M. A. Thesis

Fundamental Mathematical Concepts in
Relation to Present High School Texts.

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PLAN:

Introduction:

General Aim and Survey.

Chapter 1. Algebra.

1. Introductory: - Fundamental Principles, Fundamental Operations, Fundamental Assumptions.
2. Negative Numbers.
3. Theory of Indices by defining meanings and showing that the fundamental laws hold.

Chapter 11. Geometry.

1. Definitions - Logical ideas, - Indefinables,- Some changes in the text.
2. Continuity Axiom - Illustrations of its use in the text.

References to such High School texts as Hall and Knight's and Crawford's Algebra will be made.

Introduction:

The subject will be treated in two chapters, the first Algebra, the second Geometry.

The purpose is to treat these subjects from a correct logical view-point, chiefly for the benefit of those students who continue the study of Mathematics in the University. It is not well to develop concepts of the mathematical elements which eventually must be rejected for new concepts more in accord with established or broadly accepted usage.

For all students it is well for disciplinary and cultural values that mathematical rigor be maintained. This does not mean that we may not allow students to take as axiomatic those propositions which they may accept correctly as intuitional or that they may not take for granted, for the time being, the facts of more difficult propositions, but it does mean that in the mind of the teacher as well as in the printed text, there must be correct mathematical concepts, and that the fundamental operations and definitions be stated keeping in view the necessity for statements that are in accord with well established thought and generally accepted ideas. For example the idea of "Continuity" should not be relegated entirely to the realm of Higher Mathematics. When the student comes to a fuller consideration of "Continuity" he should do so with some correct concepts of this principle. Mathematical rigor must hot be sacrificed even in the most elementary work.

Under the topic Algebra, the fundamental principles, operations and assumptions will be treated. The introduction of the simple operations by line segment illustrations and the justification for making certain assumptions, using examples as afforded by such subjects as Geometry and Vertors, will be introduced. The methods of handling negative numbers will be pointed out. Then the subject of "Indices" will be discussed from a new basis which seems to be a more natural basis than that commonly found in texts. The meaning of surd forms written in index form will be first defined and then the fact that the fundamental laws of Algebra hold will be established in each case.

The second chapter will treat of Geometry beginning with definitions and indefinables. Some changes in the text will be suggested in accord with the logical ideas of definitions laid-down. The continuity axioms will also be introduced and a survey of the text, Bakers Theoretical Geometry, will be made noting the sections where this axiom is actually inferred.

CHAPTER I

The first numbers used are integers or whole numbers or "numbers" as they are at first called, - 1, 2, 3, 4, etc. which arise from counting objects and answer the questions, - How many? They also represent the results of measurements when the magnitudes measured are exact multiples of the unit.

When the magnitude measured is not an exact multiple of the unit of measure, other numbers called fractions are used. These are presented as problems in division and are commensurable so long as both numerator and denominator are integers, that is, are multiples of some common unit.

Numbers at first are introduced as representing magnitudes only but later as having one of two opposite senses, i.e. signed numbers. These opposite senses are called positive and negative. The idea of signed numbers is presented by referring to losses and gains or to displacements in opposite directions. Gradually the objective fades out and the subjective i.e. the properties of numbers have meaning and become the definitions of what numbers are.

The first of the fundamental operations to be considered is addition. Any two magnitudes may be represented geometrically by the lengths of two line segments say a and b. To add the lengths a and b place them on a number scale end to end and beginning at the initial point X. Call the other end position Y. Then the length XY is defined as the sum of a and b. From this we may always find a sum. The process also shows the reason for the axiom that addition is to be commutative for it matters not which length be put first and in an analagous way would show why $(a+b)+c = a + (b+c)$ or that addition is associative.

A signed number however represents a direction or sense as well as a magnitude and is represented geometrically by a "directed" segment. Consider two signed numbers a and b whose lengths are the absolute values of a and b, written $|a|$ and $|b|$ and whose directions are the same or opposite signs. The sum $a+b$ is represented by a directed segment which expresses the net result of moving in the direction b through a distance represented by $|b|$. The following cases occur.

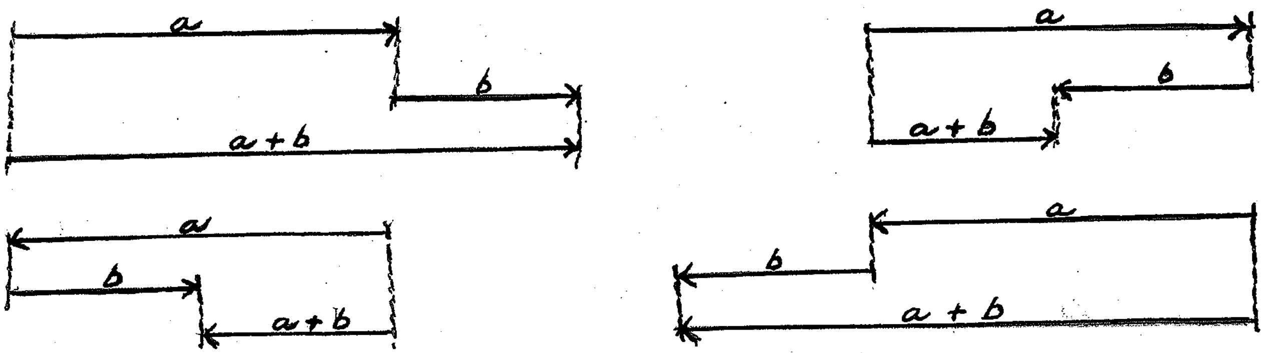


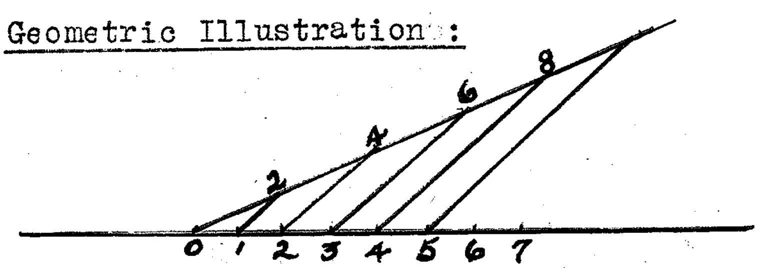
Illustration: In walking to a place four miles distant and back again the total distance walked is 10 miles but if the distance out be represented as $+5$ and the distance back as -5 the result 0 represents the distance measured from the starting point.

Multiplication is the second of the fundamental operations to be considered. The product ab of two signed numbers a and b is defined as follows:-

1. $|ab| = |a| \cdot |b|$
2. The sign ab is positive or negative according as the signs of a and b are the same or opposite.

$(+)(+) = (+)$, $(+)(-) = (-)$, $(-)(+) = (-)$, $(-)(-) = (+)$.

Geometric Illustration:

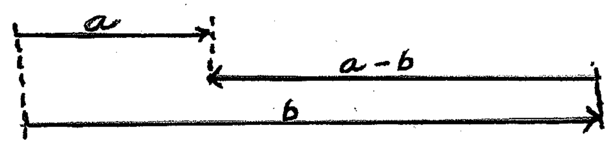


In this figure we are multiplying the figures of the scale by 2. Thus

Multiplication is equivalent to a uniform expansion of the scale away from the initial point or contraction toward the initial point according as the absolute value of the multiplier is greater or less than 1.

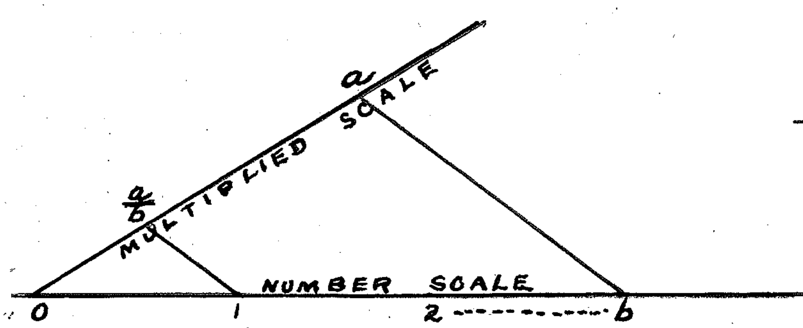
Subtraction is the third fundamental operation to deal with. ($a - b$) has the property $(a - b) + b = a$. Now this thing $(a - b)$ is found to have the same properties as positive integral numbers have whether $a < b$ or not. When $a < b$ this $(a - b)$ is called a negative number - more briefly to subtract a number b from a means to find a number x such that $x + b = a$. We then write $x = a-b$.

Geometric Representation



This diagram illustrates that $(a - b)$ is such a quantity that added to b will give a. We also see that to subtract a number b is equivalent to adding $-b$, and that therefore the present notion and the preceding notion of negative numbers are equivalent.

Division is the last of the fundamental operations to be considered. To divide a number a by a number b means to find a number x such that $bx = a$. We then write $x = \frac{a}{b}$



Geometric Illustration

Join b on the number scale to "a" on the multiplied scale.

Through 1 on number scale draw a line $\parallel$ ba. The point where this line meets the multiplied scale indicates the quotient $\frac{a}{b}$.

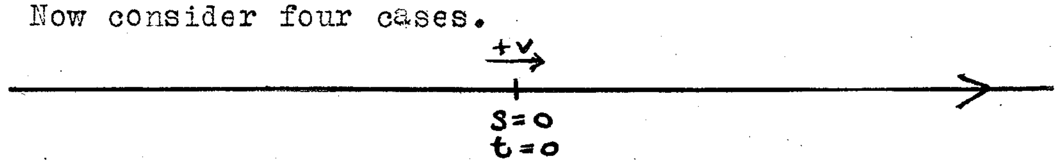
The Case: $b = o$.

Since $o \cdot x = o$ for every number x, it follows that the relation $o \cdot x = a$ cannot be satisfied by any value of x unless $a = o$ and then it is satisfied by all values of x. Hence by the definition of division the indicated quotient $x = \frac{a}{o}$ has no meaning whatever when $a \neq o$ and no definite value when $a = o$. Hence division by zero being either impossible or useless, is excluded from the legitimate operations of Algebra.

<u>An Illustration of the Law of Signs:</u>

(To show how it corresponds to concrete facts)

If a train moves at constant speed v miles per hour in t hours it travels a distance $s = vt$ miles. Here v,t,s, are unsigned numbers. At a given instant let the train be at a certain station o, let us count time from this instant ($t = o$) so that any +ve t denotes time after and -t time before ($t = o$), +s meaning the distance train to right and -s to left of o, +v meaning train moving to right -v to left. Now consider four cases.



1. v and t +ve — s is +ve as it should be.
2. v +ve, t -ve — s is -ve as it should be.
3. v -ve, t +ve — s is -ve as it should be.
4. v -ve t -ve — s is +ve as it should be.

At the basis of these operations and of all operations in Algebra are certain <u>Fundamental</u> <u>Assumptions</u> regarding positive integers which may be justified by out experience with a large number of special cases.

1. If we add a positive integer b to another positive integer a we always obtain a uniquely determined positive integer $c = a + b$ which is called the <u>sum</u> of a and b.
2. Addition is <u>commutative</u>: that is $a+b = b+a$.
3. Addition is <u>associative</u>: that is $a+(b+c) = (a+b)+c$.
4. Addition is <u>monotonic</u>: that is if $a > b$ then $a+c > b+c$.
5. If we multiply a positive integer a by another positive integer b we always obtain a uniquely determined positive integer

$c = a \times b = a \cdot b = ab$ which is called the product of a by b.

6. Multiplication is commutative: that is $a \times b = b \times a$.

7. Multiplication is associative: that is $a \times (b \times c) = (a \times b) \times c$.

8. Multiplication is monotonic: that is if $a > b$ then $a \times c$. $> b \times c$.

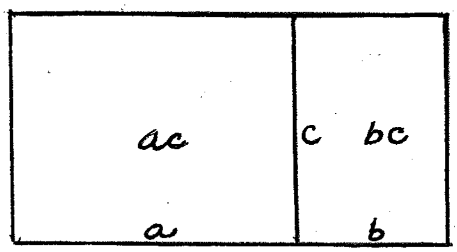
9. Multiplication is distributive: $c\ (a + b) = ca + cb$.

The laws of subtraction may be derived from the definition of subtraction and these fundamental laws. The law of factoring may be considered the reverse of (9). Moreover we showed in the previous section how these laws apply to all positive and negative rational numbers, by reference to line segments.

Since every point on a line cannot be represented by a rational number and we have illustrated these laws by line segments, they also apply to irrational numbers.

We must consider in this section why we know that these nine assumptions are true for all numbers. Their truth is not obvious but our beliaf in them is the result of experience.1. A little reasoning will show that these principles are not obvious. 2 X 3 means 3 + 3 and 3 X 2 means 2+2+2 the result being 6 in each case but this is only a particular case and not a general proof. 2. By Geometry we know that the product a b is represented as a rectangle a units one way and b units the other way. This proves to us intuitively that $ab = ba$ for the product of two positive real numbers but gives us no proof for other kinds of numbers.

Continuing the idea we get a conception of the product of three positive integers represented in a rectangular solid being commutative but not of any more factors as we cannot readily conceive of four or more dimensions.



3. By considering the adjacent figure we see geometrically the reason for $(a+b)\ c =$ ac and bc.

Euclid stated this law as follows: If there are two straight lines one of which is divided into any number of parts the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided straight line and the several parts of the divided straight line.

These proofs however depend upon the assumption that the area of a rectangle is found by multiplying the number of units in the length by the number of units in the width which corresponds to the product assumption in Algebra (5)

Thus we have not given a general conclusive proof of these assumptions. Moreover we can illustrate:

1. How the distributive law does not hold apart from our experience.

Suppose $+$ meant what we call multiplication.

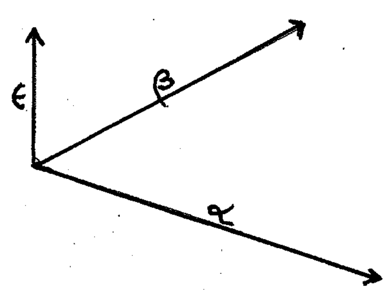
and $\times$ meant what we call addition.

Then $a\ (b+c)$ would be $a+bc$.

and $ab+ac$ would be $(a+b)(a+c)$

These results are not the same showing that with different definitions we could not use the laws.

2. How the commutative law does not hold in vector products in vector analysis.



The vector product of two vectors is defined as:

$\alpha \times \beta = \epsilon\, ab \sin(\alpha\beta)$ where ϵ is a perpendicular to the plane of $\alpha\beta$, directed as a screw would be turned from α to β. Now we see that $\beta \times \alpha$ would be $-\epsilon\, ab \sin(\alpha\beta)$.

$\therefore\ \alpha \times \beta = -\beta \times \alpha$.

Thus in vector analysis the commutative law does not hold.

References in this treatment of the subject are:

1. College Algebra   Wilcznyski and Slaught. Chapter 1.
   Secs. 2, 12, 18 etc.

2. Elementary Mathematical Analysis. Young and Morgan. Chap. 11.

3. Dr. Wilsons Notes on Vectors in Library.

The following are a few notes regarding some texts.

1. Definitions of Subtraction:

   1. C. Smith - the reverse operation to that of addition.

   2. Hall and Knight - No definition given.

   3. Crawford - Subtraction is the Inverse of addition - To subtract 4 from 7 is equivalent to finding the number which added to 4 will make 7.

   4. Schorling and Reeve - Introduces addition and subtraction by line segments i.e. geometric additions and subtraction of line segments - then states law for addition and subtraction for similar monomials.

11. Statements of Fundamental Assumptions:

Hall and Knight - states that they have established the Commutative Law and the Associative Laws for addition and Subtraction. Page 15.

Hall and Knight give as a definition of multiplication "an operation performed on one quantity which when performed on unity produces the other" and shows that $\frac{4}{5} \times \frac{3}{7} = \frac{3}{7} \times \frac{4}{5}$ following this by "the reasoning is clearly general and $\therefore a \times b = b \times a$. Thus the commutative law is established for Multiplication.

C. Smith has a similar treatment.

The Associative Law is just stated but the index law and distributive Law are stated to be established.

Crawford - Introduces the ideas of positive and negative numbers graphically, but does not anywhere state or even illustrate the Fundamental Assumptions except by one Geometrical illustration of the associative law for multiplication (Page 53).

Schorling and Reeve - Illustrates the commutative law first by Arithmetical examples resulting from equations and then by line segments (Page 38) He also introduces the commutative law for multiplication by rectangular areas (Page 85) and the distributive law is geometrically illustrated (Page 86)

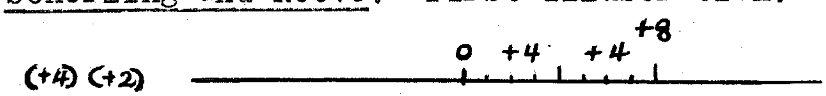
111. Method of Dealing with (-a) × (-b)

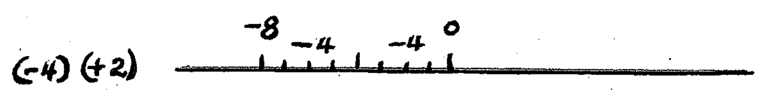
Hall and Knight (-3) × (-4) means that (-3) is to be taken 4 times and the sign changed.

Crawford - Multiplication by a negative integer means that the multiplicand is to be taken as a subtrahend as often as there are units in the multiplier ∴ -4 × -3 = -(-4) - (-4)-(-4) = 12.

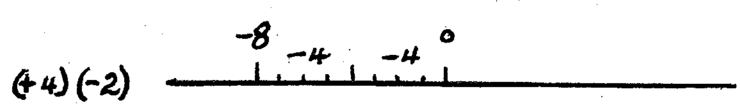
C. Smith gives the same definition of multiplication as Hall and Knight viz. to multiply one number by a second do to the first what is done to unity to obtain the second and this results in (-5) × (-4) being equal to -(-5)-(-5)-(-5)-(-5) = 20.

Schorling and Reeve: First Illustration:

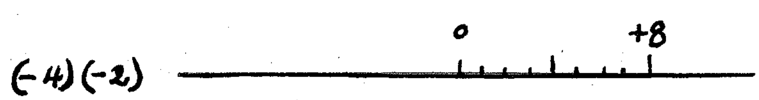




(-4) laid off twice in its own direction.

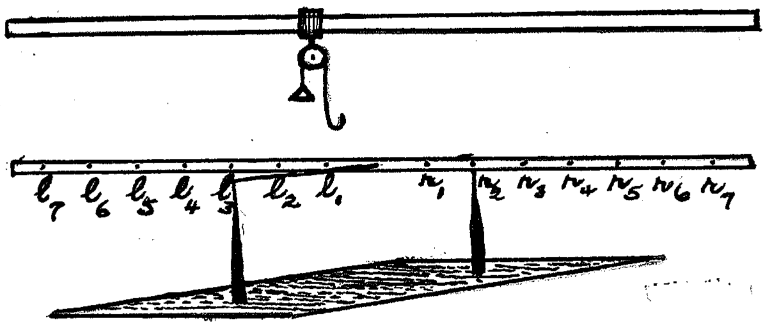


4 laid off twice opposite to own direction.



The opposite to laying off (-4) twice in its own direction.

Second Illustration: (By means of a Balance)



First develop by experiment the laws regarding the turning tendency of a lever.

(+ 2)(-4) Hang 4 downward (-) weights on second peg to right (+) The turning is clockwise or negative.

(-2)(-4) Hang two downward (-) weights on the fourth peg to the left (-) The turning is anticlockwise (+ve)

(+3) (+4) Hand three weights on hook over pulley and fasten string over pulley to 4th peg to right, etc. (-3)(+2). Hang 2 weights or hook over pulley and fasten string to third peg to left etc.

Indices

In the introduction it was stated that the subject of Indices would be treated in the natural algebraic method of defining the meaning of new forms and then showing that certain fundamental laws hold rather than by that method of assuming that the fundamental laws hold and then discovering meanings for the various forms. The treatment follows:

Theory of Indices:

A. General Proofs to establish the laws of combination in the case of all positive integral indices.

Definition: When "m" is a positive integer a^m stands for the product of m factors each equal to a.

Proposition 1. To prove that $a^m \times a^n = a^{m+n}$ (m and n being +ve integers)

By definition $a^m = a.a.a.a...$ to m factors.

$a^n = a.a.a.a...$ to n factors.

$\therefore a^m \times a^n =$ (a.a.a. to m factors)(a.a.a... to n factors)

$= a.a.a.a..$ to $(m+n)$ factors.

$= a^{m+n}$ by definition.

Similarly $a^m a^n a^x = a^{m+n+x}$ etc.

Proposition 2. To prove that $a^m \div a^n = a^{m-n}$ (M and N +ve integers) ($m > n$.)

$$\text{By definition } a^m \div a^n = \frac{a.a.a.a\ldots \text{ to m factors}}{a.a.a.a\ldots \text{ to n factors}}$$

$$= a.a.a\ldots \text{ (m-n) factors.}$$

$$= a^{m-n} \text{ by definition.}$$

Proposition 3. To prove that

$(a^m)^n = a^{mn}$ when m and n are positive integers as before.

$(a^m)^n =$ (a.a.. to m factors)(a.a. to m factors)( )

( ) ( ) to n factors.

$=$ a.a.a... ... to mn factors.

$= a^{mn}$

The definition upon which these fundamental laws have been based covers only positive integral indices. Thus fractional, zero, and negative indices have as yet no meaning. We shall attempt to assign a definition to "$a^{\frac{p}{q}}$" a^0, a^{-m} which will permit of the fundamental laws of indices holding for fractional, -zero and negative indices.

B. Fractional Integers with terms Positive Integers

1. Prove: $\left[\sqrt[q]{a^p}\right]^{tq} = a^{pt}$ (p, q and t +ve integers)

From the definition that $\sqrt[q]{a}$ is a quantity which raised to the qth. power gives a (q +ve) we have:-

$$\left[\sqrt[q]{a^p}\right]^{tq} = \left[\left(\sqrt[q]{a^p}\right)^q\right]^t = \left[a^p\right]^t = a^{pt}.$$

2. Prove

$$\sqrt[s]{a^r} = \sqrt[q]{a^p} \quad \left(\text{where } \frac{r}{s} = \frac{p}{q}, \text{ p and q being relatively prime}\right)$$

$$\left[\sqrt[q]{a^p}\right]^s = \left[\sqrt[q]{a^p}\right]^{qk} \left(\text{where } \frac{s}{q} = k, \text{ a +ve integer}\right)$$

$= a^{pk}$ as above in 1.

$= a^r$ since $\frac{r}{p}$ must also $= k$.

Hence any surd can be reduced to $\sqrt[q]{a^p}$ where p and q are relatively prime.

Also $\left(\sqrt[q]{a^p}\right)^{tq}$ behaves like $(a^k)^{tq}$

where k t q = pt or $k = \frac{p}{q}$

Therefore $\sqrt[s]{a^r} = \sqrt[q]{a^p}$ if $\frac{r}{s} = \frac{p}{q}$ or

surds are equal if the fractions

$\frac{\text{(index under sign)}}{\text{(order of root)}}$ are equal.

e.g. $\sqrt[3]{a^2} = \sqrt[6]{a^4} = \sqrt[12]{a^8}$ .... since $\frac{2}{3} = \frac{4}{6} = \frac{8}{12}$.....

3. A. (1) Prove: $\left[\sqrt[q]{a^p}\right]^m = \sqrt[q]{a^{mp}}$

This is true for raising both sides to the qth power we have a^{mp}.

(2) Prove: $\sqrt[r]{\sqrt[q]{a^p}} = \sqrt[qr]{a^p}$

This is true for raising each side to the qr th power we have a^p.

B. If $\frac{p}{q} + \frac{r}{s} = \frac{t}{u}$, all in their lowest term we have to prove

$$a^{\frac{p}{q}}\, a^{\frac{r}{s}} = a^{\frac{t}{u}} \quad \text{where } \frac{p}{q} + \frac{r}{s} = \frac{t}{u}$$

That is to prove, $\sqrt[q]{a^p}\,\sqrt[s]{a^r} = \sqrt[u]{a^t}$

or $\left[\sqrt[q]{a^p}\,\sqrt[s]{a^r}\right]^{qs} = a^t$ or $a^{ps}a^{qr} = a^t$

or $ps + qr = t$ or $\frac{p}{q} + \frac{r}{s} = \frac{t}{qs} = \frac{t}{u}$ which is true.

In words, to obtain for a product of surds, $\frac{\text{(index under root-sign)}}{\text{(order of surd)}}$ for the product it is simply necessary to add the $\frac{\text{(index under root sign)}}{\text{(order of surd)}}$ for each of the factors. For multiplication therefore this fraction, $\frac{\text{(index under root sign)}}{\text{(order of surd)}}$ behaves exactly as the index for positive integral indices. In particular for $\sqrt[q]{a^p}$ the fraction $\frac{p}{q}$ behaves exactly as does the m in a^m. We therefore write $a^{\frac{p}{q}}$ for $\sqrt[q]{a^p}$, exactly as we write a^m for a.a.a.... for convenience in multiplying. In other words, we define $a^{\frac{p}{q}}$ as $\sqrt[q]{a^p}$. We have proved that in multiplying such symbols, we add indices.

The law $a^m \div a^n = a^{m-n}$ $m > n$ follows for multiplying both sides by a^n we obtain a^n on the left and $a^{(m-n)+n} = a^m$ on the right. The law $(a^m)^n = a^{mn}$ follows immediately if n is a positive integer. For a fraction $(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}}$, and therefore the same law holds.

4. Negative Indices:

The effect of dividing by a^n when the index of the numerator is greater than n has been proved to be that of subtracting n or adding -n to the index. We therefore use a^{-n} for $\div a^n$. i.e. we define a^{-n} as $\frac{1}{a^n}$. Then all the above laws holding for a^n in the denominator; they will hold for the numerator with reversed sign. For example:

$$(a^{-n})^{\frac{p}{q}} = \left(\frac{1}{a^n}\right)^{\frac{p}{q}} = \frac{1}{a^{\frac{np}{q}}} = a^{-\frac{np}{q}}$$

5. Again, multiplying by a^n and $\frac{1}{a^n}$ or a^{-n} will be equivalent to multiplying by 1. Hence we write 1 for $a^n \times a^{-n}$ written a^{n-n} or a^0. wherever such combination occurs. Adding or subtracting the index zero will not affect the result exactly as multiplying by 1 will not affect the result. Hence with this definition 1 and a^0 behave in exactly the same manner.

CHAPTER II

This chapter on Geometry will begin with a summary of the requirements of a logical definition, as given in Creighton's Logic.

"The remedies for the obscurities and confusion of words is to be found in clear and distinct ideas. We must endeavor to go behind the words and realize clearly and distinctly in consciousness the ideas for which they stand. Now the means which logic recommends for the attainment of this end is definition. The first requirement of logical reasoning is that terms shall be Accurately defined. We may include under the general term definition (1) Intensive definition or definition in the narrower sense and (2) Extensive definition or division. To define a term is to state its connotation or to enunciate the attributes which it implies. The requirements of a logical definition are:

1. A definition should state the essential attribute of the thing to be defined. This is done by stating the genus to which the object belongs and also the peculiar marks or qualities which distinguishes it from the other members of the same class. Thus we define a triangle as a rectilineal figure (genus) having three sides (differentia)
2. A definition should not contain the name to be defined, nor any word which is directly synonymous with it.
3. The definition should be exactly equivalent to the class of object defined, that is, it must neither be too broad nor too narrow.
4. A definition should not be expressed in obscure figurative, or ambiguous language. Sometimes the words used in defining may be less familiar than the term to be explained.
5. A definition should, whenever possible be affirmative rather than negative.

A logical definition, as has been said, requires us to mention the proximate genus or next higher class to which the species to be defined belongs and also the specific or characteristic differences which distinguish it from other species. Now it is clear that there are certain cases in which these conditions cannot be fulfilled. In the first place no logical definition can be given of the highest genus, because there is no more general class to which it can be referred. The highest genus is above the sphere of logical definition.

In the light of these ideas on Definition it is possible to discuss the definitions given in Baker's Theoretical Geometry as follows;

Assuming that the chief object of definition is to make another understand by words the terms that are to be used it is necessary to use words which call up familiar ideas. No doubt Baker thought the idea of a physical body the best place to start to impart the ideas of lines yet in this the difficulty presents itself that the beginner keeps thinking of things rather than getting at the geometric concept. The first definitions given of volume and surface, not being used, are unnecessary terms in the science of plane geometry. It would be better to develop the necessary ideas from the point. No logical definition can be given of the point since there is no more general class to refer it to. This being above the sphere of definition we may say this [·] represents a point. It has neither width, length nor depth. These terms are permissible as they convey the required ideas to the student's mind. It would only be confusing to precede this with definitions concerning measuring - width - length - depth -

The next definition would be the line, a succession of points and then the straight line as having the same direction throughout its length. This is probably a highest genus notion. The idea of direction is perhaps no more simple to any student than the idea of straight itself, for he has been early accustomed to going straight to a place or making straight lines. The idea of straightness is one of the most elementary ideas and a formal definition not important, probably not possible. The other ideas regarding the dimensions, coincidence, intersection and method of naming are satisfactorily stated in the text.

In the case of the angle it would accord with logical definition to give its major outstanding property rather than a minor property. "An angle is two straight lines which meet but do not cross." This does not state anything about the length of the arms of the amount of rotation which are minor properties. It would be better to call the amount of rotation the measurement of an angle and to state that it is measured by degrees. The remarks on vertex and arms are satisfactory. The exercise on vertically opposite angles might better follow the axioms as an exercise on them. Regarding adjacent angles a definition is not given though the idea is well illustrated. An easy definition might be worded as follows. "Adjacent angles are those having a common vertex and a common arm."

Article 16 is in reality the continuity axiom, and should be listed with the axioms. This axiom and its relation to various parts of the text will be treated later.

A plane would be defined without using the word surface as that in which any two points being taken the straight line joining them would like completely within it.

The treatment of angles might be simplified by introducing a new term perigon as the angle whose magnitude is a complete revolution about a point. Then a straight angle would be one whose arms are in the same straight line but on opposite sides. It should be pointed out that a straight angle is half a perigon. Then a right angle may be defined as half a straight angle or one fourth of a perigon. Following this perpendicular should be explained both as an adjective and a noun.

Baker omits the definitions as an adjective. Then should come the measurement of angles complements and supplements, obtuse and acute angles but articles 17 and 18 might well be left over for some preliminary theorems following the axioms. This list would include:

1. Vertically opposite angle are equal.
2. Complements of equal angle are equal etc.

NOTE: (A number of the theorems given in Baker as - Right angles are equal - There can be only one perpendicular to a line are accepted intuitively and should be omitted from an elementary text altogether.)

In connection with rectilineal figures it seems scarcely necessary to go beyond what is immediately necessary in definitions. This would require definitions of plane figure, perimeter and plane rectilineal figure.

Then triangles should be defined and classified as to sides and angles. Base and Vertex and congruence of triangle should follow.

The definition of quadrilateral and diagonal should not appear until required in exercises.

The definition of polygon is unsatisfactory in that it should be either a plane figure enclosed by straight lines or a plane figure of four or more straight lines (The exclusion of the triangle is justified on account of the uniqueness of that figure. It would be necessary to define a regular polygon as a polygon having all angles equal and all sides equal. Angles and sides are independent except in the triangle)

The circle, circumference, centre, radius should be defined, but the remaining definitions regarding the circle are not used until Book IV.

Parallel lines should be left until the beginning of Book 11 and Articles 29 and 31 should be Theorems.

The rectangle, rhombus and square should be defined next, the parallelogram and trapezium being introduced in Book 11.

Some additional definitions might be used, as follows:

Following Prop. 12. B ook 1.

Distance of a point from a line means the length of the perpendicular from the point to the line.

Following Prop. 6. Book 11.

Altitude of a parallelogram - The perpendicular distance between the base and the opposite side.

Altitude of a triangle - The distance from the vertex to the base (produced if necessary)

Regarding the definition of a circle, this would harmonize better if at first defined as a curve in a plane all of whose points were the same distance from a fixed point in the plane and then in loci as the locus of all points in a plane equidistant from a fixed point in the plane.

The circumference should always be defined as the length of the circle.

References besides Baker's Geometry.

Creighton's "An Introductory Logic."

Shutts' "Plane and Solid Geometry."

For the sake of a logical rigor there is need for a better statement of the Dedekind Cut Axiom than is given in present High School texts. Baker introduces the idea by the rotation of a line segment about a point from an initial to a final position showing that from a point in a line only one perpendicular can be drawn to the line. He then generalizes with the statement that any angle can have only one line bisecting it and any line can have but one point of bisection.

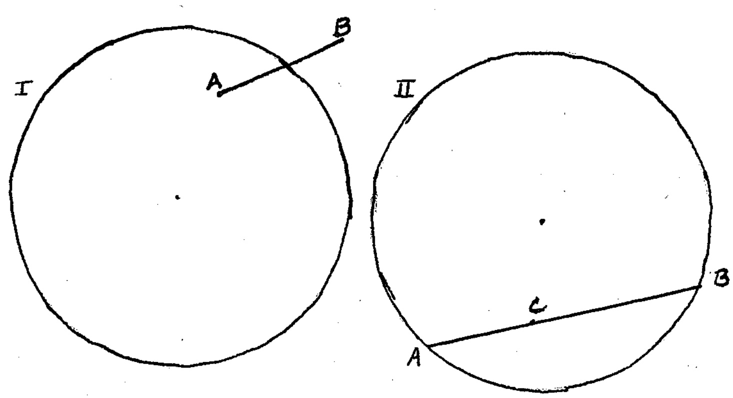
Now the Dedekind Cut Axiom states that there is a single number separating two classes of rational numbers a and A which have the properties.

1. Every a is less than every A.

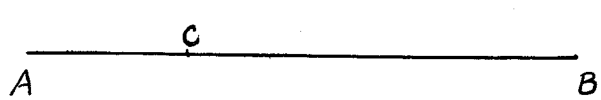
2. The difference $A_1 - a_2$ can be made as small as we please by selecting suitable numbers of the classes. Thus if a denote the numbers such that $a^2 < 2$ and A the numbers such that $A^2 > 2$, a and A satisfy the two given conditions. The Dedekind Cut Axiom states that there is a single number separating these two classes. This number we denominate $\sqrt{2}$. It has the property that $\sqrt{2} >$ every a and $\sqrt{2} <$ every A.

The notion of continuity is built upon this axiom. This notion occurs in two places in elementary geometry.

1. When a circle joins an inside and an outside point. An axiom to the effect that a line or curve joining an inside and outside point of any figure meets the boundary at least once will cover these cases. The test for inside points of a circle is that their distance from the centre shall be less than the radius and for outside points greater. Thus in fig. 1. A is inside and B outside, so the circle meets AB at least once. Also in fig. 11. a straight line which passes through an inside point meets twice, and all points of a chord except the end points are inside. This last is not proved in the texts.



2. When magnitudes are being measured there always exist any submultiples of these magnitudes. Thus for a line C may be chosen so that AC : AB as small as desirable or so that AC : AB as nearly 1 as desirable. Hence there must be one and but one point where AC is $\frac{1}{2}$ of AB.



In a similar way the application of this axiom to angles, arcs, areas etc., could be illustrated.

The places in Baker's Geometry where the need of this axiom is apparent are as follows;

1. To construct a triangle having its sides equal respectively to three given straight lines. Here the method assumes that a curve from an inside to an outside point must cut the circle. This same assumption is made in the construction of an angle equal to a given angle.

2. To bisect a given angle and a given straight line; to draw perpendiculars to a straight line; from a point in the line and a point with out the line. All of these are done by methods which assume the axiom as stated in (1) and also that there is one and only one half way position of the line dividing an angle or the point dividing an angle or the point dividing a line.

The following changes in the text would overcome these difficulties and at the same time make use of the axiom to simplify a number of proofs which in itself is a worthy end seeing that Geometry is too difficult a study for many High School students.

A new axiom or postulate should be worded in some such simple form as the revised statement above.

The proposition should be placed in the following order:-

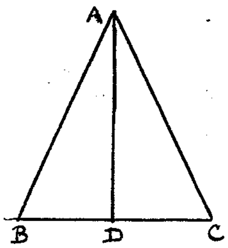
1. Proposition 1. To construct a triangle having its sides equal to three given straight lines and any two of which are greater than the third. (Refer to axiom)

2. Proposition VI. Two triangles are congruent if two sides and the included angle of one are respectively equal to two sides and the included angle of the other. The same proof should be used.

3. Proposition VII. Two triangles are congruent if two angles and the contained side of one be respectively equal to two angles and the contained side of the other.

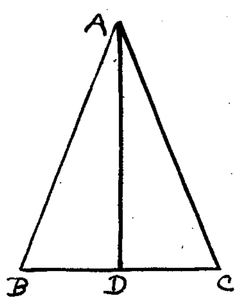
The more general case of this proposition should not be taken until the sum of the angles of a triangle has been established.

4. Proposition 11. The angles at the base of an isosceles triangle are equal. The following proof should be substituted.



Given AB = AC Prove $\angle B = \angle C$
Suppose AD to be the angle bisector of BAC (axiom) etc. This could be completed as an original exercise.

5. Proposition 111. If two angles of a triangle be equal the sides opposite them are also equal.



Given $\angle B = \angle C$ Prove AB = AC.
Use the same construction as in the previous proposition and complete as an original exercise.

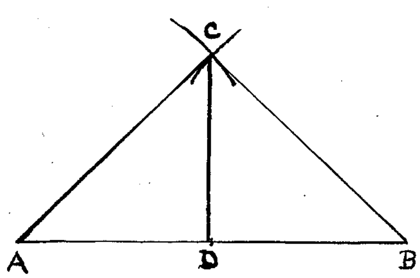
6. Proposition IV. Two triangles are congruent if three sides of the one are respectively equal to three sides of the other.

The construction and proof should be as in the text until an angle in each is proved equal. Then the proof should be completed by reference to (2) above.

7. Proposition V. To construct an angle equal to a given angle. This would be left as it is in the text.

8. Proposition IX. To bisect a given angle. This would be left as it is in the text.

9. Proposition X. To bisect a given straight line. It is required

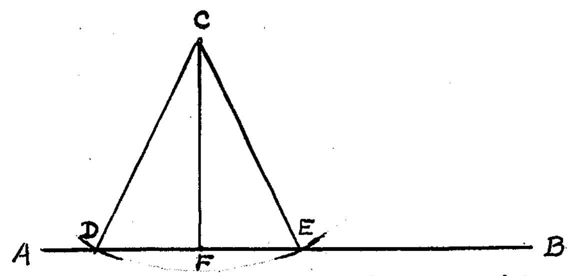


to bisect the given straight line AB.
With centres A and B and equal radii make arcs cutting at C axiom . Bisect ACB by (8) above. Complete as an exercise.

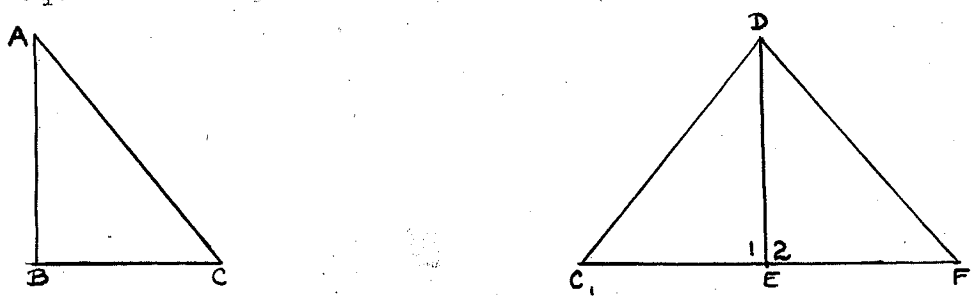
10. Proposition XI. To draw a perpendicular to a given straight line from a point in the line.

The same proof as given in the text could be used.

11. Proposition XII. To drop a perpendicular to a given straight line from a point without the line. Let AB be the given line and C the given point. With cr. c and suitable radius describe an arc cutting AB at D and E. (axiom) Bisect DCE (8) Now F is within the circle (axiom) Complete as an exercise.



12. Corollary to Proposition VIII. (Might be given after (7) above.) Two triangles are congruent if two sides of the one equal two sides of the other and a pair of corresponding angles, not the contained angles, are each equal to 90°.



Let ABC.DEF be two triangles having AB = DE, AC = DF,

and $\angle ABC = \angle DEF = 90^\circ$.

Prove $\triangle ABC \cong \triangle DEF$.

Apply ABC to DEF so that pt. A falls on pt. D and AB along DE. Then pt. B will fall on pt. E for AB = DE. Let C take the position C' opposite F. Mark angles as shown.

Now since $\angle 1 = 90^\circ$ and $\angle 2 = \angle B = 90^\circ$.

$\therefore$ $\angle 1 + \angle 2 = 180^\circ$. $\therefore$ C'EF is a straight angle.

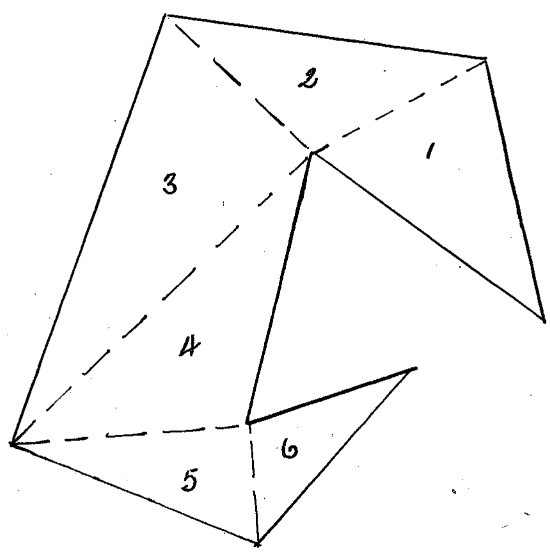
$\therefore$ C'F is a straight line and DC'F is a triangle.

Again AC = DC' = DF. $\therefore$ by (5) above $\angle C' = \angle F$.

Complete as an exercise.

NOTE: In addition to these changes more graded and suggestive exercises should be given throughout. There have not been enough simple exercises on congruent triangles.

Page 60. In proving that the sum of the interior angles of a polygon of n sides is equal to (2n - 4) right angles the proof given does not hold for polygons having re-entrant angles as in the accompanying diagram.



A proof equally easy could have been given which would have been perfectly general including polygons having re-entrant angles. The construction is to form triangles of the figure. It is readily seen that the first and final triangle use up two sides each of the polygon while the remaining triangles use one side of the polygon each. Thus we see there are n - 2 triangles all of whose angles go to form the angles of the polygon. Hence the sum of the angles of the polygon is 2(n-2) or 2n-4 right angles.

Note on Areas: No idea of what the meaning of Area is, is given in Baker. He takes for granted in Book 1 and 11 that the area of a rectangle is the product of the two dimensions and thus the idea of what measuring surface means is assumed.

It would be quite possible to introduce at the beginning of Equivalence of Areas the following sections.

1. Area is a quantity obtained in measuring surface.

   Quantity is the result of measurement and answers the question How much?

   Measurement of surface is the process of determining the number of times a unit is contained in the given surface.

2. Area of a surface is the ratio of the given surface to the unit

of the surface. The unit being a square having a given line as unit for one side as 1 sq. in. 1 sq. ft. etc.

REFERENCES: Burkhardt Chapter 111. # 23.

and notes on Dedékind axioms.

Also - Shutts. Plane and Solid Geometry.