\}: p \varepsilon X\}$, a dense subset of $\{\{p\}: p \varepsilon \beta X\}$, $Q \circ F$ is the identity on βX. (That is, on the set $\{\{p\}: p \varepsilon \beta X\}$.) Since Q and F preserve joins, so does $Q \circ F$. So if $S \varepsilon 2^{\beta X}$, we have $Q \circ F(S) = Q \circ F(\bigvee_{p\varepsilon S} \{p\})$

$= \bigvee_{p\varepsilon S} Q \circ F(\{p\}) = \bigvee_{p\varepsilon S} \{p\} = S$. Thus, $Q \circ F$ is the identity on

$2^{\beta X}$. It follows that F is one-to-one. Now 2^X is a $\vee$-complete subset of $\beta(2^X)$. Since F preserves joins, and since $F(\{x\}) = \{x\}$ for each $x \in X$, we see that $2^X \subseteq F(2^{\beta X})$. Let $L = F(2^{\beta X})$. Since L contains the pseudocompact space 2^X as a dense subset, L is itself pseudocompact. Thus F is an algebraic isomorphism of $2^{\beta X}$ onto the dense, pseudocompact sub-$\vee$-semi-lattice L, of $\beta(2^X)$. We now show that F is continuous. Now, if the topological $\vee$-semi-lattive S is a sub-$\vee$-semi-lattice of a compact $\vee$-semi-lattice, then a map G into S is continuous if, and only if, $R \circ G$ is continuous for all continuous join homomorphisms R, of S into metrizable $\vee$-semi-lattices. Indeed by 3.7, S has the weak topology generated by such R. Thus, to show $F\colon 2^{\beta X} \to L$ is continuous, we prove that if $R\colon L \to M$ is a continuous $\vee$-homomorphism of L onto a metrizable $\vee$-semi-lattice M, then $R \circ F$ is continuous. But, for any such R, M must be compact, since M, as a continuous image of L, is pseudocompact, and every pseudocompact metric space is compact. Thus $R \circ F$ is a map between compact $\vee$-semi-lattices. Since R is a continuous $\vee$-homomorphism, R preserves all joins. Since F is an algebraic isimorphism, F preserves all joins. Therefore $R \circ F$ preserves all joins. The same type of argument shows that $R \circ F$ preserves decreasing meets. Appealing to 3.6, we see that $R \circ F$ is continuous. Thus F is itself continuous. Thus $F(2^{\beta X})$ is compact. Since the image of F contains 2^X, we conclude that $F(2^{\beta X}) = \beta(2^X)$. It follows that F is a homeomorphism of $2^{\beta X}$ onto $\beta(2^X)$ fixing 2^X point wise. Therefore $2^{\beta X} = \beta(2^X)$.

3.9 Remark. It seems quite plausible that the pseudocompactness of 2^X is equivalent to that of $2^X \times 2^X$. This would establish the converse to 2.1. The author has not been able to resolve this question, and leaves it open to the reader. Even if the exact converse of 2.1 holds, it is unsatisfying in a very significant way. It does not describe the relation $\beta(2^X) = 2^{\beta X}$ in terms of properties of X. What is needed, of course, is a description of the pseudocompactness of 2^X in terms of X; we obtained some results along these lines in Chapter 2.

Now, by 5a., page 166 of [32], for any spaces S and T, $2^S \times 2^T$ is homeomorphic to 2^{S+T}, where $S+T$ denotes the free union of S and T. So if $\mathcal{P}$ is a class of topological spaces such that $X \in \mathcal{P}$ implies 2^X is pseudocompact, and such that $X,Y \in \mathcal{P}$ implies $X+Y \in \mathcal{P}$, then $X \in \mathcal{P} \Rightarrow 2^X \times 2^X$ is pseudocompact. In particular, this is true of the properties $\mathcal{D}$-compactness and $\mathcal{D}$-pseudocompactness, by the results of Chapter 2. (It is obvious that each of these properties is preserved by finite union.) We described many natural examples of these spaces in Chapter 1, and we thus have a fairly large class of spaces which satisfy the hypothesis of 3.8. In particular, we have the following corollary. (See 2.6 in Chapter 1 and 2.1 in Chapter 2.)

3.10 Corollary. Let X be normal. If all powers of X are countably compact, then $\beta(2^X) = 2^{\beta X}$. In particular, this conclusion holds if X is normal and ω_0-bounded.

3.11 Remark. The technique employed in 3.3, to impose algebraic structure on the Stone-Cech compactification, can be applied to rather general situations. For example, if we are given a completely regular topological algebra A of a given type, pseudocompactness can be used, just as in 3.3, to obtain an algebraic structure on βA of the same type, relative to which βA is a compact topological algebra, and A a dense subalgebra. This can be used to compare an algebraic compactification to the Stone-Cech compactification whenever the algebraic extension is uniquely determined. In particular, this method can be used to give a version of the Comfort-Ross Theorem 1.2 of [7]; for a totally bounded topological group G and its Weil completion $\bar{G}$, $\beta G = \bar{G}$ if, and only if, G is pseudocompact.

Chapter 4

THE G_δ-CLOSURE AND REALCOMPACTNESS OF 2^X

1. Many important properties of a completely regular space X can be described in terms of its Stone-Cech compactification βX. For example, X is pseudocompact if, and only if, every non-empty G_δ set in βX intersects X. Cech-complete spaces are precisely those spaces which are G_δ sets in their Stone-Cech compactifications. Another property of a completely regular space X which can be described in terms of the embedding of X in βX, and the one that most concerns us here, is realcompactness. To state this description, we need a definition. Let X be a subspace of a space Y. We say that X *is* G_δ*-closed in* Y if Y - X is a union of G_δ-sets in Y. This means that given a point $p \in Y - X$, we can find open subsets $G_1, G_2, \ldots,$ of Y such that $p \in \bigcap_{n \in \mathbb{N}} G_n \subseteq Y - X$.

Realcompactness can be described as follows. (See 8.8 of [15].)

1.1 Theorem. *Let X be a completely regular Hausdorff space. Then the following statements are equivalent.*

(i) *X is realcompact,*

(ii) *X is G_δ-closed in βX,*

(iii) *X is G_δ-closed in some compactification of X.*

Of course, a realcompact space need not be G_δ-closed in all of its compactifications. For example, a locally compact space is G_δ-closed in its one-point compactification if, and only if, it is σ-compact, and examples abound of realcompact, locally compact spaces which are not σ-compact.

Suppose we are given a compactification αX of a (completely regular Hausdorff) space X. If X is G_δ-closed in αX, we see, by 1.1, that X is realcompact. As we saw above, the fact that X is G_δ-closed in αX may reflect much more than the realcompactness of X, depending on the position of αX in the family of all compactifications of X.

If X is normal and T_1, then $2^{\beta X}$ is a compactification of 2^X. Since $2^{\beta X}$ is one of the most natural compactifications of 2^X, it is of some interest to describe 2^X in terms of its embedding in $2^{\beta X}$. In Chapter 3 we examined the circumstances under which 2^X is C^*-embedded in $2^{\beta X}$. In this chapter, we examine the G_δ-closure of 2^X in $2^{\beta X}$, and we characterize those spaces X such that 2^X is G_δ-closed in $2^{\beta X}$. Using 1.1, we obtain information on the realcompactness of 2^X.

2. <u>The G_δ-Closure of 2^X in $2^{\beta X}$</u>. We begin with an elementary observation.

2.1 Proposition. Let X be a subspace of Y. Let $Q_Y(X) = \{p \in Y:$ every G_δ-set in Y containing p intersects X$\}$. Then $Q_Y(X)$ is the smallest subspace of Y that contains X and is G_δ-closed in Y.

Proof: Clearly $X \subseteq Q_Y(X)$. We first show that $Q_Y(X)$ is G_δ-closed in Y. For, let $p \in Y - Q_Y(X)$. Then there is a G_δ-set H, in Y, such that $p \in H$ and $H \cap X = \phi$. Since any G_δ containing any point of $Q_Y(X)$ intersects X, we have $H \cap Q_Y(X) = \phi$. Thus $Q_Y(X)$ is G_δ-closed in Y.

Now, let S by any subspace of Y such that $X \subseteq S$ and S is G_δ-closed in Y. We show that $Q_Y(X) \subseteq S$. Let $p \in Q_Y(X)$. If $p \notin S$, since S is G_δ-closed in Y, there is a G_δ-set H, in Y, such that $p \in H$ and $H \cap S = \phi$. But since $p \in Q_Y(X)$, $H \cap X \neq \phi$, and since $X \subseteq S$, $H \cap S \neq \phi$. Thus $p \in S$. Therefore, $Q_Y(X) \subseteq S$, and so $Q_Y(X)$ is the smallest G_δ-closed subspace of Y which contains X.

We will refer to $Q_Y(X)$ in 2.1 as the G_δ-closure of X in Y. Now, in a completely regular space, a G_δ-set containing a given point contains a zero-set containing the given point. (See 3.11 of [15].) So if $X \subseteq Y$ where Y is completely regular, our G_δ conditions may be re-formulated as follows. X is G_δ-closed in Y if every point of Y - X lies in some zero-set of Y that is disjoint from X. $Q_Y(X)$ is the set of points p, such that every zero-set in Y that contains p intersects X.

We now describe the G_δ-closure of 2^X in $2^{\beta X}$ for a normal space X. Recall that we are identifying 2^X as the subspace $i(2^X)$ of $2^{\beta X}$, as in Chapter 3.

2.2 Lemma. Let X a normal, Hausdorff space. Let Q denote the G_δ-closure of 2^X in $2^{\beta X}$. Then $Q = \{F \in 2^{\beta X}: Z \in Z(\beta X),\ F \subseteq Z$ implies $F \subseteq cl_{\beta X}(Z \cap X)\}$.

Proof: Observe that $i(2^X) = \{F \in 2^{\beta X}: F = cl_{\beta X}(F \cap X)\}$. Let Q_1 denote the set described in the statement of the lemma. We will show that $2^X \subseteq Q_1 \subseteq Q$ and that Q_1 is G_δ-closed in $2^{\beta X}$, from which the assertion follows. Since $F \in 2^X$ is equivalent to $F = cl_{\beta X}(F \cap X)$, clearly $2^X \subseteq Q_1$. We now show that $Q_1 \subseteq Q$. So let $F \in Q_1$. Let $\mathcal{H}$ be any G_δ-set in $2^{\beta X}$ containing F. Write $\mathcal{H} = \bigcap_{n \in \mathbb{N}} \mathcal{G}_n$ with $\mathcal{G}_n$ open in $2^{\beta X}$ for each n. For each n, we can find open sets $G_{0,n};G_{1,n},\ldots,G_{K_n,n}$ in βX with $\bigcup_{i=1}^{K_n} G_{i,n} \subseteq G_{0,n}$ and $F \in B(G_{0,n};G_{1,n},\ldots,G_{K_n,n}) \subseteq \mathcal{G}_n$. For each n, find a zero-set Z_n in βX such that $F \subseteq Z_n \subseteq G_{0,n}$. Now $\bigcap_{n \in \mathbb{N}} Z_n = Z$ is a zero-set in βX and $F \subseteq Z$. Since $F \in Q_1$, we have $F \subseteq cl_{\beta X}(Z \cap X)$. But $cl_{\beta X}(Z \cap X) \in 2^X \cap \mathcal{H}$, and so every G_δ-set in $2^{\beta X}$ containing F meets 2^X. Therefore $F \in Q$, and so $Q_1 \subseteq Q$. We complete the proof by showing Q_1 is G_δ-closed in $2^{\beta X}$. Let $F \in 2^{\beta X} - Q_1$. Then there is a zero-set Z in βX such that $F \subseteq Z$ but

$F \notin cl_{\beta X}(Z \cap X)$. Let $\mathcal{H} = B(Z; \beta X - cl_{\beta X}(Z \cap X))$. Then clearly $\mathcal{H}$ is a G_δ in $2^{\beta X}$ and $F \in \mathcal{H}$. Obviously $\mathcal{H} \cap Q_1 = \phi$. Thus Q_1 is G_δ-closed in $2^{\beta X}$.

2.3 Theorem. <u>Let X be a normal, Hausdorff space. The following are equivalent.</u>

(i) <u>2^X is G_δ-closed in $2^{\beta X}$</u>,

(ii) <u>X is Lindelöf</u>.

Proof: (i) => (ii). Assume 2^X is G_δ-closed in $2^{\beta X}$. In the notation of the preceding lemma, this means that $2^X = Q_1$. We claim that X is Lindelöf. For the sake of contradiction, assume X is not Lindelöf. Then, there is a family $\mathcal{D}$ of closed subsets of X with the countable intersection property such that $\cap\mathcal{D} = \phi$. Let $\mathcal{D}_1$ be the family of countable intersections of members of $\mathcal{D}$. Then $\mathcal{D}_1$ is closed under countable intersection, and $\bigcap_{A\in\mathcal{D}_1} A = \phi$. Let $R = \bigcap_{A\in\mathcal{D}_1} cl_{\beta X}A$. Then, since $R \cap X = \phi$, we have $R \in 2^{\beta X} - 2^X$. Let Z be any zero-set in βX containing R. Write $Z = \bigcap_{n\in\mathbb{N}} G_n$, where G_n is open in βX. Now, for each n,

$\bigcap_{A\in\mathcal{D}_1} cl_{\beta X}A \subseteq G_n$, and so, by compactness, there is, for each n, a finite subset $\mathcal{F}_n$ of $\mathcal{D}_1$ so that $\bigcap_{A\in\mathcal{F}_n} cl_{\beta X}A \subseteq G_n$. Let $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$.

Then $\mathcal{F}$ is a countable subset of $\mathcal{D}_1$ and $\bigcap_{A\in\mathcal{F}} cl_{\beta X}A \subseteq \bigcap_{n\in\mathbb{N}} G_n = Z$.

But $\bigcap_{A\in\mathcal{F}} A \in \mathcal{D}_1$ and so $R \subseteq cl_{\beta X}(\bigcap_{A\in\mathcal{F}} A) \subseteq \bigcap_{A\in\mathcal{F}} cl_{\beta X}A \subseteq Z$. It follows easily that $R \subseteq cl_{\beta X}(Z \cap X)$. We have thus shown that, for $Z \in Z(\beta X)$, $R \subseteq Z$ implies $R \subseteq cl_{\beta X}(Z \cap X)$. This means that $R \in Q_1$. But this is nonsense, since $R \notin 2^X = Q_1$. This shows that (i) => (ii).

(ii) => (i). Assume X is Lindelöf. We show that $2^X = Q_1$, for which it suffices to show $Q_1 \subseteq 2^X$. So, let $F \in Q_1$. Then $F = \cap\{cl_{\beta X}(Z \cap X): Z \in Z(\beta X), F \subseteq Z\}$. Let $\mathcal{R} = \{Z \in Z(\beta X): F \subseteq Z\}$. We claim that $F = cl_{\beta X}[\bigcap_{Z\in\mathcal{R}} (Z \cap X)]$. If possible, let $p \in F - cl_{\beta X}[\bigcap_{Z\in\mathcal{R}} (Z \cap X)]$. Find a closed neighbourhood M of p in βX such that $M \cap [\bigcap_{Z\in\mathcal{R}} (Z \cap X)] = \phi$. Since X is Lindelöf, there is a sequence $Z_1, Z_2, \ldots,$ from $\mathcal{R}$ such that $M \cap [\bigcap_{i\in\mathbb{N}} (Z_i \cap X)] = \phi$. But $\bigcap_{i\in\mathbb{N}} Z_i \in \mathcal{R}$, and so $F \subseteq cl_{\beta X}[\bigcap_{i\in\mathbb{N}} (Z_i \cap X)]$. But this implies $p \in cl_{\beta X}[\bigcap_{i\in\mathbb{N}} (Z_i \cap X)]$, and so $M \cap [\bigcap_{i\in\mathbb{N}} (Z_i \cap X)] \neq \phi$. This contradiction proves that $F \subseteq cl_{\beta X}[\bigcap_{Z\in\mathcal{R}} (Z \cap X)]$. Since the reverse inclusion holds trivially, we conclude that $F = cl_{\beta X}[\bigcap_{Z\in\mathcal{R}} (Z \cap X)]$. Thus $F \in 2^X$, and so $Q_1 \subseteq 2^X$. Therefore 2^X is G_δ-closed in $2^{\beta X}$.

2.4 Remark. It should be observed that the results of 2.2 and 2.3 carry over to higher cardinals. Calling a set a G_m-set if it is the intersection of m open sets, (m denotes an infinite cardinal) and recalling that a space is m-Lindelöf if each of its open covers has a subcover of $\leq m$ sets, we see that, with obvious modifications, 2.2 and 2.3 hold with G_δ replaced by G_m, and Lindelöf replaced by m-Lindelöf.

3. Some Remarks on the Realcompactness of 2^X. By 2.3 and 1.1, it follows that if X is Lindelöf, then 2^X is realcompact. Indeed, if X is Lindelöf, then $2^{\beta X}$ is a compactification of 2^X in which 2^X is G_δ-closed. We now give a direct proof of this result. Recall that a completely regular space Y is realcompact if, and only if, every z-ultrafilter on Y with the countable intersection property is fixed. (See [15].)

3.1 Theorem. *Let X be Lindelöf and completely regular. Then 2^X is realcompact.*

Proof. Since a Lindelöf, completely regular space is normal, we conclude by 4.9 of [36] that 2^X is completely regular (and Hausdorff) when X is completely regular and Lindelöf. We use the above characterization of realcompactness. So let θ be a z-ultrafilter on 2^X with the countable intersection property, with X assumed to be Lindelöf. We define two families of sets as follows. We set $\alpha = \{A \in 2^X$: there exists $\mathcal{B} \in \theta$ such that $\mathcal{B} \subseteq 2^A\}$ and we set $\beta = \{A \in 2^X$: there exists $\mathcal{B} \in \theta$ such that $\mathcal{B} \subseteq B(X; A)\}$. For $A \in \beta$, we define $G_A = \{F \cap A: F \in \alpha\}$. We claim that for each

$A \in \beta$, $\mathcal{G}_A$ has the countable intersection property. Let $\{F_1, F_2, \ldots\} \subseteq \alpha$, and let $A \in \beta$. Then, for each n, there is a set $\mathcal{B}_n \in \theta$ with $\mathcal{B}_n \subseteq 2^{F_n}$, and there is a set $\mathcal{B} \in \theta$ with $\mathcal{B} \subseteq B(X; A)$. Since θ has the countable intersection property, we have $\phi \neq (\bigcap_{n\in\mathbb{N}} \mathcal{B}_n) \cap \mathcal{B} \subseteq (\bigcap_{n\in\mathbb{N}} 2^{F_n}) \cap B(X; A)$. Any element in the latter intersection is contained in $\bigcap_{n\in\mathbb{N}} F_n$ and meets A. So, in particular, $(\bigcap_{n\in\mathbb{N}} F_n) \cap A \neq \phi$. Thus, each $\mathcal{G}_A$ has the countable intersection property. Since X is Lindelöf, there is, for each $A \in \beta$, a point $p_A \in \cap\mathcal{G}_A$. Let $L = cl_X\{p_A : A \in \beta\}$. We now show that $L \in \cap\theta$, whence θ is fixed, and so 2^X is realcompact. We assume $L \notin \cap\theta$ and we will derive a contradiction. If $L \notin \cap\theta$, then there is a set $\mathcal{B} \in \theta$ such that $L \notin \mathcal{B}$. Now, since X is normal, by 0.8 in Chapter 0, the sets of the form $B(X; Z_0) \cup 2^{Z_1} \cup \ldots \cup 2^{Z_n}$, where $Z_0, Z_1, \ldots, Z_n$ are zero-sets in X, form a base for the closed sets in 2^X. Now $\mathcal{B} \in \theta$ and so $\mathcal{B}$ is a zero-set in 2^X, and is, in particular, closed. Since $L \notin \mathcal{B}$, we can find zero-sets $Z_0, Z_1, \ldots, Z_n$ in X such that $\mathcal{B} \subseteq B(X; Z_0) \cup 2^{Z_1} \cup \ldots \cup 2^{Z_n}$, and $L \notin B(X; Z_0) \cup 2^{Z_1} \cup \ldots \cup 2^{Z_n}$. Now if Z is a zero-set in X then 2^Z is a zero-set in 2^X. (See 0.7 in Chapter 0.) We cannot have any $2^{Z_i} \in \theta$, because this would put Z_i in α and would imply that $L \subseteq Z_i$, or equivalently, $L \in 2^{Z_i}$, by the construction of L. So, since θ is a z-ultrafilter, there is, for each $i = 1,2,\ldots,n$, a zero-set $\mathcal{B}_i$ in θ

such that $\mathcal{B}_i \cap 2^{Z_i} = \phi$. Letting $\mathcal{C} = \mathcal{B} \cap \bigcap_{i=1}^{n} \mathcal{B}_i$, we have $\mathcal{C} \in \theta$ and $\mathcal{C} \subseteq \mathcal{B}(X; Z_0)$. This implies $Z_0 \in \beta$, and so $p_{Z_0} \in Z_0 \cap L$. But $L \notin B(X; Z_0)$, so that $L \cap Z_0 = \phi$. This is a contradiction. We conclude that $L \in \cap\theta$.

3.2 Remark. In [55], the realcompactness of 2^X is approached by uniformities, and 3.1 can be deduced as a corollary of results proved therein.

It does not seem to be known whether 2^X is realcompact whenever X is. Of course, if 2^X is realcompact, then X is, since, (for Hausdorff X) the singletons in 2^X form a closed subspace homeomorphic to X. We have to be slightly careful in discussing the realcompactness of 2^X, since 2^X is completely regular only when X is normal. If we use the definition of realcompactness in [34], which applies in the non-completely regular setting, we can then meaningfully ask whether 2^X is realcompact when X is completely regular and realcompact.

3.3 Proposition. Let P be a closed hereditary topological property. Let X be a regular, Hausdorff space such that $2^X \in P$. If Y is a continuous-open-closed image of X, then $2^Y \in P$. (By a topological property, we mean a class of topological spaces which, whenever it contains a space X, also contains all spaces homeomorphic to X.)

Proof. Let X be regular and Hausdorff, with $2^X \in \mathcal{P}$. Let $f\colon X \to Y$ be a continuous, open, and closed surjection. Define $F\colon 2^X \to 2^Y$ by $F(A) = f(A)$, and define $G\colon 2^Y \to 2^X$ by $G(B) = f^{-1}(B)$. By 5.10.1 and 5.10.2 of [36], F and G are continuous. Let $\mathcal{Y} = G(2^Y)$. Then $F|\mathcal{Y}$ and G are mutually inverse homeomorphisms between $\mathcal{Y}$ and 2^Y, and $G \circ F$ is a retraction of 2^X onto $\mathcal{Y}$. Now, since X is regular and Hausdorff, 2^X is Hausdorff (see 4.9 of [36]). As a retract of 2^X, $\mathcal{Y}$ is therefore closed in 2^X. Thus 2^Y is homeomorphic to a closed subspace of 2^X. Since $\mathcal{P}$ is closed hereditary, $2^Y \in \mathcal{P}$.

From 3.3 we can deduce the following. Let X be completely regular. If 2^X is realcompact, then X is realcompact and every continuous-open-closed completely regular image of X is realcompact. It does not seem to be known whether realcompactness is preserved under continuous-open-closed images. A counterexample would provide an example of a realcompact space whose hyperspace is not realcompact. This question, together with the question of characterizing those spaces X for which 2^X is realcompact, we leave open to the reader.

3.4 Remark. In Chapter 2 we saw that countable compactness and pseudocompactness are not preserved in passing to the hyperspace. Most notable among the properties that 2^X enjoys when X does, are $\mathcal{D}$-compactness, ω_0-boundedness (for normal spaces), and compactness itself. Each of the latter properties is closed-hereditary and productive. (Such topological properties are called extension properties; see [23] and [52] for information on extension properties.) Realcompactness is another closed-hereditary and productive property, but, as mentioned above, it is not yet clear whether 2^X is realcompact whenever X is (even for normal X). Whatever the situation may be for realcompactness, one certainly does not expect every closed-hereditary, productive property to be preserved in passing from a (normal) space X to its hyperspace. We now give an example of a closed-hereditary, productive property which is not so preserved, using Mrowka's concept of E-compactness. (See [38].) Let us recall the definition of E-compactness. Let E be a given topological space. A space X is said to be <u>E-compact</u> if X is homeomorphic to a closed subspace of a product of copies of E. E-compactness is clearly closed-hereditary and productive. We will now give an example of a normal space E for which E-compactness is not preserved in passing to the hyperspace.

We take for our space E the space constructed by Ostaszewski in [40]. This space is countably compact, hereditarily separable, perfectly normal, and non-compact, and is constructed in [40] using certain set-theoretic assumptions which are consistent with the continuum hypothesis and the usual axioms for set theory. For this space E, we show that 2^E is not E-compact, and thus we have the desired example.

By Proposition 3.2 in [29], (which assumes the continuum hypothesis), the space W of countable ordinals can be embedded as a closed subset of 2^E. We show that W is not E-compact. Since E-compactness is closed hereditary, this shows that 2^E is not E-compact. For the sake of contradiction, suppose W is homeomorphic to a closed subspace $\bar{W}$ of a product $\prod_{i\varepsilon I} E_i$ of copies of E. Now $\bar{W}$ is ω_0-bounded, and ω_0-boundedness is preserved by continuous maps. Thus each projection $\pi_i(\bar{W})$ is ω_0-bounded. Since E is hereditarily separable, each subspace $\pi_i(\bar{W})$ is separable, and so is compact. But $\bar{W}$ is a closed subspace of the product of these compact projections, and hence must be compact. This is a contradiction, and proves our assertion.

Chapter 5

SOME CARDINAL INVARIANTS OF HYPERSPACES

1. In this chapter we are concerned with certain cardinal invariants of hyperspaces, namely, weight, character, cellularity, and π-weight. In each case, the invariant of 2^X is described by means of an equality, or inequality, in terms of invariants of X. Although these results are not directly related to the compactness-type properties we have examined in the previous chapters, in several of the theorems and examples compactness and covering conditions do play a role.

Several authors have considered cardinal invariants of 2^X in one context or another. In [36] it is shown that X is separable if, and only if, 2^X is separable, and that 2^X is second countable if, and only if, X is a compact metric space. From 1 of [27] it follows that if X has a closed discrete subset of cardinality α, then 2^X has a closed discrete subset of cardinality 2^α. Spaces X for which 2^X is Lindelöf are characterized in 2 of [27], and III of [53] describes those spaces for which 2^X is first countable.

In order to formulate the relations we will establish, let us recall the definitions of the cardinal invariants with which we are concerned. The <u>weight</u> of a space X, denoted by w(X), is the least cardinal of an open basis for X. The <u>Lindelöf number</u> of X, written L(X), is the smallest cardinal α such that every open cover of X has a subcover of cardinality $\leq \alpha$. The <u>weak</u>

covering number of X, denoted by $wc(X)$, is the least cardinal α for which each open cover of X has a subfamily with α or fewer elements whose union is dense in X. If $A \subseteq X$, the character of X at A, written $\chi(X,A)$, is the least cardinal of a base for the neighbourhoods of A in X. If $p \in X$, we write $\chi(X,p)$ instead of $\chi(X,\{p\})$. The character of X, denoted by $\chi(X)$, is defined by $\chi(X) = \sup\{\chi(X,p): p \in X\}$. The density of X, written $d(X)$, is the least cardinal of a dense subset of X. The hereditary density of X, denoted by $hd(X)$, is defined by $hd(X) = \sup\{d(Y): Y \subseteq X\}$. A family $\mathcal{P}$ of non-empty open subsets of X is called a π-basis for X, if every non-empty open subset of X contains a member of $\mathcal{P}$. The π-weight of X, denoted by $\pi(X)$, is the least cardinal of a π-basis for X. The cellularity of X, denoted by $c(X)$, is defined by $c(X) = \sup\{\alpha$: there is a family $\mathcal{G}$ of pairwise disjoint, non-empty, open subsets of X whose cardinality is $\alpha\}$. The relations we will establish may now be summarized as follows:

(i) $w(X) \leq w(2^X) \leq w(X)^{L(X)} \leq 2^{w(X)}$.

(ii) If X is normal, then $w(2^X) \leq w(X)^{wc(X)}$.

(iii) $\chi(2^X) = \bar{\chi}(X)\cdot hd(X)$, where $\bar{\chi}(X) = \sup\{\chi(X,F)$: F is a closed subset of $X\}$.

(iv) $\pi(2^X) = \pi(X)$.

(v) $c(2^X) \leq \sup\{c(X^n): n \in \mathbb{N}\} \leq 2^{c(X)}$.

Examples are given to illustrate the sharpness of these estimates, and, in particular, an example of a non-normal space for which (ii) fails is given. The relation given in (iii) is a straightforward generalization of the countable case treated in [53].

Our basic references for the cardinal invariants described in this chapter are [25], [4], and [5].

In order to avoid trivial technical difficulties, let us agree that all cardinal invariants mentioned above are infinite. If one of the invariants is finite, we agree to replace it by ω_0.

It is easy to see that if F is a closed subspace of X, then 2^F as a hyperspace has the same topology as 2^F has as a subspace of 2^X, and we will use this fact in the sequel.

2. Relations between Cardinal Invariants of 2^X and Those of X.

We first examine the weight of 2^X. For an open cover $\mathcal{G}$ of X, we define $\alpha(\mathcal{G}) = \min\{|\mathcal{H}|: \mathcal{H} \subseteq \mathcal{G}$ and $\mathcal{H}$ covers X$\}$.

2.1 Lemma. $w(2^X) \leq \sup\{w(X)^{\alpha(\mathcal{G})}: \mathcal{G}$ is an open cover of X$\}$.

Proof. Let $\alpha = \sup\{w(X)^{\alpha(\mathcal{G})}: \mathcal{G}$ is an open cover of X$\}$, and let $\mathcal{D}$ be a basis for X of cardinality w(X). If m is a cardinal number, let $\mathcal{D}(m) = \{A: A \subseteq \mathcal{D}$ and $|A| \leq m\}$, and let $\mathcal{D}_1 = \cup\{\mathcal{D}(m)$: there is an open cover $\mathcal{G}$ of X such that $\alpha(\mathcal{G}) = m\}$. Now clearly $\alpha(\mathcal{G}) \leq w(X)$ for any open cover $\mathcal{G}$ of X. Thus there are at most w(X) distinct numbers among the numbers $\alpha(G)$, and so $\mathcal{D}_1$ is really a union of at most $w(X)$ families $\mathcal{D}(m)$. Since $|\mathcal{D}(m)| = w(X)^m$, $\alpha = \sup\{|\mathcal{D}(m)|$: there is an open cover G of X such that $m = \alpha(G)\}$. Now $\alpha \geq w(X)$, and $\mathcal{D}_1$ is the union of $\leq w(X)$ families, each of whose cardinalities does not exceed α. It follows that $|\mathcal{D}_1| \leq \alpha$. Finally, we set $\mathcal{T} = \{2^{\cup A}: A \in \mathcal{D}_1\} \cup \{B(X;G): G \in \mathcal{D}\}$. We will show that the finite intersections of members of $\mathcal{T}$ form a basis for 2^X. Since $|\mathcal{T}| \leq \alpha$, it will follow that $w(2^X) \leq |\mathcal{T}| \leq \alpha$, thus proving the lemma. To do this, we show that if W is any open set in X, then 2^W is a union of members of $\mathcal{T}$, and $B(X; W)$ is a union of members of $\mathcal{T}$. This is sufficient, since the sets $B(X; U)$ and 2^V for U,V open in X, form a subbase for 2^X. So let W be open in X. Let $F \in 2^W$. Since $\mathcal{D}$ is a basis for X, there is a subfamily A of $\mathcal{D}$ such that $W = \cup A$. Choose a subfamily A_1 of A of least cardinality that covers F. Clearly, $|A_1| \leq \alpha(A \cup \{X - F\})$, and so $2^{\cup A_1} \in \mathcal{T}$. Clearly, $F \in 2^{\cup A_1} \subseteq 2^W$. Thus 2^W is a union of such sets $2^{\cup A_1}$; and each such is a member of $\mathcal{T}$. Next, let W be open in X, and let $F \in B(X; W)$. Then $F \cap W \neq \phi$. Let $x \in F \cap W$. Since $\mathcal{D}$ is a basis, there exists $G \in \mathcal{D}$ such that $x \in G \subseteq W$. It follows that $F \in B(X; G) \subseteq B(X; W)$. Therefore, $B(X; W)$ is a union of such sets $B(X; G)$, and so is a union of members of $\mathcal{T}$. This completes the proof of the lemma.

2.2 Corollary. <u>Let X be a</u> T_1 <u>space. Then</u> $w(X) \leq w(2^X) \leq w(X)^{L(X)} \leq 2^{w(X)}$.

Proof. The first inequality follows from the fact that X is homeomorphic to a subspace of 2^X. The second inequality follows from 2.1, since $\alpha(\mathcal{G}) \leq L(X)$ for all open covers $\mathcal{G}$ of X. The last inequality follows from the obvious relation $L(X) \leq w(X)$, and so $w(X)^{L(X)} \leq w(X)^{w(X)} = 2^{w(X)}$.

2.3 Corollary. Let X be compact. Then $w(X) = w(2^X)$.

Proof. If X is compact, then for any open cover $\mathcal{G}$ of X, $\alpha(\mathcal{G})$ is finite, and so $w(X)^{\alpha(\mathcal{G})} = w(X)$, and since X is a subspace of 2^X, $w(X) \leq w(2^X)$. 2.3 follows.

2.4 Corollary. If $w(X)^{L(X)} = w(X)$, then $w(X) = w(2^X)$.

Proof. This is obvious from 2.1.

2.5 Corollary. Let X be a normal, T_1 space. If $w(X) = w(\beta X)$, then $w(2^X) = w(X)$.

Proof. For a normal, T_1 space X, the mapping $F \to cl_{\beta X}F$ is an embedding of 2^X onto a dense subspace of $2^{\beta X}$. Thus $w(2^X) \leq w(2^{\beta X})$. But, by 2.3, $w(2^{\beta X}) = w(\beta X)$, and by assumption, $w(\beta X) = w(X)$. We conclude that $w(2^X) \leq w(X)$. Since the reverse inequality is always valid, we conclude that $w(X) = w(2^X)$.

Using a result of Comfort-Hager in [5], we can sharpen the estimate of 2.2 for normal spaces.

2.6 Theorem. Let X be normal and T_1. Then $w(2^X) \leq w(X)^{wc(X)}$.

Proof. In a normal space X, any open set containing a closed set F contains a cozero-set containing F, and so the sets of the form $B(W; W_1, W_2, \ldots, W_n)$, where $W, W_1, \ldots, W_n$ are cozero-sets, form a basis for 2^X. Since there are no more cozero-sets than there are continuous real-valued functions, we conclude that 2^X has a basis of cardinality $\leq |C(X)|$. By 2.2 of [5], $|C(X)| \leq w(X)^{wc(X)}$ and so, in particular, $w(2^X) \leq w(X)^{wc(X)}$. Observe that, since $wc(X) \leq L(X)$, 2.6 formally sharpens the estimate $w(X)^{L(X)}$ of 2.2.

We will now give examples to illustrate the sharpness of these estimates. From 1 of [27], and the obvious inequality $w(2^X) \leq 2^{w(X)}$, it follows that if X is a discrete space of cardinal α, then 2^X has weight 2^α. This shows that the equality $w(2^X) = w(X)^{L(X)}$ is attained for some spaces. On the other hand, the equality $w(2^X) = w(X)^{L(X)}$ fails for many familiar spaces. For example, let X be the space of countable ordinals. Then $w(X) = L(X) = \omega_1$, and so $w(X)^{L(X)} = 2^{\omega_1}$. But, since $w(\beta X) = \omega_1$, 2.5 gives $w(2^X) = \omega_1$. Regarding the last inequality in 2.2, note that the equality $w(X)^{L(X)} = 2^{w(X)}$ holds for spaces X in which $w(X) = L(X)$, (for example, discrete spaces), while the strict inequality $w(X)^{L(X)} < 2^{w(X)}$ holds, for example, for Lindelöf spaces of weight c (for example, the Sorgenfrey line). Discrete spaces show that the inequality $w(X) \leq w(2^X)$ can be strict. We have seen above, in 2.3, 2.4, and 2.5, that the equality $w(X) = w(2^X)$ may be attained.

The inequality in 2.6 may fail for non-normal spaces. For example, let X be the upper half-plane with Niemytzki's tangent disc topology. Then X is separable, and contains a closed discrete subspace of cardinality c. Since, in general, $wc(Y) \leq d(Y)$, it follows that $wc(X) = \omega_0$. Clearly, $w(X) = c$. Now from 1 of [27], it follows that 2^X contains a closed discrete subset of cardinality 2^c. It follows that $w(2^X) = 2^c = w(X)^{L(X)}$, while $w(X)^{wc(X)} = c^{\omega_0} = c$.

Furthermore, the estimate in 2.6 for normal spaces X is sharper than the general inequality $w(2^X) \leq w(X)^{L(X)}$. To see this, we use the space of 1.2 in [10]. This space, which we shall denote by Y, is constructed in [10] using the continuum hypothesis, and is countably compact, normal, separable, non-compact, and has cardinality ω_1. It follows that $wc(Y) = \omega_0$ and $w(Y) = L(Y) = \omega_1$. The estimate in 2.6 for $w(2^Y)$ is, using the continuum hypothesis, $w(Y)^{wc(Y)} = \omega_1^{\omega_0} = \omega_1$, while $w(Y)^{L(Y)} = \omega_1^{\omega_1} = 2^{\omega_1}$.

Discrete spaces are examples of normal spaces for which the relation in 2.6 is actually equality, while the countable ordinals show that the inequality in 2.6 can be strict. (This follows from the fact that the space of countable ordinals has weight and weak covering number ω_1; we have shown above that its hyperspace also has weight ω_1.)

Before turning to the character of 2^X, we will look at the π-weight of 2^X, which is very easy to handle.

2.7 Theorem. $\underline{\pi(2^X) = \pi(X)}$.

Proof. If P is a π-basis for X, then $\{2^{\cup F} \cap [\bigcap_{G\varepsilon F} B(X;\, G)]$: F is a finite subset of $P\}$ is a π-basis for 2^X. Thus $\pi(2^X) \leq \pi(X)$. For the reverse inequality, let $\{G_i: i \varepsilon I\}$ be a π-basis for 2^X. Each G_i contains a non-empty basic open set $B(G_{i,0};\, G_{i,1},\ldots,G_{i,N_i})$. Then $\{G_{i,j}: i \varepsilon I,\ j \varepsilon \{0,1,\ldots,N_i\}\}$ is a π-basis for X. Therefore a π-basis for X of least cardinality is no larger than the π-weight of 2^X. That is, $\pi(X) \leq \pi(2^X)$.

We now turn to the character of 2^X. Our result is a straightforward generalization of the countable case treated in III of [53]. Recall that, as mentioned above, $\bar{\chi}(X)$ is defined by $\bar{\chi}(X) = \sup\{\chi(X,F)$: F is a closed subset of X$\}$.

2.8 Theorem. <u>If X is a T_1 space, then $\chi(2^X) = \bar{\chi}(X)\cdot hd(X)$</u>.

Proof. Let us set $\alpha = \chi(2^X)$ and $\beta = \bar{\chi}(X)\cdot hd(X) = \max\{\bar{\chi}(X), hd(X)\}$. Let F be a closed subset of X. Since $\chi(2^X) = \alpha$, F has a neighbourhood base in 2^X of cardinality $\leq \alpha$. Let $\{G_i: i < \alpha\}$ be such a base. For each $i < \alpha$, we can find open sets

$G_{i,0}, G_{i,1}, \ldots, G_{i,N_i}$ in X so that $F \in B(G_{i,0}; G_{i,1}, \ldots, G_{i,N_i}) \subseteq \mathcal{G}_i$. Let $\mathcal{U} = \{G_{i,0}: i < \alpha\}$. We claim that $\mathcal{U}$ is a base for the neighbourhoods of F in X. For, let W be any open set in X which contains F. Then 2^W is a neighbourhood of F in 2^X, thus there is some $i < \alpha$ such that $F \in B(G_{i,0}; G_{i,1}, \ldots, G_{i,N_i}) \subseteq \mathcal{G}_i \subseteq 2^W$. This clearly implies $G_{i,0} \subseteq W$. Thus $\mathcal{U}$ forms a base as claimed. Therefore $\chi(X,F) \leq |\mathcal{U}| \leq \alpha$. Since F is an arbitrary closed subset of X, we conclude that $\bar{\chi}(X) = \sup\{\chi(X,F)$: F is a closed subset of X$\} \leq \alpha$.

We next show that $hd(X) \leq \alpha$. Let $\{\mathcal{H}_i: i < \alpha\}$ be a base for the neighbourhoods of X in 2^X. For each $i < \alpha$, find open sets $H_{i,1}, \ldots, H_{i,M_i}$ in X such that $X \in B(X; H_{i,1}, \ldots, H_{i,M_i}) \subseteq \mathcal{H}_i$. For each $i < \alpha$ and for each $j \in \{1,2,\ldots,M_i\}$ choose a point $x_{i,j} \in H_{i,j}$. Let $D = \{x_{i,j}: i < \alpha,\ j \in \{1,\ldots,M_i\}\}$. We claim that D is dense in X. For, let V be any non-empty open subset of X. Then $B(X; V)$ is a neighbourhood of X in 2^X, and so, for some $i < \alpha$, we have $B(X; H_{i,1}, \ldots, H_{i,M_i}) \subseteq \mathcal{H}_i \subseteq B(X; V)$. This implies that $H_{i,j} \subseteq V$ for some $j \in \{1,2,\ldots,M_i\}$. Therefore, for such j, $x_{i,j} \in V$. Thus $V \cap D \neq \phi$, and so D is dense in X. We conclude that $d(X) \leq |D| \leq \alpha$.

Now, let F be a closed subset of X. Then 2^F is a subspace of 2^X, and so $\chi(2^F) \leq \chi(2^X) = \alpha$. Applying the argument of the preceding paragraph to the hyperspace 2^F, we obtain $d(F) \leq \alpha$.

Thus every closed subspace of X has density $\leq \alpha$. Now X is a subspace of 2^X, and so $\chi(X) \leq \chi(2^X) = \alpha$. We show that this implies every subspace of X has density $\leq \alpha$. For, let S be any subspace of X. Then cl S is a closed subspace of X, so by the above, $d(\text{cl } S) \leq \alpha$. Let $\{p_i: i < \alpha\}$ be a dense subspace of cl S of cardinality $\leq \alpha$. Now, since $\chi(X) \leq \alpha$, every point of cl S has a neighbourhood base in X of cardinality $\leq \alpha$. For each $i < \alpha$, let $\{N_{i,j}: j < \alpha\}$ be a neighbourhood base in X at the point p_i. For each $i < \alpha$ and $j < \alpha$, choose a point $b_{i,j} \in N_{i,j} \cap S$. Let $B = \{b_{i,j}: i < \alpha, j < \alpha\}$. Then B is dense in S and $|B| \leq \alpha \cdot \alpha = \alpha$. Thus $d(S) \leq |B| \leq \alpha$. We have thus shown that every subspace of X has density $\leq \alpha$, as claimed. It follows that $hd(X) = \sup\{d(Y): Y \subseteq X\} \leq \alpha$. Together with $\bar{\chi}(X) \leq \alpha$, we conclude that $\beta \leq \alpha$. We conclude the proof of 2.8 by showing that $\alpha \leq \beta$.

Let $F \in 2^X$. Now $d(F) \leq hd(X)$, and $\chi(X) \leq \bar{\chi}(X)$. Therefore $d(F) \leq \beta$ and $\chi(X) \leq \beta$. Let $\{x_i: i < \beta\}$ be a dense subset of F of cardinality $\leq \beta$. For each $i < \beta$, let $\{G_{i,j}: j < \beta\}$ be a base for the neighbourhoods of x_i in X of cardinality $\leq \beta$. Now $\chi(X,F) \leq \bar{\chi}(X) \leq \beta$. So there is a base for the neighbourhoods of F in X of cardinality $\leq \beta$. Let $\{H_k: k < \beta\}$ be such a base. Now, let $\mathcal{U}(F) = \{2^{H_k}: k < \beta\} \cup \{B(X; G_{i,j}): i < \beta, j < \beta\}$. We claim that the finite intersections of members of $\mathcal{U}(F)$ form a base for the neighbourhoods of F in 2^X. For, let $B(V_0; V_1, \ldots, V_N)$ be a basic neighbourhood of F in 2^X. Then $F \subseteq V_0$, so there exists $k < \beta$ such that $F \subseteq H_k \subseteq V_0$. For each $r \in \{1,2,\ldots,N\}$, $F \cap V_r$ is a non-empty open subset of F. Since $\{x_i: i < \beta\}$ is dense in F, there is, for each $r \in \{1,2,\ldots,N\}$ a point $x_{i_r} \in V_r$. Since $\{G_{i,j}: j < \beta\}$ is a basis at x_i for each i, we can find, for each r, an index j_r such that $x_{i_r} \in G_{i_r,j_r} \subseteq V_r$. It follows easily that $2^{H_k} \cap \bigcap_{r=1}^{N} B(X; G_{i_r,j_r}) = B(H_k; G_{i_1,j_1},\ldots,G_{i_N,j_N})$ $\subseteq B(V_0; V_1,\ldots,V_N)$. Thus every neighbourhood of F in 2^X contains a finite intersection of members of $\mathcal{W}(F)$, and so these finite intersections, being themselves neighbourhoods of F in 2^X, form a base for the neighbourhoods of F in 2^X. Since there are no more than β such finite intersections, we conclude that $\chi(2^X,F) \leq \beta$. Since F was chosen arbitrarily in 2^X, it follows that $\alpha = \chi(2^X)$ $= \sup\{\chi(2^X,F): F \in 2^X\} \leq \beta$. This concludes the proof of 2.8.

From 2.8 we deduce Wulbert's result in III of [53], namely: If X is compact, then 2^X is first countable, if and only if, X is hereditarily separable and perfectly normal.

An interesting aspect of Theorem 2.8 is the relation between the two cardinal invariants of X used to describe the character of 2^X. A Souslin continuum, whose existence is consistent with the usual axioms for set theory, is compact, perfectly normal, and not separable (see [41]). For such a continuum S, we see that $\bar{\chi}(S) < hd(S)$. In [40], Ostaszewski, using certain set theoretic assumptions consistent with the usual axioms for set theory, constructs a non-compact space X which is countably compact, locally compact, perfectly normal, and hereditarily separable. Let K denote the one-point compactification of this space. Since X is countably compact and not compact, X is not σ-compact. Therefore, the point at infinity in K is not a G_δ in K, and thus K is not perfectly normal. However, K is hereditarily separable since X is, and we see that $hd(K) < \bar{\chi}(K)$. It follows that neither of the invariants $\bar{\chi}(X)$ or $hd(X)$ can be removed in 2.8. However, in certain models of set theory, such a simplification can be made in certain cases. For example, I. Juhasz has shown that Martin's axiom and the negation of the continuum hypothesis imply that perfectly normal compact spaces are hereditarily separable. (See 5.6 in [25].)

In light of the above remarks, it is interesting and somewhat surprising that the numbers $\bar{\chi}(X)$ and $hd(X)$ arise together in a natural way in the context of hyperspaces.

The last cardinal invariant we will examine is cellularity. We first establish the following lemma.

2.9 Lemma. <u>Let D be dense in X, and let</u> $D = \bigcup_{i \in I} D_i$. <u>Then</u> $c(X) \leq |I| \cdot \sup\{c(D_i) : i \in I\}$.

Proof. Let $\alpha = |I| \cdot \sup\{c(D_i) : i \in I\}$. For the sake of contradiction, suppose $c(X) > \alpha$. Then there is a family $\mathcal{G}$ of pairwise disjoint, non-empty, open subsets of X such that $|\mathcal{G}| = \alpha^+$. (Here α^+ denotes the least cardinal larger than α.) For each $i \in I$, let $\mathcal{G}_i = \{G \in \mathcal{G}\colon G \cap D_i \neq \phi\}$. Since D is dense in X, $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$. Thus, since $|I| \leq \alpha$, for some $i \in I$, $|\mathcal{G}_i| = \alpha^+$. Since $\{G \cap D_i\colon G \in \mathcal{G}_i\}$ is a family of pairwise disjoint, non-empty, open subsets of D_i, it follows that $|\mathcal{G}_i| \leq c(D_i)$. But $c(D_j) \leq \alpha$ for all $j \in I$. We conclude that $\alpha^+ = |\mathcal{G}_i| \leq c(D_i) \leq \alpha$, a contradiction. Thus $c(X) \leq \alpha$, as asserted in the statement of the lemma.

2.10 Theorem. <u>Let X be a T_1 space. Then</u> $c(X) \leq c(2^X) \leq \sup\{c(X^n)\colon n \in \mathbb{N}\} \leq 2^{c(X)}$.

Proof. If $\mathcal{G}$ is a family of pairwise disjoint, non-empty, open subsets of X, then $\{2^G\colon G \in \mathcal{G}\}$ is a family of pairwise disjoint, non-empty, open subsets of 2^X of the same cardinality as $\mathcal{G}$. This proves the first inequality.

For each positive integer n, let $\mathcal{F}_n(X) = \{F \in 2^X\colon |F| \leq n\}$, and let $\mathcal{F}(X) = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n(X)$. $\mathcal{F}(X)$ is dense in 2^X. So, by 2.9, $c(2^X) \leq \omega_0 \cdot \sup\{c(\mathcal{F}_n(X))\colon n \in \mathbb{N}\} = \sup\{c(\mathcal{F}_n(X))\colon n \in \mathbb{N}\}$. Now, for each n, the map $f_n\colon X^n \to \mathcal{F}_n(X)$ defined by $f_n(x_1,x_2,\ldots,x_n) = \{x_1,x_2,\ldots,x_n\}$, is a continuous surjection. (See 2.4.3 of [36].) Since cellularity is not increased by continuous maps, we have $c(F_n(X)) \leq c(X^n)$ for each n. It follows that $c(2^X) \leq \sup\{c(X^n) : n \in \mathbb{N}\}$, establishing the second inequality. The last inequality follows from 4.6 of [25].

We give two examples to illustrate the sharpness of the estimates in 2.10. If X is separable, so are 2^X and all finite powers of X, in which case all the spaces X^n and 2^X have cellularity ω_0. In this case we have $\omega_0 = c(X) = c(2^X) = \sup\{c(X^n) : n \in \mathbb{N}\} < 2^{c(X)} = 2^{\omega_0}$. If S is a Souslin line, one can show that 2^S has uncountable cellularity. Assuming the continuum hypothesis, 2.10 becomes $\omega_0 = c(S) < c(2^S) = \sup\{c(S^n) : n \in \mathbb{N}\} = 2^{c(X)} = \omega_1$. (The continuum hypothesis and the existence of a Souslin line are, together, consistent with the usual axioms for set theory; see [41].) The author has been unable to determine whether the second inequality in 2.10 can be strict.

BIBLIOGRAPHY

[1] A. R. Bernstein, A new kind of compactness for topological spaces, Fund. Math. 66 (1970), 185-193.

[2] R. L. Blair and A. W. Hager, Extensions of zero-sets and of real-valued functions, Math. Z. 136 (1974), 41-52.

[3] W. W. Comfort, A nonpseudocompact product space whose finite subproducts are pseudocompact, Math. Annal 170 (1967), 41-44.

[4] W. W. Comfort, A survey of cardinal invariants, General Topology and Its Applications 1 (1971), 163-199.

[5] W. W. Comfort and A. W. Hager, Estimates for the number of real-valued continuous functions, Trans. Amer. Math. Soc. 150 (1970), 619-631.

[6] W. W. Comfort and S. Negrepontis, The Theory of Ultrafilters Springer-Verlag, Heidelberg, Grundlehren der math. Wissenschaften, Vol. 211, 1974.

[7] W. W. Comfort and K. A. Ross, Pseudocompactness and uniform continuity in topological groups, Pac. Jour. Math. 16 (1966), 483-496.

[8] D. W. Curtis and R. M. Schori, 2^X and C(X) are homeomorphic to the Hilbert cube, Bull. Amer. Math. Soc. 80 (1974), 927-931.

[9] N. J. Fine and L. Gillman, Remote points in $\beta\underset{\sim}{R}$, Proc. Amer. Math. Soc. 13 (1962), 29-36.

[10] S. P. Franklin and M. Rajagopalan, Some examples in topology, Trans. Amer. Math. Soc. 155 (1971), 305-314.

[11] Z. Frolik, The topological product of countably compact spaces, Czech. Math. J. 85 (1960), 329-338.

[12] Z. Frolik, Sums of ultrafilters, Bull. Amer. Math. Soc. 73 (1967), 87-91.

[13] Z. Frolik, On two problems of W. W. Comfort, Comm. Math. Univ. Carol. 8, 1 (1967), 139-144.

[14] T. Ganea, Symmetrische Potenzen topologischer Räume, Math. Nachr. 11 (1954), 305-316.

[15] L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, New York, 1960.

[16] J. Ginsburg, Some results on the countable compactness and pseudocompactness of hyperspaces, to appear.

[17] J. Ginsburg, On the Stone-Cech compactification of the space of closed sets, to appear.

[18] J. Ginsburg, A note on the G_δ-closure and realcompactness of 2^X, to appear.

[19] J. Ginsburg, Some cardinal invariants of hyperspaces, to appear.

[20] J. Ginsburg and V. Saks, Some applications of ultrafilters in topology, to appear in Pac. J. Math.

[21] I. Glicksberg, Stone-Cech compactifications of products, Trans. Amer. Math. Soc. 90 (1959), 369-382.

[22] A. W. Hager, Some remarks on the tensor product of function rings, Math. Zeitschr. 92 (1966), 210-224.

[23] H. Herrlich and J. van der Slot, Properties which are closely related to compactness, Indag. Math. 29 (1967), 524-529.

[24] K. H. Hofmann and P. S. Mostert, Elements of Compact Semigroups, Charles Merrill, Columbus, Ohio, 1966.

[25] I. Juhász, Cardinal Functions in Topology, Math. Centre Tracts 34, Mathematical Centre, Amsterdam, The Netherlands, 1971.

[26] M. Katetov, A theorem on mappings, Comm. Math. Univ. Carol. 8, 3 (1967), 431-433.

[27] J. Keesling, Normality and properties related to compactness in hyperspaces, Proc. Amer. Math. Soc. 24 (1970), 760-766.

[28] J. Keesling, Normality and compactness are equivalent in hyperspaces, Bull. Amer. Math. Soc. 76 (1970), 618-619.

[29] J. Keesling, On the equivalence of normality and compactness in hyperspaces, Pac. Jour. Math. 33 (1970), 657-667.

[30] J. Keesling, Compactness related properties in hyperspaces, Lecture Notes in Math. Vol. 171, Springer-Verlag, Berlin. Heidelberg.New York 1970, p.40-43.

[31] J. L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc. 52 (1942), 22-36.

[32] K. Kuratowski, Topology I, Academic Press, 1966, 160-187.

[33] J. D. Lawson, Intrinsic topologies in topological lattices and semilattices, Pac. J. Math. 44 (1973), 593-602.

[34] M. Mandelker, Supports of continuous functions, Trans. Amer. Math. Soc. 156 (1971), 73-83.

[35] M. M. Marjanović, A pseudocompact space having no dense countably compact subspace, Glasnik. Mat. Tom 6 (26) - No. 1 - 1971, 149-150.

[36] E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951), 152-183.

[37] Kiiti Morita, Completion of hyperspaces of compact subsets and topological completion of open-closed maps, General Topology and Its Applications, 4 (1974), 217-233.

[38] S. Mrowka, Further results on E-compact spaces I, Acta Mathematica 120 (1968), 161-185.

[39] N. Noble, Countably compact and pseudocompact products, Czech. Math. J. 19 (1969), 390-397.

[40] A. J. Ostaszewski, On countably compact, perfectly normal spaces, J. London Math. Soc. (to appear).

[41] M. E. Rudin, Souslin's conjecture, Amer. Math. Mon. 76 (1969), 1113-1119.

[42] M. E. Rudin, Partial orders on the types in $\beta\underset{\sim}{N}$, Trans. Amer. Math. Soc. 155, No. 2 (1971), 353-362.

[43] V. Saks, Countably Compact Groups, Thesis, Wesleyan University, Middletown, Conn., 1972.

[44] V. Saks and R. M. Stephenson Jr., Products of m-compact spaces, Proc. Amer. Math. Soc. 28 (1971), 278-288.

[45] C. T. Scarborough and A. H. Stone, Products of nearly compact spaces, Trans. Amer. Math. Soc. 124 (1966), 131-147.

[46] Set-Valued Mappings, Selections and Topological Properties of 2^X, Lecture Notes in Mathematics, Vol. 171, Springer-Verlag, Berlin.Heidelberg.New York, 1970.

[47] D. P. Strauss, Topological lattices, Proc. London Math. Soc. 18 (1968), 217-230.

[48] L. Vietoris, Bereiche Zweiter Ordnung, Monatshefte für Mathematik und Physik vol. 33 (1923), 49-62.

[49] A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, Paris, Hermann, 1937.

[50] M. Wojdyslawski, Sur la contractilité des hyperspaces de continus localement connexes, Fund. Math. 30 (1938), 247-252.

[51] R. G. Woods, Co-absolutes of remainders of Stone-Cech compactifications, Pac. J. Math. 37 (1971), 545-560.

[52] R. G. Woods, Topological extension properties, to appear in Trans. Amer. Math. Soc.

[53] D. E. Wulbert, Subsets of first countable spaces, Proc. Amer. Math. Soc. 19 (1968), 1273-1277.

[54] P. Zenor, On the completeness of the space of compact subsets, Proc. Amer. Math. Soc. 26 (1970), 190-192.

[55] P. Zenor, On the near completeness and the realcompactness of the space of closed subsets, to appear.