Generalized Autoregressive Conditional Heteroscedastic Modeling in Finance: Inferences from Stock Prices

by

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A Thesis Submitted to

the Faculty of Graduate Studies

In Partial Fufilment of the Requirements for the Degree of

MASTER OF SCIENCE

Department of Statistics

University of Manitoba

Winnipeg, Manitoba

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES
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Generalized Autoregressive Conditional Heteroscedastic Modeling in Finance: Inferences from Stock Prices

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Abstract

Recently, there has been a growing interest in using Generalized Autoregressive Conditional Heteroscedastic (GARCH) models in the finance discipline. The discussion will begin with an overview of the GARCH(P,Q) processes and its properties, with an emphasis on the normal GARCH(1,1) model. The highlight of this thesis lies in the two numerical examples investigating the usage and performance of several GARCH models. The first example features the fitting of six GARCH models to the returns of the Standard & Poor's 100 daily index (January 1991-December 2000). The second example explores three GARCH models utilizing the risk-neutral valuation of call options via Monte Carlo simulation for the Standard & Poor's 100 daily index (January 1981-December 1993) and the Standard & Poor's 500 weekly index (January 1991-December 2000).

Dedicated to my parents and grandparents.

Acknowledgements

First off, I would like to thank my entire family for their generous love, support and encouragement through the years. A heartfelt thanks to my advisor Dr. Thavaneswaran for his kind and patient guidance. I would also like to extend my gratitudes to Dr. Paseka and Mathew McLean for their time and assistance. Last but not least, to my lovely girlfriend for her big heart, constant affection and wit that keeps me grounded at all times. Thank you all very much!

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1 Introduction

It has been well documented in financial literature that many financial times series, such as the foreign exchange rates and returns[1] on stocks, often exhibit interesting empirical properties. The following four properties have been the subject of extensive studies:

1. Financial times series are often *leptokurtic.* This means that the distribution of their returns have a higher probability mass around the tails ("fat tails") and a higher peak at the mean than that of a standard normal distribution.

2. Financial time series are often *heteroscedastic.* This means that volatility[2] is time-varying and non-constant. In other words, the volatility of returns are serially correlated.

3. The squared values of the returns exhibit a high level of correlation whereas the values of the returns do not have much correlation.

4. There exists clustering of changes in returns i.e. small changes tend to be followed by small changes and vice versa. This characteristic is also known as *volatility clustering.*

In order to illustrate the aforementioned characteristics, consider the daily closing prices for Standard & Poor's 100-share index[3] (S&P 100) recorded from January 2, 1991 to December 29, 2000 and its returns for that time period.

Figure 1(a) and Figure 1(b) illustrate the dramatic price variability over time, suggesting a heteroscedastic nature. It might also be worthwhile to note here that

[1]Unless specified otherwise, the return series y_t in this thesis are calculated using $y_t = \ln\left(\frac{p_t}{p_{t-1}}\right)$, where p_t is the observed price at time t.

[2]Volatility is a measure of the variability in price over some period of time and is typically described as the standard deviation of returns.

[3]Dividend payments are excluded here.

the price variability seen in figures 1(a) and 1(b) tend to be grouped in periods with low and high price fluctuations. Perhaps this indicates the existance of volatility clustering.

Figure 1(c) exhibits leptokurtosis with the high peak at the mean, thin midrange and "fat tails". The leptokurtic nature is also reflected on the returns' kurtosis value of 8.05, a value almost thrice as large as that of a standard normal distribution. The QQ-plot (Figure 1(d)) agrees as well.

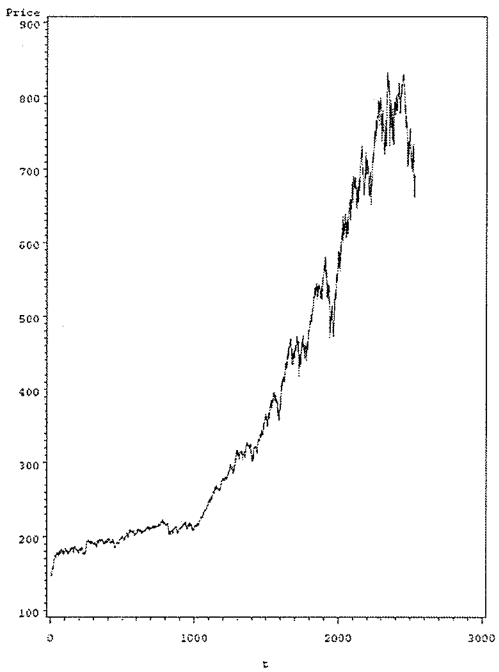
Plots of the sample autocorrelation function (SACF) of returns y_t and squared returns y_t^2 are given in Figure 2(a) and Figure 2(b) respectively. It is evident here that the majority of the sample autocorrelations for y_t^2 are significant, indicative of a dependence in the data.

Presented next is another example which also clearly exhibits the stated characteristics. It is a simple model $y_t = \epsilon_{t-1}^2\epsilon_t$ (ϵ_t is a Gaussian white noise with variance σ_ϵ^2) considered by Gouriéroux (1997) where the process y_t is (weakly) stationary having variance $Var(y_t) = 3\sigma_\epsilon^6$ and conditional variance, given past values, $Var(y_t|y_{t-1}) = \sigma_\epsilon^2\epsilon_{t-1}^4$ that are dependant on the lagged residuals. Since $E(\epsilon_t^{2n}) = \sigma_\epsilon^{2n}\dfrac{(2n)!}{2^n n!}$ and $E(y_t^4) = 315\sigma_\epsilon^{12}$, the kurtosis is $K^{(y)} = 35$, which is substantial in value. As for the correlation of y_t and y_t^2 processes,

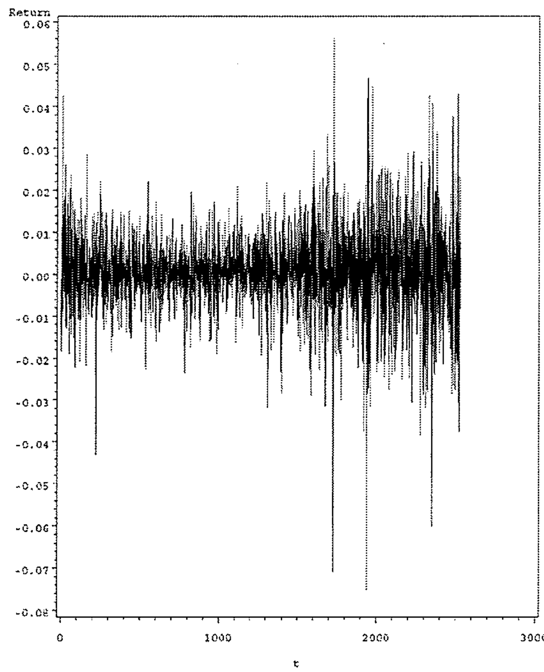
$$\rho_k^y = 0 \text{ for } k > 0,$$

$$\rho_k^{y^2} = \begin{cases} 1 & \text{if } k = 0, \\ 0.114285 & \text{if } k = 1, \\ 0 & \text{if } k \geq 1. \end{cases}$$

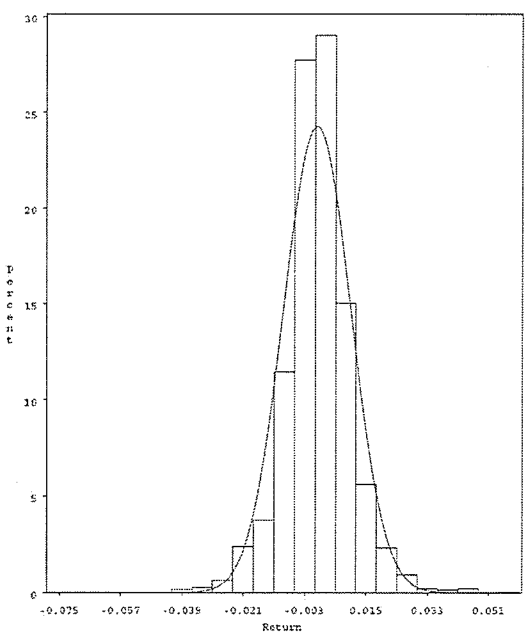
This clearly shows that even a simple model can generate very high peakedness, dependency of variance with its lagged residuals and correlated y_t^2 process.



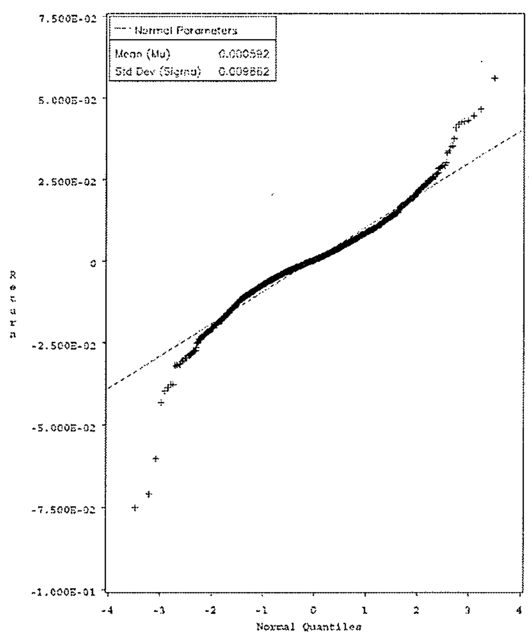
(a) Daily closing prices p_t



(b) Returns y_t

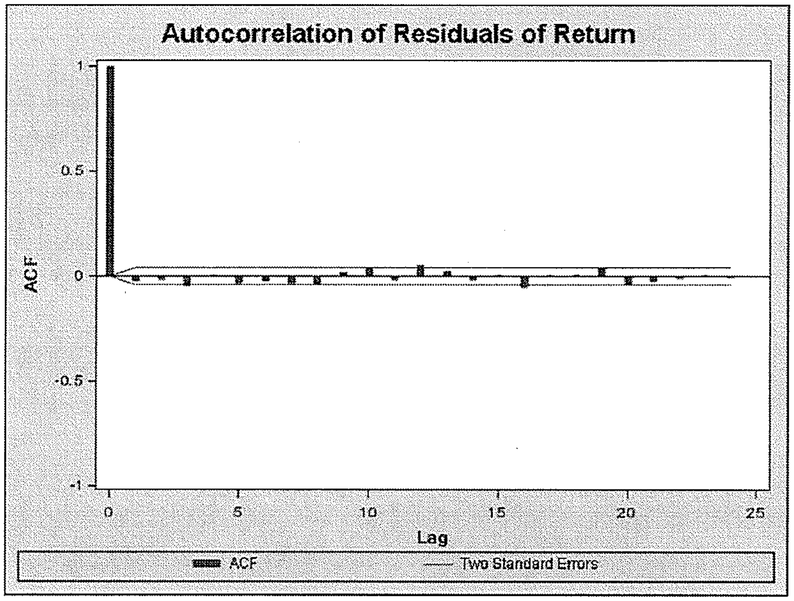


(c) Probability distribution curve of y_t

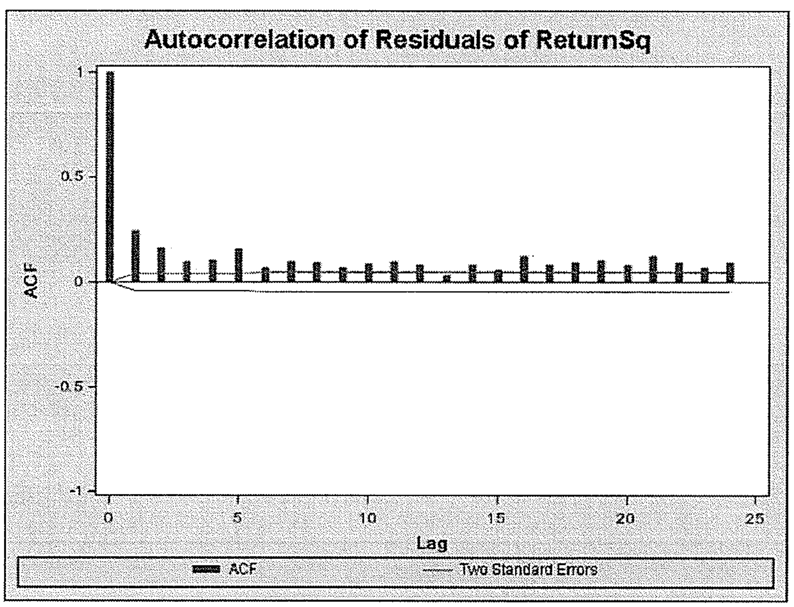


(d) QQ-plot of y_t

Figure 1: Preliminary graphs for the S&P 100 dataset



(a) SACF of returns y_t



(b) SACF of squared returns y_t^2

Figure 2: SACF for the S&P 100 dataset

General time series models, such as the Autoregressive Moving Average (ARMA) models, fail to capture the above phenomena the majority of the time. Additionally, general time series models have a significant limitation - the assumption of *homoscedasticity* (constant volatility). In this thesis, several Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models that have the ability to successfully capture all the above characteristics often seen in financial time series will be discussed. Option pricing based on GARCH processes, following in the spirit of the papers writen by Duan (1995), Duan *et al.* (2006), and Hafner and Herwartz (2001) will be explored as well.

For further insight on ARMA and GARCH models, refer to Abraham (1983), Appadoo *et al.* (2005), Appadoo *et al.* (2006), Ghahramani and Thavaneswaran (2007), Ruppert (2004), Tsay (2005) and Wei (2006).

2 GARCH (P,Q) Model

In 1986, Tim Bollerslev introduced the GARCH model, which is an extension to the ARCH model pioneered by Robert Engle (1982). The general GARCH(P,Q) model for the time series y_t is given by

$$y_t|Y_{t-1} \sim (0, h_t),$$

$$y_t = \sqrt{h_t} Z_t, \tag{2.1}$$

$$h_t = \omega + \sum_{i=1}^{P} \alpha_i y_{t-i}^2 + \sum_{j=1}^{Q} \beta_j h_{t-j}, \tag{2.2}$$

where $P > 0$, $Q \geq 0$, $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $Y_{t-1} = (y_1, y_2, \ldots, y_{t-1})$ and Z_t is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and unit variance (i.e. $Z_t \overset{i.i.d.}{\sim} (0,1)$). It is easy to see that $\{y_t\}$ will reduce to an ARCH(P) process when $Q = 0$, and $\{y_t\}$ is just white noise when $P = Q = 0$.

There is an alternative representation for the GARCH(P,Q) model as described in equations (2.1) and (2.2). By letting $u_t = y_t^2 - h_t$ (u_t is known as a martingale difference sequence[4]), the GARCH(P,Q) model can be interpreted as an ARMA(R,Q) representation in y_t^2:

$$\begin{aligned}
& & y_t^2 - u_t &= \omega + \sum_{i=1}^{P} \alpha_i y_{t-i}^2 + \sum_{j=1}^{Q} \beta_j h_{t-j} \\
&\Leftrightarrow & y_t^2 - u_t &= \omega + \sum_{i=1}^{P} \alpha_i y_{t-i}^2 + \sum_{j=1}^{Q} \beta_j [y_{t-j}^2 - u_{t-j}] \\
&\Leftrightarrow & (1 - \sum_{i=1}^{P} \alpha_i B^i - \sum_{j=1}^{Q} \beta_j B^j) y_t^2 &= \omega + u_t - \sum_{j=1}^{Q} \beta_j B^j u_{t-j} \\
&\Leftrightarrow & \phi(B) y_t^2 &= \omega + \beta(B) u_t
\end{aligned} \tag{2.3}$$

[4] $E(u_t) = 0$ and $Cov(u_t, u_{t-j}) = 0$ for $j \geq 1$.

where

$$\phi(B) = 1 - \sum_{i=1}^{R} \phi_i B^i, \qquad \beta(B) = 1 - \sum_{i=1}^{Q} \beta_i B^i,$$

$$\phi_i = \alpha_i + \beta_i, \qquad R = max(P, Q).$$

Stationary assumptions for y_t^2 having the ARMA(R,Q) representation will be made here, and they are as follows:

1. All zeroes of the polynomial $\phi(B)$ lie outside the unit circle. This assumption is needed as it ensures the u_t's are uncorrelated with zero mean and finite variance, and that the process y_t^2 is weakly stationary.

2. There exists a sequence of constants $\{\psi_i\}$ such that $\sum_{i=0}^{\infty} \psi_i^2 < \infty$, where the ψ_i's are obtained from the relation $\psi(B)\phi(B) = \beta(B)$ with $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$. Refer to Appendix I to see how the ψ-weights can be obtained for any stationary process.

Consequently, the autocorrelation function (ACF) of y_t^2 will be exactly identical as that for a stationary ARMA(R,Q) process (Thavaneswaran *et al.* (2005b)). This implies that the kth lag ACF of y_t^2 can be calculated using

$$\rho_k^{y^2} = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+k}}{\sum_{i=0}^{\infty} \psi_i^2}.$$

2.1 Kurtosis of GARCH(P,Q)

The following theorem calculates the kurtosis for a GARCH process in terms of the ψ weights (Thavaneswaran *et al.* (2005a)).

Theorem 2.1. *Under the stationarity assumptions and finite fourth moment, the kurtosis $K^{(y)}$ of the process (2.3) is given by*

$$K^{(y)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{i=0}^{\infty}\psi_i^2}$$

and the variance of the y_t^2 process is

$$Var(y_t^2) = \sigma_u^2 \sum_{i=0}^{\infty}\psi_i^2$$

where $\sigma_u^2 = \frac{\sigma_y^4(K^{(y)} - 1)}{\sum_{i=0}^{\infty}\psi_i^2}$ *and* $\sigma_y^2 = Var(y_t) = \frac{\omega}{1 - \phi_1 - \phi_2 - \ldots - \phi_R}$.

Proof. Let $Var(u_t) = \sigma_u^2$ and recall that Z_t have zero mean, unit variance and finite forth moments. Taking expectations of (2.1) yields the following unconditional mean and variance of returns:

$$E(y_t) = E(\sqrt{h_t}Z_t) = E(\sqrt{h_t})E(Z_t) = 0,$$
$$Var(y_t) = E(y_t^2) = E(h_t Z_t^2) = E(h_t)E(Z_t^2) = E(h_t).$$

From (2.3), we can see that

$$\begin{aligned} & \phi(B)y_t^2 = \omega + \beta(B)u_t \\ \Leftrightarrow \quad & y_t^2 = \frac{\omega}{\phi(B)} + \frac{\beta(B)}{\phi(B)}u_t = \frac{\omega}{\phi(B)} + \sum_{i=0}^{\infty}\psi_i u_{t-i} \\ \Leftrightarrow \quad & Var(y_t^2) = \sigma_u^2 \sum_{i=0}^{\infty}\psi_i^2. \end{aligned}$$

Now, observe that

$$\sigma_u^2 = Var(y_t^2 - h_t) = E(y_t^4) - E(h_t^2) = E(h_t^2 Z_t^4) - E(h_t^2) = E(h_t^2)[E(Z_t^4) - 1],$$

which results in

$$Var(y_t^2) = E(h_t^2)[E(Z_t^4) - 1]\sum_{i=0}^{\infty}\psi_i^2. \tag{2.4}$$

Moreover, by the definition of a variance, equation (2.3) also leads to

$$\begin{aligned} Var(y_t^2) &= E(y_t^4) - [E(y_t^2)]^2 \\ &= E(h_t^2 Z_t^4) - [E(h_t)]^2 \\ &= E(h_t^2)E(Z_t^4) - [E(h_t)]^2. \end{aligned} \tag{2.5}$$

Equating equations (2.4) and (2.5), we have

$$\begin{aligned} E(h_t^2)E(Z_t^4) - [E(h_t)]^2 &= E(h_t^2)[E(Z_t^4) - 1]\sum_{i=0}^{\infty}\psi_i^2 \\ \Leftrightarrow \qquad \frac{E(h_t^2)}{[E(h_t)]^2} &= \frac{1}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{i=0}^{\infty}\psi_i^2}. \end{aligned}$$

Now the corresponding kurtosis is

$$\begin{aligned} K^{(y)} = \frac{E[(y_t - E(y_t))^4]}{[Var(y_t)]^2} &= \frac{E(y_t^4)}{[E(y_t^2)]^2} \\ &= \frac{E(h_t^2)E(Z_t^4)}{[E(h_t)]^2[E(Z_t^2)]^2} \\ &= \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{i=0}^{\infty}\psi_i^2}, \end{aligned}$$

which completes the first half of Theorem 2.1. For the proof of the second half, notice that from (2.3),

$$Var(y_t) = \frac{\omega}{1 - \phi_1 - \phi_2 - \ldots - \phi_R} = \sigma_y^2,$$

and that, from the previous derivation,

$$K^{(y)} = \frac{E(y_t^4)}{[E(y_t^2)]^2} = \frac{E(y_t^4)}{\sigma_y^4},$$

so that

$$Var(y_t^2) = E(y_t^4) - [E(y_t^2)]^2 = \sigma_y^4[K^{(y)} - 1] = \sigma_u^2 \sum_{i=0}^{\infty} \psi_i^2,$$

which, in turn, produces

$$\sigma_u^2 = \frac{\sigma_y^4(K^{(y)} - 1)}{\sum_{i=0}^{\infty} \psi_i^2}.$$

□

To illustrate Theorem 2.1, the $K^{(y)}$ and σ_u^2 for the ARCH(1) (or equivalantly GARCH(1,0)) model of the form

$$y_t = \sqrt{h_t} Z_t,$$
$$h_t = \omega + \alpha_1 y_{t-1}^2,$$

where $Z_t \overset{i.i.d.}{\sim} N(0,1)$ are simply $K^{(y)} = \dfrac{3(1-\alpha_1^2)}{1-3\alpha_1^2}$ and $\sigma_u^2 = \sigma_y^4(K^{(y)} - 1)(1 - \alpha_1^2)$ respectively.

2.2 Estimation of GARCH(P,Q)

A commonly used method for estimation is the maximum likelihood estimation (MLE) method. In this thesis, MLE will be used to provide an appropriate estimate for the GARCH parameters, namely $\Theta = (\omega, \alpha_1, \ldots, \alpha_P, \beta_1, \ldots, \beta_Q)$.

Consider the conditional density of observation t

$$f(y_t|Y_{t-1}) = \frac{f(Z_t)}{\sqrt{h_t}}.$$

The likelihood function is the product of $f(y_t|Y_{t-1})$ from a set of n observed values $y_1, y_2, \ldots, y_n$, that is

$$L(\Theta) = f(y_1|Y_0) f(y_2|Y_1) \ldots f(y_n|Y_{n-1}).$$

Maximizing this likelihood, or equivalently, in logarithmic form,

$$\ln L(\Theta) = \sum_{t=1}^{n} \left[-\frac{1}{2} \ln(h_t) + \ln(f(Z_t)) \right],$$

provides the maximum likelihood estimate of all the parameters.

2.3 Forecasts based on GARCH(P,Q)

Using methods similar to those used for the ARMA process, predicting $y^2_{n+\ell}$ ($\ell \geq 1$) based on past observations can be easily done. Based on observations y_1, y_2, ..., y_n, let $y^2_n(\ell)$ be the ℓ-steps-ahead forecast of $y^2_{n+\ell}$.

The linear filter representation of the GARCH model (2.3) in terms of ψ-weights is given by

$$\begin{aligned} & \phi(B) y^2_{n+\ell} = \omega + \beta(B) u_{n+\ell} \\ \Leftrightarrow \quad & y^2_{n+\ell} = \frac{\omega}{\phi(B)} + u_{n+\ell} + \psi_1 u_{n+\ell-1} + \ldots + \psi_{\ell-1} u_{n+1} + \psi_\ell u_n + \ldots . \end{aligned}$$

Hence, the ℓ-step-ahead forecast of $y^2_{n+\ell}$ based on n observations can be represented as

$$y_n^2(\ell) = \frac{\omega}{\phi(B)} + \epsilon_0 u_n + \epsilon_1 u_{n-1} + \epsilon_2 u_{n-2} + \ldots$$

where ϵ_j ($j \geq 0$) are constants chosen to minimize the mean squared error. By expressing the mean squared error as

$$\begin{aligned}
E[(y_{n+\ell}^2 - y_n^2(\ell))^2] &= E[(\{u_{n+\ell} + \psi_1 u_{n+\ell-1} + \ldots + \psi_{\ell-1} u_{n+1} + \psi_\ell u_n + \psi_{\ell+1} u_{n-1} + \ldots\} - \\
&\qquad \{\epsilon_0 u_n + \epsilon_1 u_{n-1} + \epsilon_2 u_{n-2} + \ldots\})^2] \\
&= E[(\{u_{n+\ell} + \psi_1 u_{n+\ell-1} + \ldots + \psi_{\ell-1} u_{n+1}\} + \\
&\qquad \{(\psi_\ell - \epsilon_0) u_n + (\psi_{\ell+1} - \epsilon_1) u_{n-1} + \ldots\})^2] \\
&= E(u_{n+\ell}^2) + \psi_1^2 E(u_{n+\ell-1}^2) + \ldots + \psi_{\ell-1}^2 E(u_{n+1}^2) + \\
&\qquad (\psi_\ell - \epsilon_0)^2 E(u_n^2) + (\psi_{\ell+1} - \epsilon_1)^2 E(u_{n-1}^2) + \ldots \\
&= \sigma_u^2 [1 + \psi_1^2 + \ldots + \psi_{\ell-1}^2] + \sigma_u^2 \sum_{j=0}^{\infty} (\psi_{\ell+j} - \epsilon_j)^2,
\end{aligned}$$

the mean squared error is minimized when $\psi_{\ell+j} = \epsilon_j$ for $j \geq 0$ since u_t is a martingale. Therefore, the ℓ-steps-ahead minimum mean squared error (MMSE) forecast of $y_{n+\ell}^2$ is

$$y_n^2(\ell) = \frac{\omega}{\phi(B)} + \psi_\ell u_n + \psi_{\ell+1} u_{n-1} + \psi_{\ell+2} u_{n-2} + \ldots.$$

Using conditional expectations, it can be further shown that

$$\begin{aligned}
y_n^2(\ell) &= E(y_{n+\ell}^2 | y_n, \ldots, y_1) \\
&= \frac{\omega}{\phi(B)} + \psi_\ell u_n + \psi_{\ell+1} u_{n-1} + \psi_{\ell+2} u_{n-2} + \ldots
\end{aligned}$$

following from the fact that

$$E(u_{n+j}|y_n,\ldots,y_1) = \begin{cases} u_{n+j}, & \text{if } j \leq 0 \\ 0, & \text{if } j > 0 \end{cases}.$$

Similarly, the ℓ-steps-ahead forecast for the conditional variances given a history of returns, denoted by $h_n(\ell)$, can be obtained with ease as

$$h_n(\ell) = Var(y_{n+\ell}|y_n,\ldots,y_1) = E(y^2_{n+\ell}|y_n,\ldots,y_1).$$

2.4 Forecast errors of GARCH(P,Q)

Based on n observations y_1, y_2, $\ldots$, y_n, let $y^2_n(\ell)$ be the ℓ-steps-ahead MMSE forecast of $y^2_{n+\ell}$ and let $e_n(\ell) = y^2_n(\ell) - y^2_{n+\ell}$ be the corresponding forecast error. The following theorem (Thavaneswaran *et al.* (2005a)) formulates the ℓ-steps-ahead forecast error variance $Var(e_n(\ell))$ in terms of the kurtosis $K(y)$ and the ψ weights.

Theorem 2.2. *The ℓ-steps-ahead forecast error variance $Var(e_n(\ell))$ for any GARCH(P,Q) model is given by*

$$Var(e_n(\ell)) = \frac{\left(\frac{\omega}{1-\phi_1-\phi_2-\ldots-\phi_R}\right)^2 (K^{(y)}-1)}{\sum_{j=0}^{\infty}\psi_j^2}\left[1+\sum_{j=1}^{\ell-1}\psi_j^2\right]$$

where $\phi_i = \alpha_i + \beta_i$ and $R = max(P,Q)$.

Proof. For a stationary ARMA process, the variance of ℓ-steps-ahead forecast error with error variance σ_u^2 is

$$Var(e_n(\ell)) = \sigma_u^2 \left[1 + \sum_{j=1}^{\ell-1} \psi_j^2\right].$$

Using the results of Theorem 2.1, we can clearly see that Theorem 2.2 holds. □

2.5 Normal GARCH (1,1) model

The simplest of the nontrivial GARCH processes is the normal GARCH(1,1) model, for which

$$\begin{aligned} &y_t|Y_{t-1} \sim N(0, h_t), \\ &y_t = \sqrt{h_t}Z_t, &(2.6)\\ &h_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1}, &(2.7)\end{aligned}$$

where $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $Z_t \overset{i.i.d.}{\sim} N(0,1)$. The strengths and weaknesses of GARCH models can be investigated by focusing on this simple model.

By using the martingale difference equation $u_t = y_t^2 - h_t$, equations (2.6) and (2.7) can be interpreted as

$$\begin{aligned} & y_t^2 - u_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1(y_{t-1}^2 - u_{t-1}) \\ \Leftrightarrow \quad & y_t^2 - \alpha_1 y_{t-1}^2 - \beta_1 y_{t-1}^2 = \omega + u_t - \beta_1 u_{t-1} \\ \Leftrightarrow \quad & (1 - \phi_1 B)y_t^2 = \omega + (1 - \beta_1 B)u_t \qquad (2.8)\end{aligned}$$

so that

$$(1 - \phi_1 B)(1 + \psi_1 B + \psi_2 B^2 + \ldots) = (1 - \beta_1 B),$$

from which we obtain

$$\psi_0 = 1, \quad \psi_1 = \alpha_1, \quad \psi_2 = \alpha_1(\alpha_1 + \beta_1),$$
$$\psi_3 = \alpha_1(\alpha_1 + \beta_1)^2, \quad \dots, \quad \psi_i = \alpha_1(\alpha_1 + \beta_1)^{i-1}$$

and

$$\sum_{i=0}^{\infty} \psi_i^2 = 1 + \alpha_1^2 + \alpha_1^2(\alpha_1 + \beta_1)^2 + \dots = 1 + \frac{\alpha_1^2}{1 - \phi_1^2}.$$

2.5.1 Kurtosis

Recall that for any random variable Y with finite fourth moments, the kurtosis is defined by $K^{(y)} = \frac{E[(Y - E(Y))^4]}{[Var(Y)]^2}$. With stationarity conditions in place, the following theorem provides the kurtosis of y_t for a normal GARCH(1,1) process.

Corollary 2.1. *Consider the GARCH(1,1) model in equations (2.6) and (2.7). The kurtosis of y_t is given by*

$$K^{(y)} = \frac{3(1 - \phi_1^2)}{1 - \phi_1^2 - 2\alpha_1^2}$$

where $\phi_1 = \alpha_1 + \beta_1 < 1$, $\sigma_u^2 = \frac{\sigma_y^4(K^{(y)} - 1)(1 - \phi_1^2)}{1 - \phi_1^2 + \alpha_1^2}$ and $\sigma_y^2 = \frac{\omega}{1 - \phi_1}$.

Proof. The kurtosis, according to Theorem 2.1, is

$$K^{(y)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{i=0}^{\infty} \psi_i^2} = \frac{3}{3 - 2\left[1 + \frac{\alpha_1^2}{1 - \phi_1^2}\right]}$$
$$= \frac{3(1 - \phi_1^2)}{1 - \phi_1^2 - 2\alpha_1^2},$$

which corresponds to the result found in Bollerslev (1986). Moreover,

$$\sigma_u^2 = \frac{\sigma_y^4(K^{(y)}-1)}{\sum_{i=0}^{\infty}\psi_i^2} = \frac{\sigma_y^4(K^{(y)}-1)}{\left[1+\frac{\alpha_1^2}{1-\phi_1^2}\right]} = \frac{\sigma_y^4(K^{(y)}-1)(1-\phi_1^2)}{1-\phi_1^2+\alpha_1^2}$$

and the unconditional variance of y_t is $\sigma_y^2 = \frac{\omega}{1-\phi_1} = \frac{\omega}{1-\alpha_1-\beta_1}$. □

It can be shown that if $1-\phi_1^2-2\alpha_1^2 > 0$, then $K^{(y)} > 3$. This consequently implies that the normal GARCH(1,1) process has a heavier tail distribution than that of a standard normal distribution. See Appendix II for an alternate proof of Corollary 2.1.

2.5.2 Estimation

From n observed values $Y_n = (y_1, y_2, \ldots, y_n)$, the estimates of the normal GARCH(1,1) parameters, $\hat{\Theta} = (\hat{\omega}, \hat{\alpha}_1, \hat{\beta}_1)$, can be obtained by maximizing the conditional likelihood function

$$\begin{aligned} L(\Theta) &= f(y_1|Y_0)f(y_2|Y_1)\ldots f(y_n|Y_{n-1}) \\ &= \prod_{t=1}^{n}\left(\frac{1}{\sqrt{2\pi h_t}}\right)\exp\left(-\frac{y_t^2}{2h_t}\right) \end{aligned}$$

or equivalently, by maximizing the log-likehood function

$$\begin{aligned} \ln L(\Theta) &= -\frac{1}{2}\sum_{t=1}^{n}\left(\ln(2\pi)+\ln(h_t)+\frac{y_t^2}{h_t}\right) \\ &= -\frac{1}{2}\left(n\ln(2\pi)+\sum_{t=1}^{n}[\ln(h_t)+Z_t^2]\right), \end{aligned}$$

where y_t and h_t are as equated in (2.6) and (2.7) respectively.

2.5.3 Forecasts

Equation (2.8) can be rewritten as $y_t^2 = \omega + \phi_1 y_{t-1}^2 + u_t - \beta_1 u_{t-1}$. Taking conditional expectations, the one-step-ahead ($\ell = 1$) forecast given past observations can be expressed as

$$\begin{aligned} y_n^2(1) &= E(y_{n+1}^2 | y_n, \ldots, y_1) \\ &= E(\omega + \phi_1 y_n^2 + u_{n+1} - \beta_1 u_n | y_n, \ldots, y_1) \\ &= \omega + \phi_1 y_n^2 \end{aligned}$$

since $E(y_n^2 | y_n, \ldots, y_1) = y_n^2$ and $E(u_{n+1} | y_n, \ldots, y_1) = E(u_n | y_n, \ldots, y_1) = 0$.

Similarly, the ℓ-step-ahead forecast, for $\ell \geq 2$, can be shown to satisfy

$$\begin{aligned} y_n^2(2) &= E(y_{n+2}^2 | y_n, \ldots, y_1) = \omega + \phi_1 y_n^2(1), \\ y_n^2(3) &= E(y_{n+3}^2 | y_n, \ldots, y_1) = \omega + \phi_1 y_n^2(2), \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ y_n^2(\ell) &= E(y_{n+\ell}^2 | y_n, \ldots, y_1) = \omega + \phi_1 y_n^2(\ell - 1). \end{aligned}$$

By repeated substitution, we get

$$y_n^2(\ell) = \omega + \phi_1 B y_n^2(\ell),$$

implying that

$$\lim_{\ell \to \infty} y_n^2(\ell) = \frac{\omega}{1 - \phi_1}.$$

The conditional variance forecasts are obtained in a similar fashion.

2.5.4 Forecast errors

Based on the unconditional variance σ_y^2, the error variance σ_u^2 and the ψ-weights for a normal GARCH(1,1) process as well as Theorem 2.2, the ℓ-steps-ahead forecast error variance is

$$\begin{aligned} Var(e_n(\ell)) &= \sigma_u^2 \left[1 + \sum_{j=1}^{\ell-1} \psi_j^2\right] \\ &= \frac{\sigma_y^4(K^{(y)}-1)(1-\phi_1^2)}{1-\phi_1^2+\alpha_1^2}\left[1+\sum_{j=1}^{\ell-1}[\alpha_1(\alpha_1+\beta_1)^{j-1}]^2\right]. \end{aligned} \tag{2.9}$$

For $|\alpha_1 + \beta_1| < 1$, equation (2.9) converges to

$$Var(e_n(\ell)) = \frac{\sigma_y^4(K^{(y)}-1)(1-\phi_1^2)}{1-\phi_1^2+\alpha_1^2}\left[1+\frac{\alpha_1^2}{1-(\alpha_1+\beta_1)^2}\right].$$

2.5.5 Simulation

Using SAS[5], ten samples of the normal GARCH(1,1) model, following equations (2.6) and (2.7), were simulated with $\omega = 0.1$ and various combinations of α_1 and β_1 values that ranged from 0.1 to 0.9 and 0.05 to 0.9 respectively, under the additional requirement that $\alpha_1 + \beta_1 \leq 1$. Each sample was generated using 2000 observations with an initial 'burn' of 500 observations to ensure stability of the process. The kurtosis values of y_t, as well as all significant k-th lag sample autocorrelations of both y_t and y_t^2 were then tabulated in Microsoft Excel (figures 3, 4 and 5).

The majority of sample autocorrelations for y_t^2 shows strong presence of significance and an exponential decaying pattern, whereas the sample autocorrelations for y_t are mostly not significant (except when $\alpha + \beta$ is close to or at the nonstationary boundary i.e. $\alpha + \beta = 1$) with no obvious pattern. See Figure 6(a) and Figure 6(b)

[5]See Appendix VII for the full SAS code used in the simulation presented here. SAS codes for the simulations presented in later sections are given in Appendix VII as well.

Normal GARCH(1,1) simulation											
Sample	β_1	α_1	$K^{(y)}$	α_1	$K^{(y)}$	α_1	$K^{(y)}$	α_1	$K^{(y)}$	α_1	$K^{(y)}$
1			3.0722		3.3917		5.5014		18.9659		41.8930
2			3.3233		3.5398		4.7107		8.7146		14.6698
3			3.2218		3.5822		5.2068		9.6451		17.0817
4			3.1662		3.6661		6.5620		13.0010		17.3573
5	0.05	0.1	3.1724	0.25	3.4856	0.5	5.2097	0.75	11.2517	0.9	29.7147
6			3.1831		3.5563		5.0205		12.8543		62.2158
7			3.1319		3.5578		7.1129		21.0929		39.4813
8			3.1451		3.4551		4.5210		7.2972		13.2502
9			3.0599		3.3751		6.1927		27.5492		58.8962
10			3.0953		3.5832		7.8273		20.4985		35.9411
1			3.0733		3.4033		5.7566		22.7730		52.0790
2			3.3238		3.5512		4.8652		10.1798		18.9301
3			3.2255		3.6062		5.3744		10.6237		19.6967
4			3.1672		3.6905		6.7101		12.9953		27.0762
5	0.1	0.1	3.1703	0.25	3.4886	0.5	5.2798	0.75	13.3298	0.9	37.6979
6			3.1847		3.5639		5.1485		20.1862		114.9206
7			3.1326		3.5779		7.3807		22.3403		41.3759
8			3.1455		3.4586		4.5626		7.9701		16.8770
9			3.0611		3.3983		6.5888		29.8903		60.7106
10			3.0963		3.6074		7.8763		20.2988		36.6729
1			3.0794		3.4659		8.1611		81.1582		
2			3.3292		3.6217		5.9954		20.6280		
3			3.2394		3.7002		6.1537		16.3982		
4			3.1752		3.7949		7.1909		87.7949		
5	0.25	0.1	3.1650	0.25	3.5127	0.5	5.8632	0.75	23.3133		
6			3.1905		3.6052		6.7872		101.0627		
7			3.1354		3.6581		8.4689		29.0454		
8			3.1503		3.4918		4.9737		16.1860		
9			3.0709		3.5115		8.8207		46.1437		
10			3.1013		3.6831		8.0317		21.6362		
1			3.1078		4.0449		97.2126				
2			3.3567		3.9871		23.3757				
3			3.2721		4.0180		20.9194				
4			3.2127		4.1317		63.4937				
5	0.5	0.1	3.1683	0.25	3.7440	0.5	18.6928				
6			3.2123		4.0141		37.0021				
7			3.1443		3.9695		17.0697				
8			3.1787		3.7366		22.2556				
9			3.1216		4.1463		35.2988				
10			3.1136		3.8323		8.6652				
1			3.3042		33.9020						
2			3.4383		38.2519						
3			3.3657		11.0649						
4			3.3139		33.6499						
5	0.75	0.1	3.2911	0.25	14.5737						
6			3.4019		8.4787						
7			3.2387		9.1003						
8			3.3209		13.0249						
9			3.3327		13.4853						
10			3.1493		5.8686						
1			12.7964								
2			9.9866								
3			5.4209								
4			21.9512								
5	0.9	0.1	9.4458								
6			5.5283								
7			5.4654								
8			6.0175								
9			5.1764								
10			4.8088								

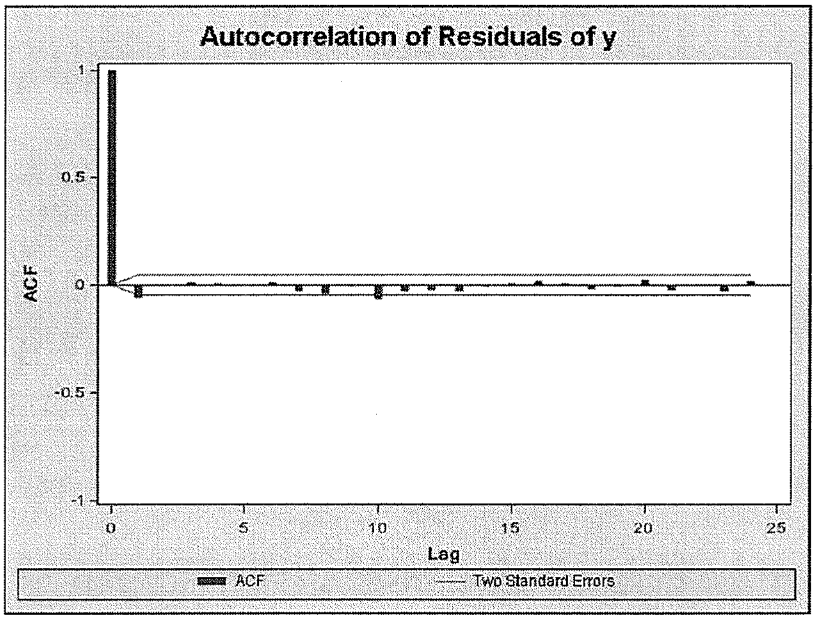
Figure 3: Kurtosis values of y_t for the simulated normal GARCH(1,1) model

Normal GARCH(1,1) simulation											
Sample	β_1	α_1	$\rho_k(y)$	α_1	$\rho_k(y)$	α_1	$\rho_k(y)$	α_1	$\rho_k(y)$	α_1	$\rho_k(y)$
1			None		1		1		None		10
2			None		None		6		6		5-6
3			None		None		None		None		1-2, 20
4			11, 22, 24		11, 22, 24		11		1		1, 4
5	0.05	0.1	None	0.25	None	0.5	None	0.75	4-5	0.9	5
6			None		None		2		2		None
7			None		None		None		None		None
8			7		7		None		None		2
9			6		6		6		6-7		1, 3-4, 7
10			None		None		None		1		1-2
1			None		1		1, 10		10		2-3, 5, 8, 10-11, 13
2			None		None		6		5-6		6, 19
3			None		None		None		20		1-2, 20
4			11, 22, 24		11, 22, 24		11		1		1, 4
5	0.1	0.1	None	0.25	None	0.5	5	0.75	4-5	0.9	5
6			None		None		2		None		1
7			None		None		None		None		None
8			7		None		None		None		2, 11
9			6		6		6		1, 7, 9		1-4, 7, 9
10			None		None		None		1-2, 4		1-2, 4
1			None		1		1, 10		2-3, 5, 7-8, 10, 11, 13, 16-18, 21		
2			None		None		6		6, 11, 17, 19-20		
3			None		None		20		20		
4			11, 22, 24		11, 22, 24		1, 5, 11		4, 9, 12-13, 17		
5	0.25	0.1	None	0.25	None	0.5	5	0.75	5-6		
6			None		None		2		1		
7			None		None		14		6, 14		
8			None		None		None		5, 7-8, 11		
9			6		6		6-7		1, 3, 7, 9		
10			None		None		2		1-2, 4		
1			None		None		1-5, 8, 12-14, 16, 18, 21				
2			None		None		1-2, 15, 19, 21				
3			None		None		6, 9, 13, 20				
4			11, 22, 24		5, 11, 24		4, 6, 9, 12, 13, 15-17				
5	0.5	0.1	None	0.25	None	0.5	1, 4, 5, 10-11, 14-15, 19, 24				
6			None		2		1, 11, 16				
7			None		None		3, 6, 8, 14				
8			None		7		1, 3, 5, 7, 9, 17, 20				
9			6		6		1, 5, 7, 9, 11, 20				
10			None		None		2, 4, 6, 16				
1			None		4, 5, 8, 12, 16, 18, 21-23						
2			None		4, 9, 10, 12, 14, 15, 19, 21, 22						
3			None		5, 9,13, 18, 20						
4			11, 22, 24		1, 3, 5, 6, 8, 9, 11, 14-17, 22						
5	0.75	0.1	None	0.25	1, 4-6, 10-11, 14, 18-19, 24						
6			None		None						
7			14		9, 11, 14						
8			7		1, 3, 5, 7, 20						
9			6		5, 6, 15, 18, 20, 23						
10			None		2, 6						
1			2, 4, 8, 11, 12, 16, 24								
2			2, 5, 21, 22								
3			4, 5, 13, 20								
4			1, 3, 8, 9, 11, 15-18, 21-23								
5	0.9	0.1	1, 4-6, 8, 10, 24								
6			2, 18								
7			11, 14, 17								
8			7								
9			6, 18, 20								
10			6, 7								

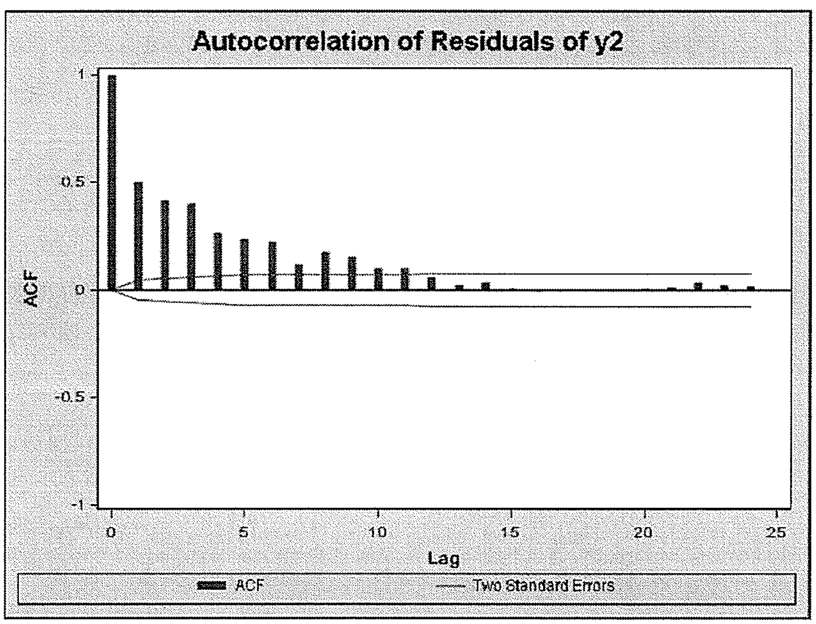
Figure 4: Significant k-th lag SACF of y_t for the simulated normal GARCH(1,1) model

Normal GARCH(1,1) simulation											
Sample	β_1	α_1	$\rho_k(y^2)$	α_1	$\rho_k(y^2)$	α_1	$\rho_k(y^2)$	α_1	$\rho_k(y^2)$	α_1	$\rho_k(y^2)$
1			1		1-2		1-4, 8		1-6, 8		1-6, 8-9
2			None		1-2		1-3, 16		1-4, 11-12		1-5, 11-12, 16
3			1		1		1-2		1-3		1-4
4			1, 11		1-2, 11		1-3		1-3		1-5
5	0.05	0.1	1	0.25	1-2,	0.5	1-3	0.75	1-6	0.9	1-6
6			1		1-2, 22		1-2		1-2, 4, 6		1-2
7			1, 19		1-2		1-3		1-5		1-5
8			1, 22		1		1-2		1-4		1-7
9			1		1-2		1-4, 7-8		1-7		1-7
10			1, 5		1-2		1-2		1-4		1-4
1			1		1-2		1-6, 8		1-10		1-6, 8-11
2			1		1-2		1-3, 16		1-5, 11-12, 16		1-5, 11, 12, 16, 17
3			1		1		1-2		1-4		1-6
4			1		1-2, 11		1-3		1-4		1-5
5	0.1	0.1	1	0.25	1-2	0.5	1-3	0.75	1-7	0.9	1-6
6			1		1-2, 22		1-2		1-2, 6		1-2
7			1, 19		1-2		1-3		1-5		1-5
8			1, 22		1		1-2		1-7		1-8
9			1		1-2		1-4, 7, 8		1-8		1-8
10			1, 5		1-2		1-3		1-4		1-4
1			1, 2		1-3, 8		1-11		1-21, 23		
2			1, 2		1-3		1-5, 12, 16, 20		1-6, 9-12, 14-21, 23		
3			1		1-2		1-4		1-9		
4			1, 11		1-2, 11		1-4		1-6, 9		
5	0.25	0.1	1	0.25	1-3	0.5	1-7	0.75	1-7		
6			1		1-2, 22		1-2, 4, 6		1-2		
7			1, 19		1-3		1-5		1-7		
8			1, 22		1-2		1-6		1-10		
9			1, 3		1-3		1-9		1-9		
10			1, 5		1-3		1-4		1-6		
1			1-3		1-9		1-21, 23				
2			1-3		1-5, 16		All				
3			1		1-3		1-14				
4			1, 2, 11		1-4, 11		1-13				
5	0.5	0.1	1, 2	0.25	1-5	0.5	1-11, 13-16, 18-19				
6			1, 2		1-2, 4, 6		1-2, 4, 6, 7				
7			1, 2, 19		1-6		1-12				
8			1, 4, 22		1-7		1-21				
9			1, 3		1-5, 7-8		1-10				
10			1		1-4		1-15				
1			1-6, 8, 18		All						
2			1-5, 12		1-17, 19-21, 23-24						
3			1-3, 10		1-22						
4			1-4, 8, 11		All						
5	0.75	0.1	1-5, 8	0.25	All						
6			1, 2, 6, 11, 13		1-21, 23, 24						
7			1-4, 6, 19		All						
8			1-7, 10, 17		All						
9			1-4, 7		All						
10			1, 3		All						
1			All								
2			All								
3			All								
4			All								
5	0.9	0.1	All								
6			All								
7			All								
8			All								
9			All								
10			All								

Figure 5: Significant k-th lag SACF of y_t^2 for the simulated normal GARCH(1,1) model



(a) The 1st and the 10th lag are weakly significant for y_t



(b) An exponentially decaying SACF with the 1st up to the 11th significant lags for y_t^2

Figure 6: Sample 1's SACF of y_t and y_t^2 based on the simulated normal GARCH(1,1) process with $\omega = 0.1$, $\alpha = 0.5$ and $\beta = 0.25$

for an example. As seen in Figure 3, the kurtosis values for all simulated processes exhibit values larger than 3, which indicates a leptokurtic nature. Additionally, the kurtosis values have a common theme of aquiring even larger values when $\alpha + \beta$ is in the close proximity of 1, the nonstationary boundary value. The simulation work done here does suggest that the normal GARCH(1,1) model possesses the characteristics mentioned in the introductory chapter.

3 Other Variations of GARCH Models and the Standard Stochastic Volatility Model

The GARCH family is capable of incorporating nonnormal conditional distributions as well as varying specifications for the conditional mean and conditional variance. A remark by Bollerslev *et al.* (1994) puts this into perspective:

> *The richness of the family of parametric ARCH models is both a blessing and a curse. It certainly complicates the search for the 'true' model, and leaves quite a bit of arbitrariness in the model selection stage. On the other hand, the flexibility of the ARCH class of models means that in the analysis of structural economic models with time varying volatility, there is a good chance that an appropriate parametric ARCH model can be formulated that will make the analysis tractable.*

In this section, three GARCH models, namely GARCH(P,Q) with conditional t-distribution, GARCH-in-mean (GARCH-M) and GJR-GARCH, and the standard stochastic volatility (SV) model will be briefly discussed. The inclusion of SV models in our discussion is justified by the fact that SV and GARCH models have many similar capabilities.

3.1 GARCH models with conditional t-distribution

The normality assumption, $y_t|Y_{t-1} \sim N(0, h_t)$ and thus $Z_t \sim N(0,1)$, may not be a viable option if the estimated standardized residuals $\hat{Z}_t$ derived from observed returns and all related parameter estimates, exhibits nonnormal behaviour (e.g. $K^{(\hat{Z})} \neq 3$). If a situation like this arises, an alternative to the normal distribution could be the standardized t-distribution, $Z_t \sim t(\nu)$, amongst others.

Determined by the degrees-of-freedom, $\nu > 2$, the conditional density of y_t, for a standardized t-distribution, is

$$f(y_t|Y_{t-1}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)h_t}}\left[1+\frac{Z_t^2}{\nu-2}\right]^{-\frac{\nu+1}{2}},$$

where the gamma function[6], $\Gamma(a)$ for $a > 0$, is defined as

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx.$$

Note that with the parameterization, $Var(Z_t) = 1$ for al $\nu > 2$, justifying the name standardized t distribution. The parameter estimation for a t-distributed GARCH(P,Q) model will now involve maximizing the log-likelihood function

$$\ln L(\Theta) = n \ln\left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)}}\right) - \frac{1}{2}\sum_{t=1}^{n}\left[\ln(h_t) - (\nu+1)\ln\left(1+\frac{Z_t^2}{\nu-2}\right)\right]$$

where $\Theta = (\omega, \nu, \alpha_1, \ldots, \alpha_Q, \beta_1, \ldots, \beta_P)$.

3.2 GARCH-M model

A GARCH-M model is being considered when the conditional mean term μ_t for

$$y_t|Y_{t-1} \sim (\mu_t, h_t)$$
$$y_t = \mu_t + \sqrt{h_t}Z_t$$

[6]Other notable results for the gamma function are $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma(a+1) = a\Gamma(a)$, and $\Gamma(b) = (b-1)!$ for positive integers b.

is a function of the conditional variance h_t that follows a GARCH process. A plausible specification of μ_t may take the form of

$$\mu_t = r + \lambda\sqrt{h_t}$$

which follows an intuitive notion that expected returns relate to a positive risk-free interest rate r and risk (measured by h_t). For most assets, λ should be positive to reflect that an increase in risk increases the expected returns.

3.3 GJR-GARCH model

It is well known that future volatility of financial markets have the tendency to react differently in the event of a rise in price versus the event of a fall in price (asymmetric volatility phenomenon). As investigated by Nelson (1991), the impact of a fall in the US stock market on the volatility for the following day is much larger than that of a rise of the same magnitude. Glosten, Jagannathan, and Runkle (1993) present the GJR-GARCH model to address this phenomenon.

The GJR-GARCH incorporates additional information at time $t-i$ with a weighted y_{t-i}^2 using the indicator variable

$$I_{t-i} = \begin{cases} 1 & \text{if } y_{t-i} \leq 0 \\ 0 & \text{if } y_{t-i} > 0 \end{cases}$$

to describe the volatility asymmetry. For a GJR-GARCH(P,Q) model, the conditional variance will take the form of

$$h_t = \omega + \sum_{i=1}^{P}(\alpha_i + \alpha_i^* I_{t-i})y_{t-i}^2 + \sum_{j=1}^{Q}\beta_j h_{t-j}$$

where the parameters are usually constrained by $\omega > 0$, $\alpha_i > 0$, $\alpha_i + \alpha_i^* > 0$, and $\beta_j \geq 0$.

3.4 The standard stochastic volatility model

Due to frequent changes in volatility, it is appropriate to model volatility by a random variable. From discrete-time returns data, volatility cannot be observed directly as it is a latent variable that is not traded. In order to intepret volatility as a variable that can be modeled and predicted, SV models specify a stochastic process for volatility. This approach differs from ARCH models, which specify a process for the conditional variance of returns.

The growth of SV literature is less rapid than that of the comparable ARCH literature. The greater popularity of ARCH models is simply due to its ease of estimation via maximum likelihood, whereas the estimation of SV models is not trivial due to the existence of the latent volatility. Nevertheless, SV models are a natural choice for modeling random volatility.

The most widely used SV model is perhaps the standard SV model of Taylor (1986) where a lognormal specification for volatility follows an AR(1) process

$$y_t = \sigma_t Z_t$$
$$\ln(\sigma_t) = \alpha + \beta \ln(\sigma_{t-1}) + \eta_t$$

with $|\beta| < 1$, $Z_t \overset{i.i.d.}{\sim} N(0,1)$, $\eta_t \overset{i.i.d.}{\sim} N(0,\sigma_\eta^2)$, $\ln(\sigma_t) \sim N\left(\frac{\alpha}{1-\beta}, \frac{\sigma_\eta^2}{1-\beta^2}\right)$, and the processes $\{\sigma_t\}$ and $\{Z_t\}$ are stochastically independent.

Using the properties of a lognormal distribution[7], the second and the fourth

[7]See Appendix III.

moments of y_t are given by

$$E(y_t^2) = E(\sigma_t^2) = \exp\left(\frac{\alpha}{1-\beta} + \frac{\sigma_\eta^2}{2[1-\beta^2]}\right)$$
$$E(y_t^4) = E(\sigma_t^4)E(Z_t^4) = 3\exp\left(\frac{2\alpha}{1-\beta} + \frac{2\sigma_\eta^2}{1-\beta^2}\right).$$

With the above results, the kurtosis of y_t is

$$K^{(y)} = \frac{E(y_t^4)}{[E(y_t^2)]^2} = 3\exp\left(\frac{\sigma_\eta^2}{1-\beta^2}\right)$$

and the variance of y_t^2 is

$$\begin{aligned} Var(y_t^2) &= E[(\sigma_t^2 Z_t^2)^2] - [E(\sigma_t^2 Z_t^2)]^2 \\ &= (Var(\sigma_t^2) + [E(\sigma_t^2)]^2)E(Z_t^4) - [E(\sigma_t^2)]^2 \\ &= [E(\sigma_t^2)]^2 \left[3\left(\frac{Var(\sigma_t^2)}{[E(\sigma_t^2)]^2} + 1\right) - 1\right] \\ &= [E(\sigma_t^2)]^2 \left[3\exp\left(\frac{\sigma_\eta^2}{1-\beta^2}\right) - 1\right]. \end{aligned}$$

Moreover, the ACF of y_t^2 can be obtained provided that $E(Z_t^4) < \infty$ and $|\beta| < 1$. Following Jacquier *et al.* (1994), the autocovariance of y_t^2 can be expressed as

$$Cov(y_t^2, y_{t-k}^2) = Cov(\sigma_t^2, \sigma_{t-k}^2) = [E(\sigma_t^2)]^2 \left[\exp\left(\frac{\sigma_\eta^2 \beta^k}{1-\beta^2}\right) - 1\right].$$

Hence, the ACF of y_t^2 at lag k is then

$$\begin{aligned}
\rho_k^{y^2} &= \frac{Cov(y_t^2, y_{t-k}^2)}{\sqrt{Var(y_t^2)Var(y_{t-k}^2)}} \\
&= \frac{\exp\left(\dfrac{\sigma_\eta^2 \beta^k}{1-\beta^2}\right) - 1}{3\exp\left(\dfrac{\sigma_\eta^2}{1-\beta^2}\right) - 1} \\
&\approx \frac{\exp\left(\dfrac{\sigma_\eta^2}{1-\beta^2}\right) - 1}{3\exp\left(\dfrac{\sigma_\eta^2}{1-\beta^2}\right) - 1}\beta^k
\end{aligned}$$

for any positive integer k.

Sometimes the assumption of a heavy-tailed distribution for the error process $\{Z_t\}$ may be more appropriate than a Gaussian assumption. For instance, if $\{Z_t\}$ assumes a standardized t-distribution with variance 1 and degrees-of-freedom $\nu > 4$, then $E(Z_t^4) = \dfrac{3(\nu - 2)}{\nu - 4}$ as opposed to $E(Z_t^4) = 3$ for a normal distribution.

4 Example: Standard & Poor 100-Share Index

4.1 Dataset description

With its induction in 1983, the S&P 100 measures performances of 100 major U.S. companies across diverse industry groups. As previously mentioned, the S&P 100 dataset used in this thesis covers the period of January 1991 to December 2000 inclusive. There are a total of $n = 2531$ price index observations, p_t, with per observation recorded at the close of each trading day over the 10 year period. The dataset, which commences at the price of $p_0 = 153$ and terminates at $p_{2530} = 686$, attained a minimum price of 146 in 1991 and a maximum of 833 in 2000.

As for the returns, $y_t = \ln\left(\frac{p_t}{p_{t-1}}\right)$, the first return is observed at $y_1 = -0.013$ and the last at $y_{2531} = -0.008$. The average return is a small positive value of 0.0006. Table 1 summarizes some of the basic statistical measure of y_t. Preliminary graphs and SACF of y_t can be reexamined in figures 1(a)- 1(c) and 2(a)- 2(b) respectively.

	Values
Mean	0.0005924
Variance	0.00009726
Skewness	-0.2855339
Kurtosis	8.0452076
Minimum	-0.0751646
Maximum	0.0560616

Table 1: Basic statistical measures of y_t for the S&P 100 dataset

4.2 Fitting a normal GARCH(1,1) model

Recall that returns y_t exhibit the characteristics outlined in the introductory chapter. This indicates that a GARCH process may be a good candidate to model y_t. In this section, the S&P 100 dataset will be fitted following the normal GARCH(1,1) volatility model. By adapting methods presented in Chapter 2, the S&P 100 dataset can be modeled with little difficulty using SAS.

Returns (y_t), conditional variances (h_t), and standardized residuals (Z_t) are connected by equations (2.6) and (2.7), where parameters[8] ω, α_1 and β_1 are estimated with the MLE method. Using SAS, the estimated values are presented in Table 2.

Parameter	Values	Std. Error	t value	Approx. Pr > $\|t\|$
$\hat{\omega}$	4.971 x 10^{-7}	1.2622 x 10^{-7}	3.94	< .0001
$\hat{\alpha}_1$	0.0510	0.004561	11.19	< .0001
$\hat{\beta}_1$	0.9454	0.005066	186.59	< .0001

Table 2: Parameter estimates for the S&P 100 dataset based on the normal GARCH(1,1) model.

Observing the t-ratios of the parameters, it suggests that all three parameters are significant. Hence, the estimated conditional variance is

$$\hat{h}_t = 0.0000004971 + 0.051 y_{t-1}^2 + 0.9454 \hat{h}_{t-1}. \tag{4.1}$$

The maximum likelihood estimates of the unconditional variance and kurtosis of y_t are then respectively $\hat{\sigma}_y^2 = \dfrac{\hat{\omega}}{1-\hat{\alpha}_1-\hat{\beta}_1} = 0.000138$ and $\hat{K}^{(y)} = \dfrac{3(1-\hat{\phi}_1^2)}{1-\hat{\phi}_1^2-2\hat{\alpha}_1^2} =$ 10.861806, which are relatively close to what was recorded in Table 1.

Initiated by the variance of the complete sample of returns ($\hat{h}_1 = 0.00009726$), $\hat{h}_t$ and $\hat{Z}_t$ for $t = 1, \ldots, 2530$ are recursively computed and then tabulated. Displayed in the table below are some of the values.

Figure 7(a) depicts the time series of volatility estimates $\sqrt{\hat{h}_t}$ from 1991 to 2000. The plotted volatility estimates, with respective median and mean values of 0.00832

[8] h_1 can be treated as an additional model parameter to be included in the estimation procedure. In this thesis, h_1 will simply be assigned the value of the sample variance of returns and not be estimated.

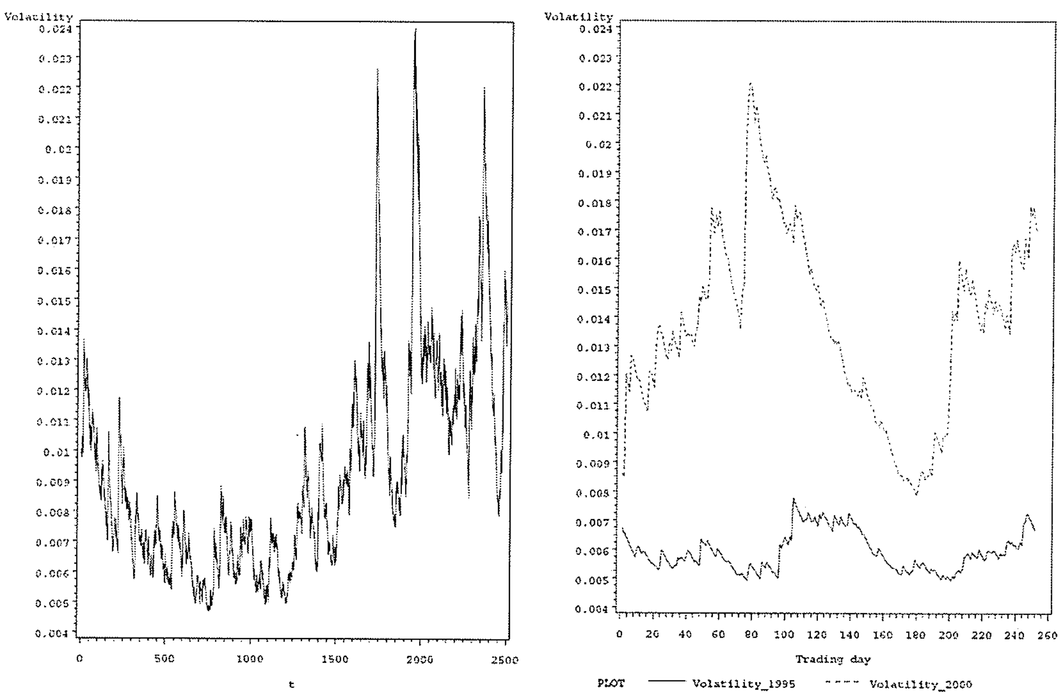
Date	t	y_t	$\hat{h}_t$	$\sqrt{\hat{h}_t}$	$\hat{Z}_t$
02-Jan-91	0	−0.012599			
03-Jan-91	1	−0.001057	0.00009726	0.00986229	−1.2775
04-Jan-91	2	−0.018345	0.00010055	0.01002727	−0.1054
07-Jan-91	3	−0.002562	0.00009561	0.00977806	−1.8762
08-Jan-91	4	−0.014276	0.00010805	0.01039478	−0.2464
⋮	⋮	⋮	⋮	⋮	⋮
22-Dec-00	2526	0.023153	0.00030694	0.01751971	1.3215
26-Dec-00	2527	0.004026	0.00031802	0.01783303	0.2257
27-Dec-00	2528	0.006066	0.00030198	0.01737747	0.3491
28-Dec-00	2529	0.001605	0.00028786	0.01696651	0.0946
29-Dec-00	2530	−0.008472	0.00027277	0.01651586	−0.5129

Table 3: Values of the estimated conditional variance $\hat{h}_t$, estimated volatility $\sqrt{\hat{h}_t}$, and the estimated standardized residuals $\hat{Z}_t$, based on $\hat{\omega}$, $\hat{\alpha}_1$ and $\hat{\beta}_1$ from Table 2.

and 0.00931, range from a minumum value of 0.00468 (December 27, 1993) to a maximum value of 0.02389 (September 14, 1998). The second half of the returns series appears to show higher volatility than the first half. The clustering phenomenon is evident here with a consistently high estimated volatility for some periods and then low for some. To further emphasize this point, Figure 7(b) shows in more detail a year with high volatility (2000), represented by a dashed line, and a year with low volatility (1995), represented by a solid line. With an estimated value of $\hat{\beta}_1 = 0.9454$, it is no surprise that large volatility are clustered together, and likewise with low volatility.

4.3 Diagnostic checks and forecasts for the normal GARCH(1,1) model

To ensure the fitted normal GARCH(1,1) model is appropriate, the examination of the standardized residuals, $\hat{Z}_t = \dfrac{y_t}{\sqrt{\hat{h}_t}}$, is needed. The sample mean and variance of $\hat{Z}_t$, calculated to be 0.0602 and 0.99671, are very close to the theoretical values of 0

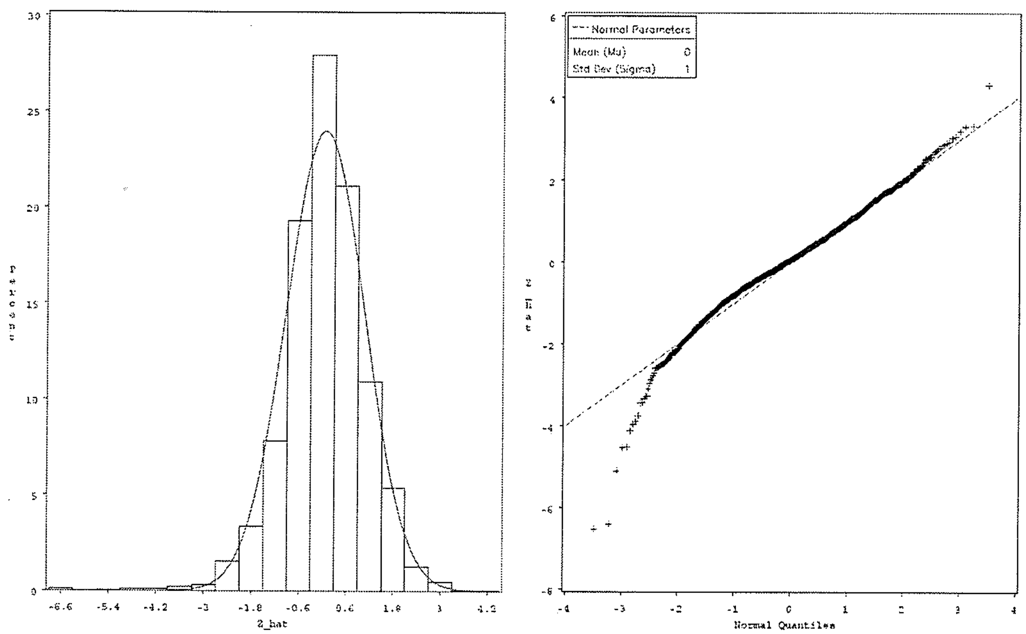


(a) Volatility 1991-2000

(b) Volatility in 1995 and 2000

Figure 7: S&P 100 volatility estimates based on the normal GARCH(1,1) model

and 1 respectively. However, the skewness (−0.4596) and kurtosis (5.6012) values of $\hat{Z}_t$ indicates that the model assumption of normality here may not be ideal. This is also reflected in the probability distribution curve and QQ-plot as seen in figures 8(a) and 8(b). It seems that a heavy-tailed distribution would be a better choice over the current standard normal distribution.



(a) Probability distribution curve of $\hat{Z}_t$

(b) QQ-plot of $\hat{Z}_t$

Figure 8: Graphs of $\hat{Z}_t$ for the S&P 100 based on the normal GARCH(1,1) model

Figures 9(a) and 9(b) provides the SACF of the standardized residuals and its squared counterpart. These ACFs are examined to see if there are any evidence of serial correlation or conditional heteroscedasticity in the terms $\hat{Z}_t$. Here, the ACFs fail to suggest any significant form of correlation or heteroscedastic nature in the standardized residual series.

Overall, the GARCH(1,1) model with the estimated conditional variance described in equation (4.1) appears to be an adequate fit. Aside from the visual diagnostics presented thus far, other form of diagnostics such as the Ljung-Box Q-statistic (Ljung and Box, 1978) can help to further investigate model adequacy.

To forecast the volatility of returns of the S&P 100 index, refer to the recursive equations from Section 2.5.3. Table 4 shows the volatility forecasts for the next 5 trading days based on the GARCH(1,1) model where $\hat{h}_{2530} = 0.00027277$. Figure 10 shows volatility forecasts for 253 trading days into the future, assuming that h_{2530} to be a low of 0.00002 and a high of 0.0006. As $\ell \to \infty$, the volatility forecast, $\sqrt{\hat{h}_n(\ell)}$, will converge to the value of 0.0117509. The rate of convergence is dependant on the combination of α and β parameters.

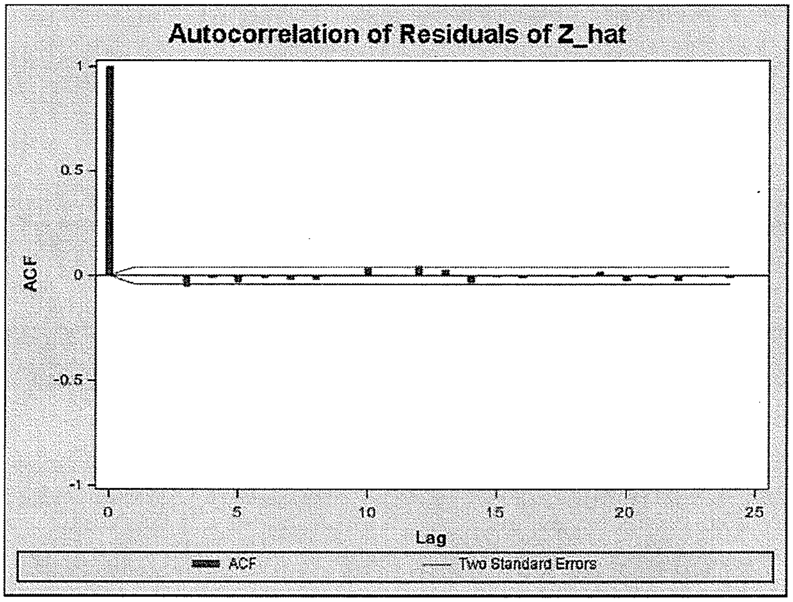
Horizon (days)	1	2	3	4	5	∞
Volatility	0.0165012	0.0164865	0.0164719	0.0164574	0.0164428	0.0117509

Table 4: Volatility forecasts based on the normal GARCH(1,1) model for the S&P 100 dataset.

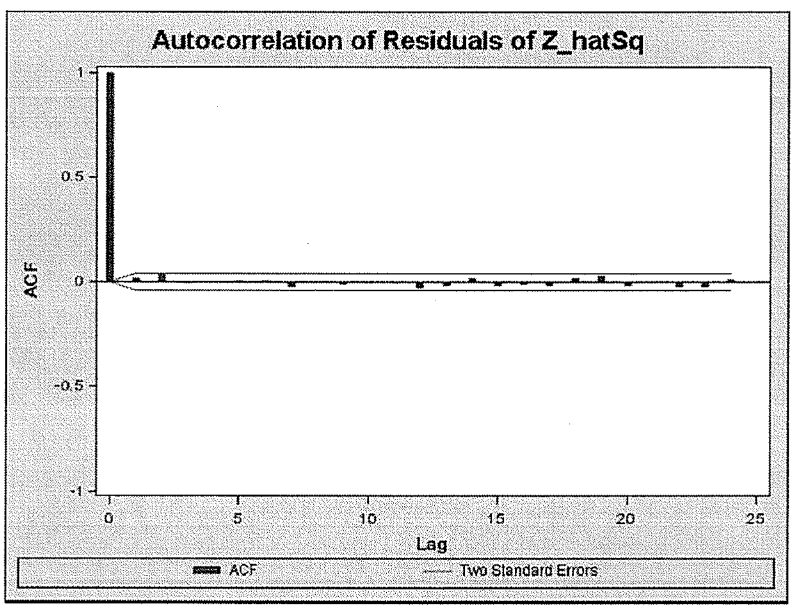
4.4 Comparing the GARCH(1,1) models

It is commonplace in practice to compare the forecasting performance of different models. The choices of models used in an empirical study are largely dependant on a number of constraints such as expertise level of researcher, the amount of time allocated to research and software availability, just to name a few.

Table 5 lists the parameter estimates and their respective standard errors (in brackets) for the S&P 100 dataset from five different volatility models along with the normal GARCH(1,1) model estimates discussed earlier. The five models being considered are as follows: conditional t-distributed GARCH(1,1) model with a constant mean r; normal GARCH(1,1)-M model with conditional mean $\mu_t = r + \lambda\sqrt{h_t}$; normal GJR-GARCH(1,1) model with a constant mean r; and GJR-MA(1)-GARCH(1,1)-M



(a) SACF of $\hat{Z}_t$



(b) SACF of $\hat{Z}_t^2$

Figure 9: SACF of $\hat{Z}_t$ and $\hat{Z}_t^2$ for the S&P 100 based on the normal GARCH(1,1) model

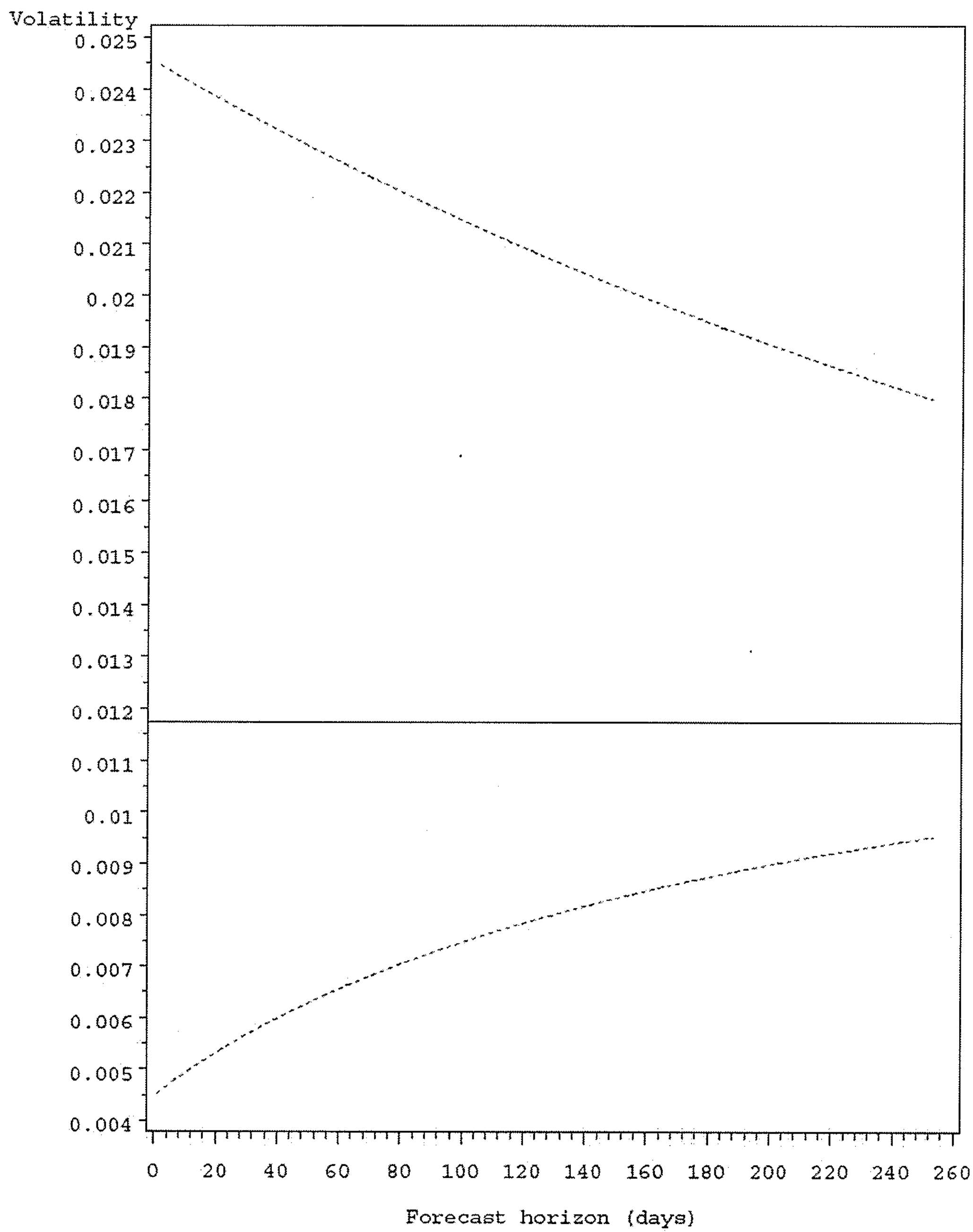


Figure 10: Volatility forecasts for the S&P 100 based on the normal GARCH(1,1) model

for two conditional distributions, normal and t.

The GJR-MA(1)-GARCH(1,1)-M model is defined by

$$
\begin{aligned}
y_t|Y_{t-1} &\sim (\mu_t, h_t), \\
y_t &= \mu_t + e_t = \mu_t + \sqrt{h_t} Z_t, \\
\mu_t &= r + \lambda\sqrt{h_t} + \theta e_{t-1}, \\
h_t &= \omega + (\alpha_1 + \alpha_1^* I_{t-1}) e_{t-1}^2 + \beta_1 h_{t-1}, \\
I_{t-1} &= \begin{cases} 1 & \text{if } e_{t-1} \leq 0 \\ 0 & \text{if } e_{t-1} > 0 \end{cases},
\end{aligned}
$$

which connects returns y_t (ignoring dividends), conditional means μ_t, conditional variances h_t, residuals e_t, and standardized residuals Z_t. The GARCH(1,1)-M and GJR-GARCH(1,1) models are as described in Chapter 3.

Due to complications with SAS[9], the parameter estimates and respective standard errors[10] for the normal GJR-MA(1)-GARCH(1,1)-M model were obtained using Microsoft Excel[11] as discussed in Taylor (2005). The results for the conditional t-distributed GJR-MA(1)-GARCH(1,1)-M model estimates were directly taken from Taylor (2005) as well. The parameter estimates for the remaining models were obtained using SAS.

A commonly used method to select an appropriate model from a fixed set of models is to compare either the Akaike information criterion (AIC) or the Schwarz Bayesian information criterion (BIC) obtained from fitting the considered models.

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[9]See Appendix IV.

[10]See Appendix V.

[11]All Excel files for this thesis can be obtained at http://seetonglim.thesis.googlepages.com.

Parameters	Normal GARCH	t-GARCH	GARCH-M	GJR-GARCH	GJR-MA(1) -GARCH-M	t-GJR-MA(1) -GARCH-M
$\hat{r}$		0.000654	-0.000305	0.000424	-0.000607	-0.000256
		(0.000142)	(0.000502)	(0.000154)	(0.000545)	(0.000445)
$\hat{\lambda}$			0.1221		0.1377	0.1112
			(0.0648)		(0.0690)	(0.05777)
$\hat{\theta}$					0.0111	-0.0111
					(0.0203)	(0.0207)
$\hat{\omega}$ x 10^{-7}	4.971	1.759	5.6673	9.056	0.1150	0.0752
	(1.2622)	(1.158)	(1.3787)	(2.561)	(0.0338)	(0.0214)
$\hat{\alpha_1}$	0.0510	0.028263	0.0544	0.095817	0.0108	0.0123
	(0.00456)	(0.00635)	(0.00473)	(0.0148)	(0.0097)	(0.0103)
$\hat{\alpha_1^*}$				-0.08386	0.0869	0.0784
				(0.0157)	(0.0218)	(0.0153)
$\hat{\beta}_1$	0.9454	0.956618	0.9413	0.937381	0.9324	0.9398
	(0.00507)	(0.00955)	(0.005304)	(0.00995)	(0.0115)	(0.0088)
$\hat{\nu}$		6.032156				6.616
		(0.7086)				(0.771)
Log MLE	8400.312	8481.417	8409.939	8428.155	8430.94	8495.13
AIC	-16794.62	-16952.83	-16809.88	-16846.31	-16847.88	-16974.26
SBC	-16777.12	-16923.65	-16780.70	-16817.13	-16807.03	-16927.57

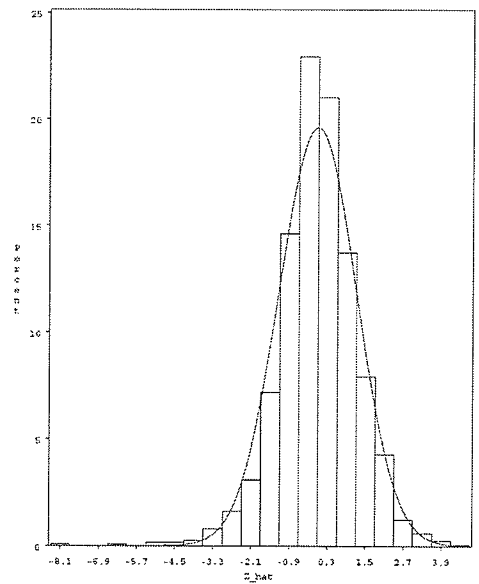
Table 5: Various GARCH(1,1) models' parameter estimates and respective standard errors (in brackets) estimated from the S&P 100 dataset.

The AIC and BIC are computed as follows:

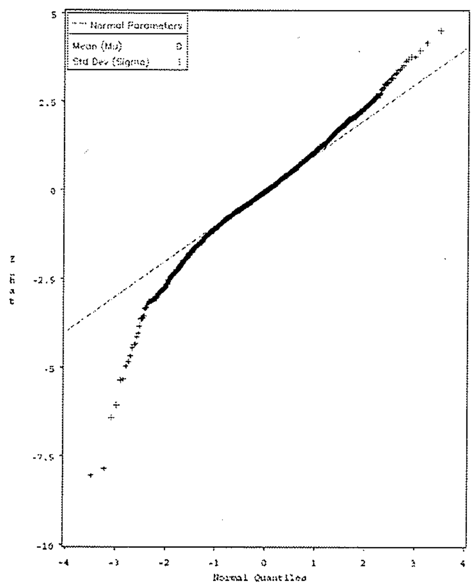
$$AIC = -2\ln L(\hat{\Theta}) + 2P,$$
$$BIC = -2\ln L(\hat{\Theta}) + P\ln(n),$$

where $\hat{\Theta}$ is the value of the likelihood function evaluated at the parameter estimates, P is the number of estimated parameters and n is the number of observations. A model with a smaller AIC (or BIC) value indicates a better fitting model. By this token the conditional t-distributed GJR-MA(1)-GARCH(1,1)-M model is the best here, although the simpler conditional t-distributed GARCH(1,1) fits nearly just as well.

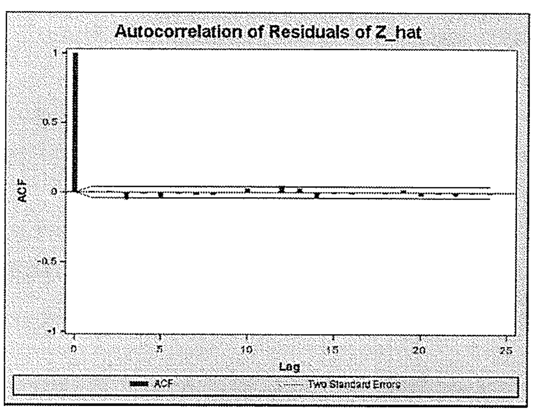
The diagnostic graphs for the five GARCH models are displayed in figures 11 to 15. The diagnostic results here are similar to those previously discussed for the normal GARCH(1,1). With the exception for the conditional heteroscedasticity of $\hat{Z}_t$ for the normal GJR-GARCH(1,1) model indicated by Figure 13(d), the other models appear to provide an adequate fit with no sign of serial correlation. Figure 16 and Figure 17 are the projected volatilities based on the parameter estimates from Table 5. The spikes for the three GJR models are more pronounced in comparison with the non-GJR models. Overall, the estimated volatilities from all six models resembles one another quite closely.



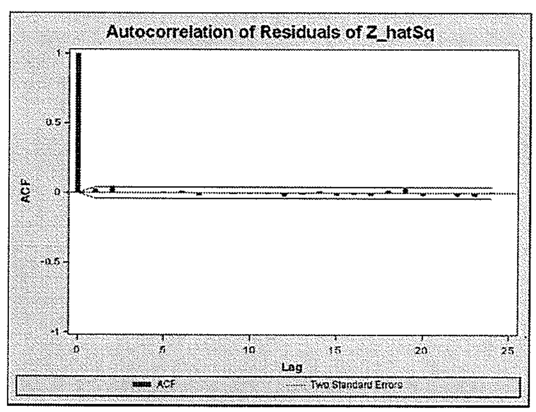
(a) Probability distribution curve of $\hat{Z}_t$



(b) QQ-plot of $\hat{Z}_t$

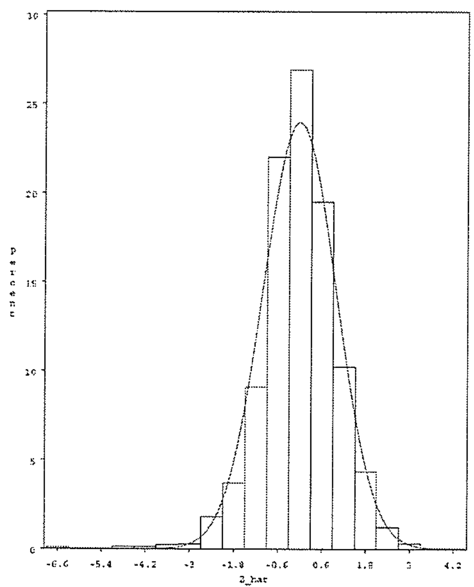


(c) SACF of $\hat{Z}_t$

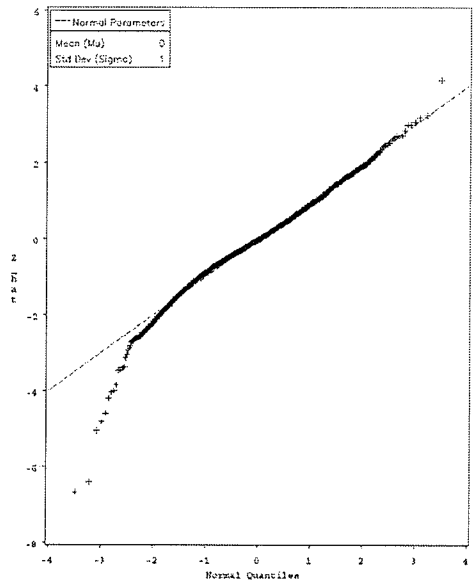


(d) SACF of $\hat{Z}_t^2$

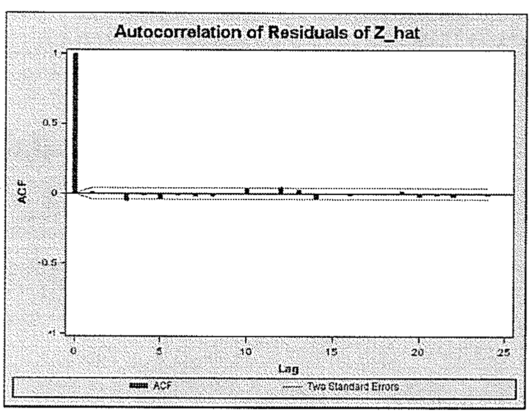
Figure 11: Diagnostic graphs for the conditional t-distributed GARCH(1,1) model



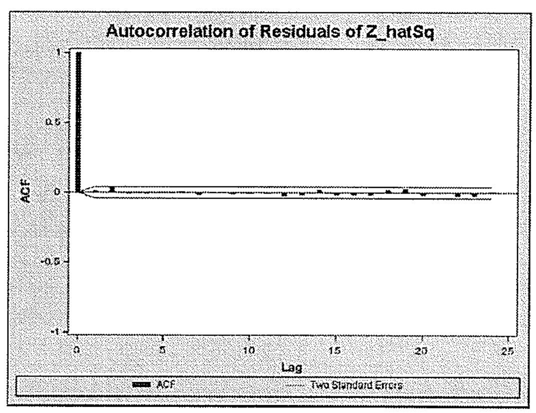
(a) Probability distribution curve of $\hat{Z}_t$



(b) QQ-plot of $\hat{Z}_t$

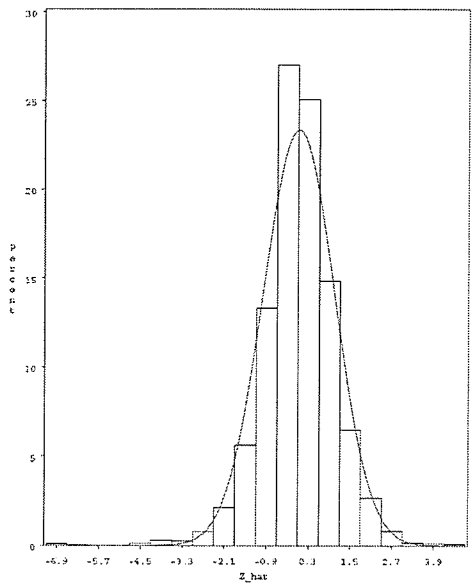


(c) SACF of $\hat{Z}_t$

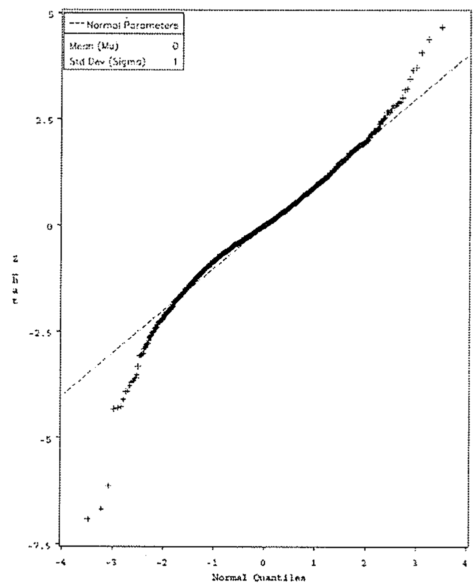


(d) SACF of $\hat{Z}_t^2$

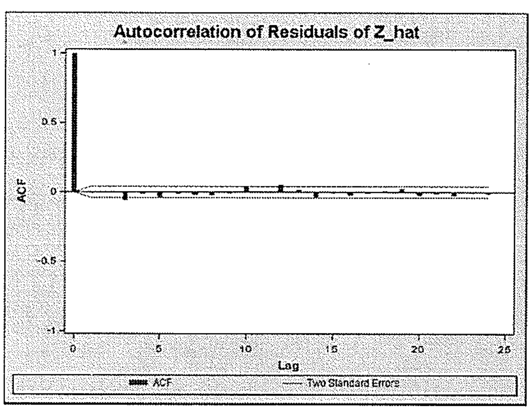
Figure 12: Diagnostic graphs for the normal GARCH(1,1)-M model



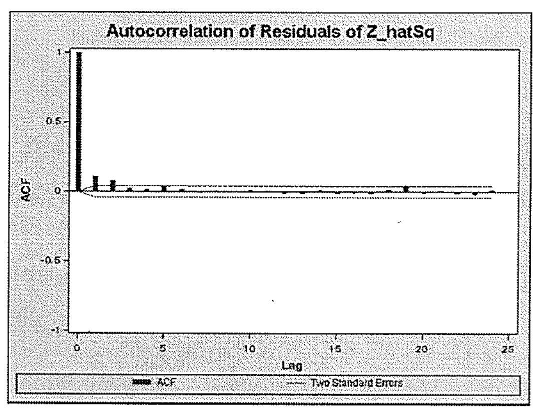
(a) Probability distribution curve of $\hat{Z}_t$



(b) QQ-plot of $\hat{Z}_t$

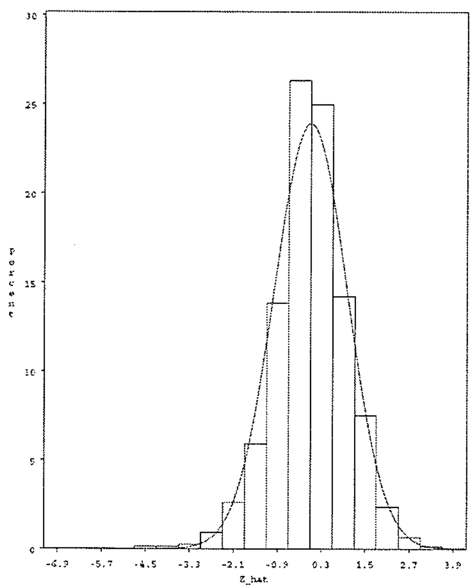


(c) SACF of $\hat{Z}_t$

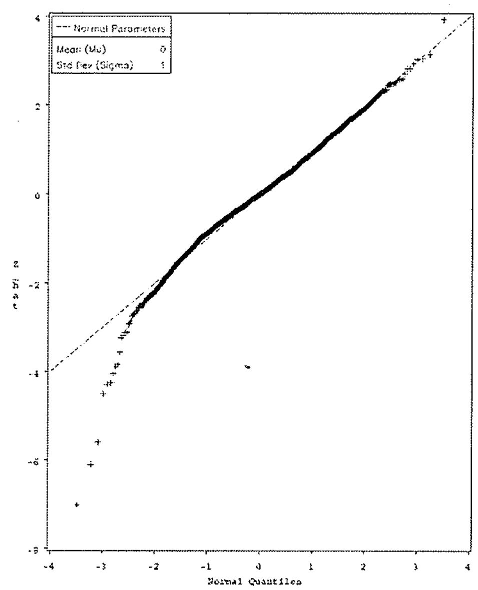


(d) SACF of $\hat{Z}_t^2$

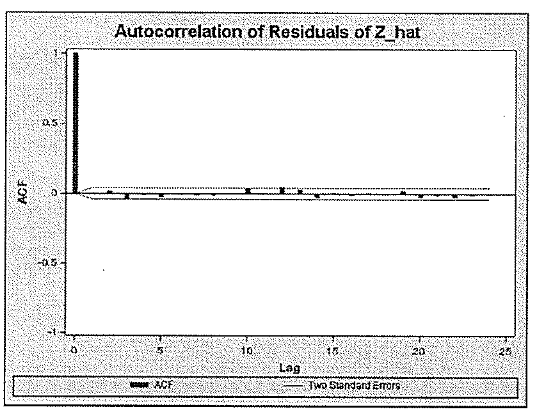
Figure 13: Diagnostic graphs for the normal GJR-GARCH(1,1) model



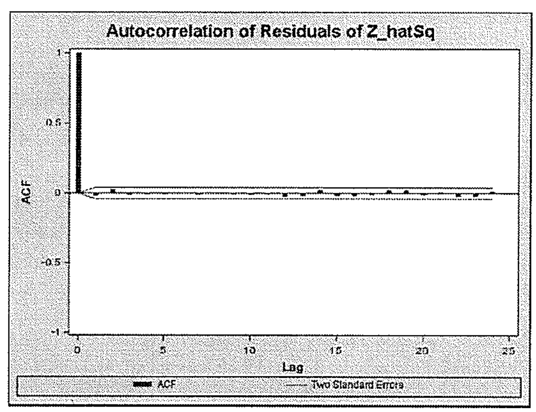
(a) Probability distribution curve of $\hat{Z}_t$



(b) QQ-plot of $\hat{Z}_t$

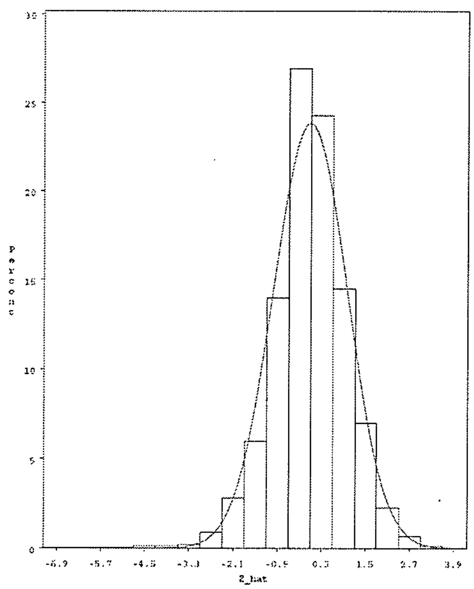


(c) SACF of $\hat{Z}_t$

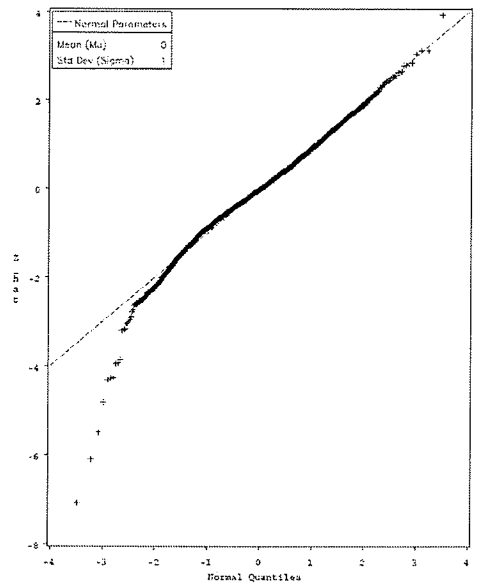


(d) SACF of $\hat{Z}_t^2$

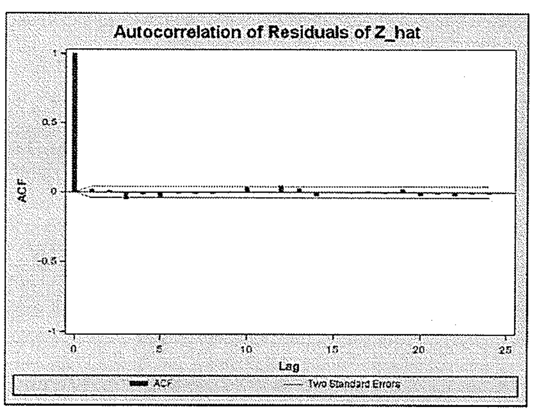
Figure 14: Diagnostic graphs for the normal GJR-MA(1)-GARCH(1,1)-M model



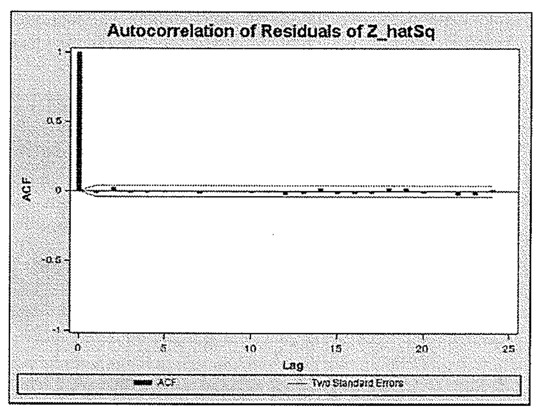
(a) Probability distribution curve of $\hat{Z}_t$



(b) QQ-plot of $\hat{Z}_t$



(c) SACF of $\hat{Z}_t$



(d) SACF of $\hat{Z}_t^2$

Figure 15: Diagnostic graphs for the conditional t-distributed GJR-MA(1)-GARCH(1,1)-M model

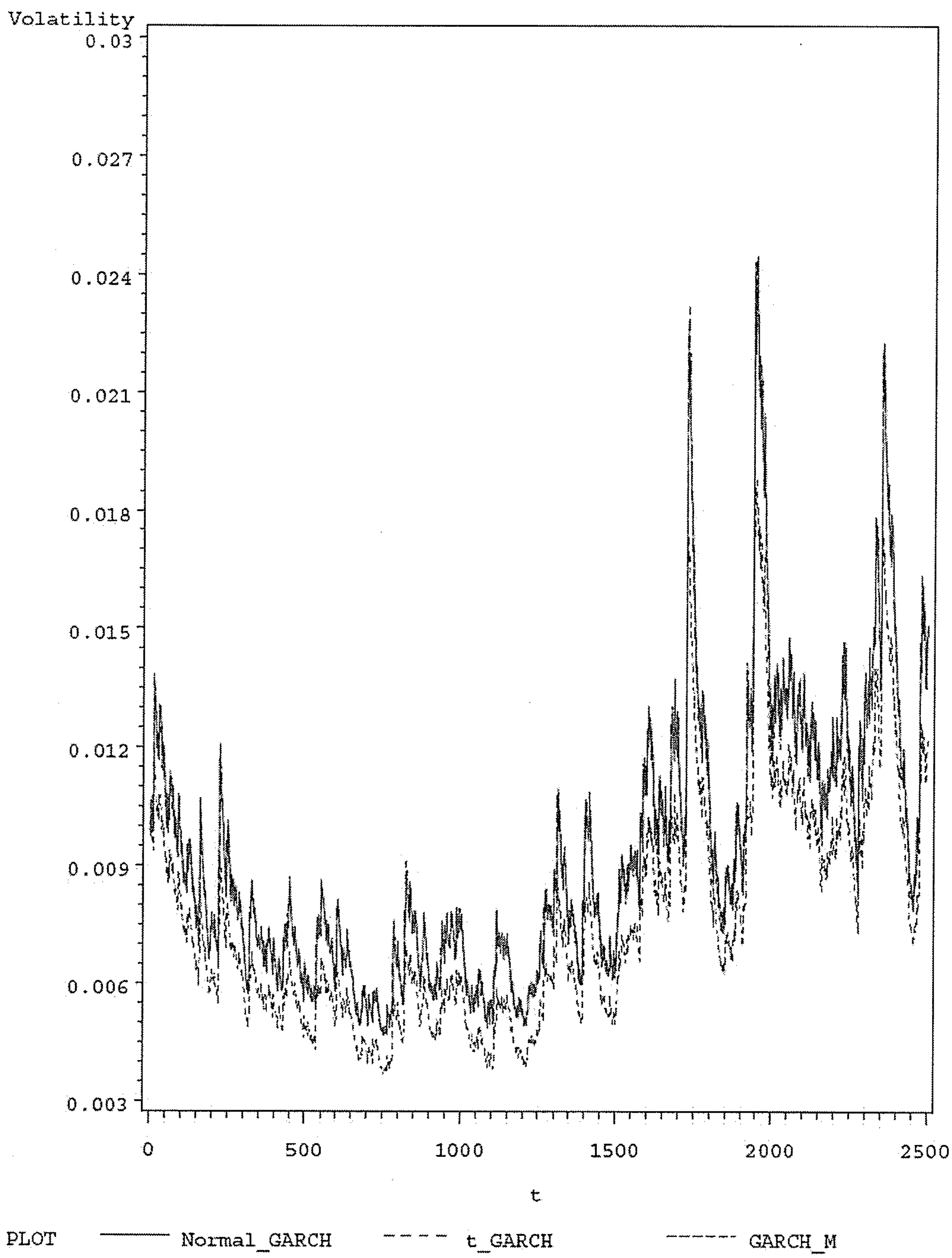


Figure 16: Estimated volatility for the normal GARCH(1,1), conditional t-distributed GARCH(1,1) and normal GARCH(1,1)-M models

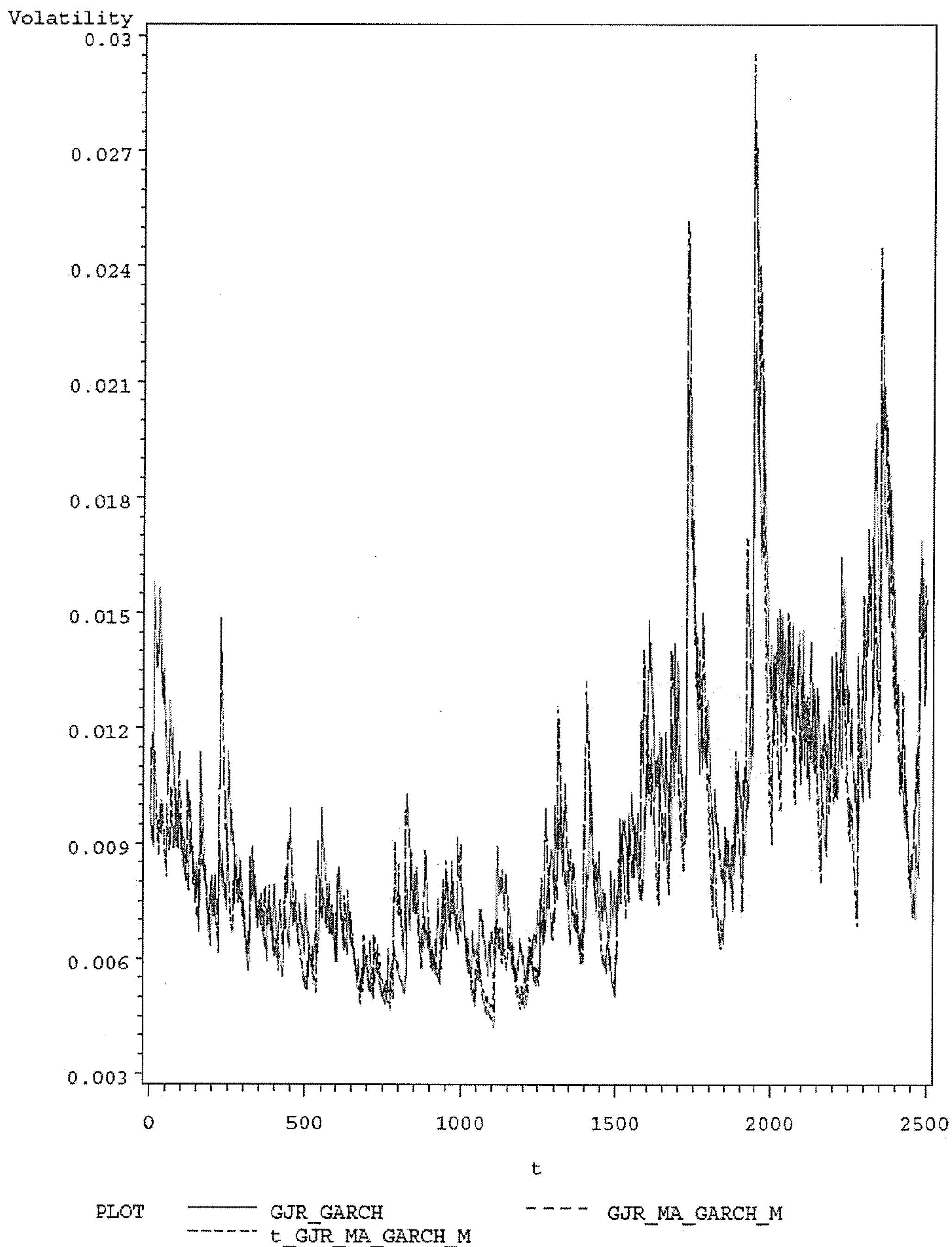


Figure 17: Estimated volatility for the normal GJR-GARCH(1,1), normal GJR-MA(1)-GARCH(1,1)-M and conditional t-distributed GJR-MA(1)-GARCH(1,1)-M models

5 Option Pricing with GARCH Models

5.1 Risk-neutral measure

Following the GARCH frameworks discussed in the previous chapters, consider the structure for the real-world measure[12] **P**:

$$
\begin{aligned}
y_t | Y_{t-1} &\overset{\mathbf{P}}{\sim} N(\mu_t, h_t), \\
y_t &= \mu_t + e_t = \mu_t + \sqrt{h_t} Z_t, \\
\mu_t &= r - \frac{1}{2} h_t + \lambda \sqrt{h_t}, \\
h_t &= \omega + \sum_{i=1}^{P} \alpha_i e_{t-i}^2 + \sum_{j=1}^{Q} \beta_j h_{t-j},
\end{aligned}
$$

where $y_t = \ln(p_t) - \ln(p_{t-1})$ are the log returns that define the information sets $Y_t = \{y_{t-j}, j \geq 0\}$; r denotes the continuously compounded risk-free rate; λ is the constant market price of risk; and $Z_t = \dfrac{y_t - \mu_t}{\sqrt{h_t}} \underset{i.i.d.}{\overset{\mathbf{P}}{\sim}} N(0,1)$ is the standardized residual. The typical parameter restrictions apply as well: $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$ and $\sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j < 1$.

Definition 1. *A pricing measure* $\boldsymbol{Q}$ *is said to satisfy the locally risk-neutral valuation relationship if*

- *measure* $\boldsymbol{Q}$ *is mutually absolutely continuous with respect to measure* $\boldsymbol{P}$*,*
- $\dfrac{p_t}{p_{t-1}} | Y_{t-1}$ *distributes lognormally under measure* $\boldsymbol{Q}$*,*
- $E^Q \left[\dfrac{p_t}{p_{t-1}} | Y_{t-1} \right] = e^r$*, and*

[12] Sometimes known as the physical measure, the real-world measure assume that a more risky asset, on average, command a higher rate of return than a less risky asset. In contrast, the risk-neutral measure assume that no extra compensation is required for additional risk.

- $Var^{Q}\left[\ln\left(\frac{p_t}{p_{t-1}}\right)|Y_{t-1}\right] = Var^{P}\left[\ln\left(\frac{p_t}{p_{t-1}}\right)|Y_{t-1}\right]$ *almost surely with respect to measure **P**.*

Fair option prices can be obtained in the GARCH setting using a risk-neutral measure **Q**, as defined in the definition above by Duan (1995), with the following theorem.

Theorem 5.1. *The locally risk-neutral valuation relationship implies that, under pricing **Q**,*

$$
\begin{aligned}
y_t|Y_{t-1} &\overset{Q}{\sim} N(\mu_t, h_t),\\
y_t &= \mu_t + \tilde{e}_t = \mu_t + \sqrt{h_t}\tilde{Z}_t,\\
\mu_t &= r - \frac{1}{2}h_t,\\
h_t &= \omega + \sum_{i=1}^{P} \alpha_i(\tilde{e}_{t-i} - \lambda\sqrt{h_{t-i}})^2 + \sum_{j=1}^{Q} \beta_j h_{t-j},
\end{aligned}
$$

with $\tilde{Z}_t = Z_t + \lambda \underset{i.i.d.}{\overset{Q}{\sim}} N(0,1)$.

For the popular GARCH(1,1) model, the conditional variance under measure **P** is

$$h_t \overset{\mathbf{P}}{=} \omega + [\alpha_1 Z_{t-1}^2 + \beta_1]h_{t-1}$$

whereas under measure **Q** it is

$$h_t \overset{\mathbf{Q}}{=} \omega + [\alpha_1(\tilde{Z}_{t-1} - \lambda)^2 + \beta_1]h_{t-1}.$$

Theorem 5.2. *For a GARCH(1,1) model under pricing measure **Q**,*

- *the stationary variance of* $\tilde{e}_t$ *equals to* $\dfrac{\omega}{1-(1+\lambda^2)\alpha_1 - \beta_1}$,
- $\tilde{e}_t$ *is leptokurtic, and*

- $Cov^Q\left[\frac{\tilde{e}_t}{\sqrt{h_t}}, h_{t+1}\right] = \frac{-2\lambda\omega\alpha_1}{1-(1+\lambda^2)\alpha_1-\beta_1}$,

if $|\lambda| < \sqrt{(1-\alpha_1-\beta_1)/\alpha_1}$.

Similar results can be obtained for returns with different GARCH specifications. This simply requires replacing the residual variable, e_t, in the conditional variance under measure **P** by $\tilde{e}_t - \lambda\sqrt{h_t}$ with everything else unaltered. For instance, the GJR-GARCH(1,1) model leads to

$$h_t \stackrel{\mathbf{P}}{=} \omega + [\alpha_1 Z_{t-1}^2 + \alpha_1^* \max(0, -Z_t)^2 + \beta_1]h_{t-1}$$
$$h_t \stackrel{\mathbf{Q}}{=} \omega + [\alpha_1(\tilde{Z}_{t-1}-\lambda)^2 + \alpha_1^* \max(0, -(\tilde{Z}_{t-1}-\lambda))^2 + \beta_1]h_{t-1}$$

as seen in Duan *et al.* (2006). To ensure that the conditional variances remains positive for the GJR-GARCH(1,1) specification under measure **P**, the parameter restrictions are $\omega > 0, \alpha_1 > 0, \alpha_1^* > 0$ and $\beta_1 \geq 0$. Note as well that the stationarity conditions for the GJR-GARCH(1,1) model differs under measure **P**, which is $\alpha_1 + \alpha_1^*/2 + \beta_1 < 1$, from that under **Q**, which is $(\alpha_1 + \alpha_1^* N(\lambda))(1+\lambda^2) + \alpha_1^*\lambda n(\lambda) + \beta_1 < 1$ where $N(.)$ and $n(.)$ stand for the standard normal distribution and density functions, respectively.

Another noteworthy specification for pricing options is the AR(1)-GJR-GARCH(1,1) model used in Hafner and Herwartz (2001). With returns defined as relative price changes here, i.e. $y_t = \frac{p_t - p_{t-1}}{p_{t-1}}$, the structure under measure **P** follows

$$y_t|Y_{t-1} \stackrel{\mathbf{P}}{\sim} N(\mu_t, h_t),$$
$$y_t = \mu_t + e_t = \mu_t + \sqrt{h_t}Z_t,$$
$$\mu_t = v + \xi y_{t-1},$$
$$h_t = \omega + [\alpha_1 Z_{t-1}^2 + \alpha_1^* \max(0, -Z_t)^2 + \beta_1]h_{t-1},$$

where v and ξ are constant parameters. Under the risk-neutral measure, the model takes the form of

$$\begin{aligned}
&y_t|Y_{t-1} \overset{Q}{\sim} N(\mu_t, h_t),\\
&y_t = \mu_t + \tilde{e}_t = \mu_t + \sqrt{h_t}\tilde{Z}_t,\\
&\mu_t = r,\\
&h_t = \omega + [\alpha_1(\tilde{Z}_{t-1} - \lambda_{t-1})^2 + \alpha_1^* \max(0, -(\tilde{Z}_{t-1} - \lambda_{t-1}))^2 + \beta_1]h_{t-1},\\
&\lambda_t = \frac{v + \xi y_{t-1} - r}{\sqrt{h_t}},
\end{aligned}$$

where r is a constant risk-free interest rate and $\tilde{Z}_t = Z_t + \lambda_t$.

5.2 Black-Scholes and GARCH option pricing models

Developed in the early 1970s by Fischer Black, Myron Scholes, and Robert Merton, the Black-Scholes (BS) model is influential in the valuation of option prices. The price of a European call[13] option on a stock with no dividend payments at time t based on the BS formulation is

$$\begin{aligned}
C_t^{BS} &= p_t N(d_1) - Ke^{-r(T-t)}N(d_2),\\
d_1 &= \frac{\ln(p_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},\\
d_2 &= d_1 - \sigma\sqrt{T-t},
\end{aligned} \tag{5.1}$$

where p_t is the price of the underlying asset at time t, K is the strike price, r is the risk-free rate (continuously compounded), σ is the stock price volatility, and $T-t$ is the time to maturity of the option.

[13]Generally, a European call (put) option is the right to purchase (sell) a particular asset for a specified amount at the time of maturity. For an American option, the right can be exercised at any time during its lifespan.

Following GARCH processes, the terminal asset price under the measure **Q** specification, for log returns with maturity at time T, is

$$
\begin{aligned}
p_T &= p_t \exp\left[(T-t)r - \frac{1}{2}\sum_{s=t+1}^{T} h_s + \sum_{s=t+1}^{T} \tilde{e_s}\right] \\
&= p_t \exp(y_{t+1} + y_{t+2} \ldots + y_T),
\end{aligned}
$$

and for returns defined as relative price changes, the terminal asset price follows

$$
\begin{aligned}
p_T &= p_t \prod_{s=t+1}^{T} (1 + r + \tilde{e_s}) \\
&= p_t(1+y_{t+1})(1+y_{t+2})\ldots(1+y_T).
\end{aligned}
$$

With p_T at hand, the discounted theoretical fair price of a European call option with exercise price K can be obtained with

$$
C_t = D^* E^{\mathbf{Q}}[max(p_T - K, 0)|Y_t] \tag{5.2}
$$

where the discounting factor is $D^* = e^{-r(T-t)}$ for log returns or $D^* = (1+r)^{-(T-t)}$ for returns defined as relative price changes. Using Monte Carlo simulations of returns, equation (5.2) can be approximated by

$$
\hat{C}_t = \frac{D^*}{N}\sum_{i=1}^{N} max(p_{i,T} - K, 0)
$$

for N simulations delivering terminal asset prices $\{p_{i,T}, 1 \leq i \leq N\}$.

5.3 Implied volatility and delta

In practice, implied volatility and delta are two important parameters in option pricing. This dedicated section will cover a brief overview of them.

Implied volatility is the σ value that determines the current market price based on the Black-Scholes formulation, given that the exercise price, current price, risk-free rate and expiration date are known. In a sense, implied volatility may be viewed as the amount of volatility the market is currently observing.

There are currently no explicit formulas of σ expressed as a function of K, p_t, r, T and C_t. However, $\sigma^{implied}$ can be obtained using iterative search procedures such as interpolation or Newton-Raphson, among others. Plotting $\sigma^{implied}$ versus the moneyness ratio, $\frac{p_t}{K}$, for a specified maturity date, produces a U-shaped graph known as a volatility smile[14].

Belonging to the set of "Greek letters"[15], the delta of an option measures the sensitivity of the option price to changes in price of the underlying asset. The delta measurement is an important element especially in hedging, which is a risk management strategy. In general, delta is defined as the first partial derivative of C_t with respect to p_t

$$\Delta_t = \frac{dC_t}{dp_t}.$$

For a European call option determined by the BS formula, it can be easily shown

[14]See Hull (2006) for a thorough description on volatility smiles.

[15]The Greek letters consists of delta, theta, gamma, rho and vega. Theta, vega and rho respectively measures the sensitivity of C_t to changes with respect to t, r and σ. Gamma measure sensitivity of delta with respect to p_t.

that the delta on a stock with no dividends is

$$\begin{aligned}\Delta_t^{BS} &= N(d_1) + p_t n(d_1)\frac{dd_1}{dp_t} - Ke^{-r(T-t)}n(d_2)\frac{dd_2}{dp_t} \\ &= N(d_1),\end{aligned}$$

with d_1 defined as in equation (5.1). Incorporating the GARCH specification under measure $\mathbf{Q}$, Duan (1995) derived the delta at time t to be

$$\Delta_t = D^* E^{\mathbf{Q}}\left[\frac{p_T}{p_t} 1_{(p_T \geq K)} | Y_t\right],$$

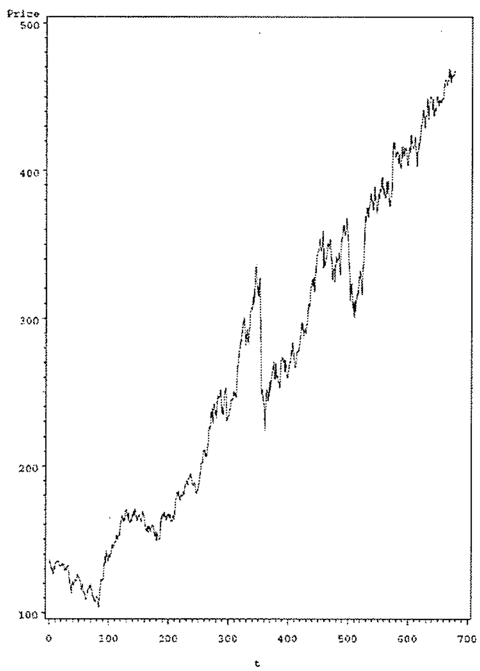
where $1_{(p_T \geq K)}$ is an indicator function and D^* is the discounting factor. As discussed previously for equation (5.2), the GARCH delta here can be approximated via Monte Carlo simulation.

5.4 Data analysis

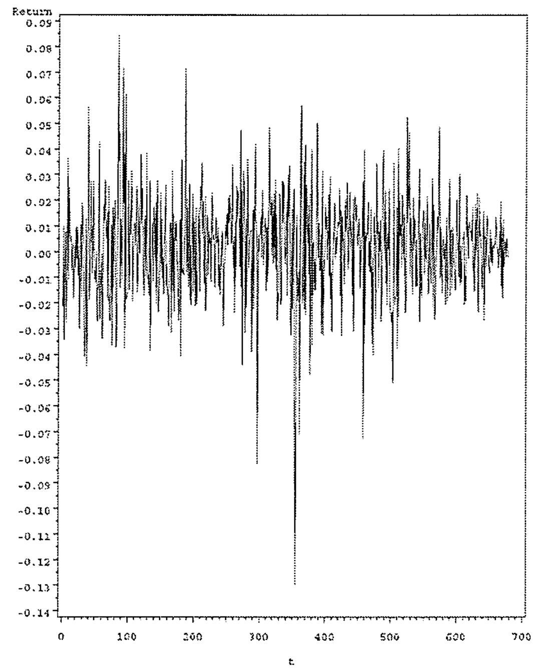
The main objective here is to explore the use of GARCH processes in pricing options. This involves parameter estimation, simulating and comparing call prices and deltas, and evaluating the performance of various models through volatility smiles.

Three different GARCH models, specifically the GARCH(1,1), GJR-GARCH(1,1) and the AR(1)-GJR-GARCH(1,1) as described in Section 5.1, will be used in the analysis of two datasets: the S&P 100 daily index series from January 2, 1991 to December 29, 2000 (as previously used) and S&P 500[16] weekly index series from January 2, 1981 to December 27, 1993. Verification of the four characteristics mentioned in the introductory chapter for the S&P 500 index can be visually inspected in figures 18 and 19.

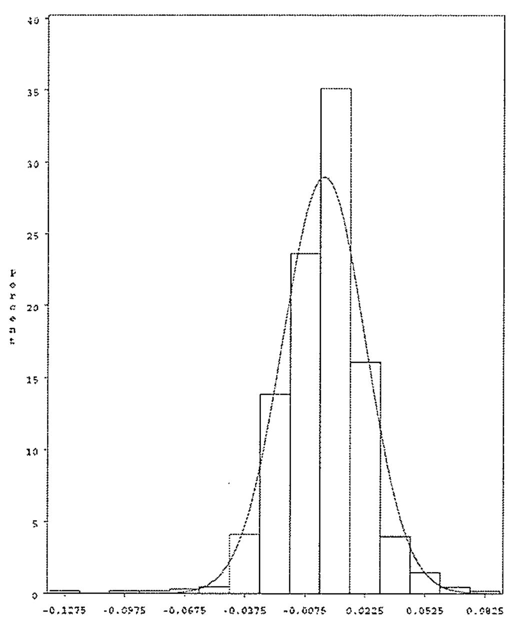
[16]The S&P 500 index is based on the performances of 500 major U.S. companies and widely regarded as the best gauge of the U.S. market.



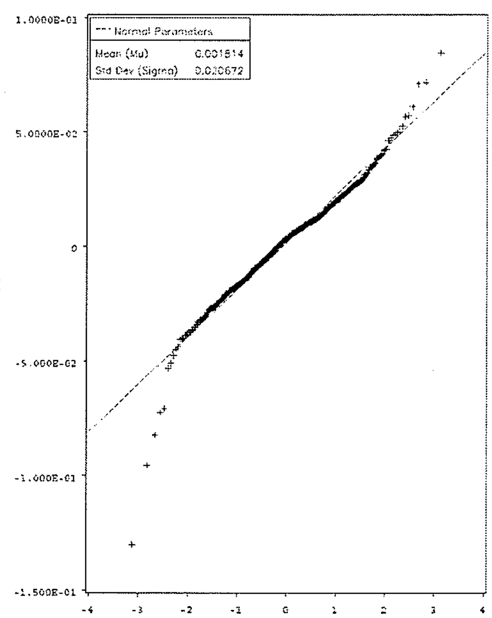
(a) Weekly closing prices p_t



(b) Returns y_t

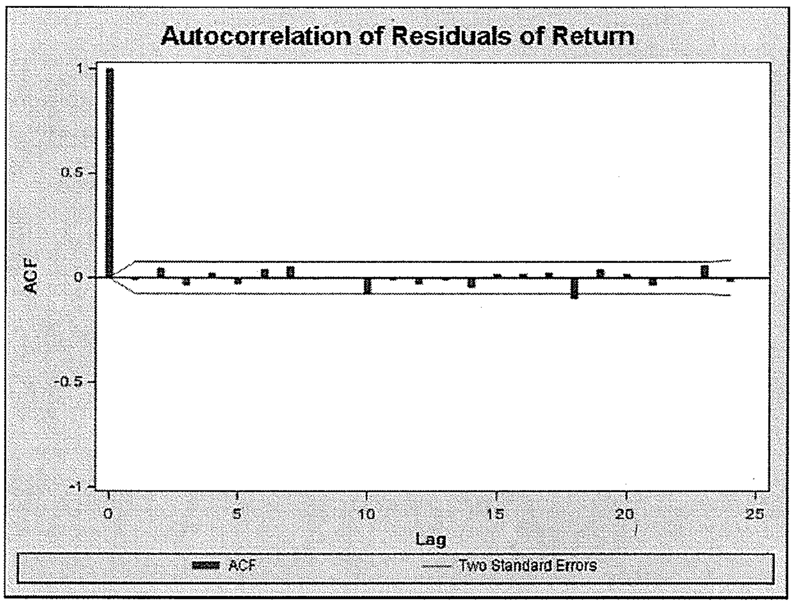


(c) Probability distribution curve of y_t

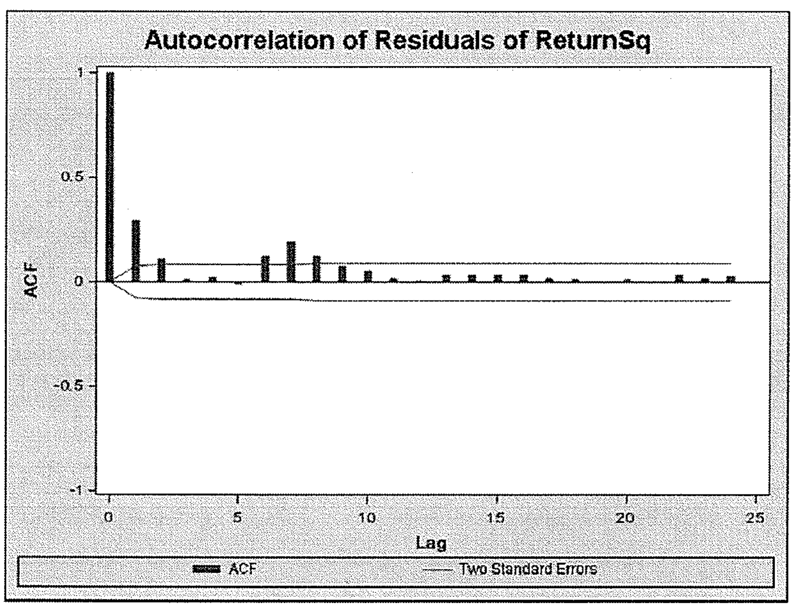


(d) QQ-plot of y_t

Figure 18: Preliminary graphs for the S&P 500 dataset



(a) SACF of returns y_t



(b) SACF of squared returns y_t^2

Figure 19: SACF for the S&P 500 dataset

5.4.1 Parameter estimation

Firstly, the GARCH(1,1), GJR-GARCH(1,1) and the AR(1)-GJR-GARCH(1,1) models under measure $\mathbf{P}$ are fitted to the indices to obtain the parameter estimates. For simplicity, dividend payments are ignored and r assumes the value of 0%. Once again, I use Microsoft Excel's Solver tool to estimate model parameters (see the results in Table 6). I also attempted to estimate the parameters using SAS and a Bayesian estimation software called WinBUGS[17], but with little success due to convergence issues. Note that the unconditional variance σ_y^2 for the GARCH models under measure $\mathbf{P}$ are as follows:

$$\text{GARCH(1,1)}: \ \sigma_y^2 \stackrel{\mathbf{P}}{=} \frac{\omega}{1-\alpha_1-\beta_1},$$
$$\text{GJR-GARCH(1,1)}: \ \sigma_y^2 \stackrel{\mathbf{P}}{=} \frac{\omega}{1-\alpha_1-0.5\alpha^*-\beta_1},$$
$$\text{AR(1)-GJR-GARCH(1,1)}: \ \sigma_y^2 \stackrel{\mathbf{P}}{=} \frac{\omega}{(1-\xi^2)(1-\alpha_1-0.5\alpha^*-\beta_1)}.$$

Most of the estimates are similar across the three models for the S&P 100 and for the S&P 500 dataset as well. Clearly, β_1 has a large influence on the conditional variance h_t. High persistency of shocks in volatility is evident for the GARCH(1,1) model as $\hat{\alpha_1}+\hat{\beta_1}$ is close to 1. The same can be said for both the GJR-GARCH(1,1) and AR(1)-GJR-GARCH(1,1) models as their respective $\hat{\alpha_1} + 0.5\hat{\alpha_1}^* + \hat{\beta_1}$ values range from 0.9493 to 0.9955.

5.4.2 Monte Carlo simulation

With exercise price set at $K = \$1$, half a million simulation runs ($N = 500,000$) corresponding to different maturities, moneyness ratios and initial conditional volatilities h_1 are carried out to obtain GARCH call prices and deltas. The Excel tables in

[17]Go to http://www.mrc-bsu.cam.ac.uk/bugs/ for more information on WinBUGS.

	S&P 100			S&P 500		
Parameters	GARCH	GJR-GARCH	AR(1)-GJR-GARCH	GARCH	GJR-GARCH	AR(1)-GJR-GARCH
$\hat{v}$			0.000438			0.002070
$\hat{\xi}$			0.009783			-0.078682
$\hat{\lambda}$	0.089998	0.068756		0.126089	0.108192	
$\hat{\omega}$ x 10^{-7}	5.598	9.798	8.903	0.016626	0.021365	0.019279
$\hat{\alpha_1}$	0.053597	0.011391	0.010487	0.120538	0.067039	0.067851
$\hat{\alpha_1}^*$		0.084140	0.088370		0.090386	0.096315
$\hat{\beta}_1$	0.941952	0.935815	0.937335	0.844191	0.837096	0.839995
$\hat{\sigma}_y^2$ x 10^{-4}	1.257700	0.913678	1.114030	4.713810	4.216257	4.409108
Log MLE	8409.56	8430.02	8432.34	1710.31	1712.81	1714.62

Table 6: Parameter estimates under measure **P**.

figures 20 and 21 reflects the simulation results for the S&P 100 daily index series. For the S&P 500 weekly index series, refer to the tables in figures 22 and 23. Prices based on the Black-Scholes formulation are included in the tables as well.

Observing from deep out-of-the-money[18] options to deep in-the-money options, the disparity between the GARCH option prices and the BS prices generally decrease in magnitude. The same can't be said when the comparison is done across the $\frac{h_1}{\sigma_y^2}$ scale. The price disparity becomes larger in magnitude for higher valued initial conditional volatility. Moreover, the BS model almost always underprices for deep out-of-the-money options and the underpricing is more pronounced for deep out-of-the-money options with shorter maturity times. The comparison between the GARCH deltas and the BS deltas exhibit similar patterns to the price comparison.

The tables in Figure 24 and Figure 27 report the implied volatilities for the GARCH call prices seen in Figure 20 and in Figure 22 respectively. For the S&P 500 index, results obtained from a stochastic volatility pricing model by Heston (1993) are displayed along with the GARCH models for comparison. The simulated call prices[19] and the implied volatilities for Heston's model are tabulated in Figure 26.

Featured in Figure 25 are graphs of the implied volatilities for the S&P 100 index, and likewise, the implied volatilities for the S&P 500 index appears in Figure 28. Clearly, all plotted graphs exhibits the characteristic U-shaped smiles, or in some instances smirks. The concavity of the volatility smiles flattens out considerably as time to maturity increases in duration.

Focusing on the volatility smiles for the S&P 100 index, one can see that the GJR-GARCH(1,1) smiles and that of the AR(1)-GJR-GARCH(1,1) are strikingly similar.

[18]A call option is out-of-the-money when $p_t < K$, in-the-money when $p_t > K$ and at-the-money when $p_t = K$.

[19]I would like to thank Dr. Paseka for providing me with all the simulated call prices for Heston's model. See Appendix VI for an overview of Heston's model. The parameter estimates: $\hat{R} = 0.1299$, $\hat{c} = 10.98$, $\hat{\epsilon} = 0.4152$ and $\hat{a}/\hat{c} = 0.02253$, and the volatility risk premium $\lambda = 2.52$ from Eraker (2004) were used in the simulation exercise.

			GARCH(1,1)					
T-t	p_t/K		$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
(days)		C_t^{BS}	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias
30	0.8	0.0188	0.31344	1570.92%	0.73823	3835.44%	1.53722	8094.78%
	0.9	10.2636	8.2744	-19.38%	13.90203	35.45%	21.69397	111.37%
	1.0	245.0151	207.52	-15.30%	232.3	-5.19%	259.28	5.82%
	1.1	1016.7609	1014.5	-0.22%	1022.70	0.58%	1033.6	1.66%
	1.2	2000.2874	2001.5	0.06%	2003.00	0.14%	2005.5	0.26%
90	0.8	6.1932	6.78175	9.50%	11.82241	90.89%	19.14837	209.18%
	0.9	85.5797	62.31172	-27.19%	82.99168	-3.02%	107.73	25.88%
	1.0	424.2447	368.66	-13.10%	405.84	-4.34%	446.8	5.32%
	1.1	1112.8253	1090.6	-2.00%	1116.40	0.32%	1146.6	3.04%
	1.2	2020.5922	2023.8	0.16%	2035.40	0.73%	2050.8	1.49%
180	0.8	40.9049	32.76411	-19.90%	45.91624	12.25%	62.8045	53.54%
	0.9	203.5342	157.65	-22.54%	188.5	-7.39%	224.24	10.17%
	1.0	599.6893	530.81	-11.49%	573.41	-4.38%	621	3.55%
	1.1	1251.5987	1208.2	-3.47%	1244.6	-0.56%	1286.3	2.77%
	1.2	2090.0898	2079.6	-0.50%	2103.2	0.63%	2131.9	2.00%

(a) Based on GARCH(1,1) estimates

			GJR-GARCH(1,1)					
T-t	p_t/K		$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
(days)		C_t^{BS}	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias
30	0.8	0.0010	0.27193	26813.08%	0.49506	48896.39%	0.85622	84640.62%
	0.9	4.0777	6.5199	59.89%	9.19315	125.45%	12.7601	212.92%
	1.0	208.8430	213.96	2.45%	226.8	8.60%	241.38	15.58%
	1.1	1007.4422	1024.4	1.68%	1030.10	2.25%	1037.1	2.94%
	1.2	2000.0359	2004.8	0.24%	2006.60	0.33%	2009	0.45%
90	0.8	1.8328	2.9977	63.56%	4.57376	149.55%	6.82376	272.31%
	0.9	52.1250	49.70487	-4.64%	58.76793	12.74%	69.75403	33.82%
	1.0	361.6436	383.13	5.94%	400.68	10.79%	420.8	16.36%
	1.1	1071.7836	1123.5	4.83%	1137.30	6.11%	1153.4	7.62%
	1.2	2008.1919	2046.9	1.93%	2055.20	2.34%	2065.3	2.84%
180	0.8	18.9869	16.33132	-13.99%	20.38518	7.36%	25.0252	31.80%
	0.9	140.5957	140.7	0.07%	153.18	8.95%	167.97	19.47%
	1.0	511.2658	554.98	8.55%	573.05	12.08%	593.96	16.17%
	1.1	1178.2929	1262.2	7.12%	1278.8	8.53%	1298.2	10.18%
	1.2	2049.0292	2132.2	4.06%	2145	4.68%	2160.1	5.42%

(b) Based on GJR-GARCH(1,1) estimates

			AR(1)-GJR-GARCH(1,1)					
T-t	p_t/K		$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
(days)		C_t^{BS}	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias
30	0.8	0.0068	0.22203	3167.40%	0.40729	5893.68%	0.70858	10327.47%
	0.9	7.3944	6.11016	-17.37%	8.67953	17.38%	12.0735	63.28%
	1.0	230.6007	212.78	-7.73%	225.84	-2.06%	240.43	4.26%
	1.1	1012.5494	1023.7	1.10%	1029.3	1.65%	1036.1	2.33%
	1.2	2000.1390	2004.4	0.21%	2006.1	0.30%	2008.4	0.41%
90	0.8	4.0303	2.46877	-38.74%	3.81882	-5.25%	5.73234	42.23%
	0.9	71.5228	46.72791	-34.67%	55.44957	-22.47%	65.85524	-7.92%
	1.0	399.3005	377.13	-5.55%	394.45	-1.21%	413.98	3.68%
	1.1	1095.7429	1118.1	2.04%	1131.40	3.25%	1146.7	4.65%
	1.2	2014.8425	2043.2	1.41%	2050.9	1.79%	2060.2	2.25%
180	0.8	31.1024	13.80487	-55.61%	17.31896	-44.32%	21.78620	-29.95%
	0.9	177.7763	131.78	-25.87%	143.51	-19.27%	157.18	-11.59%
	1.0	564.4599	540.51	-4.24%	557.75	-1.19%	577.37	2.29%
	1.1	1221.7425	1247.3	2.09%	1262.9	3.37%	1280.8	4.83%
	1.2	2072.3941	2119.6	2.28%	2131.2	2.84%	2144.7	3.49%

(c) Based on AR(1)-GJR-GARCH(1,1) estimates

Figure 20: Simulated call prices for different maturities, excercise prices and initial conditional volatilities for the S&P 100 daily index. Biases are as a percentage of the Black-Scholes' prices. Prices are recorded as 10,000 times.

T-t (days)	p_t / K	Δ^{BS}	GARCH(1,1)					
			$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
			Δ	% bias	Δ	% bias	Δ	% bias
30	0.8	0.0002	0.000921724	483.72%	0.001954873	1138.01%	0.003852340	2339.66%
	0.9	0.0460	0.029062	-36.87%	0.042648	-7.36%	0.059090	28.35%
	1.0	0.5123	0.51552	0.64%	0.51673	0.87%	0.51797	1.12%
	1.1	0.9432	0.96167	1.96%	0.94754	0.46%	0.93182	-1.21%
	1.2	0.9986	0.99739	-0.13%	0.99504	-0.36%	0.99157	-0.71%
90	0.8	0.0205	0.015703	-23.28%	0.023854	16.54%	0.034438	68.25%
	0.9	0.1744	0.12913	-25.94%	0.15155	-13.08%	0.174570	0.13%
	1.0	0.5212	0.52522	0.77%	0.52710	1.13%	0.52928	1.55%
	1.1	0.8287	0.87013	5.00%	0.85243	2.86%	0.83547	0.82%
	1.2	0.9614	0.97154	1.06%	0.96169	0.03%	0.95033	-1.15%
180	0.8	0.0796	0.55653	599.24%	0.069776	-12.33%	0.085808	7.81%
	0.9	0.2660	0.22329	-16.05%	0.24196	-9.03%	0.26101	-1.87%
	1.0	0.5300	0.53459	0.87%	0.53674	1.27%	0.53935	1.77%
	1.1	0.7607	0.79805	4.90%	0.79805	4.90%	0.77418	1.77%
	1.2	0.9009	0.92533	2.71%	0.91339	1.38%	0.90067	-0.03%

(a) Based on GARCH(1,1) estimates

T-t (days)	p_t / K	Δ^{BS}	GJR-GARCH(1,1)					
			$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
			Δ	% bias	Δ	% bias	Δ	% bias
30	0.8	0.0000	0.000731512	6324.34%	0.001231919	10719.05%	0.002014587	17592.65%
	0.9	0.0235	0.024134	2.69%	0.031169	32.62%	0.039472	67.95%
	1.0	0.5104	0.54235	6.25%	0.54323	6.42%	0.54411	6.60%
	1.1	0.9676	0.95209	-1.60%	0.94519	-2.32%	0.93757	-3.10%
	1.2	0.9998	0.99383	-0.59%	0.99188	-0.79%	0.98953	-1.02%
90	0.8	0.0079	0.008362448	6.41%	0.011477	46.04%	0.015638	98.98%
	0.9	0.1321	0.12919	-2.20%	0.14089	6.66%	0.15417	16.71%
	1.0	0.5181	0.56494	9.04%	0.56625	9.30%	0.5676	9.56%
	1.1	0.8635	0.86731	0.44%	0.86129	-0.26%	0.85449	-1.05%
	1.2	0.9801	0.95982	-2.07%	0.95547	-2.51%	0.95061	-3.01%
180	0.8	0.0469	0.041148	-12.23%	0.047127	0.53%	0.054227	15.67%
	0.9	0.2244	0.24639	9.80%	0.25517	13.72%	0.26489	18.05%
	1.0	0.5256	0.58012	10.38%	0.58178	10.70%	0.58357	11.04%
	1.1	0.7903	0.80959	2.45%	0.80611	2.01%	0.80232	1.53%
	1.2	0.9313	0.91535	-1.72%	0.91164	-2.11%	0.90750	-2.56%

(b) Based on GJR-GARCH(1,1) estimates

T-t (days)	p_t / K	Δ^{BS}	AR(1)-GJR-GARCH(1,1)					
			$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
			Δ	% bias	Δ	% bias	Δ	% bias
30	0.8	0.0001	0.000610254	855.95%	0.001083411	1597.13%	0.001763572	2662.59%
	0.9	0.0364	0.023303	-36.05%	0.030331	-16.76%	0.038673	6.13%
	1.0	0.5115	0.54256	6.07%	0.54337	6.22%	0.54422	6.39%
	1.1	0.9533	0.95256	-0.08%	0.94548	-0.82%	0.93787	-1.62%
	1.2	0.9993	0.99411	-0.52%	0.99227	-0.70%	0.98989	-0.94%
90	0.8	0.0147	0.007348597	-49.96%	0.010225	-30.38%	0.013927	-5.17%
	0.9	0.1581	0.12531	-20.76%	0.13706	-13.33%	0.15009	-5.09%
	1.0	0.5200	0.56368	8.41%	0.56486	8.63%	0.56631	8.91%
	1.1	0.8418	0.86901	3.23%	0.86262	2.47%	0.85570	1.65%
	1.2	0.9693	0.96126	-0.83%	0.95704	-1.27%	0.95218	-1.77%
180	0.8	0.0662	0.036886	-44.25%	0.042675	-35.50%	0.049128	-25.75%
	0.9	0.2504	0.23885	-4.61%	0.24754	-1.14%	0.25721	2.72%
	1.0	0.5282	0.57731	9.29%	0.57872	9.56%	0.58040	9.88%
	1.1	0.7715	0.81091	5.11%	0.80709	4.61%	0.80331	4.12%
	1.2	0.9128	0.91822	0.59%	0.91431	0.16%	0.91008	-0.30%

(c) Based on AR(1)-GJR-GARCH(1,1) estimates

Figure 21: Simulated deltas for different maturities, excercise prices and initial conditional volatilities for the S&P 100 daily index. Biases are as a percentage of the Black-Scholes' deltas.

			GARCH(1,1)					
T-t	p_t / K		$h_t/\sigma_y^2 = 0.64$		$h_t/\sigma_y^2 = 1.00$		$h_t/\sigma_y^2 = 1.44$	
(weeks)		C_t^{BS}	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias
4	0.8	0.0000	0.1338	1379605.40%	0.146652	1512131.37%	0.16356	1686481.58%
	0.9	1.0344	3.73393	260.98%	3.84376	271.60%	3.98043	284.81%
	1.0	173.2184	166.12	-4.10%	166.72	-3.75%	167.45	-3.33%
	1.1	1002.2574	1006.9	0.46%	1007.0	0.47%	1007.2	0.49%
	1.2	2000.0014	2000.9	0.04%	2000.90	0.04%	2000.9	0.04%
12	0.8	0.2885	2.68275	830.01%	2.77538	862.13%	2.89291	902.87%
	0.9	26.0993	30.31653	16.16%	30.73782	17.77%	31.25204	19.74%
	1.0	299.9757	288.35	-3.88%	289.26	-3.57%	290.36	-3.21%
	1.1	1038.4684	1048.4	0.96%	1048.9	1.00%	1049.6	1.07%
	1.2	2002.0820	2011.0	0.45%	2011.20	0.46%	2011.5	0.47%
24	0.8	6.1818	12.37085	100.12%	12.60263	103.87%	12.88945	108.51%
	0.9	85.5127	83.93793	-1.84%	84.61939	-1.04%	85.44740	-0.08%
	1.0	424.1295	410.32	-3.26%	411.39	-3.00%	412.67	-2.70%
	1.1	1112.7445	1120.2	0.67%	1121.1	0.75%	1122.1	0.84%
	1.2	2020.5631	2036.8	0.80%	2037.3	0.83%	2037.8	0.85%

(a) Based on GARCH(1,1) estimates

			GJR-GARCH(1,1)					
T-t	p_t / K		$h_t/\sigma_y^2 = 0.64$		$h_t/\sigma_y^2 = 1.00$		$h_t/\sigma_y^2 = 1.44$	
(weeks)		C_t^{BS}	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias
4	0.8	0.0000	0.10593	6083307.72%	0.10943	6284307.69%	0.11385	6538141.94%
	0.9	0.6359	3.31532	421.39%	3.35300	427.32%	3.39997	434.70%
	1.0	163.8231	169.73	3.61%	169.94	3.73%	170.21	3.90%
	1.1	1001.4831	1008.4	0.69%	1008.4	0.69%	1008.5	0.70%
	1.2	2000.0004	2001.1	0.05%	2001.1	0.05%	2001.2	0.06%
12	0.8	0.1484	1.69897	1045.13%	1.72521	1062.82%	1.75812	1085.00%
	0.9	20.6285	26.38384	27.90%	26.52619	28.59%	26.69980	29.43%
	1.0	283.7099	293.91	3.60%	294.23	3.71%	294.62	3.85%
	1.1	1031.1670	1056.2	2.43%	1056.40	2.45%	1056.6	2.47%
	1.2	2001.2808	2014.3	0.65%	2014.40	0.66%	2014.5	0.66%
24	0.8	4.1693	7.85147	88.31%	7.91668	89.88%	7.99730	91.81%
	0.9	72.5294	76.14754	4.99%	76.36986	5.30%	76.64079	5.67%
	1.0	401.1415	417.10	3.98%	417.46	4.07%	417.89	4.18%
	1.1	1096.9729	1134.7	3.44%	1134.9	3.46%	1135.3	3.49%
	1.2	2015.2298	2046.6	1.56%	2046.8	1.57%	2047.0	1.58%

(b) Based on GJR-GARCH(1,1) estimates

			AR(1)-GJR-GARCH(1,1)					
T-t	p_t / K		$h_t/\sigma_y^2 = 0.64$		$h_t/\sigma_y^2 = 1.00$		$h_t/\sigma_y^2 = 1.44$	
(weeks)		C_t^{BS}	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias	$\hat{C}_t$	% bias
4	0.8	0.0000	1.43318	40481078.71%	1.51862	42894391.70%	1.62386	45866969.64%
	0.9	0.7766	13.36763	1621.24%	13.74283	1669.55%	14.19426	1727.68%
	1.0	167.5273	215.31	28.52%	216.38	29.16%	217.65	29.92%
	1.1	1001.7623	1026.1	2.43%	1026.6	2.48%	1027.3	2.55%
	1.2	2000.0007	2006.4	0.32%	2006.7	0.33%	2007.0	0.35%
12	0.8	0.1951	10.89178	5483.07%	11.28035	5682.25%	11.75081	5923.40%
	0.9	22.7098	64.36794	183.44%	65.39317	187.95%	66.61052	193.31%
	1.0	290.1230	373.31	28.67%	374.98	29.25%	376.95	29.93%
	1.1	1033.9630	1113.0	7.64%	1114.3	7.77%	1115.8	7.91%
	1.2	2001.5639	2043.7	2.11%	2044.50	2.15%	2045.5	2.20%
24	0.8	4.9008	32.67819	566.80%	33.38916	581.30%	34.23071	598.48%
	0.9	77.5584	148.16	91.03%	149.63	92.93%	151.37	95.17%
	1.0	410.2051	530.88	29.42%	532.85	29.90%	535.16	30.46%
	1.1	1103.1017	1234.2	11.88%	1236.0	12.05%	1238.0	12.23%
	1.2	2017.2224	2114.4	4.82%	2115.7	4.88%	2117.3	4.96%

(c) Based on AR(1)-GJR-GARCH(1,1) estimates

Figure 22: Simulated call prices for different maturities, excercise prices and initial conditional volatilities for the S&P 500 weekly index. Biases are as a percentage of the Black-Scholes' prices. Prices are recorded as 10,000 times.

T-t (weeks)	p_t / K	Δ^{BS}	GARCH(1,1)					
			$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
			Δ	% bias	Δ	% bias	Δ	% bias
4	0.8	0.0000	0.000426726	274580.03%	0.000453314	291694.51%	0.000487945	313986.20%
	0.9	0.0081	0.014004	73.03%	0.014240	75.95%	0.014596	80.35%
	1.0	0.5087	0.51302	0.86%	0.51303	0.86%	0.51308	0.87%
	1.1	0.9867	0.98059	-0.62%	0.98029	-0.65%	0.97994	-0.68%
	1.2	1.0000	0.99882	-0.12%	0.99877	-0.12%	0.9987	-0.13%
12	0.8	0.0017	0.006040344	255.62%	0.006209422	265.57%	0.006409114	277.33%
	0.9	0.0864	0.075586	-12.51%	0.076164	-11.85%	0.076830	-11.07%
	1.0	0.5150	0.52212	1.38%	0.52218	1.39%	0.52225	1.41%
	1.1	0.9040	0.91612	1.34%	0.91560	1.28%	0.91505	1.22%
	1.2	0.9931	0.98706	-0.61%	0.98685	-0.63%	0.98660	-0.65%
24	0.8	0.0204	0.023509	15.02%	0.023805	16.47%	0.024149	18.15%
	0.9	0.1743	0.1544	-11.41%	0.15499	-11.07%	0.15567	-10.68%
	1.0	0.5212	0.52873	1.44%	0.52882	1.46%	0.52891	1.48%
	1.1	0.8288	0.85014	2.58%	0.84972	2.53%	0.84924	2.47%
	1.2	0.9614	0.95987	-0.16%	0.95955	-0.19%	0.95920	-0.23%

(a) Based on GARCH(1,1) estimates

T-t (weeks)	p_t / K	Δ^{BS}	GJR-GARCH(1,1)					
			$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
			Δ	% bias	Δ	% bias	Δ	% bias
4	0.8	0.0000	0.000335741	1081486.78%	0.000348678	1123163.22%	0.000361731	1165213.35%
	0.9	0.0055	0.013097	139.73%	0.013210	141.80%	0.013338	144.14%
	1.0	0.5082	0.52013	2.35%	0.52014	2.35%	0.52015	2.35%
	1.1	0.9904	0.97773	-1.28%	0.97763	-1.29%	0.97749	-1.30%
	1.2	1.0000	0.99846	-0.15%	0.99844	-0.16%	0.99841	-0.16%
12	0.8	0.0010	0.004219871	338.37%	0.004265651	343.13%	0.004334765	350.31%
	0.9	0.0741	0.074229	0.13%	0.074447	0.42%	0.074704	0.77%
	1.0	0.5142	0.53625	4.29%	0.53626	4.29%	0.53628	4.30%
	1.1	0.9155	0.91095	-0.50%	0.91080	-0.51%	0.91061	-0.54%
	1.2	0.9953	0.98379	-1.16%	0.98371	-1.17%	0.98364	-1.17%
24	0.8	0.0151	0.017926	18.87%	0.018017	19.47%	0.018122	20.17%
	0.9	0.1594	0.15869	-0.42%	0.15889	-0.29%	0.15919	-0.11%
	1.0	0.5201	0.54602	4.99%	0.54604	5.00%	0.54607	5.00%
	1.1	0.8408	0.84848	0.91%	0.84838	0.90%	0.84824	0.88%
	1.2	0.9688	0.95439	-1.48%	0.95429	-1.49%	0.95418	-1.50%

(b) Based on GJR-GARCH(1,1) estimates

T-t (weeks)	p_t / K	Δ^{BS}	AR(1)-GJR-GARCH(1,1)					
			$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
			Δ	% bias	Δ	% bias	Δ	% bias
4	0.8	0.0000	0.002776647	4592516.12%	0.002882328	4767314.09%	0.003020483	4995824.55%
	0.9	0.0064	0.034463	436.30%	0.035067	445.70%	0.035717	455.81%
	1.0	0.5084	0.52641	3.55%	0.52646	3.56%	0.52654	3.57%
	1.1	0.9890	0.95415	-3.52%	0.95359	-3.58%	0.95298	-3.64%
	1.2	1.0000	0.99295	-0.70%	0.99277	-0.72%	0.99261	-0.74%
12	0.8	0.0012	0.017574	1343.68%	0.018060	1383.60%	0.018586	1426.81%
	0.9	0.0790	0.12190	54.38%	0.12275	55.46%	0.12373	56.70%
	1.0	0.5145	0.54554	6.03%	0.54563	6.05%	0.54573	6.07%
	1.1	0.9110	0.87570	-3.87%	0.87507	-3.94%	0.87448	-4.00%
	1.2	0.9945	0.96477	-2.99%	0.96438	-3.03%	0.96388	-3.08%
24	0.8	0.0171	0.048600	184.27%	0.049176	187.64%	0.049929	192.04%
	0.9	0.1653	0.21705	31.28%	0.21782	31.75%	0.21870	32.28%
	1.0	0.5205	0.55899	7.39%	0.55910	7.41%	0.55925	7.44%
	1.1	0.8360	0.81755	-2.20%	0.81717	-2.25%	0.81668	-2.31%
	1.2	0.9659	0.92527	-4.21%	0.92487	-4.25%	0.92435	-4.30%

(c) Based on AR(1)-GJR-GARCH(1,1) estimates

Figure 23: Simulated deltas for different maturities, excercise prices and initial conditional volatilities for the S&P 500 weekly index. Biases are as a percentage of the Black-Scholes' deltas.

Compared to the other two models, the GARCH(1,1) overprices for out-of-the-money calls and underprices when calls are in-the-money. The overpricing (underpricing) by the GARCH(1,1) model is much more evident for longer (shorter) maturity options. In addition, the GARCH implied volatilities here appear to be sensitive to changes in the initial conditional variance h_1. This can be visually verified by examining the vertical placement of the smiles for each of the three maturities.

For the S&P 500 index, the AR(1)-GJR-GARCH(1,1) and Heston smiles notably stands out whereas the GARCH(1,1) and the GJR-GARCH(1,1) smiles holds a close resemblance. Though the smiles of the GARCH(1,1) and the GJR-GARCH(1,1) look similar, the GARCH(1,1) overprices for out-of-the-money calls and underprices for in-the-money calls. The Heston model underprices almost always and the AR(1)-GJR-GARCH(1,1) model however overprices regardless of situation. Unlike what was observed for the S&P 100 index volatility smiles, the impact of h_1 on the GARCH smiles here are marginal.

5.4.3 Simulated versus observed

To investigate the performance of the GARCH models, comparisons will be made between the GARCH volatility smiles and the observed volatility smiles for a randomly selected day in our sample. For the S&P 100 daily index, the randomly selected date is October 27, 1993. The random date for the S&P 500 weekly index is February 17, 1993. The available observed market data are rather limited and do not offer information on deep out-of- and in-the-money options with extended time to maturity. Hence, the discussions here will be restricted to short maturity near the money options, i.e. $\frac{p_t}{K} \approx 1$. As before, Heston's model will also be included in the evaluation only for the S&P 500 index. Keep in mind that the parameter estimates for the GARCH models were derived only from time series information, which may

T-t (days)	p_t / K	GARCH(1,1)					
		$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
		$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$
30	0.8	0.31344	0.01383	0.738230015	0.01500	1.53722	0.01622
	0.9	8.2744	0.01077	13.90203	0.01192	21.69397	0.01314
	1.0	207.52	0.00950	232.3	0.01063	259.28	0.01187
	1.1	1014.5	0.01087	1022.7	0.01201	1033.6	0.01324
	1.2	2001.5	0.01322	2003	0.01437	2005.5	0.01560
90	0.8	6.78175	0.01137	11.82241	0.01242	19.14837	0.01354
	0.9	62.31172	0.01010	82.99168	0.01110	107.73	0.01218
	1.0	368.66	0.00974	405.84	0.01073	446.8	0.01181
	1.1	1090.6	0.01035	1116.4	0.01135	1146.6	0.01244
	1.2	2023.8	0.01154	2035.4	0.01254	2050.8	0.01363
180	0.8	32.76411	0.01067	45.91624	0.01152	62.8045	0.01245
	0.9	157.65	0.01002	188.5	0.01083	224.24	0.01173
	1.0	530.81	0.00992	573.41	0.01072	621	0.01161
	1.1	1208.2	0.01025	1244.6	0.01106	1286.3	0.01196
	1.2	2079.6	0.01083	2103.2	0.01167	2131.9	0.01259

(a) Based on GARCH(1,1) estimates

T-t (days)	p_t / K	GJR-GARCH(1,1)					
		$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
		$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$
30	0.8	0.27193	0.01366	0.49506027	0.01442	0.85622	0.01523
	0.9	6.5199	0.01032	9.19315	0.01098	12.7601	0.01171
	1.0	213.96	0.00979	226.8	0.01038	241.38	0.01105
	1.1	1024.4	0.01222	1030.1	0.01287	1037.1	0.01360
	1.2	2004.8	0.01530	2006.6	0.01602	2009	0.01680
90	0.8	2.9977	0.01015	4.57376	0.01074	6.82376	0.01138
	0.9	49.70487	0.00942	58.76793	0.00991	69.75403	0.01047
	1.0	383.13	0.01013	400.68	0.01059	420.8	0.01112
	1.1	1123.5	0.01161	1137.3	0.01211	1153.4	0.01267
	1.2	2046.9	0.01338	2055.2	0.01392	2065.3	0.01452
180	0.8	16.33132	0.00930	20.38518	0.00969	25.0252	0.01009
	0.9	140.7	0.00956	153.18	0.00990	167.97	0.01030
	1.0	554.98	0.01038	573.05	0.01072	593.96	0.01111
	1.1	1262.2	0.01145	1278.8	0.01180	1298.2	0.01222
	1.2	2132.2	0.01260	2145	0.01298	2160.1	0.01342

(b) Based on GJR-GARCH(1,1) estimates

T-t (days)	p_t / K	AR(1)-GJR-GARCH(1,1)					
		$h_1/\sigma_y^2 = 0.64$		$h_1/\sigma_y^2 = 1.00$		$h_1/\sigma_y^2 = 1.44$	
		$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$
30	0.8	0.22203	0.01343	0.40729	0.01416	0.70858	0.01494
	0.9	6.11016	0.01021	8.67953	0.01087	12.0735	0.01158
	1.0	212.78	0.00974	225.84	0.01034	240.43	0.01100
	1.1	1023.7	0.01214	1029.3	0.01278	1036.1	0.01350
	1.2	2004.4	0.01511	2006.1	0.01583	2008.4	0.01662
90	0.8	2.46877	0.00991	3.81882	0.01048	5.73234	0.01109
	0.9	46.72791	0.00926	55.44957	0.00974	65.85524	0.01028
	1.0	377.13	0.00997	394.45	0.01043	413.98	0.01094
	1.1	1118.1	0.01141	1131.4	0.01190	1146.7	0.01244
	1.2	2043.2	0.01312	2050.9	0.01364	2060.2	0.01422
180	0.8	13.80487	0.00902	17.31896	0.00940	21.78620	0.00981
	0.9	131.78	0.00931	143.51	0.00964	157.18	0.01001
	1.0	540.51	0.01011	557.75	0.01043	577.37	0.01080
	1.1	1247.3	0.01112	1262.9	0.01146	1280.8	0.01185
	1.2	2119.6	0.01221	2131.2	0.01257	2144.7	0.01297

(c) Based on AR(1)-GJR-GARCH(1,1) estimates

Figure 24: Implied volatilities based on simulated call prices for the S&P 100 daily index.

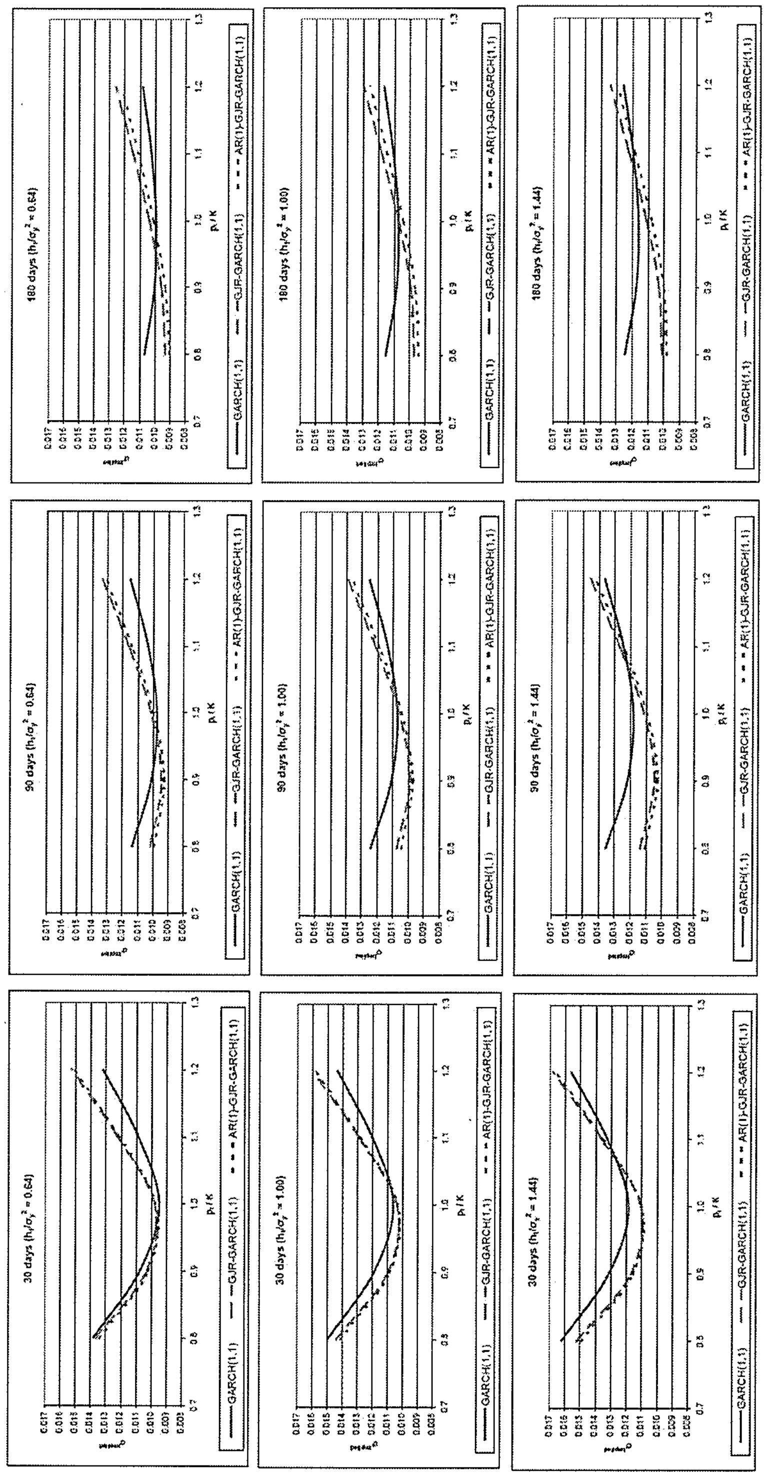


Figure 25: Volatility smiles based on simulated call prices for the S&P 100 daily index.

T-t (weeks)	p_t / K	C_t^{BS}	Heston $h_1/\sigma_y^2 = 0.64$ $\hat{C}_t$	% bias	$h_1/\sigma_y^2 = 1.00$ $\hat{C}_t$	% bias	$h_1/\sigma_y^2 = 1.44$ $\hat{C}_t$	% bias
	0.8	0.0000	4.56E-07	-98.35%	8.82E-06	-68.08%	1.19E-04	330.64%
	0.9	1.3982	0.0436	-96.88%	0.24	-82.84%	0.934	-33.20%
4	1.0	179.7491	114	-36.58%	142	-21.00%	170	-5.42%
	1.1	1002.9305	1000.627523	-0.23%	1001.89153	-0.10%	1004.53817	0.16%
	1.2	2000.0029	2000.001913	0.00%	2000.01148	0.00%	2000.051873	0.00%
	0.8	0.4345	0.00528	-98.78%	1.97E-02	-95.47%	6.68E-02	-84.63%
	0.9	30.2665	1.53	-94.94%	3.75	-87.61%	8.11	-73.20%
12	1.0	311.2815	158	-49.24%	194	-37.68%	231	-25.79%
	1.1	1043.9386	1007.908657	-3.45%	1014.592056	-2.81%	1024.47309	-1.86%
	1.2	2002.8132	2000.466747	-0.12%	2001.090235	-0.09%	2002.309566	-0.03%
	0.8	7.8977	0.0512	-99.35%	0.125	-98.42%	0.296	-96.25%
	0.9	94.9601	4.1	-95.68%	7.97	-91.61%	14.5	-84.73%
24	1.0	440.1066	188	-57.28%	223	-49.33%	260	-40.92%
	1.1	1124.1256	1016.20958	-9.60%	1025.625063	-8.76%	1038.442685	-7.62%
	1.2	2024.8136	2001.875553	-1.13%	2003.396502	-1.06%	2005.870375	-0.94%

(a) Simulated call prices

T-t (weeks)	p_t / K	Heston $h_1/\sigma_y^2 = 0.64$ $\hat{C}_t$	$\sigma^{Implied}$	$h_1/\sigma_y^2 = 1.00$ $\hat{C}_t$	$\sigma^{Implied}$	$h_1/\sigma_y^2 = 1.44$ $\hat{C}_t$	$\sigma^{Implied}$
	0.8	4.56E-07		8.82E-06	0.02164	0.000119	0.02383
	0.9	0.0436	0.01611	0.24	0.01859	0.934	0.02145
4	1.0	114	0.01429	142	0.01780	170	0.02131
	1.1	1000.627523	0.01855	1001.89153	0.02120	1004.53817	0.02408
	1.2	2000.001913	0.02207	2000.01148	0.02435	2000.051873	0.02685
	0.8	0.00528	0.01645	0.0197	0.01779	0.0668	0.01932
	0.9	1.53	0.01316	3.75	0.01492	8.11	0.01697
12	1.0	158	0.01143	194	0.01404	231	0.01672
	1.1	1007.908657	0.01528	1014.592056	0.01721	1024.47309	0.01934
	1.2	2000.466747	0.01853	2001.090235	0.02018	2002.309566	0.02199
	0.8	0.0512	0.01340	0.125	0.01432	0.296	0.01539
	0.9	4.1	0.01070	7.97	0.01196	14.5	0.01344
24	1.0	188	0.00962	223	0.01141	260	0.01331
	1.1	1016.20958	0.01245	1025.625063	0.01383	1038.442685	0.01535
	1.2	2001.875553	0.01516	2003.396502	0.01632	2005.870375	0.01760

(b) Implied volatilities

Figure 26: Heston's closed-form stochastic volatility pricing model for the S&P 500 weekly index. Prices are recorded as 10,000 times.

T-t (weeks)	p_t / K	GARCH(1,1) $h_1/\sigma_y^2 = 0.64$ $\hat{C}_t$	$\sigma^{implied}$	$h_1/\sigma_y^2 = 1.00$ $\hat{C}_t$	$\sigma^{implied}$	$h_1/\sigma_y^2 = 1.44$ $\hat{C}_t$	$\sigma^{implied}$
4	0.8	0.1338	0.03527	0.146652	0.03553	0.16356	0.03585
	0.9	3.73393	0.02582	3.84376	0.02594	3.98043	0.02608
	1.0	166.12	0.02082	166.72	0.02090	167.45	0.02099
	1.1	1006.9	0.02583	1007	0.02590	1007.2	0.02602
	1.2	2000.9	0.03425	2000.9	0.03425	2000.9	0.03425
12	0.8	2.68275	0.02741	2.77538	0.02753	2.89291	0.02768
	0.9	30.31653	0.02254	30.73782	0.02262	31.25204	0.02272
	1.0	288.35	0.02087	289.26	0.02094	290.36	0.02102
	1.1	1048.4	0.02317	1048.9	0.02324	1049.6	0.02333
	1.2	2011	0.02745	2011.2	0.02753	2011.5	0.02766
24	0.8	12.37085016	0.02424	12.60263	0.02431	12.88945	0.02441
	0.9	83.93793	0.02157	84.61939	0.02163	85.4474	0.02171
	1.0	410.32	0.02100	411.39	0.02106	412.67	0.02112
	1.1	1112.744453	0.02171	1121.1	0.02231	1122.1	0.02239
	1.2	2036.8	0.02449	2037.3	0.02457	2037.8	0.02464

(a) Based on GARCH(1,1) estimates

T-t (weeks)	p_t / K	GJR-GARCH(1,1) $h_1/\sigma_y^2 = 0.64$ $\hat{C}_t$	$\sigma^{implied}$	$h_1/\sigma_y^2 = 1.00$ $\hat{C}_t$	$\sigma^{implied}$	$h_1/\sigma_y^2 = 1.44$ $\hat{C}_t$	$\sigma^{implied}$
4	0.8	0.10593	0.03464	0.10943	0.03472	0.11385	0.03483
	0.9	3.31532	0.02536	3.353	0.02541	3.39997	0.02546
	1.0	169.73	0.02127	169.94	0.02130	170.21	0.02133
	1.1	1008.4	0.02676	1008.4	0.02676	1008.5	0.02681
	1.2	2001.1	0.03498	2001.1	0.03498	2001.2	0.03531
12	0.8	1.69897	0.02595	1.72521	0.02599	1.75812	0.02605
	0.9	26.38384	0.02177	26.52619	0.02180	26.6998	0.02183
	1.0	283.7098797	0.02053	294.23	0.02130	294.62	0.02132
	1.1	1056.2	0.02422	1056.4	0.02425	1056.6	0.02428
	1.2	2014.3	0.02872	2014.4	0.02875	2014.5	0.02879
24	0.8	7.851470888	0.02251	7.91668	0.02254	7.9973	0.02257
	0.9	76.14754	0.02087	76.36986	0.02089	76.64079	0.02091
	1.0	417.1	0.02135	417.46	0.02137	417.89	0.02139
	1.1	1134.7	0.02327	1134.9	0.02329	1135.3	0.02331
	1.2	2046.6	0.02586	2046.8	0.02589	2047	0.02591

(b) Based on GJR-GARCH(1,1) estimates

T-t (weeks)	p_t / K	AR(1)-GJR-GARCH(1,1) $h_1/\sigma_y^2 = 0.64$ $\hat{C}_t$	$\sigma^{implied}$	$h_1/\sigma_y^2 = 1.00$ $\hat{C}_t$	$\sigma^{implied}$	$h_1/\sigma_y^2 = 1.44$ $\hat{C}_t$	$\sigma^{implied}$
4	0.8	1.43318	0.04408	1.51862	0.04437	1.62386	0.04471
	0.9	13.36763	0.03237	13.74283	0.03255	14.19426	0.03277
	1.0	215.31	0.02699	216.38	0.02712	217.65	0.02728
	1.1	1026.1	0.03402	1026.6	0.03418	1027.3	0.03440
	1.2	2006.4	0.04367	2006.7	0.04397	2007	0.04426
12	0.8	10.89178	0.03355	11.28035	0.03375	11.75081	0.03398
	0.9	64.36794	0.02794	65.39317	0.02808	66.61052	0.02824
	1.0	373.31	0.02702	374.98	0.02714	376.95	0.02729
	1.1	1113	0.03073	1114.3	0.03086	1115.8	0.03102
	1.2	2043.7	0.03602	2044.5	0.03618	2045.5	0.03637
24	0.8	32.67819	0.02921	33.38916	0.02935	34.23071	0.02951
	0.9	148.16	0.02675	149.63	0.02686	151.37	0.02699
	1.0	530.88	0.02718	532.85	0.02728	535.16	0.02740
	1.1	1234.2	0.02967	1236	0.02977	1238	0.02990
	1.2	2114.4	0.03297	2115.7	0.03308	2117.3	0.03322

(c) Based on AR(1)-GJR-GARCH(1,1) estimates

Figure 27: Implied volatilities based on simulated call prices for the S&P 500 weekly index.

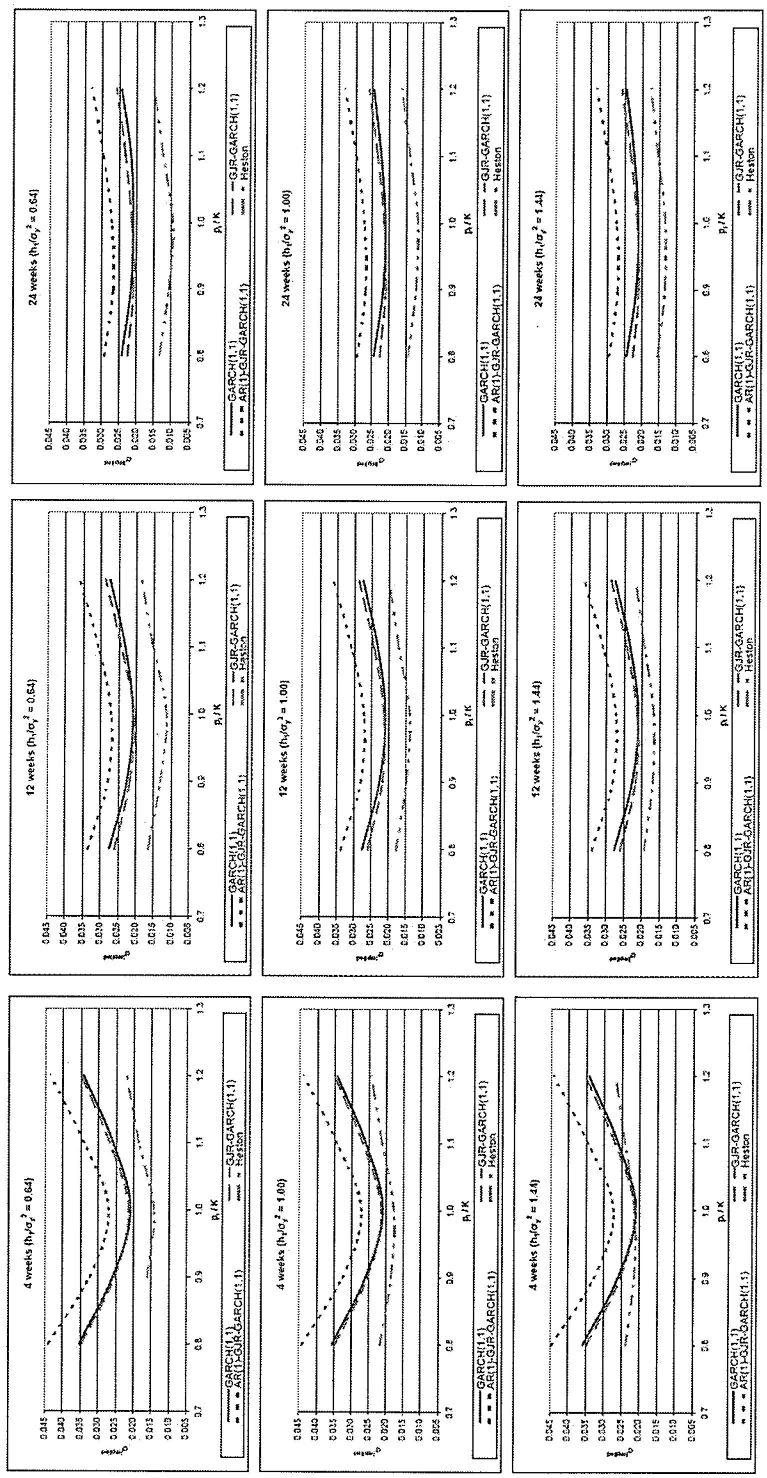


Figure 28: Volatility smiles based on simulated call prices for the S&P 500 weekly index.

result in a poor fit between the simulated volatility smiles and the observed volatility smile.

The simulation process to obtain call prices on a specific date will be initiated with h_1 assuming the estimate $\hat{h}_t$ value where t is the time being considered. For instance, $h_1 = 3.00 \text{ x } 10^{-5}$ will be used to simulate the GJR-GARCH(1,1) call prices on October 27, 1993. Refer to figures 29 and 31 for all the h_1 values along with the p_t, K and $T - t$ values implemented in the simulation exercise here, as well as the resulting simulated call prices and implied volatilities for the three GARCH models. The figures that immediately follow display the corresponding volatility smiles.

The volatility smiles in Figure 30 indicate that all three GARCH models overprice options relative to the observed prices regardless of maturity. The disparity between the observed values and the GARCH models generally decline as we proceed horizontally from left to right on the $\frac{p_t}{K}$ axis, with the GARCH(1,1) model being the closest to the observed. Although graphically the disparity may seem large, the differences of the implied volatilites are actually no larger than 0.01.

Figure 32 features volatility smiles of calls with a time to maturity of approximately 4 weeks and 8 weeks. Unlike previously seen in Figure 30, only the AR(1)-GJR-GARCH(1,1) model have the tendency to overprice options. The GARCH(1,1) and GJR-GARCH(1,1) models however mostly overprice out-of-the-money options and underprice in-the-money options. The Heston model, on the other hand, underprices options for both stated maturities. Once again, the disparities between the observed and GARCH implied volatilities is no larger than 0.01 in magnitude.

Observed						GARCH(1,1)		GJR-GARCH(1,1)		AR(1)-GJR-GARCH(1,1)	
K	P_t	P_t / K	T-t (days)	C_t	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$
395	425.73	1.08	24	30.75	0.0059759	31.0628	0.00900825	31.5973	0.0111287	31.5301	0.0109089
400	425.73	1.06	24	25.88	0.006700	26.2992	0.0087301	26.9817	0.0107288	26.8992	0.0105191
405	425.73	1.05	24	21	0.0062267	21.6977	0.00848817	22.5388	0.0103552	22.4394	0.0101545
410	425.67	1.04	24	16.5	0.0066192	17.2960	0.00827973	18.2869	0.0100075	18.1690	0.0098128
415	425.68	1.03	24	11.875	0.005769	13.3235	0.00811578	14.4230	0.0097037	14.2874	0.0095127
420	425.65	1.01	24	7.69	0.0051862	9.81989	0.00800489	10.9540	0.0094468	10.8036	0.0092569
425	425.65	1.00	24	4.44	0.0049406	6.93686	0.00794839	8.01302	0.0092441	7.85439	0.0090532
430	425.68	0.99	24	2.095	0.0046382	4.70563	0.00794548	5.64382	0.0091012	5.48562	0.0089069
435	425.65	0.98	24	0.78	0.0044374	3.05407	0.00799046	3.80463	0.0090137	3.65650	0.0088143
440	425.16	0.97	24	0.25	0.0045179	1.83488	0.00809434	2.37494	0.0089893	2.24688	0.0087826
445	424.78	0.95	24	0.095	0.0048234	1.08599	0.00824122	1.45660	0.0090395	1.35255	0.008824
450	425.19	0.94	24	0.095	0.0056961	0.69071	0.00839488	0.95290	0.0091296	0.86880	0.0089052
380	425.73	1.12	87	46.75	0.0083756	47.2348	0.00922177	48.9373	0.0115049	48.6823	0.011201
385	425.73	1.11	87	42	0.0080473	42.7012	0.00907786	44.5669	0.0112615	44.2899	0.010967
390	425.73	1.09	87	37.5	0.0079667	38.2994	0.00894634	40.3181	0.0110286	40.0194	0.010742
395	425.73	1.08	87	33	0.0077138	34.0614	0.00883074	36.2102	0.0108059	35.8902	0.010527
400	425.73	1.06	87	28.5	0.0073283	30.0168	0.00872901	32.2639	0.0105934	31.9242	0.010322
405	425.73	1.05	87	24.13	0.0069421	26.1969	0.00864087	28.5014	0.0103917	28.1437	0.010126
410	425.26	1.04	87	20.375	0.0071569	22.2972	0.00855994	24.6051	0.0101835	24.2316	0.009924
415	425.86	1.03	87	16.125	0.006232	19.4324	0.008506	21.7016	0.0100248	21.3172	0.009769
420	425.68	1.01	87	12.815	0.0061734	16.3363	0.00845765	18.5010	0.009852	18.1093	0.009600
425	425.42	1.00	87	9.315	0.0057542	13.5387	0.0084254	15.5398	0.0096913	15.1459	0.009442
430	425.62	0.99	87	6.505	0.005350	11.3069	0.00840908	13.1213	0.0095584	12.7300	0.009311
435	425.82	0.98	87	4.505	0.005183	9.36248	0.00840486	10.9610	0.0094373	10.5775	0.009190
440	425.68	0.97	87	2.75	0.004930	7.57269	0.00841297	8.92241	0.0093219	8.55326	0.009075
445	425.75	0.96	87	1.595	0.0047325	6.13385	0.00843182	7.24412	0.0092245	6.89331	0.008976
450	425.78	0.95	87	0.845	0.0045584	4.92417	0.00846041	5.80455	0.0091395	5.47655	0.008889
455	425.39	0.93	87	0.44	0.0045309	3.84689	0.00850372	4.49810	0.0090617	4.19932	0.008809
380	425.73	1.12	115	47.25	0.008042	48.1959	0.00921393	50.4508	0.0114442	50.0832	0.0111114
390	425.73	1.09	115	38.13	0.007610	39.6148	0.008998	42.1377	0.0110257	41.7238	0.0107101
400	425.73	1.06	115	29.38	0.0070985	31.7069	0.008825	34.3838	0.0106424	33.9297	0.010342
410	425.73	1.04	115	21.19	0.0065347	24.6558	0.00869731	27.3197	0.0102959	26.8351	0.0100077
420	425.41	1.01	115	13.875	0.0060607	18.4207	0.00861094	20.8689	0.0099777	20.3699	0.0096994
430	425.63	0.99	115	8.125	0.0055524	13.5935	0.00857107	15.6760	0.0097165	15.1812	0.0094444
440	425.28	0.97	115	3.875	0.0051178	9.56229	0.00856527	11.1507	0.0094795	10.6821	0.009211
450	425.13	0.94	115	1.5	0.0047695	6.61162	0.00859466	7.69526	0.009286	7.27129	0.0090177
					h_1	2.792014E-05		3.003261E-05		2.994508E-05	

Figure 29: Observed and simulated call prices and its respective implied volatilities for the S&P 100 daily index on October 27, 1993.

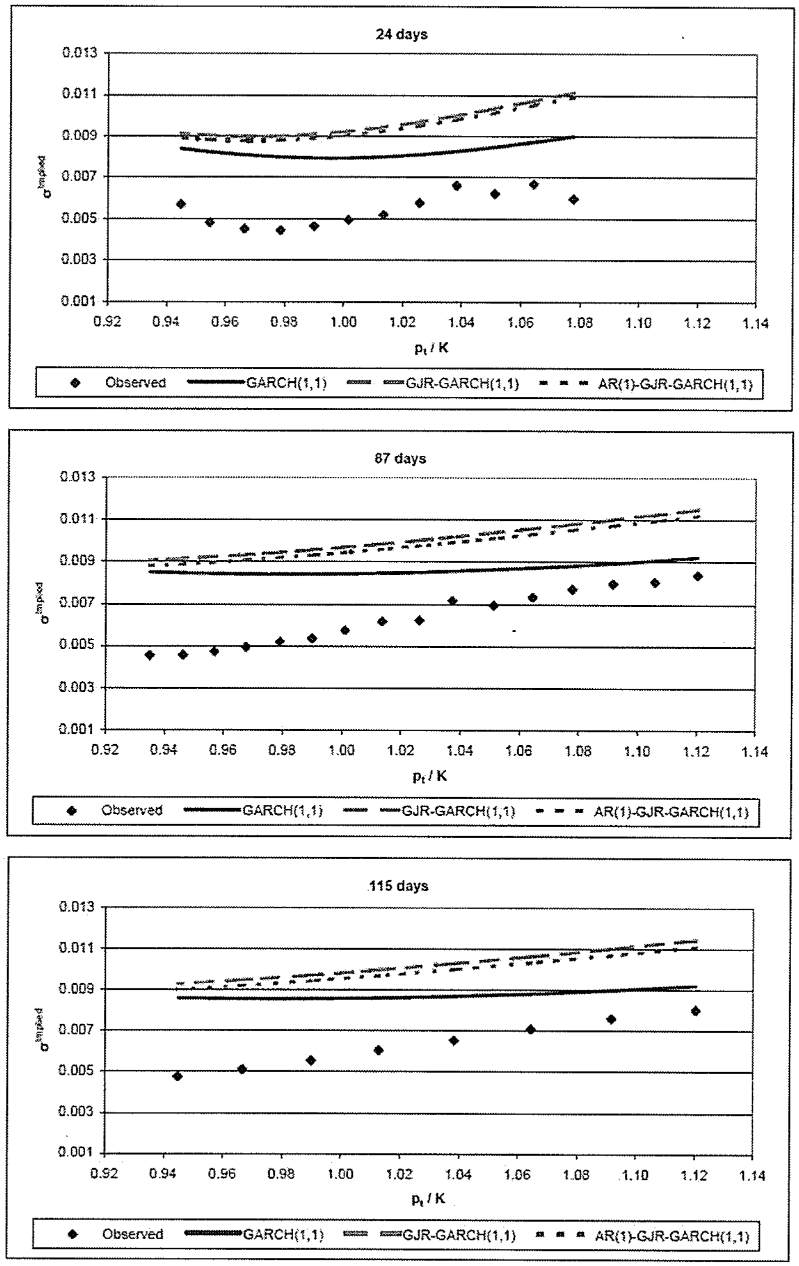


Figure 30: Observed and simulated volatility smiles for the S&P 100 daily index on October 27, 1993.

Observed						GARCH(1,1)		GJR-GARCH(1,1)		AR(1)-GJR-GARCH(1,1)		Heston	
K	p_t	p_t / K	T-t (weeks)	C_t	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$	$\hat{C}_t$	$\sigma^{implied}$
420	432.39	1.03	4	17.25	0.028942	15.1160	0.021506	15.3031	0.022192	17.0188	0.028168	14.3413	0.018536
425	432.38	1.02	4	13.25	0.026573	11.4939	0.021087	11.6871	0.021699	13.5565	0.027516	10.5772	0.018134
430	432.34	1.01	4	9.625	0.024431	8.40053	0.020845	8.57900	0.021368	10.5284	0.027072	7.3596	0.017788
435	432.38	0.99	4	6.4375	0.022188	5.95650	0.020783	6.10002	0.021202	8.03823	0.026850	4.8437	0.017526
440	432.39	0.98	4	3.875	0.020216	4.09283	0.020901	4.18919	0.021203	6.02770	0.026853	2.9822	0.017357
445	432.55	0.97	4	2.125	0.018721	2.78845	0.021164	2.83930	0.021346	4.51771	0.027054	1.7573	0.017287
450	432.38	0.96	4	0.9375	0.017291	1.82748	0.021595	1.83988	0.021649	3.30427	0.027464	0.9405	0.017307
425	431.46	1.02	8	15.25	0.024303	13.6088	0.020808	13.8878	0.021404	16.5893	0.027137	11.3356	0.015895
430	431.46	1.00	8	11.5	0.022135	10.8000	0.020692	11.0365	0.021179	13.7837	0.026842	8.3189	0.015574
435	431.08	0.99	8	8.5	0.021172	8.25347	0.020661	8.42720	0.021021	11.1469	0.026642	5.6881	0.015324
440	431.08	0.98	8	6.125	0.020264	6.33881	0.020728	6.44656	0.020961	9.07522	0.026571	3.8373	0.015194
445	431.08	0.97	8	4.1875	0.019392	4.81361	0.020874	4.85818	0.020978	7.34227	0.026609	2.5011	0.015163
					h_1	2.327286E-04		2.296489E-04		2.205753E-04		2.216E-02	

Figure 31: Observed and simulated call prices and its respective implied volatilities for the S&P 500 weekly index on February 17, 1993.

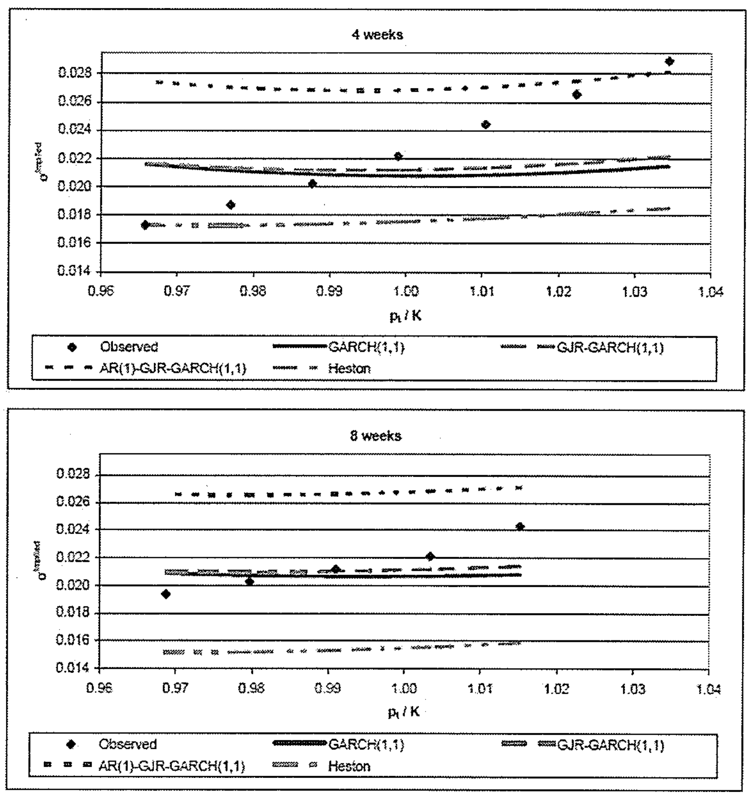


Figure 32: Observed and simulated volatility smiles for the S&P 500 weekly index on February 17, 1993.

6 Appendices

6.1 Appendix I: ψ-weights for a stationary ARMA(p,q) process

Any stationary ARMA(p,q) process can be written as

$$\begin{aligned} & Z_t - \mu = \phi_1(Z_{t-1}-\mu) + \ldots + \phi_p(Z_{t-p}-\mu) + a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q} \\ \Leftrightarrow \quad & \Phi(B)(Z_t-\mu) = \Theta(B)a_t \\ \Leftrightarrow \quad & Z_t - \mu = \Psi(B)a_t \end{aligned}$$

where a_t is a white noise process and $\Psi(B) = \dfrac{\Theta(B)}{\Phi(B)}$ (or equivalently $1+\psi_1 B+\psi_2 B^2+\ldots = \dfrac{1-\theta_1 B-\theta_2 B^2-\ldots-\theta_q B^q}{1-\phi_1 B-\phi_2 B^2-\ldots-\phi_p B^p}$). Equating coeeficients of B^j in $\Psi(B) = \dfrac{\Theta(B)}{\Phi(B)}$ will then yield the ψ-weights.

As an example, consider an ARMA(1,1) process $(1-\phi_1 B)(Z_t-\mu) = (1-\theta_1 B)a_t$. Equating coeeficients of B^j, we have

$$\begin{aligned} & 1+\psi_1 B+\psi_2 B^2+\ldots = \frac{1-\theta_1 B}{1-\phi_1 B} \\ \Leftrightarrow \quad & (1+\psi_1 B+\psi_2 B^2+\ldots)(1-\phi_1 B) = 1-\theta_1 B \\ \Leftrightarrow \quad & 1+(\psi_1-\phi_1)B+(\psi_2-\phi_1\psi_1)B^2+(\psi_3-\phi_1\psi_2)B^3+\ldots = 1-\theta_1 B, \end{aligned}$$

will reveal the ψ-weights as

$$\begin{aligned} B\colon\ & \psi_1-\phi_1 = -\theta_1 && \Longrightarrow \psi_1 = \phi_1-\theta_1 \\ B^2\colon\ & \psi_2-\phi_1\psi_1 = 0 && \Longrightarrow \psi_2 = \phi_1\psi_1 = \phi_1(\phi_1-\theta_1) \\ & \vdots && \quad \vdots \\ B^j\colon\ & \psi_j-\phi_1\psi_{j-1} = 0 && \Longrightarrow \psi_j = \phi_1\psi_{j-1} = \phi_1^{j-1}(\phi_1-\theta_1) \text{ for } j>0. \end{aligned}$$

6.2 Appendix II: Alternative proof of Corollary 2.1

First, since $y_t^2 = h_t Z_t^2$, we have that

$$E(y_t^2) = E(h_t)E(Z_t^2) = E(h_t).$$

The mean of h_t and h_t^2 are given by

$$\begin{aligned} E(h_t) &= E(\omega + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1}) \\ &= \omega + \alpha_1 E(y_{t-1}^2) + \beta_1 E(h_{t-1}) \\ &= \omega + (\alpha_1 + \beta_1)E(h_{t-1}), \end{aligned}$$

so that

$$\begin{aligned} E(h_t) - (\alpha_1 + \beta_1)E(h_{t-1}) &= \omega \\ \Leftrightarrow \qquad E(h_t) &= \frac{\omega}{1 - (\alpha_1 + \beta_1)}. \end{aligned}$$

For h_t^2, we have instead

$$\begin{aligned} E(h_t^2) &= E[(\omega + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1})^2] \\ &= \omega^2 + 2\omega\alpha_1 E(y_{t-1}^2) + 2\omega\beta_1 E(h_{t-1}) + \alpha_1^2 E(y_{t-1}^4) \\ &\quad + 2\alpha_1\beta_1 E(y_{t-1}^2 h_{t-1}) + \beta_1^2 E(h_{t-1}^2) \\ &= \omega^2 + 2\omega\alpha_1 E(h_{t-1}) + 2\omega\beta_1 E(h_{t-1}) + 3\alpha_1^2 E(h_{t-1}^2) \\ &\quad + 2\alpha_1\beta_1 E(h_{t-1}^2) + \beta_1^2 E(h_{t-1}^2), \end{aligned}$$

so that

$$E(h_t^2) - (3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2)E(h_{t-1}^2) = \omega^2 + 2\omega(\alpha_1 + \beta_1)E(h_{t-1}),$$

and, finally

$$\begin{aligned} E(h_t^2) &= \frac{\omega^2 + 2\omega(\alpha_1+\beta_1)\left[\dfrac{\omega}{1-(\alpha_1+\beta_1)}\right]}{1-(3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2)} \\ &= \frac{\omega^2(1+\alpha_1+\beta_1)}{[1-3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2)][1-(\alpha_1+\beta_1)]}. \end{aligned}$$

Hence, the kurtosis is

$$\begin{aligned} K^{(y)} &= \frac{E[(y_t - E(y_t))^4]}{[Var(y_t)]^2} = \frac{E(y_t^4)}{[E(y_t^2)]^2} = \frac{E(h_t^2)E(Z_t^4)}{[E(h_t)E(Z_t^2)]^2} = \frac{3E(h_t^2)}{[E(h_t)]^2} \\ &= \frac{\dfrac{3\omega^2(1+\alpha_1+\beta_1)}{[1-3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2)][1-(\alpha_1+\beta_1)]}}{\left[\dfrac{\omega}{1-(\alpha_1+\beta_1)}\right]^2} \\ &= \frac{3[1+\alpha_1+\beta_1][1-(\alpha_1+\beta_1)]}{1-3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2} \\ &= \frac{3(1-\phi_1^2)}{1-\phi_1^2 - 2\alpha_1^2}, \end{aligned}$$

where $\phi_1 = \alpha_1 + \beta_1$.

6.3 Appendix III: Lognormal asset pricing

Definition 2. *X is said to have a lognormal distribution with parameters μ and σ, if $\ln(X)$ has a normal distribution with mean μ and variance σ^2. In other words, X has the distribution of $e^{\mu+\sigma Z}$, where Z is a standard normal random variable.*

Let X be a lognormal variable with mean μ and variance σ^2, Z be a standard normal variable with cumulative distribution $N(z)$, and K be a constant. The following properties applies:

1. $E(X^n) = E(e^{n(\mu+\sigma Z)}) = e^{n\mu+(n\sigma)^2/2}$

2. $Var(X) = Var(e^{\mu+\sigma Z}) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

3. $E[\max(X - K, 0)] = E[\max(e^{\mu+\sigma Z} - K, 0)] = e^{\mu+\sigma^2/2}N(d) - KN(d-\sigma)$ where $d = \dfrac{\ln(\frac{1}{K}) + \mu + \sigma^2}{\sigma}$.

Proof. Since $X = e^{\mu+\sigma Z}$ and the moment generating function of Z is $M_Z(t) = E(e^{tZ}) = e^{t^2/2}$, the nth moment about the origin for X is

$$
\begin{aligned}
E(X^n) &= E(e^{n(\mu+\sigma Z)}) = e^{n\mu}E(e^{n\sigma Z}) \\
&= e^{n\mu}M_Z(n\sigma) = e^{n\mu}e^{(n\sigma)^2/2} \\
&= e^{n\mu+(n\sigma)^2/2},
\end{aligned}
$$

and the variance for X is

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 = e^{2\mu+2\sigma^2} - (e^{\mu+\sigma^2/2})^2 \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\
&= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).
\end{aligned}
$$

Now let A denote the event that $X = e^{\mu+\sigma Z} \geq K$. Therefore, we can write

$$
E[\max(e^{\mu+\sigma Z} - K, 0)] = E((e^{\mu+\sigma Z} - K)I_A) = E(e^{\mu+\sigma Z}I_A) - KE(I_A).
$$

However, we can calculate

$$
\begin{aligned}
E(e^{\mu+\sigma Z} I_A) &= \int_A e^{\mu+\sigma Z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \int_A e^{\mu+\sigma^2/2} \frac{1}{\sqrt{2\pi}} e^{-(z-\sigma)^2/2} dz \\
&= e^{\mu+\sigma^2/2} N\left(\frac{\ln(\frac{1}{K}) + \mu}{\sigma} + \sigma\right) = e^{\mu+\sigma^2/2} N(d),
\end{aligned}
$$

and

$$
\begin{aligned}
E(I_A) &= P(e^{\mu+\sigma Z} \geq K) \\
&= P(Z \geq \frac{\ln(K) - \mu}{\sigma}) \\
&= N\left(\frac{\ln(\frac{1}{K}) + \mu}{\sigma}\right) = N(d - \sigma),
\end{aligned}
$$

with $d = \dfrac{\ln(\frac{1}{K}) + \mu + \sigma^2}{\sigma}$. □

6.4 Appendix IV: SAS estimation issue

One possibility as to why SAS fails to obtain parameter estimates could be due to its inability to apply or recognize appropriate constraints at various local maximum points. To illustrate this issue, see the following example:

- Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables from $N(\mu, \sigma^2)$ where $\mu = \sigma^2 = \theta$.
- Consider the estimation of θ with the sample mean $\overline{X}$ and sample variance S^2.
- Using results from Example 2.8 from Shao (2003)

$$
\sqrt{n}(\bar{X} - \theta, S^2 - \theta) \xrightarrow{d} N(0, \Sigma)
$$

where

$$\Sigma = \begin{pmatrix} \theta & \mu_3 \\ \mu_3 & \mu_4 - \theta^2 \end{pmatrix}$$

and $\mu_k = E[(X-\theta)^k]$, $k = 3, 4$.

Since $X_i \overset{i.i.d.}{\sim} N(\theta, \theta)$, thus $\mu_3 = 0$ and $\mu_4 - \theta^2 = \theta(3-\theta)$. Hence $\hat{\theta} = \overline{X}$ is a better estimate when $0 \leq \theta < 2$, and $\hat{\theta} = S^2$ when $2 < \theta \leq 3$.

6.5 Appendix V: Estimating the standard errors

Aside from obtaining the parameter estimates by maximizing the log likelihood function, it is also of interest to estimate the standard errors of the parameter estimates. Here, Θ denotes the p parameters of the model so that $\Theta = (\theta_1, \ldots, \theta_p)'$ is a p x 1 vector; Θ_0 denotes the true value of Θ; the conditional mean and variance, μ_t and h_t, of the return y_t are known at time $t-1$ from information Y_{t-1} and are assumed to be differentiable; and the standardized residual $Z_t = \dfrac{y_t - \mu_t}{\sqrt{h_t}}$ are i.i.d. observations from a distribution whose density function is $f(Z|\Theta)$.

The MLE, denoted by $\hat{\Theta}$, maximizes $\ln L(\Theta)$ from n observations by solving the p equations

$$\sum_{t=1}^{n} s_t(\theta_i) = 0, \;\; 1 \leq i \leq p$$

where $s_t(\Theta)$ is known as the score vector (p x 1) derived from the partial derivatives of the logarithms of the conditional densities $l_t(\Theta)$, that is

$$l_t(\Theta) = \ln[f(y_t|Y_{t-1}, \Theta)] = -\frac{1}{2}\ln[h_t(\Theta)] + \ln[f(Z_t(\Theta))],$$

where

$$l_t(\Theta) = \ln[f(y_t|Y_{t-1}, \Theta)] = -\frac{1}{2}\ln[h_t(\Theta)] + \ln[f(Z_t(\Theta))].$$

Suppose that of the p parameters, the first m appear in μ_t and h_t, with the remaining parameters (if any) defining the density function $f(Z|\Theta)$. The general formula for the first m terms in the score vector is

$$s_t(\theta_i) = \frac{a(Z_t)Z_t}{\sqrt{h_t}}\left(\frac{d\mu_t}{d\theta_i}\right) + \frac{a(Z_t)Z_t^2 - 1}{2h_t}\left(\frac{dh_t}{d\theta_i}\right), 1 \le i \le m$$

with the function $a(.)$ determined by the density function of the standardized residuals Z_t. For example, when Z_t is normal, $a(Z_t) = 1$ and when Z_t is the standardized t-distribution with ν degrees-of-freedom, $a(Z_t) = \dfrac{\nu+1}{\nu - 2 + Z_t^2}$.

The analytic standard errors for non-normal conditional distributions can be calculated by estimating A_0 (or B_0 in place of A_0) of

$$\sqrt{n}(\hat{\Theta} - \Theta_0) \xrightarrow{D} N(0, A_0^{-1}),$$

and by estimating $A_0^{-1}B_0A_0^{-1}$ for normal distributions from the asymptotic result of

$$\sqrt{n}(\hat{\Theta} - \Theta_0) \overset{a}{\sim} N(0, A_0^{-1}B_0A_0^{-1})$$

where the elements in the p x p matrices of A_0 and B_0 can be estimated by

$$\hat{A}_{i,j} = -\frac{1}{n}\sum_{t=1}^{n}\frac{d^2 l_t(\Theta)}{d\theta_i d\theta_j} = -\frac{1}{n}\left(\frac{d^2 \ln L(\Theta)}{d\theta_i d\theta_j}\right)$$

$$\hat{B}_{i,j} = -\frac{1}{n}\sum_{t=1}^{n} s_t(\theta_i)s_t(\theta_j).$$

The following is an alternative estimate of A_0, given by Bollerslev and Wooldridge (1992), that can be used in place of the above to avoid second-order derivatives:

$$\hat{A}_{i,j}^{BW} = \frac{1}{n}\sum_{t=1}^{n}\left[\frac{1}{h_t}\left(\frac{d\mu_t}{d\theta_i}\right)\left(\frac{d\mu_t}{d\theta_j}\right) + \frac{1}{2h_t^2}\left(\frac{dh_t}{d\theta_i}\right)\left(\frac{dh_t}{d\theta_j}\right)\right].$$

For illustration, consider the GJR-MA(1)-GARCH(1,1)-M model defined and estimated previously with residuals e_{t-1}. When Z_t is normal and $\Theta = (r, \lambda, \theta, \omega, \alpha_1, \alpha_1^*, \beta_1)'$, $m = p = 7$, the recursive formulae for the partial derivatives are

$$\frac{d\mu_t}{d\Theta} = (1, \sqrt{h_t}, e_{t-1}, 0, 0, 0, 0)' - \theta\frac{d\mu_{t-1}}{d\Theta} + \frac{\lambda}{2\sqrt{h_t}}\frac{dh_t}{d\Theta} \tag{6.1}$$

$$\frac{dh_t}{d\Theta} = (0, 0, 0, 1, e_{t-1}^2, I_{t-1}e_{t-1}^2, h_{t-1})' - 2(\alpha_1 + \alpha_1^* I_{t-1})e_{t-1}\frac{d\mu_{t-1}}{d\Theta} + \beta_1\frac{dh_{t-1}}{d\Theta}. \tag{6.2}$$

When Z_t is t-distributed, an additional parameter ν defines the density of the standardized residuals. The equations (6.1) and (6.2) will suffice to define the first seven terms of the vector as the eighth term is zero. However, the final term in the score vector is

$$\frac{dl_t(\Theta)}{d\nu} = \frac{d}{d\nu}[\ln c(\nu)] - \frac{\ln(x_t)}{2} + \frac{(\nu+1)Z_t^2}{2x_t(\nu-2)^2},$$

where

$$c(\nu) = \frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\sqrt{\pi(\nu-2)}}$$

and

$$x_t = 1 + \frac{Z_t^2}{\nu-2}.$$

6.6 Appendix VI: Heston's option pricing formula

Heston (1993) provided a closed-form stochastic volatility option pricing formula that can be evaluated rapidly. The fair price of a call option under Heston's formulation is

$$C_t^H = Se^{-qT}P_1 - Ke^{-rT}P_2,$$

where S is the initial asset price, T is the expiry time, K is the exercise price, q is the dividend yield and r is the risk-free interest rate.

The terms P_1 and P_2 are two calculations of the probability that $Y_T = \ln(S_T)$ exceeds $\ln(K)$ when the state vector $(Y_t, V_t)'$ has initial value $(Y_0, V_0)'$ and dynamics

$$\begin{aligned} Y_t &= \ln(S_t), \\ V_t &= \sigma_t^2, \\ dY &= (R + uV)dt + \sqrt{V}dW, \\ dV &= (a - cV)dt + \epsilon\sqrt{V}dZ, \end{aligned}$$

where a, c, R, u and ϵ are parameters and the correlation between dW and dZ is ρ. The term P_2 is the probability obtained for the risk-neutral measure $\mathbf{Q}$ when $R = r - q$, $u = -0.5$, and $c = b$, which gives the price dynamics

$$\begin{aligned} dY &\overset{\mathbf{Q}}{=} (r - q - 0.5V)dt + \sqrt{V}d\tilde{W}, \\ dV &\overset{\mathbf{Q}}{=} (a - bV)dt + \epsilon\sqrt{V}d\tilde{Z}, \end{aligned}$$

while P_1 is the probability obtained for measure $\mathbf{Q}^*$ when $R = r - q$, $u = 0.5$, and

$c = b - \rho\epsilon$, which gives

$$dY \overset{\mathbf{Q^*}}{=} (r - q + 0.5V)dt + \sqrt{V}d\tilde{W},$$
$$dV \overset{\mathbf{Q^*}}{=} (a - [b - \rho\epsilon]V)dt + \epsilon\sqrt{V}d\tilde{Z}.$$

The conditional probability that Y_T exceeds $\ln(K)$ is given by a standard inversion formula:

$$P(Y_T \geq \ln(K)|Y_0, V_0) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\varphi\ln(K)}g(\varphi)}{i\varphi}\right]d\varphi,$$

with $i = \sqrt{-1}$ and $Re[.]$ representing the real part of a complex number (see Kendall, Stuart, and Ord (1987)). Defined for all real numbers φ, probabilities obtained from the conditional characteristic function of Y_T, denoted by $g(\varphi)$, follows

$$g(\varphi) = e^{C + DV_0 + i\varphi Y_0},$$

where

$$C = RT\varphi i + a\epsilon^{-2}\left[hT - 2\ln\left(\frac{1 - ke^{wT}}{1 - k}\right)\right],$$
$$D = \frac{h(1 - e^{wT})}{\epsilon^2(1 - ke^{wT})},$$
$$w = \sqrt{(\rho\epsilon\varphi i - c)^2 - \epsilon^2(2u\varphi i - \varphi^2)},$$
$$h = c - \rho\epsilon\varphi i + w,$$
$$k = \frac{h}{h - 2w}.$$

6.7 Appendix VII: SAS codes

```
/*********************************************/
/* SIMULATION OF A NORMAL GARCH(1,1) PROCESS */
/*********************************************/

%LET start = 2.5;
%LET num_obs = 2000;
%LET omega = 0.1;
%LET alpha1 = 0.5;
%LET beta1 = 0.25;

DATA NormalGarch_1_1;

DO sample=1 TO 10;
      lag_y = &start;
      lag_h = &start;

      DO    t = -500 TO &num_obs;
            h = &omega + &alpha1 * lag_y**2 + &beta1 * lag_h;
            y = SQRT(h) * RANNOR(12345);
            y2 = y**2;
            lag_h = h;
            lag_y = y;

            IF t>0 THEN OUTPUT;
      END;
END;
RUN;

ODS PDF FILE="D:\NormalGARCH(1,1).pdf";
ODS GRAPHICS ON;

PROC UNIVARIATE DATA=NormalGarch_1_1;
 VAR y;
 BY sample;
RUN;

PROC AUTOREG DATA=NormalGarch_1_1;
      MODEL y = / GARCH=(P=1,Q=1);
      BY sample;
RUN;
QUIT;

ODS GRAPHICS OFF;
ODS PDF CLOSE;
```

```
/*****************/
/* INPUT DATASET */
/*****************/
PROC IMPORT DATAFILE='D:\SP100.txt' OUT=SP100;
   GETNAMES=YES;
RUN;

/****************************************************/
/* FIT NORMAL GARCH(1,1) TO GET PARAMETER ESTIMATES */
/****************************************************/
PROC AUTOREG DATA=SP100;
   MODEL Return = / GARCH=(P=1,Q=1) NOINT;
RUN;

/**********************************************/
/* FIT GARCH(1,1)-M TO GET PARAMETER ESTIMATES */
/**********************************************/
PROC AUTOREG DATA=SP100;
   MODEL Return = / GARCH=(P=1,Q=1, MEAN=SQRT);
RUN;

/***********************************************/
/* FIT t GARCH(1,1) TO GET PARAMETER ESTIMATES */
/***********************************************/
PROC MODEL DATA=SP100;
   PARMS omega .01 alpha1 .01 beta .9 df 6;

   /* Mean model */
   Return = r;
   /* Variance model */
   H.Return = omega + alpha1*XLAG(RESID.Return**2,MSE.Return)
                + beta*XLAG(H.Return,MSE.Return);
   /* Error distribution */
      ERRORMODEL Return ~ t(H.Return, df);
   /* Fit the model */
   FIT Return / METHOD = MARQUARDT FIML;
RUN;

/**********************************************/
/* FIT GJR-GARCH(1,1) TO GET PARAMETER ESTIMATES */
/**********************************************/
PROC MODEL DATA=SP100;
   PARMS omega .01 alpha1 .01 alpha2 .01 beta .9;

   /* Mean model */
   Return = r;
   /* Variance model */
   IF ZLAG(RESID.Return) > 0 THEN
      H.Return = omega + alpha1*XLAG(RESID.Return**2,MSE.Return)
                 + beta*XLAG(H.Return,MSE.Return);
   ELSE
      H.Return = omega + (alpha1 +
                 alpha2)*XLAG(RESID.Return**2,MSE.Return) +
                 beta*XLAG(H.Return,MSE.Return);
   /* Fit the model */
   FIT Return / METHOD = MARQUARDT FIML;
RUN;
QUIT;
```

```
/***************************************************/
/* NORMAL GARCH(1,1) CALL OPTION & DELTA SIMULATION */
/***************************************************/
%LET nobs = 500000;       /* # of simulations */
%LET T = 4;               /* time-to-maturity */
%LET r = 0;               /* One-period risk-free rate */
%LET p = 0.8;             /* Initial asset price */
%LET K = 1;               /* Strike price */
%LET omega = 0.000016626;
%LET alpha = 0.120538286;
%LET beta = 0.844190832;
%LET lambda = 0.126088592;

DATA sim;
   value = 0; S = 0; callSum = 0; callPrice = 0; deltaSum=0; delta=0;
   deltaPrice=0;

   DO j = 1 TO &nobs;
      z = 0; lz = 0; h = 0; y = 0; sum = 0; info=0;
      lh = 0.64*(&omega/(1-&alpha-&beta));

      /* GARCH(1,1) under pricing measure Q */
      DO i= -100 TO &T;
         z = rannor(696336);
         h = &omega + (&alpha*(lz - &lambda)**2 + &beta)*lh;
         y = &r - 0.5*h + sqrt(h)*z;
         lz = z; lh = h;
            IF i > 0 THEN sum = sum + y;
            IF i > 0 THEN OUTPUT;
      END;

      S = &p*exp(sum);  /* Asset price at expiry */

      /* Calculates the call price */
      value = max(0, S - &K);
      callSum = callSum + value;
      callPrice = exp(-&r*&T)*callSum/&nobs;

      /* Calculates the delta */
      IF S >= &K THEN info = 1;
      delta = info*S/&p;
      deltaSum = deltaSum + delta;
      deltaPrice = exp(-&r*&T)*deltaSum/&nobs;
   END;
RUN;

DATA results;
   SET sim; IF j=&nobs && i=&T;
RUN;

PROC PRINT DATA=results;
   VAR callPrice deltaPrice;
RUN;
QUIT;
```

```
/**************************************************/
/* GJR-GARCH(1,1) CALL OPTION & DELTA SIMULATION  */
/**************************************************/
%LET nobs = 500000;      /* # of simulations */
%LET T = 4;              /* time-to-maturity */
%LET r = 0;              /* One-period risk-free rate */
%LET p = 0.8;            /* Initial asset price */
%LET K = 1;              /* Strike price */
%LET omega = 0.000021365;
%LET alpha = 0.067038568;
%LET alpha_star = 0.090386339;
%LET beta = 0.837095581;
%LET lambda = 0.108192440;

DATA sim;
   value = 0; S = 0; callSum = 0; callPrice = 0; deltaSum=0; delta=0;
   deltaPrice=0;

   DO j = 1 TO &nobs;
      z = 0; lz = 0; h = 0; y = 0; sum = 0; info=0;
      lh = 0.64*(&omega/(1-&alpha-0.5*&alpha_star-&beta));

      /* GJR-GARCH(1,1) under pricing measure Q */
      DO i= -100 TO &T;
         z = rannor(696336);
         h = &omega + (&alpha*(lz - &lambda)**2
             + &alpha_star*(max(0, -lz + &lambda))**2 + &beta)*lh;
         y = &r - 0.5*h + sqrt(h)*z;
         lz = z; lh = h;
            IF i > 0 THEN sum = sum + y;
            IF i > 0 THEN OUTPUT;
      END;

      S = &p*exp(sum);  /* Asset price at expiry */

      /* Calculates the call price */
      value = max(0, S - &K);
      callSum = callSum + value;
      callPrice = exp(-&r*&T)*callSum/&nobs;

      /* Calculates the delta */
      IF S >= &K THEN info = 1;
      delta = info*S/&p;
      deltaSum = deltaSum + delta;
      deltaPrice = exp(-&r*&T)*deltaSum/&nobs;
   END;
RUN;

DATA results;
   SET sim; IF j=&nobs && i=&T;
RUN;

PROC PRINT DATA=results;
   VAR callPrice deltaPrice;
RUN;
QUIT;
```

```
/***********************************************************/
/* AR(1)-GJR-GARCH(1,1) CALL OPTION & DELTA SIMULATION   */
/***********************************************************/
%LET nobs = 500000;       /* # of simulations */
%LET T = 4;               /* time-to-maturity */
%LET r = 0;               /* One-period risk-free rate */
%LET p = 0.8;             /* Initial asset price */
%LET K = 1;               /* Strike price */
%let v = 0.002069294;
%let xi = -0.078682487;         /* AR parameter */
%let omega = 0.000019279;
%let alpha = 0.067850576;
%let alpha_star = 0.096314808;
%let beta = 0.839995117;

DATA sim;
   value = 0; S = 0; callProd = 0; callPrice = 0; deltaSum=0; delta=0;
   deltaPrice=0;

   DO j = 1 TO &nobs;
      z = 0; lz = 0; h = 0; y = 0; prod = 1; info=0;
      lh = 0.64*(&omega/((1-&alpha-0.5*&alpha_star-&beta)*(1-&xi**2)));
      lambda=0; llambda=(&v - &r)/sqrt(lh);

      /* AR(1)-GJR-GARCH(1,1) under pricing measure Q */
      DO i= -100 TO &T;
         z = rannor(696336);
         h = &omega + (&alpha*(lz - llambda)**2
             + &alpha_star*(max(0, -lz + llambda))**2 + &beta)*lh;
         y = &r + sqrt(h)*z;
         lambda = (&v + &xi*y - &r)/sqrt(h);
         lz = z; lh = h; llambda = lambda;
            IF i > 0 THEN prod = prod*(1+y);
            IF i > 0 THEN OUTPUT;
      END;

      S = &p*prod;       /* Asset price at expiry */

      /* Calculates the call price */
      value = max(0, S - &K);
      callProd = callProd + value;
      callPrice = (1+&r)**(-&T)*callProd/&nobs;

      /* Calculates the delta */
      IF S >= &K THEN info = 1;
      delta = info*S/&p;
      deltaSum = deltaSum + delta;
      deltaPrice = (1+&r)**(-&T)*deltaSum/&nobs;
   END;
RUN;

DATA results;
   SET sim; IF j=&nobs && i=&T;
RUN;

PROC PRINT DATA=results;
   VAR callPrice deltaPrice;
RUN;
QUIT;
```

References

[1] Abraham, B. (1983). *Statistical Methods for Forecasting.* John Wiley & Sons, U.S.A.

[2] Appadoo, S.S., Ghahramani, M. and Thavaneswaran, A. (2005). *Moment Properties of Some Time Series Models.* Math. Scientist 30, 50-63.

[3] Appadoo, S.S., Thavaneswaran, A. and Singh, J. (2006). *Random Coeffecient GARCH Models.* Applied Mathematics Letters 19, 824-829.

[4] Bollerslev, T. (1986). *Generalized Autoregressive Conditional Heteroskedasticity.* Journal of Econometrics 31, 307-327.

[5] Bollerslev, T., Engle R.F. and Nelson, D.B. (1994). *ARCH Models.* Handbook of Econometrics 4, 2959-3038.

[6] Bollerslev, T. and Woolridge, J.M. (1992). *Quasi-Maximum Likelihood Estimation and Inference In Dynamic Models with Time Varying Covariances.* Econometric Reviews 11, 143-172.

[7] Duan, J. (1995). *The GARCH Option Pricing Model.* Mathematical Finance 5, 13-32.

[8] Duan, J., Gauthier, G., Sasseville, C. and Simonato, J. (2006). *Approximating the GJR-GARCH and EGARCH Option Pricing Models Analytically.* Journal of Computational Finance 9 (3).

[9] Engle, R.F. (1982). *Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation.* Econometrica 50, 987-1008.

[10] Eraker, B. (2004). *Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices.*. Journal of Finance 59 (3), 1367-1404.

[11] Ghahramani, M. and Thavaneswaran, A. (2007). *Identification of ARMA Models with GARCH Errors.* Math. Scientist 32, 60-69.

[12] Glosten, L.R., Jagannathan, R. and Runkle, D. (1993). *Relationship Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks.* Journal of Finance 48, 1779-1801.

[13] Gouriéroux, C. (1997). *ARCH Models and Financial Applications.* Springer, New York.

[14] Hafner C.M. and Herwartz, H. (2001). *Option Pricing Under Linear Autoregressive Dynamics, Heteroskedasticity, and Conditional Leptokurtosis.* Journal of Empirical Finance 8, 1-34.

[15] Heston, S.L. (1993). *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.* Review of Financial Studies 6, 327-343.

[16] Hull, J.C. (2006). *Options, Futures, and Other Derivatives.* 6th edition. Prentice Hall, New Jersey.

[17] Jacquier, E., Polson, N.G. and Rossi, P.E. (1994). *Bayesian Analysis of Stochastic Volatility Models.* Journal of Business & Economic Statistics 12, 371-389.

[18] Kendall, M.G., Stuart, A. and Ord, J.K. (1983). *The Advanced Theory of Statistics, Vol. 3.* 4th edition. Charles Griffin, London.

[19] Ljung, G.M. and Box, G.E.P. (1978). *On a Measure of Lack of Fit in Time Series Models.* Biometrika 65, 297-303.

[20] Nelson, D.B. (1991). *Conditional Heteroskedasticity In Asset Returns: A New Approach.* Econometrica 59, 347-370.

[21] Ruppert, D. (2004). *Statistics and Finance: An Introduction.* Springer, New York.

[22] Shao, J. (2003). *Mathematical statistics* 2nd edition. Springer-Verlag, New York.

[23] Taylor, Stephen J. (1986). *Modelling Financial Time Series.* Wiley, Chichester.

[24] Taylor, Stephen J. (2005). *Asset Price Dynamics, Volatility, and Prediction.* Princeton University Press, New Jersey.

[25] Thavaneswaran, A., Appadoo, S.S. and Peiris, S. (2005a). *Forecasting Volatility.* Statistics & Probability Letters 75, 1-10.

[26] Thavaneswaran, A., Appadoo, S.S. and Samanta, M. (2005b). *Random Coeffecient GARCH Models.* Mathematical and Computing Modelling 41, 723-733.

[27] Tsay, Ruey S. (2005). *Analysis of Financial Time Series.* 2nd edition. John Wiley & Sons, U.S.A.

[28] Wei, William W.S. (2006). *Time Series Analysis: Univariate and Multivariate Methods.* 2nd edition. Addison-Wesley, U.S.A.