THE COMPARISON OF SOME RATIO ESTIMATORS FOR SMALL SAMPLES

by

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ABSTRACT

One of the main objectives of a sample survey is the estimation of the population mean or total of a character 'y' attached to the units in the population. The ratio method of estimation provides a powerful technique for obtaining increased precision whenever information on an auxiliary character 'x' positively correlated with 'y' is available. Since the classical ratio estimator is biased, considerable attention has been given in recent years to the development of wholly unbiased or approximately unbiased ratio estimators. The relative efficiencies of these ratio estimators have been previously investigated by assuming a linear regression of y on x and gamma distribution for x. In this thesis, various x-populations are employed to investigate relative efficiencies of the estimators empirically, assuming linear regression of y on x. The results obtained indicate that the relative efficiencies are fairly insensitive to the distribution of x-values.

Two classical estimators of variance and a 'jack-knife' variance estimator are also considered and the relative biases and relative stabilities of these variance estimators are investigated empirically.

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CHAPTER I

INTRODUCTION

1.1 Objective

Sample surveys are conducted to obtain estimates of various unknown population parameters of interest. Here we confine ourselves to estimation of the population mean or total of a character under study 'y'. Ratio estimators incorporate the knowledge of an auxiliary variable 'x', which is positively correlated with the character 'y'.

In such situations, the classical ratio estimator is often used. However, it is biased and, in surveys where small or moderate samples within many strata make it appropriate to use the 'separate' ratio estimators, the bias relative to the standard error could be large. For these types of surveys, modifications to the classical ratio estimator lead to wholly or approximately unbiased ratio estimators. This thesis considers the performance of some modified estimators and investigates the stability of three variance estimators under different population conditions.

1.2 Review of Literature

For simple random sampling of size n, drawn without replacement from a finite population of size N, the classical ratio estimator of the population mean $\bar{Y}$ is given by

$$\bar{y}_r = \frac{\bar{y}}{\bar{x}} \bar{X} = r\bar{X} \tag{1.1}$$

where $\bar{X}$ is the known population mean of the auxiliary variable 'x'; $\bar{y}$ and $\bar{x}$ are the sample means of 'y' and 'x' respectively; and

$$r = \frac{\bar{y}}{\bar{x}} \tag{1.2}$$

is the customary estimator of the ratio

$$R = \frac{\bar{Y}}{\bar{X}} . \tag{1.3}$$

The classical ratio estimator has a bias almost of order $O(n^{-1})$ for large n.

Several modifications to the classical ratio estimator leading to approximately or wholly unbiased estimators have been proposed in the literature. Beale's (1962) approximately unbiased estimator of $\bar{Y}$ is given by

$$\bar{y}_B = r\bar{X} \frac{1 + \left(\frac{1}{n} - \frac{1}{N}\right) \frac{s_{xy}}{\bar{x}\,\bar{y}}}{1 + \left(\frac{1}{n} - \frac{1}{N}\right) \frac{s_x^2}{\bar{x}^2}} \tag{1.4}$$

while Tin's (1965) modification of this estimator is

$$\bar{y}_T = r\bar{X} \left[1 + \left(\frac{1}{n} - \frac{1}{N}\right)\left(\frac{s_{xy}}{\bar{x}\,\bar{y}} - \frac{s_x^2}{\bar{x}^2}\right)\right] \tag{1.5}$$

where

$$s_x^2 = \frac{n(\Sigma x^2)_s - (\Sigma x)_s^2}{n(n-1)} \tag{1.6}$$

$$s_{xy} = \frac{n(\Sigma xy)_s - (\Sigma x)_s(\Sigma y)_s}{n(n-1)} \tag{1.7}$$

where $(\Sigma x)_s$ is the sample total for 'x', i.e.,

$$(\Sigma x)_s = \sum_{i=1}^{n} x_i. \tag{1.8}$$

A wholly unbiased ratio estimator, proposed by Hartley and Ross (1954), is

$$\bar{y}_H = \bar{r}\,\bar{X} + \frac{(N-1)n}{N(n-1)}\,(\bar{y} - \bar{r}\bar{x}) \tag{1.9}$$

where

$$\bar{r} = n^{-1}(\Sigma[y/x])_s. \tag{1.10}$$

Two estimators based on splitting up the sample at random into 'g'($\geq$ 2) groups, each of size 'm', where n = mg, are due to Quenouille (1956) and Mickey (1959). Mickey's wholly unbiased estimator of $\bar{Y}$ is

$$\bar{y}_{M(g)} = \bar{r}'_g\,\bar{X} + \frac{(N - n + m)g}{N}\,(\bar{y} - \bar{r}'_g\,\bar{x}) \tag{1.11}$$

where

$$\bar{r}'_g = \sum_{j=1}^{g} \bar{r}'_j/g \tag{1.12}$$

where $\bar{r}'_j$ is the classical ratio estimator computed from the sample after omitting the j^{th} group, i.e.,

$$\bar{r}'_j = \frac{\bar{y}'_j}{\bar{x}'_j} \tag{1.13}$$

where

$$\bar{y}'_j = \frac{n\bar{y} - m\bar{y}_j}{n-m} \; ; \quad \bar{x}'_j = \frac{n\bar{x} - m\bar{x}_j}{n-m} \quad . \tag{1.14}$$

and $\bar{y}_j$, $\bar{x}_j$ are the j^{th} group sample means. Note that $\bar{y}_H = \bar{y}_{M(g)}$ when $n = 2$.

Quenouille's method of bias reduction leads to the approximately unbiased estimator

$$\bar{y}_{Q(g)} = [\omega_g r - (\omega_g - 1)\bar{r}'_g] \; \bar{X} \tag{1.15}$$

where

$$\omega_g = g \left[1 - \frac{(n - m)}{N}\right]. \tag{1.16}$$

The approximately unbiased estimators have bias almost of order $O(n^{-2})$ for large n (terms of order $O(N^{-1})$ do not appear).

Turning to the variance estimators, the classical estimator of the variance of $\bar{y}_r$ is

$$v_1(\bar{y}_r) = \left[\frac{1}{n} - \frac{1}{N}\right] (s_y^2 - 2rs_{xy} + r^2 s_x^2). \tag{1.17}$$

Another variance estimator (Cochran, 1963) is

$$v_2(\bar{y}_r) = \frac{\bar{X}^2}{\bar{x}^2} \left[\frac{1}{n} - \frac{1}{N}\right] (s_y^2 - 2rs_{xy} + r^2 s_x^2). \tag{1.18}$$

Tukey's (1958) 'jack-knife' variance estimator of $\bar{y}_r$ is

$$v_{3g}(\bar{y}_r) = \bar{X}^2 \left[1 - \frac{n}{N}\right] \frac{(g - 1)}{g} \sum_{j=1}^{g} (\bar{r}'_j - \bar{r}'_g)^2 \quad . \tag{1.19}$$

For the comparison of the various estimators, several approaches have been used. Tin compared $\bar{y}_r$, $\bar{y}_B$, $\bar{y}_T$, $\bar{y}_H$ and $\bar{y}_{Q(2)}$ for large n without assuming any model. Rao (1965), Rao and Webster (1966) and Rao (1967) compared $\bar{y}_r$, $\bar{y}_T$, $\bar{y}_{Q(g)}$, $\bar{y}_{M(g)}$ and $\bar{y}_H$ under two models assuming infinite populations, viz the regression of y on x is linear with constant error variance and

(1) $x \sim N(\mu, \sigma^2)$, i.e., x is normally distributed with mean μ and variance σ^2;

(2) $x \sim \Gamma(h)$, i.e., x is distributed as a gamma with parameter h.

The results under model (2) are exact for any sample size n.

Tin showed that $\bar{y}_B$ and $\bar{y}_T$ were better than the other estimators with regard to bias, efficiency and approach to normality.

Using model (1), Rao has shown that

A) $g = n$ is the optimum choice for $\bar{y}_{M(g)}$ and $\bar{y}_{Q(g)}$.

B) the asymptotic variance of $\bar{y}_{M(g)}$ (with $g = n$) is slightly smaller than that of $\bar{y}_r$, but is slightly larger than the mean square error of $\bar{y}_{Q(g)}$ (with $g = n$).

Using model (2), Rao and Webster demonstrated that

A) $g = n$ is the optimum choice for $\bar{y}_{M(g)}$ and $\bar{y}_{Q(g)}$.

B) $\bar{y}_{M(g)}$ is considerably better than $\bar{y}_H$ for $n > 2$ and only slightly better than $\bar{y}_r$ for $n \geq 8$.

C) $\bar{y}_{Q(g)}$ (with $g = n$) and $\bar{y}_T$ are better than $\bar{y}_r$, $\bar{y}_H$ and $\bar{y}_{M(g)}$.

CHAPTER II

APPROACH TO THE PROBLEM

We assume a relationship between the auxiliary variable 'x' and the character of interest 'y'. However, instead of making a specific distributional assumption on x, a wide variety of live and synthetic x-populations will be employed. The purpose of this approach is to study whether these results will be similar to those obtained under the assumption of a gamma distribution for x (which leads to elegant analytical results). With regard to the estimators used when splitting up the sample into groups, we confine ourselves here to $g = n$, as previous investigations indicate that $g = n$ is an optimum choice (and also to reduce computer time).

Mickey's and Quenouille's estimators, along with Tukey's variance estimator, for $g = n$, reduce to

$$\bar{y}_M = \bar{X}\,\bar{r}'_n + \frac{(N - n + 1)}{N}\, n\,(\bar{y} - \bar{r}'_n\,\bar{x}) \tag{2.1}$$

$$\bar{y}_Q = \bar{X}\,[\omega r - (\omega - 1)\,\bar{r}'_n] \tag{2.2}$$

and

$$v_3(\bar{y}_r) = \bar{X}^2 \left(1 - \frac{n}{N}\right) \frac{(n - 1)}{n} \sum_{j=1}^{n} (\bar{r}'_j - \bar{r}'_n)^2 \tag{2.3}$$

where

$$\bar{r}'_n = \frac{(\Sigma r')_s}{n} \tag{2.4}$$

$$\bar{y}_j' = \frac{n\bar{y} - y_j}{n-1} \; ; \quad \bar{x}_j' = \frac{n\bar{x} - x_j}{n-1} \tag{2.5}$$

$$\omega = n \left[1 - \frac{(n-1)}{N}\right]. \tag{2.6}$$

In developing the above approach, it was originally planned to develop a general relationship between y and x, such as

$$y_i = \alpha + \beta x_i^{p+1} + \gamma x_i^{2(p+1)} + u_i \qquad i = 1, 2, \ldots, N \tag{2.7}$$

for general p, where u_i is the error term such that

$$\varepsilon(u_i^2 | x_i) = \delta x_i^t, \quad \delta > 0; \quad i = 1, 2, \ldots, N \tag{2.8}$$

$$\varepsilon(u_i u_j | x_i, x_j) = 0 \qquad \text{for } i \neq j = 1, 2, \ldots, N \tag{2.9}$$

where ε denotes the conditional expectation under the model for given x.

This approach, however, was found to be too ambitious, even for the classical ratio estimator, as the average mean square error ε(MSE) involved too many parameters. To illustrate this, consider the average MSE of $\bar{y}_r$ under the model

$$\varepsilon(\text{MSE } \bar{y}_r) = \xi \left[\frac{\bar{y}}{\bar{x}} \bar{X} - \bar{Y}\right]^2$$

$$= \xi \left[\frac{\{n\alpha + \beta(\Sigma x^{p+1})_s + \gamma(\Sigma x^{2(p+1)})_s + n\bar{u}\}}{n\bar{x}} \bar{X} - \frac{(N\alpha + \beta\sum^N x_i^{p+1} + \gamma\sum^N x_i^{2(p+1)} + N\bar{U})}{N}\right]^2 \tag{2.10}$$

where $\bar{u}$ and $\bar{U}$ are the sample mean and the population mean of the errors u_i respectively and $\xi = \varepsilon E$ where E is the expectation over all the $\binom{N}{n}$ possible.

samples for a given finite population. Therefore, (2.10) reduces to

$$\varepsilon(\text{MSE } \bar{y}_r) = E\left[\alpha\left(\frac{\bar{X}}{\bar{x}} - 1\right) + \beta\left(\frac{\bar{X}}{\bar{x}}\frac{(\Sigma x^{p+1})_s}{n} - \frac{\overset{N}{\Sigma} x_i^{p+1}}{N}\right)\right.$$

$$\left. + \gamma\left(\frac{\bar{X}}{\bar{x}}\frac{(\Sigma x^{2(p+1)})_s}{n} - \frac{\overset{N}{\Sigma} x_i^{2(p+1)}}{N}\right)\right]^2$$

$$+ \delta\, E\left[\frac{\bar{X}^2}{\bar{x}^2}\frac{(\Sigma x^t)_s}{n^2} + \frac{\overset{N}{\Sigma} x_i^t}{N^2} - 2\frac{\bar{X}}{\bar{x}}\frac{(\Sigma x^t)_s}{nN}\right] \tag{2.11}$$

because

$$n^2\, \varepsilon(\bar{u}^2) = \varepsilon[(\Sigma u)_s^2] = \delta(\Sigma x^t)_s \tag{2.12}$$

$$N^2\, \varepsilon(\bar{U}^2) = \varepsilon\left[\overset{N}{\Sigma} u_i\right]^2 = \delta \overset{N}{\Sigma} x_i^t \tag{2.13}$$

$$nN\, \varepsilon(\bar{u}\bar{U}) = \varepsilon\left[(\Sigma u)_s(\overset{N}{\Sigma} u_i)\right] = \delta(\Sigma x^t)_s \tag{2.14}$$

It is seen that (2.11) involves α^2, β^2, γ^2, $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$ as well as δ, p, t.

We therefore confine ourselves to a linear regression of y on x, viz

$$y_i = \alpha + \beta x_i + u_i \qquad i = 1, 2, \ldots, N \tag{2.15}$$

assume that u_i's satisfy (2.8) and (2.9). Then

$$\varepsilon(\text{MSE } \bar{y}_r) = \xi\left[\frac{(\alpha + \beta\bar{x} + \bar{u})}{\bar{x}}\bar{X} - (\alpha + \beta\bar{X} + \bar{U})\right]^2$$

$$= \alpha^2 E\left[\frac{(\overset{N}{\Sigma x_i}) - (\Sigma x)_s}{(\Sigma x)_s}\right]^2$$

$$+ \delta E\left[\frac{(\Sigma x^t)_s(\overset{N}{\Sigma x_i})\{(\overset{N}{\Sigma x_i})-2(\Sigma x)_s\}+(\overset{N}{\Sigma x_i^t})(\Sigma x)_s^2}{N^2(\Sigma x)_s^2}\right] \qquad (2.16)$$

which involves only α^2, δ and t. Noting that

$$s_{xy} = \beta s_x^2 + s_{xu} \qquad (2.17)$$

$$s_y^2 = \beta^2 s_x^2 + 2\beta s_{xu} + s_u^2 \qquad (2.18)$$

where

$$s_{xu} = \frac{n(\Sigma xu)_s - (\Sigma x)_s(\Sigma u)_s}{n(n-1)} \qquad (2.19)$$

$$s_u^2 = \frac{n(\Sigma u^2)_s - (\Sigma u)_s^2}{n(n-1)} \qquad (2.20)$$

we have, for Beale's estimator

$$\varepsilon MSE(\bar{y}_B) = \xi\left[\frac{\{nN\bar{x}\bar{y} + (N-n)(\beta s_x^2 + s_{xu})\}\bar{X}}{nN\bar{x}^2 + s_x^2(N-n)} - \bar{Y}\right]^2$$

$$= \xi\left[\alpha\left(\frac{nN\bar{x}\bar{X}}{nN\bar{x}^2 + s_x^2(N-n)} - 1\right)\right.$$

$$\left. + \bar{u}\left(\frac{nN\bar{x}\bar{X}}{nN\bar{x}^2 + s_x^2(N-n)}\right) - s_{xu}\left(\frac{(N-n)\bar{X}}{nN\bar{x}^2 + s_x^2(N-n)}\right) - \bar{U}\right]^2 \qquad (2.21)$$

Similarly, for Tin's estimator

$$\varepsilon MSE(\bar{y}_T) = \xi\left[\alpha\left(\frac{\bar{X}}{\bar{x}}\left\{1 - \frac{(N-n)s_x^2}{nN\bar{x}^2}\right\} - 1\right)\right.$$

$$\left. + \bar{u}\left(\frac{\bar{X}}{\bar{x}}\left\{1 - \frac{(N-n)s_x^2}{nN\bar{x}^2}\right\}\right) + s_{xu}\frac{(N-n)\bar{X}}{nN\bar{x}^2} - \bar{U}\right]^2. \qquad (2.22)$$

For the Hartley-Ross unbiased estimator

$$\varepsilon V(\bar{y}_H) = \xi\left[\alpha\left\{\left(\frac{\bar{X}}{n} - \frac{(N-1)\bar{x}}{N(n-1)}\right)(\Sigma x^{-1})_s + \frac{(N-1)n}{N(n-1)} - 1\right\}\right.$$

$$\left. + (\Sigma ux^{-1})_s\left(\frac{\bar{X}}{n} - \frac{(N-1)\bar{x}}{N(n-1)}\right) + \bar{u}\,\frac{(N-1)n}{N(n-1)} - \bar{U}\right]^2 \qquad (2.23)$$

where V denotes the variance over all samples from a given finite population. Finally, for the estimators due to Mickey and Quenouille, we have

$$\varepsilon V(\bar{y}_M) = \xi\left[\alpha\left\{(\Sigma\bar{x}_j'^{-1})_s\left(\frac{\bar{X}}{n} - \frac{(N-n+1)\bar{x}}{N}\right) + \frac{(N-n+1)n}{N} - 1\right\}\right.$$

$$\left. + (\Sigma\bar{u}_j'\bar{x}_j'^{-1})_s\left(\frac{\bar{X}}{n} - \frac{(N-n+1)\bar{x}}{N}\right) + \bar{u}\,\frac{(N-n+1)n}{N} - \bar{U}\right]^2. \qquad (2.24)$$

$$\varepsilon MSE(\bar{y}_Q) = \xi\left[\alpha\left(\frac{n\bar{X}(N-n+1)}{N\bar{x}} - \frac{(N-n)(n-1)\bar{X}}{nN}(\Sigma\bar{x}_j'^{-1})_s - 1\right)\right.$$

$$\left. - (\Sigma\bar{u}_j'\bar{x}_j'^{-1})_s\,\frac{(N-n)(n-1)\bar{X}}{nN} + \bar{u}\left(\frac{n\bar{X}(N-n+1)}{N\bar{x}}\right) - \bar{U}\right]^2. \qquad (2.25)$$

Now, using the formulae given in Appendicies I and II, we get the following average MSE's and average variances:

$$\varepsilon \mathrm{MSE}(\bar{y}_B) = \alpha^2 E\left[\frac{(\Sigma x)_s\{n(\Sigma^N x_i) - N(\Sigma x)_s\} - n(N-n)s_x^2}{N(\Sigma x)_s^2 + n(N-n)s_x^2}\right]$$

$$+ \delta E\left[\frac{n^2(N-n)^2(\Sigma x^{t+2})_s(\Sigma^N x_i)^2}{N^2(n-1)^2[N(\Sigma x)_s^2 + n(N-n)s_x^2]^2}\right]$$

$$+ \delta E\left[\frac{2n(N-n)(\Sigma x^{t+1})_s(\Sigma^N x_i)\{[N(n-2)+n](\Sigma x)_s(\Sigma^N x_i)-(n-1)[N(\Sigma x)_s^2+n(N-n)s_x^2]\}}{N^2(n-1)^2[N(\Sigma x)_s^2 + n(N-n)s_x^2]^2}\right]$$

$$+ \delta E\left[\frac{[N(n-2)+n](\Sigma x^t)_s(\Sigma x)_s(\Sigma^N x_i)\{[N(n-2)+n](\Sigma x)_s(\Sigma^N x_i)-2(n-1)[N(\Sigma x)_s^2+n(N-n)s_x^2]\}}{N^2(n-1)^2[N(\Sigma x)_s^2 + n(N-n)s_x^2]^2}\right]$$

$$+ \frac{\delta(\Sigma^N x_i^t)}{N^2} . \qquad (2.26)$$

$$\varepsilon \mathrm{MSE}(\bar{y}_T) = \alpha^2 E\left[\frac{n(\Sigma^N x_i)[N(\Sigma x)_s^2-n(N-n)s_x^2]-N^2(\Sigma x)_s^3}{N^2(\Sigma x)_s^3}\right]^2$$

$$+ \delta E\left[\frac{n^2(N-n)^2(\Sigma x^{t+2})_s(\Sigma^N x_i)^2}{N^4(n-1)^2(\Sigma x)_s^4}\right] + \delta E\left[\frac{2n(N-n)(\Sigma x^{t+1})_s(\Sigma^N x_i)}{N^2(n-1)(\Sigma x)_s^2}\right.$$

$$\left.\left\{\frac{(\Sigma^N x_i)\{[N(n-2)+n](\Sigma x)_s^2 - n(n-1)(N-n)s_x^2\}}{N^2(n-1)(\Sigma x)_s^3} - \frac{1}{N}\right\}\right]$$

$$+ \delta E\left[\frac{(\Sigma^N x_i)(\Sigma x^t)_s[N(\Sigma x)_s^2 - n(N-n)s_x^2]}{N^2(\Sigma x)_s^3}\right.$$

$$\left\{ \frac{(\overset{N}{\Sigma} x_i)\{(\Sigma x)_s^2[N(n-3) + 2n] - n(n-1)(N-n)s_x^2\}}{N^2(n-1)(\Sigma x)_s^3} - \frac{2}{N} \right\}$$

$$+ \left. \frac{(\overset{N}{\Sigma x_i})(\Sigma x^t)_s(N-n)\{(\Sigma x_i)(N-n) + 2N(n-1)(\Sigma x)_s\}}{N^4(n-1)^2(\Sigma x)_s^2} \right] + \frac{\delta(\overset{N}{\Sigma} x_i^t)}{N^2} \qquad (2.27)$$

$$\varepsilon V(\bar{y}_H) = \alpha^2 E\left[\left\{ \frac{(n-1)(\overset{N}{\Sigma} x_i) - (N-1)(\Sigma x)_s}{nN(n-1)} \right\}(\Sigma 1/x)_s + \frac{N-n}{N(n-1)}\right]^2$$

$$+ \delta E\left[(\Sigma x^{t-2})_s \left\{ \frac{(n-1)(\overset{N}{\Sigma} x_i) - (N-1)(\Sigma x)_s}{nN(n-1)} \right\}^2\right]$$

$$+ 2\delta E\left[(\Sigma x^{t-1})_s \left\{ \frac{(n-1)(\overset{N}{\Sigma} x_i) - (N-1)(\Sigma x)_s}{nN(n-1)} \right\}\left\{\frac{N-n}{N(n-1)}\right\}\right]$$

$$+ \delta E\left[\frac{(N-1)(N-2n+1)(\Sigma x^t)_s}{N^2(n-1)^2}\right] + \frac{\delta(\overset{N}{\Sigma} x_i^t)}{N^2} \qquad (2.28)$$

$$\varepsilon V(\bar{y}_M) = \alpha^2 E\left[\frac{(n-1)\{(\overset{N}{\Sigma} x_i)-(N-n+1)(\Sigma x)_s\}}{nN}\left\{\Sigma\left[\frac{1}{(\Sigma x)_s - x_i}\right]\right\}_s + \frac{(N-n)(n-1)}{N}\right]^2$$

$$+ \delta E\left[\frac{(\Sigma x^t)_s(N-n+1)(N-n-1)}{N} + \left\{ \frac{(\overset{N}{\Sigma} x_i)-(\Sigma x)_s(N-n+1)}{nN} \right\}^2\right.$$

$$\left.\left\{ \Sigma\left[\frac{(\Sigma x^t)_s - x_i^t}{[(\Sigma x)_s - x_i]^2}\right] + \sum_{i \neq j}\sum\left[\frac{(\Sigma x^t)_s - x_i^t - x_j^t}{[(\Sigma x)_s - x_i][(\Sigma x)_s - x_j]}\right] \right\}_s\right]$$

$$+ 2\delta E\left[(N-n)\left\{\frac{(\overset{N}{\Sigma x_i})-(\Sigma x)_s(N-n+1)}{nN^2}\right\}\left\{\Sigma\left[\frac{(\Sigma x^t)_s-x_i^t}{(\Sigma x)_s-x_i}\right]\right\}_s\right] + \frac{\delta\overset{N}{\Sigma} x_i^t}{N^2} \qquad (2.29)$$

$$\varepsilon MSE(\bar{y}_Q) = \alpha^2 E\left[\frac{n^2(N-n+1)(\overset{N}{\Sigma x_i})-N^2(\Sigma x)_s}{N^2(\Sigma x)_s} - \left\{\frac{(N-n)(n-1)^2(\overset{N}{\Sigma x_i})}{nN^2}\right\}\right.$$

$$\left.\left\{\Sigma\left[\frac{1}{(\Sigma x)_s-x_i}\right]\right\}_s\right]^2 + \delta E\left[(\Sigma x^t)_s\left\{\frac{n(N-n+1)(\overset{N}{\Sigma x_i})}{N^2(\Sigma x)_s} - \frac{2}{N}\right\}\left\{\frac{n(N-n+1)(\overset{N}{\Sigma x_i})}{N^2(\Sigma x)_s}\right\}\right]$$

$$+ \delta\left\{\frac{(N-n)(n-1)(\overset{N}{\Sigma x_i})}{nN^2}\right\}^2 E\left[\left\{\Sigma\left[\frac{(\Sigma x^t)_s-x_i^t}{[(\Sigma x)_s-x_i]^2}\right] + \sum_{i\neq j}\Sigma\left[\frac{(\Sigma x^t)_s-x_i^t-x_j^t}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j]}\right]\right\}_s\right]$$

$$- 2\delta E\left[\left\{\frac{(\overset{N}{\Sigma x_i})(N-n)(n-1)[n(\overset{N}{\Sigma x_i})(N-n+1)-N(\Sigma x)_s}{(\Sigma x)_s nN^4}\right\}\left\{\Sigma\left[\frac{(\Sigma x^t)_s-x_i^t}{(\Sigma x)_s-x_i}\right]\right\}_s\right]$$

$$+ \delta\,\frac{\overset{N}{\Sigma} x_i^t}{N^2} \qquad (2.30)$$

We now derive the average expectations and average MSE's of the three variance estimators. Under our model, (1.17), (1.18) and (1.19) reduce to

$$v_1(\bar{y}_r) = \frac{(N-n)}{nN}\left[s_x^2\,\frac{(\alpha+\bar{u})^2}{\bar{x}^2} - 2s_{xu}\,\frac{(\alpha+\bar{u})}{\bar{x}} + s_u^2\right] \qquad (2.31)$$

$$v_2(\bar{y}_r) = \frac{\bar{X}^2}{\bar{x}^2}\,\frac{(N-n)}{nN}\left[s_x^2\,\frac{(\alpha+\bar{u})}{\bar{x}^2} - 2s_{xu}\,\frac{(\alpha+\bar{u})}{\bar{x}} + s_u^2\right] \qquad (2.32)$$

$$v_3(\bar{y}_r) = \bar{X}^2 \frac{(N-n)(n-1)}{n^2N}\Bigg[(n-1)\left\{\Sigma\left(\frac{\alpha+\bar{u}_j'}{\bar{x}_j'}\right)^2\right\}_s$$

$$-\left\{\sum_{i\neq j}\sum\left(\frac{(\alpha+\bar{u}_i')(\alpha+\bar{u}_j')}{\bar{x}_i'\bar{x}_j'}\right)\right\}_s\Bigg]$$

$$= \bar{X}^2 \frac{(N-n)(n-1)}{n^2N}\Bigg[\alpha^2\left[(n-1)\left\{\Sigma\left(\frac{1}{\bar{x}_j'^2}\right)\right\}_s - \left\{\sum_{i\neq j}\sum\left(\frac{1}{\bar{x}_i'\bar{x}_j'}\right)\right\}_s\right]$$

$$+ (n-1)\left\{\Sigma\left(\frac{\bar{u}_j'^2}{\bar{x}_j'^2}\right)\right\}_s - \left\{\sum_{i\neq j}\sum\left(\frac{\bar{u}_i'\bar{u}_j'}{\bar{x}_i'\bar{x}_j'}\right)\right\}_s\Bigg] + \text{terms involving } \alpha \text{ and } \bar{u}_i' \text{ whose expectation is zero.} \tag{2.33}$$

Therefore

$$\xi v_1(\bar{y}_r) = \frac{(N-n)}{nN(n-1)}\Bigg\{n\alpha^2 E\left[\frac{n(\Sigma x^2)_s-(\Sigma x)_s^2}{(\Sigma x)_s^2}\right]$$

$$+ \delta E\left[(\Sigma x^t)_s\left\{\frac{(\Sigma x^2)_s+(\Sigma x)_s^2}{(\Sigma x)_s^2}\right\}\right] - 2\delta E\left[\frac{(\Sigma x^{t+1})_s}{(\Sigma x)_s}\right]\Bigg\} \tag{2.34}$$

$$\xi v_2(\bar{y}_r) = \frac{\bar{X}^2}{\bar{x}^2}\,\xi v_1(\bar{y}_r) \tag{2.35}$$

$$\xi v_3(\bar{y}_r) = \bar{X}^2 \frac{(N-n)(n-1)}{n^2N}\Bigg\{(n-1)^2\alpha^2 E\left[(n-1)\left\{\Sigma\left(\frac{1}{[(\Sigma x)_s-x_i]^2}\right)\right\}_s\right.$$

$$\left.-\left\{\sum_{i\neq j}\sum\left(\frac{1}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j]}\right)\right\}_s\right] + \delta E\left[\frac{(n-1)}{n}\left\{\Sigma\left(\frac{(\Sigma x^t)_s-x_i^t}{[(\Sigma x)_s-x_i]^2}\right)\right\}_s\right.$$

$$-\left\{\sum_{i\neq j}\sum\left[\frac{(\Sigma x^t)_s-x_i^t-x_j^t}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j]}\right]\right\}_s\Bigg]\Bigg\}. \tag{2.36}$$

Turning to the evaluations of the average MSE's of the variance estimators, we further assume that the u_i are independently and normally distributed so that

$$\varepsilon(u_i^4|x_i) = 3\delta^2 x_i^{2t} \tag{2.37}$$

$$\varepsilon(u_i^2u_j^2|x_i,\ x_j) = \delta^2 x_i^t x_j^t \qquad i \neq j = 1,\ 2,\ldots,N. \tag{2.38}$$

Now

$$v_1^2(\bar{y}_r) = \frac{(N-n)^2}{n^2N^2}\Bigg[s_x^4\ \frac{(\alpha+\bar{u})^4}{\bar{x}^4} + 4s_{xu}^2\ \frac{(\alpha+\bar{u})^2}{\bar{x}^2} + s_u^4$$

$$- 4s_x^2s_{xu}\ \frac{(\alpha+\bar{u})^3}{\bar{x}^3} + 2s_x^2s_u^2\ \frac{(\alpha+\bar{u})^2}{\bar{x}^2} - 4s_u^2s_{xu}\ \frac{(\alpha+\bar{u})}{\bar{x}}\Bigg]$$

$$= \frac{(N-n)^2}{n^2N^2}\Bigg[\frac{s_x^4}{\bar{x}^4}\ (\alpha^4+6\alpha^2\bar{u}^2+\bar{u}^4) + \frac{4s_{xu}^2}{\bar{x}^2}\ (\alpha^2+\bar{u}^2) + s_u^4 - \frac{4s_x^2s_{xu}}{\bar{x}^3}\ (3\alpha^2\bar{u}+\bar{u}^3)$$

$$+ \frac{2s_x^2s_u^2}{\bar{x}^2}\ (\alpha+\bar{u}^2) - 4\ \frac{s_{xu}s_u^2}{\bar{x}}\ \bar{u}\Bigg] + \text{terms whose average expectation is zero.} \tag{2.39}$$

$$v_3^2(\bar{y}_r) = \frac{\bar{X}^4(N-n)^2(n-1)^2}{n^4N^2}\{\alpha^4[(n-1)(\Sigma\bar{x}_j'^{-2})_s-(\sum_{i\neq j}\Sigma\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s]^2$$

$$+ (n-1)^2(\Sigma\bar{u}_j'^2\bar{x}_j'^{-2})_s^2 + (\sum_{i\neq j}\Sigma\bar{u}_i'\bar{u}_j'\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s^2 + \alpha^2[4(n-1)^2(\Sigma\bar{u}_j'\bar{x}_j'^{-2})_s^2$$

$$+ (\sum_{i\neq j}\Sigma\{\bar{u}_i'+\bar{u}_j'\}\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s^2 - 4(n-1)(\Sigma\bar{u}_j'\bar{x}_j'^{-2})_s(\sum_{i\neq j}\Sigma\{\bar{u}_i'+\bar{u}_j'\}\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s]$$

$$- 2(n-1)(\Sigma\bar{u}_j'^2\bar{x}_j'^{-2})_s(\sum_{i\neq j}\Sigma\bar{u}_i'\bar{u}_j'\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s + 2\alpha^2[(n-1)^2(\Sigma\bar{x}_j'^{-2})_s(\Sigma\bar{u}_j'^2\bar{x}_j'^{-2})_s$$

$$- (n-1)(\Sigma\bar{x}_j'^{-2})_s(\sum_{i\neq j}\Sigma\bar{u}_i'\bar{u}_j'\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s - (n-1)(\sum_{i\neq j}\Sigma\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s(\Sigma\bar{u}_j'^2\bar{x}_j'^{-2})_s$$

$$+ (\sum_{i\neq j}\Sigma\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s(\sum_{i\neq j}\Sigma\bar{u}_i'\bar{u}_j'\bar{x}_i'^{-1}\bar{x}_j'^{-1})_s]\} + \text{terms whose average expectation is zero.} \qquad (2.40)$$

Using the formulae given in Appendices I and II, we get, after considerable simplification,

$$\xi v_1^2(\bar{y}_r) = \frac{(N-n)^2}{n^2N^2}\left\{\alpha^4 E\left[\frac{n^4 s_x^4}{(\Sigma x)_s^4}\right] + 2\alpha^2\delta E\left[\frac{2n^2(\Sigma x^{t+2})_s}{(n-1)^2(\Sigma x)_s^2}\right.\right.$$

$$\left.\left.- \frac{2n(\Sigma x^{t+1})_s}{(n-1)(\Sigma x)_s}\left(\frac{2}{n-1} + \frac{3ns_x^2}{(\Sigma x)_s^2}\right) + (\Sigma x^t)_s\left[\frac{2}{(n-1)^2} + \frac{ns_x^2}{(\Sigma x)_s^2}\left\{\frac{3ns_x^2}{(\Sigma x)_s^2} + \frac{6}{n-1} + 1\right\}\right]\right]\right.$$

$$+ \delta^2 E\left[\frac{4(\Sigma x^t)_s(\Sigma x^{t+2})_s}{(n-1)^2(\Sigma x)_s^2} - \frac{8(\Sigma x^{2t+1})_s}{(n-1)^2(\Sigma x)_s} + (\Sigma x^t)_s^2\left(\frac{3s_x^4}{(\Sigma x)_s^4} + \frac{n^2+2n+3}{n^2(n-1)^2}\right.\right.$$

$$\left. + \frac{2(n+3)s_x^2}{n(n-1)(\Sigma x)_s^2}\right) + \frac{2(\Sigma x^{2t})_s}{n-1}\left(\frac{n+2}{n(n-1)} + \frac{2s_x^2}{(\Sigma x)_s^2}\right) + \frac{4(\Sigma x^{t+1})_s}{(n-1)(\Sigma x)_s}$$

$$\left.\left.\left(\frac{2(\Sigma x^{t+1})_s - (\Sigma x)_s(\Sigma x^t)_s}{(n-1)(\Sigma x)_s} - \frac{3(\Sigma x^t)_s}{n(n-1)} - \frac{3s_x^2(\Sigma x^t)_s}{(\Sigma x)_s^2}\right)\right]\right\}. \qquad (2.41)$$

$$\xi v_2^2(\bar{y}_r) = \frac{\bar{X}^4}{\bar{x}^4}\,\xi v_1^2(\bar{y}_r). \qquad (2.42)$$

$$\xi(v_3^2) = \frac{\bar{X}^4 (N-n)^2 (n-1)^2}{n^4 N^2} \left\{ (n-1)^4 \alpha^4 E \left[(n-1) \left\{ \Sigma \left(\frac{1}{[(\Sigma x)_s - x_j]^2} \right) \right\}_s \right. \right.$$

$$- \left\{ \sum_{i \neq j} \Sigma \left(\frac{1}{[(\Sigma x)_s - x_i][(\Sigma x) - x_j]} \right) \right\}_s \Bigg]^2 + \delta^2 E \left[3(n-1)^2 \left\{ \Sigma \left(\frac{(\Sigma x^t)_s^2 - 2x_j^t (\Sigma x^t)_s + x_j^{2t}}{[(\Sigma x^t)_s - x_j]^4} \right) \right\}_s \right.$$

$$+ [2+(n-1)^2] \left\{ \sum_{i \neq j} \Sigma \left(\frac{(\Sigma x^t)_s [3(\Sigma x^t)_s - 5x_i^t - 5x_j^t] + 2[x_i^{2t} + x_j^{2t}] + 5x_i^t x_j^t}{[(\Sigma x)_s - x_i^t]^2 [(\Sigma x)_s - x_j^t]^2} \right) \right\}_s$$

$$- 12(n-1) \left\{ \sum_{i \neq j} \Sigma \left(\frac{(\Sigma x^t)_s [(\Sigma x^t)_s - 2x_i^t - x_j^t] + x_i^{2t} + x_i^t x_j^t}{[(\Sigma x)_s - x_i]^3 [(\Sigma x)_s - x_j]} \right) \right\}_s$$

$$- 2(n-3) \left\{ \sum_{i \neq j \neq k} \Sigma\,\Sigma \left(\frac{(\Sigma x^t)_s [3(\Sigma x^t)_s - 5x_i^t - 3x_j^t - 3x_k^t] + 2[x_i^{2t} + x_j^t x_k^t] + 3x_i^t [x_j^t + x_k^t]}{[(\Sigma x)_s - x_i]^2 [(\Sigma x)_s - x_j][(\Sigma x)_s - x_k]} \right) \right\}_s$$

$$\left. + \left\{ \sum_{i \neq j \neq k \neq \ell} \Sigma\,\Sigma\,\Sigma \left(\frac{3(\Sigma x^t)_s [(\Sigma x^t)_s - x_i^t - x_j^t - x_k^t - x_\ell^t] + 2x_i^t [x_j^t + x_k^t + x_\ell^t] + 2x_j^t [x_k^t + x_\ell^t] + 2x_k^t + x_\ell^t}{[(\Sigma x)_s - x_i][(\Sigma x)_s - x_j][(\Sigma x)_s - x_k][(\Sigma x)_s - x_\ell]} \right) \right\}_s \right]$$

$$+ \alpha^2 \delta E \left[6(n-1)^4 \left\{ \Sigma \left(\frac{(\Sigma x^t)_s - x_j^t}{[(\Sigma x)_s - x_j]^4} \right) \right\}_s + 4[(n-1)^2 + (n-1)^4] \right.$$

$$\left\{ \sum_{i \neq j} \Sigma \left(\frac{(\Sigma x^t)_s - x_i^t - x_j^t}{[(\Sigma x)_s - x_i]^2 [(\Sigma x)_s - x_j]^2} \right) \right\}_s + 2(n-1)^2 \left\{ \sum_{i \neq j} \Sigma \left(\frac{4(\Sigma x^t)_s - 3[x_i^t + x_j^t]}{[(\Sigma x)_s - x_i][(\Sigma x)_s - x_j]^2} \right) \right\}_s$$

$$+ 2(n-1)^4 \left\{ \sum_{i \neq j} \Sigma \left(\frac{(\Sigma x^t)_s - x_j^t}{[(\Sigma x)_s - x_i]^2 [(\Sigma x)_s - x_j]^2} \right) \right\}_s$$

$$- 8(n-1)^3\left\{\sum_{i\neq j}\sum\left[\frac{2(\Sigma x^t)_s-2x_i^t-x_j^t}{[(\Sigma x)_s-x_i]^3[(\Sigma x)_s-x_j]}\right]\right\}_s - 4(n-1)^3$$

$$\left\{\sum_{i\neq j}\sum\left[\frac{(\Sigma x^t)_s-x_i^t-x_j^t}{[(\Sigma x)_s)-x_i]^3[(\Sigma x)_s-x_j]}\right]\right\}_s - 4(n-1)^3\left\{\sum_{i\neq j}\sum\left[\frac{(\Sigma x^t)_s-x_j^t}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j]^3}\right]\right\}_s$$

$$+ 4(n-1)^2\left\{\sum_{i\neq j\neq k}\sum\sum\left[\frac{4(\Sigma x^t)_s-3x_i^t-2[x_j^t+x_k^t]}{[(\Sigma x)_s-x_i]^2[(\Sigma x)_s-x_j][(\Sigma x)_s-x_k]}\right]\right\}_s$$

$$- 4(n-1)^3\left\{\sum_{i\neq j\neq k}\sum\sum\left[\frac{2(\Sigma x^t)_s-2x_i^t-x_j^t-x_k^t}{[(\Sigma x)_s-x_i]^2[(\Sigma x)_s-x_j][(\Sigma x)_s-x_k]}\right]\right\}_s$$

$$- 2(n-1)^3\left\{\sum_{i\neq j\neq k}\sum\sum\left[\frac{(\Sigma x^t)_s-x_i^t-x_j^t}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j][(\Sigma x)_s-x_k]^2}\right]\right\}_s$$

$$- 2(n-1)^3\left\{\sum_{i\neq j\neq k}\sum\sum\left[\frac{(\Sigma x^t)_s-x_j^t}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j]^2[(\Sigma x)_s-x_k]}\right]\right\}_s$$

$$+ 8(n-1)^2\left\{\sum_{i\neq j\neq k}\sum\sum\left[\frac{(\Sigma x^t)_s-x_i^t-x_j^t}{[(\Sigma x)_s-x_i]^2[(\Sigma x)_s-x_j][(\Sigma x)_s-x_k]}\right]\right\}_s$$

$$+ 2(n-1)^2\left\{\sum_{i\neq j\neq k\neq \ell}\sum\sum\sum\left[\frac{2(\Sigma x^t)_s-x_i^t-x_j^t-x_k^t-x_\ell^t}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j][(\Sigma x)_s-x_k][(\Sigma x)_s-x_\ell]}\right]\right\}_s$$

$$+ 2(n-1)^2\left\{\sum_{i\neq j\neq k\neq \ell}\sum\sum\sum\left[\frac{(\Sigma x^t)_s-x_i^t-x_j^t}{[(\Sigma x)_s-x_i][(\Sigma x)_s-x_j][(\Sigma x)_s-x_k][(\Sigma x)_s-x_\ell]}\right]\right\}_s\Bigg]\Bigg\} \quad (2.42)$$

The identities given in Appendix III were used in the evaluation of the multiple-fold summations involved in $\xi(v_3^2)$.

The average bias of $v_i(\bar{y}_r)$ $(i = 1, 2, 3)$ as an estimator of $V(\bar{y}_r)$ is given by

$$\varepsilon B(v_i) = \xi v_i(\bar{y}_r) - \varepsilon V(\bar{y}_r). \tag{2.43}$$

The average MSE of $v_i(\bar{y}_r)$ is given by

$$\varepsilon MSE(v_i) = \xi[v_i(\bar{y}_r) - V(\bar{y}_r)]^2. \tag{2.44}$$

However, the evaluation of $\varepsilon MSE(v_i)$ is difficult. We therefore employ the alternative criterion

$$\begin{aligned} \varepsilon MSE^*(v_i) &= \xi[v_i(\bar{y}_r) - \varepsilon V(\bar{y}_r)]^2 \\ &= \xi[v_i(\bar{y}_r)]^2 - 2[\xi v_i(\bar{y}_r)][\varepsilon V(\bar{y}_r)] + [\varepsilon V(\bar{y}_r)]^2 \end{aligned} \tag{2.45}$$

which is readily evaluable by using the formulae for $\xi[v_i(\bar{y}_r)]^2$ and $\xi v_i(\bar{y}_r)$ derived above.

CHAPTER III

EMPIRICAL STUDY

In this thesis, we examine only six populations and three sample sizes ($n = 4$, 6 and 8). A more extensive study, however, will appear in a Technical Report of the Department of Statistics, University of Manitoba. Moreover, the conclusions presented here are similar to those derived from the more extensive study. Table 1 shows the source, nature of the character x, the population size N and the coefficient of variation of x, C_x. The populations are arranged according to increasing order of C_x, where N ranges from 49 to 270 and C_x ranges from 0.42 to 1.01.

To compute the coefficients of α^2 and δ, in the formulae for average MSE's or average variances of the estimators given in Chapter II, we need to draw all the $\binom{N}{n}$ possible samples of x-values. However, $\binom{N}{n}$ is very large for the population given here and, due to limitations on available computer time, we adopted the following procedure for computing the coefficients: From a given population of size N 2,000 independent samples, each of a given size n, were drawn at random without replacement and

$$\binom{N}{n}^{-1} \sum_{s=1}^{\binom{N}{n}} h(s) \text{ is approximated by } \sum_{s=1}^{2000} h(s)/2000,$$ where $h(s)$ is a function of the x-values in the sample s. Some preliminary calculations indicate that this approximation is very satisfactory for the comparison of the estimators or variance estimators.

TABLE 1

Description of the Population (Arranged in Increasing Order of C_x)

Pop. No.	Source	Nature of the Character	N	C_x
1	Murthy (1967) p. 131	Length of Timber	176	0.42
2	Biometrika (1959) Vol 46 p. 178	Gamma (h = 2)	100	0.59
3	Sukhatme (1970) p. 256	No. of Villages in a Circle	89	0.61
4	Biometrika (1959) Vol 46 p. 178	Log Normal	100	0.75
5	Kish (1965) p. 625	No. of Dwellings	270	0.99
6	Cochran (1963) p. 156	Size of Cities in U.S. in 1920	49	1.01

Comparison of the estimators

We denote the coefficients of α^2 and δ in the average MSE of the estimators by A_r, A_B, A_T, A_H, A_M, A_Q and by D_r, D_B, D_T, D_H, D_M, D_Q respectively, where the subscript in A or D indicates the estimator. Tables 2, 3 and 4 give the values of the ratios $E_{1\alpha} = A_H/A_M$, $E_{2\alpha} = A_r/A_T$, $E_{3\alpha} = A_B/A_T$, $E_{4\alpha} = A_M/A_T$ and $E_{5\alpha} = A_Q/A_T$ and of the ratios for $E_{1\delta},\ldots,E_{5\delta}$, where $E_{i\delta}$ is obtained from $E_{i\alpha}$ by replacing A by D $(i = 1,\ldots,5)$. The following conclusions may be drawn from these tables: (1) Mickey's unbiased estimator is superior to the Hartley-Ross estimator as $E_{1\alpha}$ is considerably greater than 1 (especially when C_x is large), $E_{1\delta} > 1$ for $t = 0, 1$ ($E_{1\delta}$ is considerable for $t = 0$); for $t = 2$ we have $0.96 \leq E_{1\delta} \leq 1.00$ for the first four populations (with moderate C_x) and $0.91 \leq E_{1\delta} \leq 0.96$ for the populations 5 and 6 (with larger C_x). (2) As expected, the efficiencies of Beale's and Tin's are virtually equal, expecially when C_x is not large. (3) Tin's estimator is significantly better than Mickey's as $E_{4\alpha} > 1$ and, for all t, $E_{4\delta} \geq 1$. (4) For $t = 1$ and 2, Tin's estimator is better than Quenouille's as $E_{5\delta} > 1$ and $0.99 \leq E_{5\alpha} \leq 2.61$; for $t = 0$, $0.97 \leq E_{5\delta} \leq 1.67$. (5) Tin's estimator is slightly more efficient than the classical estimator as $E_{2\alpha} > 1$ and, for $t = 0$, $E_{2\delta} > 1$. However, this efficiency is reversed for $t = 1$ and 2, since, for $t = 1$, $0.95 \leq E_{2\delta} \leq 1.00$, and, for $t = 2$, $0.82 \leq E_{2\delta} \leq 0.98$. (6) As n increases, all the E-values (excepting E_1) tend to 1.

TABLE 2

Values of: $E_{1\alpha} = A_H/A_M$, $E_{2\alpha} = A_r/A_T$, $E_{3\alpha} = A_B/A_T$, $E_{4\alpha} = A_M/A_T$, $E_{5\alpha} = A_Q/A_T$
$E_{1\delta} = D_H/D_M$, $E_{2\delta} = D_r/D_T$, $E_{3\delta} = D_B/A_T$, $E_{4\delta} = D_M/D_T$, $E_{5\delta} = D_Q/D_T$

$n = 4$

Coefficient		Pop. No.	E_1	E_2	E_3	E_4	E_5	Pop. No.	E_1	E_2	E_3	E_4	E_5
α^2		1	1.96	1.35	1.02	1.22	0.99	4	2.30	1.22	1.00	1.48	1.11
δ	$t = 0$	1	1.13	1.10	1.01	1.02	0.99	4	1.79	1.19	1.02	1.06	0.99
δ	$t = 1$	1	1.01	0.99	1.00	1.00	1.01	4	1.10	0.97	0.99	1.06	1.05
δ	$t = 2$	1	0.99	0.92	0.99	1.01	1.02	4	0.96	0.84	0.97	1.16	1.12
α^2		2	2.88	1.28	1.01	1.30	1.02	5	6.09	1.75	1.09	2.58	2.61
δ	$t = 0$	2	1.59	1.16	1.01	1.02	0.99	5	7.85	1.44	1.06	1.51	1.67
δ	$t = 1$	2	1.04	0.98	0.99	1.01	1.02	5	1.52	0.95	0.98	1.12	1.23
δ	$t = 2$	2	0.98	0.88	0.98	1.05	1.05	5	0.91	0.82	0.95	1.27	1.27
α^2		3	1.55	1.16	1.00	1.25	1.05	6	9.06	1.09	0.98	1.72	1.23
δ	$t = 0$	3	1.14	1.14	1.01	1.01	0.99	6	11.58	1.16	1.01	1.12	1.01
δ	$t = 1$	3	1.02	0.98	0.99	1.02	1.02	6	1.30	0.97	0.99	1.14	1.07
δ	$t = 2$	3	0.97	0.87	0.98	1.06	1.06	6	0.95	0.82	0.96	1.37	1.18

TABLE 3

Values of: $E_{1\alpha} = A_H/A_M$, $E_{2\alpha} = A_r/A_T$, $E_{3\alpha} = A_B/A_T$, $E_{4\alpha} = A_M/A_T$, $E_{5\alpha} = A_Q/A_T$

$E_{1\delta} = D_H/D_M$, $E_{2\delta} = D_r/D_T$, $E_{3\delta} = D_B/D_T$, $E_{4\delta} = D_M/D_T$, $E_{5\delta} = D_Q/D_T$

$n = 8$

Coefficient		Pop. No.	E_1	E_2	E_3	E_4	E_5	Pop. No.	E_1	E_2	E_3	E_4	E_5
α^2		1	1.99	1.14	1.00	1.06	1.00	4	2.90	1.12	1.00	1.15	1.01
δ	$t = 0$		1.06	1.05	1.00	1.00	1.00		1.50	1.13	1.01	1.00	0.98
	$t = 1$		1.01	1.00	1.00	1.00	1.00		1.06	0.99	1.00	1.01	1.01
	$t = 2$		1.00	0.97	1.00	1.00	1.00		0.97	0.89	0.99	1.04	1.05
α^2		2	2.95	1.15	1.00	1.10	0.99	5	13.88	1.43	1.03	1.44	1.14
δ	$t = 0$		1.28	1.08	1.00	1.00	0.99		8.39	1.26	1.03	1.05	0.97
	$t = 1$		1.03	0.99	1.00	1.00	1.00		1.37	0.98	1.00	1.02	1.03
	$t = 2$		0.99	0.94	1.00	1.01	1.01		0.93	0.86	0.98	1.08	1.09
α^2		3	1.87	1.12	1.00	1.09	0.99	6	8.77	1.02	0.99	1.23	1.08
δ	$t = 0$		1.09	1.08	1.00	1.00	0.99		8.20	1.14	1.01	0.99	0.97
	$t = 1$		1.02	0.99	1.00	1.00	1.00		1.19	0.98	1.00	1.02	1.02
	$t = 2$		0.98	0.93	1.00	1.01	1.01		0.95	0.85	0.98	1.11	1.09

TABLE 4

Values of: $E_{1\alpha} = A_H/A_M$, $E_{2\alpha} = A_r/A_T$, $E_{3\alpha} = A_B/A_T$, $E_{4\alpha} = A_M/A_T$, $E_{5\alpha} = A_Q/A_T$,

$E_{1\delta} = A_H\ A_M$, $E_{2\delta} = A_r\ A_T$, $E_{3\delta} = A_B\ A_T$, $E_{4\delta} = A_M\ A_T$, $E_{5\delta} = A_Q\ A_T$

$n = 12$

Coefficient	Pop. No.	E_1	E_2	E_3	E_4	E_5	Pop. No.	E_1	E_2	E_3	E_4	E_5
α^2		1.94	1.09	1.00	1.04	1.00		3.33	1.08	1.00	1.09	1.00
δ t = 0	1	1.04	1.03	1.00	1.00	1.00	4	1.34	1.09	1.00	1.00	0.99
δ t = 1		1.01	1.00	1.00	1.00	1.00		1.05	0.99	1.00	1.00	1.00
δ t = 2		1.00	0.98	1.00	1.00	1.00		0.98	0.92	1.00	1.02	1.02
α^2		2.88	1.08	1.00	1.06	1.00		17.14	1.29	1.02	1.19	0.97
δ t = 0	2	1.18	1.05	1.00	1.00	1.00	5	6.20	1.17	1.01	1.01	0.97
δ t = 1		1.02	1.00	1.00	1.00	1.00		1.26	0.99	1.00	1.00	1.01
δ t = 2		1.00	0.96	1.00	1.00	1.00		0.96	0.90	0.99	1.03	1.04
α^2		1.98	1.08	1.00	1.05	1.00		7.74	1.05	1.00	1.11	1.01
δ t = 0	3	1.06	1.05	1.00	1.00	1.00	6	5.59	1.10	1.00	0.99	0.98
δ t = 1		1.01	1.00	1.00	1.00	1.00		1.13	0.99	1.00	1.01	1.01
δ t = 2		0.99	0.95	1.00	1.00	1.01		0.96	0.89	0.99	1.04	1.04

The above conclusions are remarkably similar to the analytical results obtained by Rao and Rao (1971) under the assumption of a gamma distribution for x and infinite population size N.

Comparison of the variance estimators

Tables 5, 6 and 7 give the values of the coefficients of α^2 and δ in the biases of v_1, v_2 and v_3 as estimators of $V(\bar{y}_r)$. We denote the coefficients of α^2 by $-C_{1\alpha}$, $-C_{2\alpha}$ and $C_{3\alpha}$ and of δ by $-C_{1\delta}$, $-C_{2\delta}$ and $C_{3\delta}$ for v_1, v_2 and v_3 respectively. We draw the following conclusions from these tables: (1) v_1 consistently underestimates $V(\bar{y}_r)$ whereas v_3 consistently overestimates $V(\bar{y}_r)$ for all t and v_2 for t = 0 only. (2) $C_{1\alpha} > C_{2\alpha}$ and $C_{1\delta} > C_{2\delta}$ for t = 1 and $C_{1\delta} + C_{2\delta} > 0$ for t = 0 so that $|B(v_1)| > |B(v_2)|$ for t = 0 and 1. For t = 2, $C_{1\delta} < C_{2\delta}$ so that the comparison of $|B(v_1)|$ with $|B(v_2)|$ depends on the value of α^2/δ unless $\alpha \doteq 0$ (i.e., regression approximately through the origin). (3) $C_{3\alpha} > C_{2\alpha}$ and $C_{3\delta} > -C_{2\delta}$ for t = 0, $C_{3\delta} > C_{2\delta}$ for t = 1 so that $|B(v_2)| < |B(v_3)|$ for t = 0 and 1; for t = 2, $C_{3\delta} < C_{2\delta}$ so that the comparison depends on α^2/δ. (4) For $n \geq 8$, $C_{3\delta} < C_{1\delta}$ for all t and $C_{3\alpha} < C_{1\alpha}$ or $C_{3\alpha} \doteq C_{1\alpha}$ (excepting for n = 8 and population 5) so that v_3 may be preferable to v_1 with regard to absolute bias.

We now turn to the mean square errors of the variance estimators v_1, v_2 and v_3. Here we confine ourselves to examining only the δ^2 term in the m.s.e. (denoted by F_1, F_2 and F_3) given in Tables 8, 9 and 10 (assuming $\alpha \doteq 0$). It is clear from these tables that

TABLE 5

Coefficients of α^2 and δ in $B(v_1)$, $B(v_2)$, $B(v_3)$

(Original Values Multiplied by 1000

Except for Those Indicated by a *)

$n = 4$

Coefficients		Pop. No.	$-C_1$	$-C_2$	C_3	Pop. No.	$-C_1$	$-C_2$	C_3
α^2			19	- 1	22		167	107	89
δ	$t = 0$	1	30	- 17	41	4	142	- 43	158
	$t = 1$		166	101	131		52	23	35
	$t = 2$		1197	1778	443		31	44	5
α^2			41	10	57		714	303	2370
δ	$t = 0$	2	53	- 27	78	5	382	- 152	1255
	$t = 1$		75	44	60		3449	1925	4138
	$t = 2$		151	223	42		75*	116*	21*
α^2			44	22	42		214	183	59
δ	$t = 0$	3	62	- 24	72	6	185	- 35	164
	$t = 1$		158	78	104		11*	4*	6*
	$t = 2$		555	766	109		1069*	1484*	9*

TABLE 6

Coefficients of α^2 and δ in $B(v_1)$, $B(v_2)$, $B(v_3)$

(Original Values Multiplied by 1000

Except for Those Indicated by a *)

$n = 8$

Coefficient		Pop. No.	$-C_1$	$-C_2$	C_3	Pop. No.	$-C_1$	$-C_2$	C_3
α^2		1	3	- 1	2	4	40	27	16
δ	t = 0		7	- 3	5		27	- 9	20
	t = 1		44	22	23		13	6	7
	t = 2		350	413	135		11	15	3
α^2		2	6	- 1	10	5	108	33	173
δ	t = 0		10	- 5	9		55	- 23	65
	t = 1		18	10	11		872	449	627
	t = 2		44	60	17		30*	40*	13*
α^2		3	12	6	5	6	47	41	44
δ	t = 0		14	- 5	9		27	- 9	27
	t = 1		41	19	19		2371	1332	1470
	t = 2		170	209	48		406*	620*	83*

TABLE 7

Coefficient of α^2 and δ in $B(v_1)$, $B(v_2)$ and $B(v_3)$

(Original Values Multiplied by 1000

Except for Those Indicated by a *)

$n = 12$

Coefficient		Pop. No.	$-C_1$	$-C_2$	C_3	Pop. No.	$-C_1$	$-C_2$	C_3
α^2			1	0.2	0.3		11	6	8
δ	$t = 0$	1	2	- 1	2	4	8	- 4	6
	$t = 1$		16	9	9		5	3	3
	$t = 2$		130	177	59		5	7	2
α^2			2	- 0.2	3		39	11	36
δ	$t = 0$	2	3	- 2	3	5	18	- 8	15
	$t = 1$		5	4	4		341	185	217
	$t = 2$		14	26	8		14*	19	6
α^2			3.2	1	3.0		27	22	12
δ	$t = 0$	3	4	- 2	3	6	13	- 3	8
	$t = 1$		14	8	8		1219	661	494
	$t = 2$		68	95	24		235*	312*	39*

TABLE 8

Values of F_1, F_2 and F_3 (Coefficients of δ^2) in MSE(v_1), MSE(v_2) and MSE(v_3) Respectively (Original Values Multiplied by 1000)

$n = 4$

	Pop. No.	F_1	F_2	F_3	Pop. No.	F_1	F_2	F_3
t = 0		45	123	179		72	525	1105
t = 1	1	2224	2460	3994	4	22	32	75
t = 2		187*	133*	228*		31	12	32
t = 0		49	194	335		0.2*	7*	107*
t = 1	2	160	189	378	5	50*	89*	605*
t = 2		1266	714	1550		1.3†	0.4†	1.5†
t = 0		50	181	277		86	518	887
t = 1	3	527	643	1192	6	592*	921*	1920*
t = 2		14*	8*	17*		305†	79†	231†

* Original values as is.

† Original values multiplied by 10^{-5}.

TABLE 9

Values of F_1, F_2 and F_3 (Coefficients of δ^2) in MSE(v_1), MSE(v_2) and MSE(v_3) Respectively (Original Values Multiplied by 1000)

n = 8

	Pop. No.	F_1	F_2	F_3	Pop. No.	F_1	F_2	F_3
t = 0		4	7	8		5	22	27
t = 1	1	237	252	306	4	2.5	3	5
t = 2		22*	18*	23*		5	2	6
t = 0		4	9.8	10.9		9	72	284
t = 1	2	17	19	26	5	5628	7321	20302
t = 2		164	113	190		0.2†	0.1†	0.3†
t = 0		4	11	12		5	21	28
t = 1	3	56	63	88	6	61*	70*	139*
t = 2		1846	1229	2083		47†	19†	57†

* Original values as is.

† Original values multiplied by 10^{-5}.

TABLE 10

Values of F_1, F_2, and F_3 (Coefficients of δ^2) in MSE(v_1), MSE(v_2), and MSE(v_3) Respectively (Original Values Multiplied by 1000)

n = 12

	Pop. No.	F_1	F_2	F_3	Pop. No.	F_1	F_2	F_3
t = 0		1.1	1.78	1.8		1	3.6	3.9
t = 1	1	65	67	76	4	0.68	0.73	1
t = 2		6150	5098	6160		1.5	0.9	1.7
t = 0		1	2.06	2.14		1.8	10	13
t = 1	2	4.7	4.8	6	5	1575	1849	3478
t = 2		4.8	35	49		7121*	3994*	9502*
t = 0		1	2.1	2.2		1	3.8	4.4
t = 1	3	15	16	20	6	15*	17*	29*
t = 2		538	387	571		13†	7†	17†

* Original values as is.

† Original values multiplied by 10^{-5}.

$F_1 < F_2 < F_3$ for t = 0 and 1 whereas $F_2 < F_1 < F_3$ for t = 2 (F_1/F_2 is close to 1 for t = 1 excepting population five, but F_3 is considerably larger than F_1 for all t especially when $n \leq 8$ and C_x is large). Consequently, v_3 is the least stable; v_1 is more stable than v_2 when t = 0 and 1 and, for t = 2, v_2 is more stable than v_1. It may be noted, however, that v_2 has smaller absolute bias than v_1 for t = 0 or 1 and the converse result for t = 2.

The above conclusions are again similar to those obtained by Rao and Rao (1971) assuming the gamma distribution for x and infinite N.

Concluding Remark

An important conclusion from this study is that the results on efficiency of estimators (and to a lesser extent on the variance estimators) are fairly insensitive to the distribution of x-values in the population. This result is very encouraging because one could get analytical results, by assuming a gamma distribution for x, along the lines of Rao and Rao (1971).

APPENDIX I

SOME IDENTITIES USED IN THE DERIVATIONS

1. $\varepsilon[\bar{u}^2] = \varepsilon\left[\frac{(\Sigma u)_s^2}{n^2}\right]$

2. $\varepsilon[\bar{U}^2] = \varepsilon\left[\frac{(\overset{N}{\Sigma} u_i)^2}{N^2}\right]$

3. $\varepsilon[\bar{u}\bar{U}] = \varepsilon\left[\frac{(\Sigma u)_s(\overset{N}{\Sigma} u_i)}{nN}\right]$

4. $\varepsilon[\bar{u}^4] = \varepsilon\left[\frac{(\Sigma u)_s^4}{n^4}\right]$

5. $\varepsilon[s_{xu}^2] = \varepsilon\left[\frac{n^2(\Sigma xu)_s^2 + (\Sigma x)_s^2(\Sigma u)_s^2 - 2n(\Sigma x)_s(\Sigma xu)_s(\Sigma u)_s}{n^2(n-1)^2}\right]$

6. $\varepsilon[s_{xu}\bar{u}] = \varepsilon\left[\frac{n(\Sigma xu)_s(\Sigma u)_s - (\Sigma x)_s(\Sigma u)_s^2}{n^2(n-1)}\right]$

7. $\varepsilon[s_{xu}\bar{U}] = \varepsilon\left[\frac{n(\Sigma xu)_s(\overset{N}{\Sigma} u_i) - (\Sigma x)_s(\Sigma u)_s(\overset{N}{\Sigma} x_i)}{nN(n-1)}\right]$

8. $\varepsilon[s_{xu}^2\bar{u}^2] = \varepsilon\left[\frac{n^2(\Sigma xu)_s^2(\Sigma u)_s^2 + (\Sigma x)_s^2(\Sigma u)_s^4 - 2n(\Sigma x)_s(\Sigma xu)_s(\Sigma u)_s^3}{n^4(n-1)^2}\right]$

9. $\varepsilon[s_{xu}\bar{u}^3] = \varepsilon\left[\frac{n(\Sigma xu)_s(\Sigma u)_s^3 - (\Sigma x)_s(\Sigma u)_s^4}{n^4(n-1)}\right]$

10. $\varepsilon[s_u^2] = \varepsilon\left[\frac{n(\Sigma u^2)_s - (\Sigma u)_s^2}{n(n-1)}\right]$

11. $\varepsilon[s_u^4] = \varepsilon\left[\frac{n^2(\Sigma u^2)_s^2 + (\Sigma u)_s^4 - 2n(\Sigma u^2)_s(\Sigma u)_s^2}{n^2(n-1)^2}\right]$

12. $\varepsilon[s_u^2\bar{u}^2] = \varepsilon\left[\frac{n(\Sigma u^2)_s(\Sigma u)_s^2 - (\Sigma u)_s^4}{n^3(n-1)}\right]$

13. $\varepsilon[s_{xu}s_u^2\bar{u}] = \varepsilon\left[\frac{n^2(\Sigma xu)_s(\Sigma u^2)_s(\Sigma u)_s - n(\Sigma xu)_s(\Sigma u)_s^3}{n^3(n-1)^2}\right.$

$$\left. - \frac{n(\Sigma x)_s(\Sigma u^2)_s(\Sigma u)_s^2 + (\Sigma x)_s(\Sigma u)_s^4}{n^3(n-1)^2}\right]$$

14. $\varepsilon[\bar{u}(\Sigma u/x)_s] = \varepsilon\left[\frac{(\Sigma u)_s(\Sigma u/x)_s}{n}\right] = \frac{(\Sigma\varepsilon[u^2/x])_s}{n}$

15. $\varepsilon[\bar{U}(\Sigma u/x)_s] = \frac{(\Sigma\varepsilon[u^2/x])_s}{N}$

16. $\varepsilon[(\Sigma u/x)_s^2] = (\Sigma\varepsilon[u^2/x^2])_s$

17. $\varepsilon[\Sigma(\bar{u}_j'/\bar{x}_j')_s]^2 = (\Sigma\varepsilon[\bar{u}_j'^2/\bar{x}_j'^2])_s + (\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_i'\bar{u}_j'/\bar{x}_i'\bar{x}_j'])_s$

18. $\varepsilon[\bar{u}(\Sigma\bar{u}_j'/\bar{x}_j')_s] = (\Sigma\varepsilon[\bar{u}\ \bar{u}_j'/\bar{x}_j'])_s$

19. $\varepsilon[\bar{U}(\Sigma\bar{u}_j'/\bar{x}_j')_s] = (\Sigma\varepsilon[\bar{U}\ \bar{u}_j'/\bar{x}_j'])_s$

20. $\varepsilon[(\Sigma\bar{u}_j'^2/\bar{x}_j'^2)_s^2] = (\Sigma\varepsilon[\bar{u}_j'^4/\bar{x}_j'^4])_s + (\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_i'^2\bar{u}_j'^2/\bar{x}_i'^2\bar{x}_j'^2])_s$

21. $\varepsilon[(\Sigma\Sigma\bar{u}_i'\bar{u}_j'/\bar{x}_i'\bar{x}_j')_s^2] = 2(\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_i'^2\bar{u}_j'^2/\bar{x}_i'^2\bar{x}_j'^2])_s$

$$+ 4(\sum_{i\neq j\neq k}\Sigma\ \Sigma\varepsilon[\bar{u}_i'^2\bar{u}_j'\bar{u}_k'/\bar{x}_i'^2\bar{x}_j'\bar{x}_k'])_s + (\sum_{i\neq j\neq k\neq \ell}\Sigma\ \Sigma\ \Sigma\varepsilon[\bar{u}_i'\bar{u}_j'\bar{u}_k'\bar{u}_\ell'/\bar{x}_i'\bar{x}_j'\bar{x}_k'\bar{x}_\ell'])_s$$

22. $\varepsilon[(\Sigma\bar{u}_j'/\bar{x}_j'^2)_s^2] = (\Sigma\varepsilon[\bar{u}_j'^2/\bar{x}_j'^4])_s + (\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_i'\bar{u}_j'/\bar{x}_i'^2\bar{x}_j'^2])_s$

23. $\varepsilon[(\sum_{i\neq j}\Sigma\{\bar{u}_i'+\bar{u}_j'\}/\bar{x}_i'\bar{x}_j')_s^2] = 2(\sum_{i\neq j}\Sigma\varepsilon[\{\bar{u}_i'+\bar{u}_j'\}^2/\bar{x}_i'^2\bar{x}_j'^2])_s$

$+ 4(\sum_{i\neq j\neq k}\Sigma\,\Sigma\varepsilon[\{\bar{u}_i'+\bar{u}_j'\}\{\bar{u}_i'+\bar{u}_k'\}/\bar{x}_i'^2\bar{x}_j'\bar{x}_k'])_s$

$+ (\sum_{i\neq j\neq k\neq \ell}\Sigma\,\Sigma\,\Sigma\varepsilon[\{\bar{u}_i'+\bar{u}_j'\}\{\bar{u}_k'+\bar{u}_\ell'\}/\bar{x}_i'\bar{x}_j'\bar{x}_k'\bar{x}_\ell'])_s$

24. $\varepsilon[(\Sigma\bar{u}_j'/\bar{x}_j'^2)_s(\sum_{i\neq j}\Sigma\{\bar{u}_i'+\bar{u}_j'\}/\bar{x}_i'\bar{x}_j')_s] = 2(\sum_{i\neq j}\Sigma\varepsilon[\{\bar{u}_i'^2+\bar{u}_i'\bar{u}_j'\}/\bar{x}_i'^3\bar{x}_j'])_s$

$+ (\sum_{i\neq j\neq k}\Sigma\,\Sigma\varepsilon[\bar{u}_i'(\bar{u}_j'+\bar{u}_k')/\bar{x}_i'^2\bar{x}_j'\bar{x}_k'])_s$

25. $\varepsilon[(\Sigma\bar{u}_j'^2/\bar{x}_j'^2)_s(\sum_{i\neq j}\Sigma\bar{u}_i'\bar{u}_j'/\bar{x}_i'\bar{x}_j')_s] = 2(\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_i'^3\bar{u}_j'/\bar{x}_i'^3\bar{x}_j'])_s$

$+ (\sum_{i\neq j\neq k}\Sigma\,\Sigma\varepsilon[\bar{u}_i'^2\bar{u}_j'\bar{u}_k'/\bar{x}_i'^2\bar{x}_j'\bar{x}_k'])_s$

26. $\varepsilon[(\Sigma 1/\bar{x}_j'^2)_s(\Sigma\bar{u}_j'^2/\bar{x}_j'^2)_s] = (\Sigma\varepsilon[\bar{u}_j'^2/\bar{x}_j'^4])_s + (\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_j'^2/\bar{x}_i'^2\bar{x}_j'^2])_s$

27. $\varepsilon[(\Sigma 1/\bar{x}_j'^2)_s(\sum_{i\neq j}\Sigma\bar{u}_i'\bar{u}_j'/\bar{x}_i'\bar{x}_j')_s] = 2(\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_i'\bar{u}_j'/\bar{x}_i'^3\bar{x}_j'])_s$

$+ (\sum_{i\neq j\neq k}\Sigma\,\Sigma\varepsilon[\bar{u}_i'\bar{u}_j'/\bar{x}_i'\bar{x}_j'\bar{x}_k'^2])_s$

28. $\varepsilon[(\sum_{i\neq j}\Sigma 1/\bar{x}_i'\bar{x}_j')_s(\Sigma\bar{u}_j'^2/\bar{x}_j'^2)_s] = 2(\sum_{i\neq j}\Sigma\varepsilon[\bar{u}_j'^2/\bar{x}_i'\bar{x}_j'^3])_s$

$+ (\sum_{i\neq j\neq k}\Sigma\,\Sigma\varepsilon[\bar{u}_j'^2/\bar{x}_i'\bar{x}_j'^2\bar{x}_k'])_s$

29. $$\varepsilon[(\sum_{i\neq j}\sum 1/\bar{x}_i'\bar{x}_j')_s(\sum_{i\neq j}\sum \bar{u}_i'\bar{u}_j'/\bar{x}_i'\bar{x}_j')_s] = 2(\sum_{i\neq j}\sum \varepsilon[\bar{u}_i'\bar{u}_j'/\bar{x}_i'^2\bar{x}_j'^2])_s$$

$$+ 4(\sum_{i\neq j\neq k}\sum\sum \varepsilon[\bar{u}_i'\bar{u}_j'/\bar{x}_i'^2\bar{x}_j'\bar{x}_k'])_s + (\sum_{i\neq j\neq k\neq \ell}\sum\sum\sum \varepsilon[\bar{u}_i'\bar{u}_j'/\bar{x}_i'\bar{x}_j'\bar{x}_k'\bar{x}_\ell'])_s$$

APPENDIX II

Some formulae used in the derivations (assuming the model).

The model is

$$y_i = \alpha + \beta x_i + u_i$$

where

$$\varepsilon(u_i | x_i) = 0,$$

$$\varepsilon(u_i^2 | x_i) = \delta x_i^t,$$

$$\varepsilon(u_i u_j | x_i, x_j) = 0 \quad i \neq j = 1, 2, \ldots, N.$$

1. $\varepsilon[(\Sigma u)_s^2] = \delta(\Sigma x^t)_s$ 2. $\varepsilon[(\overset{N}{\Sigma} u_i)] = \delta(\overset{N}{\Sigma} x_i^t)$

3. $\varepsilon[(\Sigma u)_s(\overset{N}{\Sigma} u_i)] = \delta(\Sigma x^t)_s$ 4. $\varepsilon[(\Sigma xu)_s^2] = \delta(\Sigma x^{t+2})_s$

5. $\varepsilon[(\Sigma xu)_s(\Sigma u)_s] = \delta(\Sigma x^{t+1})_s$

6. $\varepsilon[(\Sigma xu)_s(\overset{N}{\Sigma} u_i)] = \delta(\Sigma x^{t+1})_s$

7. $\varepsilon[(\Sigma u)_s(\overset{N}{\Sigma} u_i)] = \delta(\Sigma x^t)_s$

8. $\varepsilon[(\Sigma u^2)_s] = \delta(\Sigma x^t)_s$

9. $\varepsilon\left[\frac{u_i^2}{x_i}\right] = \delta x_i^{t-1}$

10. $\varepsilon\left[\frac{u_i^2}{x_i^2}\right] = \delta x_i^{t-2}$

11. $$\varepsilon\left[\frac{\bar{u}_j'^2}{\bar{x}_j'^2}\right] = \delta\left[\frac{(\Sigma x^t)_s - x_j^t}{\{(\Sigma x)_s - x_i\}^2}\right]$$

12. $$\varepsilon\left[\frac{\bar{u}_i'\bar{u}_j'}{\bar{x}_i'\bar{x}_j'}\right] = \delta\left[\frac{(\Sigma x^t)_s - x_i^t - x_j^t}{\{(\Sigma x)_s - x_i\}\{(\Sigma x)_s - x_j\}}\right]$$

13. $$\varepsilon\left[\bar{u}\,\frac{\bar{u}_j'}{\bar{x}_j'}\right] = \delta\left[\frac{(\Sigma x^t)_s - x_j^t}{(\Sigma x)_s - x_j}\right]$$

14. $$\varepsilon\left[\bar{U}\,\frac{\bar{u}_j'}{\bar{x}_j'}\right] = \delta\left[\frac{(\Sigma x^t)_s - x_j^t}{(\Sigma x)_s - x_j}\right]$$

15. $$\varepsilon\left[\frac{\bar{u}_j'^2}{\bar{x}_j'^4}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - x_j^t}{\{(\Sigma x)_s - x_j\}^4}\right]$$

16. $$\varepsilon\left[\frac{\bar{u}_i'\bar{u}_j'}{\bar{x}_i'^2\bar{x}_j'^2}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - (x_i^t + x_j^t)}{\{(\Sigma x)_s - x_i\}\{(\Sigma x)_s - x_j\}}\right]$$

17. $$\varepsilon\left[\frac{(\bar{u}_i' + \bar{u}_j')^2}{\bar{x}_i'^2\bar{x}_j'^2}\right] = (n-1)^2\delta\left[\frac{4(\Sigma x^t)_s - 3(x_i^t + x_j^t)}{\{(\Sigma x)_s - x_i\}^2\{(\Sigma x)_s - x_j\}^2}\right]$$

18. $$\varepsilon\left[\frac{(\bar{u}_i' + \bar{u}_j')(\bar{u}_i' + \bar{u}_k')}{\bar{x}_i'^2\ \bar{x}_j'\ \bar{x}_k'}\right] = (n-1)^2\delta\left[\frac{4(\Sigma x^t)_s - 3x_i^t - 2(x_j^t + x_k^t)}{\{(\Sigma x)_s - x_i\}^2\{(\Sigma x)_s - x_j\}\{(\Sigma x)_s - x_k\}}\right]$$

19. $$\varepsilon\left[\frac{(\bar{u}_i' + \bar{u}_j')(\bar{u}_k' + \bar{u}_\ell')}{\bar{x}_i'\ \bar{x}_j'\ \bar{x}_k'\ \bar{x}_\ell'}\right] = 2(n-1)^2\delta\left[\frac{2(\Sigma x^t)_s - (x_i^t + x_j^t + x_k^t + x^t)}{\{(\Sigma x)_s - x_i\}\{(\Sigma x)_s - x_j\}\{(\Sigma x)_s - x_k\}\{(\Sigma x)_s - x_\ell\}}\right]$$

20. $$\varepsilon\left[\frac{\bar{u}_i'^2 + \bar{u}_i'\bar{u}_j'}{\bar{x}_i'^3\ \bar{x}_j'}\right] = (n-1)^2\delta\left[\frac{2(\Sigma x^t)_s - (2x_i^t + x_j^t)}{\{(\Sigma x)_s - x_i\}^3\{(\Sigma x)_s - x_j\}}\right]$$

21. $$\varepsilon\left[\frac{\bar{u}_i'(\bar{u}_j' + \bar{u}_k')}{\bar{x}_i'^2\ \bar{x}_j'\ \bar{x}_k'}\right] = (n-1)^2\delta\left[\frac{2(\Sigma x^t)_s - (2x_i^t + x_j^t + x_k^t)}{\{(\Sigma x)_s-x_i\}^2\{(\Sigma x)_s-x_j\}\{(\Sigma x)_s-x_k\}}\right]$$

22. $$\varepsilon\left[\frac{\bar{u}_j'^2}{\bar{x}_i'^2\ \bar{x}_j'^2}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - x_j^t}{\{(\Sigma x)_s - x_i\}^2\{(\Sigma x)_s - x_j\}^2}\right]$$

23. $$\varepsilon\left[\frac{\bar{u}_i'\ \bar{u}_j'}{\bar{x}_i'\ \bar{x}_j'\ \bar{x}_k'^2}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - (x_i^t + x_j^t)}{\{(\Sigma x)_s-x_i\}\{(\Sigma x)_s-x_j\}\{(\Sigma x)_s-x_k\}^2}\right]$$

24. $$\varepsilon\left[\frac{\bar{u}_j'^2}{\bar{x}_i'\ \bar{x}_j'^3}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - x_j^t}{\{(\Sigma x)_s - x_i\}\{(\Sigma x)_s - x_j\}^2}\right]$$

25. $$\varepsilon\left[\frac{\bar{u}_j'^2}{\bar{x}_i'\ \bar{x}_j'^2\ \bar{x}_k'}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - x_j^t}{\{(\Sigma x)_s-x_i\}\{(\Sigma x)_s-x_j\}^2\{(\Sigma x)_s-x_k\}}\right]$$

26. $$\varepsilon\left[\frac{\bar{u}_i'\ \bar{u}_j'}{\bar{x}_i'^2\ \bar{x}_j'^2}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - (x_i^t + x_j^t)}{\{(\Sigma x)_s - x_i\}^2\{(\Sigma x)_s - x_j\}^2}\right]$$

27. $$\varepsilon\left[\frac{\bar{u}_i'\ \bar{u}_j'}{\bar{x}_i'^2\ \bar{x}_j'\ \bar{x}_k'}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - (x_i^t + x_j^t)}{\{(\Sigma x)_s-x_i\}^2\{(\Sigma x)_s-x_j\}\{(\Sigma x)_s-x_k\}}\right]$$

28. $$\varepsilon\left[\frac{\bar{u}_i'\ \bar{u}_j'}{\bar{x}_i'\bar{x}_j'\bar{x}_k'\bar{x}_\ell'}\right] = (n-1)^2\delta\left[\frac{(\Sigma x^t)_s - (x_i^t + x_j^t)}{\{(\Sigma x)_s-x_i\}\{(\Sigma x)_s-x_j\}\{(\Sigma x)_s-x_k\}\{(\Sigma x)_s-x_\ell\}}\right]$$

Assuming further that the errors u_i are independently and normally distributed so that

$$\varepsilon(u_i^4|x_i) = 3\delta^2 x_i^{2t}$$

and

$$\varepsilon(u_i^2\, u_j^2|x_i,\, x_j) = \delta^2 x_i^t x_j^t,$$

we get

29. $\varepsilon[(\Sigma u)_s^4] = 3\delta^2(\Sigma x^t)_s^2$

30. $\varepsilon[(\Sigma xu)_s^2(\Sigma u)_s^2] = \delta^2[(\Sigma x^t)_s(\Sigma x^{t+2})_s + 2(\Sigma x^{t+1})_s^2]$

31. $\varepsilon[(\Sigma u^2)_s^2] = \delta^2[(\Sigma x^t)_s^2 + 2(\Sigma x^{2t})_s]$

32. $\varepsilon[(\Sigma xu)_s(\Sigma u)_s^3] = 3\delta^2(\Sigma x^t)_s(\Sigma x^{t+1})_s$

33. $\varepsilon[(\Sigma u^2)_s(\Sigma u)_s^2] = \delta^2[(\Sigma x^t)_s^2 + 2(\Sigma x^{2t})_s]$

34. $\varepsilon[(\Sigma xu)_s(\Sigma u^2)_s(\Sigma u)_s] = \delta^2[(\Sigma x^t)_s(\Sigma x^{t+1})_s + 2(\Sigma x^{2t+1})_s]$

35. $$\varepsilon\left[\frac{\bar{u}_j'^4}{\bar{x}_j'^4}\right] = 3\delta^2\left[\frac{\{(\Sigma x^t)_s - x_j^t\}^2}{\{(\Sigma x)_s - x_j\}^4}\right]$$

36. $$\varepsilon\left[\frac{\bar{u}_i'^2\bar{u}_j'^2}{\bar{x}_i'^2\bar{x}_j'^2}\right] = \delta^2\left[\frac{(\Sigma x^t)_s\{3(\Sigma x^t)_s-5(x_i^t+x_j^t)\}+2(x_i^{2t}+x_j^{2t})+5x_i^t x_j^t}{\{(\Sigma x)_s - x_i\}^2\{(\Sigma x)_s - x_j\}^2}\right]$$

37. $$\varepsilon\left[\frac{\bar{u}_i'^2\bar{u}_j'\bar{u}_k'}{\bar{x}_i'^2\bar{x}_j'\bar{x}_k'}\right] = \delta^2\left[\frac{(\Sigma x^t)_s\{3(\Sigma x^t)_s-5x_i^t-3(x_j^t+x_k^t)\}+2x_i^{2t}+3x_i^t(x_j^t+x_k^t)+2x_j^t x_k^t}{\{(\Sigma x)_s - x_i\}^2\{(\Sigma x)_s - x_j\}\{(\Sigma x)_s - x_k\}}\right]$$

38. $$\varepsilon\left[\frac{\bar{u}_i'^{3}\bar{u}_j'}{\bar{x}_i'^{3}\bar{x}_j'}\right] = 3\delta^2\left[\frac{(\Sigma x^t)_s\{(\Sigma x^t)_s - 2x_i^t - x_j^t\} + x_i^{2t} + x_i^t x_j^t}{\{(\Sigma x)_s - x_i\}^3\{(\Sigma x)_s - x_j\}}\right]$$

39. $$\varepsilon\left[\frac{\bar{u}_i'\bar{u}_j'\bar{u}_k'\bar{u}_\ell'}{\bar{x}_i'\bar{x}_j'\bar{x}_k'\bar{x}_\ell'}\right] = \delta^2\left[\frac{3(\Sigma x^t)_s\{(\Sigma x^t)_s-(x_i^t+x_j^t+x_k^t+x_\ell^t)\}+2x_i^t(x_j^t+x_k^t+x_\ell^t)}{\{(\Sigma x)_s-x_i\}\{(\Sigma x)_s-x_j\}\{(\Sigma x)_s-x_k\}\{(\Sigma x)_s-x_\ell\}}\right.$$

$$\left.+\frac{2x_j^t(x_k^t+x_\ell^t)+2x_k^t x_\ell^t}{\{(\Sigma x)_s-x_i\}\{(\Sigma x)_s-x_j\}\{(\Sigma x)_s-x_k\}\{(\Sigma x)_s-x_\ell\}}\right]$$

APPENDIX III

The following relationships have been used in the computation of the multiple-summations in $\varepsilon(\text{MSE } v_3)$:

1. $$\sum_{i \neq j}\sum a_i b_j = (\Sigma a_i)(\Sigma b_i) - \Sigma a_i b_i$$

2. $$\sum_{i \neq j \neq k}\sum\sum a_i b_j c_k = (\Sigma a_i)(\Sigma b_i)(\Sigma c_i) - [(\Sigma a_i)(\Sigma b_i c_i) + (\Sigma b_i)(\Sigma a_i c_i) + (\Sigma c_i)(\Sigma a_i b_i)] + 2\Sigma a_i b_i c_i$$

3. $$\begin{aligned}\sum_{i \neq j \neq k \neq \ell}\sum\sum\sum a_i b_j c_k d_\ell &= (\Sigma a_i)(\Sigma b_i)(\Sigma c_i)(\Sigma d_i) + 2[(\Sigma a_i)(\Sigma b_i c_i d_i) \\ &+ (\Sigma b_i)(\Sigma a_i c_i d_i) + (\Sigma c_i)(\Sigma a_i b_i d_i) + (\Sigma d_i)(\Sigma a_i b_i c_i)] \\ &- [(\Sigma a_i)(\Sigma b_i)(\Sigma c_i d_i) + (\Sigma a_i)(\Sigma c_i)(\Sigma b_i d_i) + (\Sigma a_i)(\Sigma d_i)(\Sigma b_i c_i) \\ &+ (\Sigma b_i)(\Sigma c_i)(\Sigma a_i d_i) + (\Sigma b_i)(\Sigma d_i)(\Sigma a_i c_i) + (\Sigma c_i)(\Sigma d_i)(\Sigma a_i b_i)] \\ &+ (\Sigma a_i b_i)(\Sigma c_i d_i) + (\Sigma a_i c_i)(\Sigma b_i d_i) + (\Sigma a_i d_i)(\Sigma b_i c_i) - 6\Sigma a_i b_i c_i d_i .\end{aligned}$$

APPENDIX IV

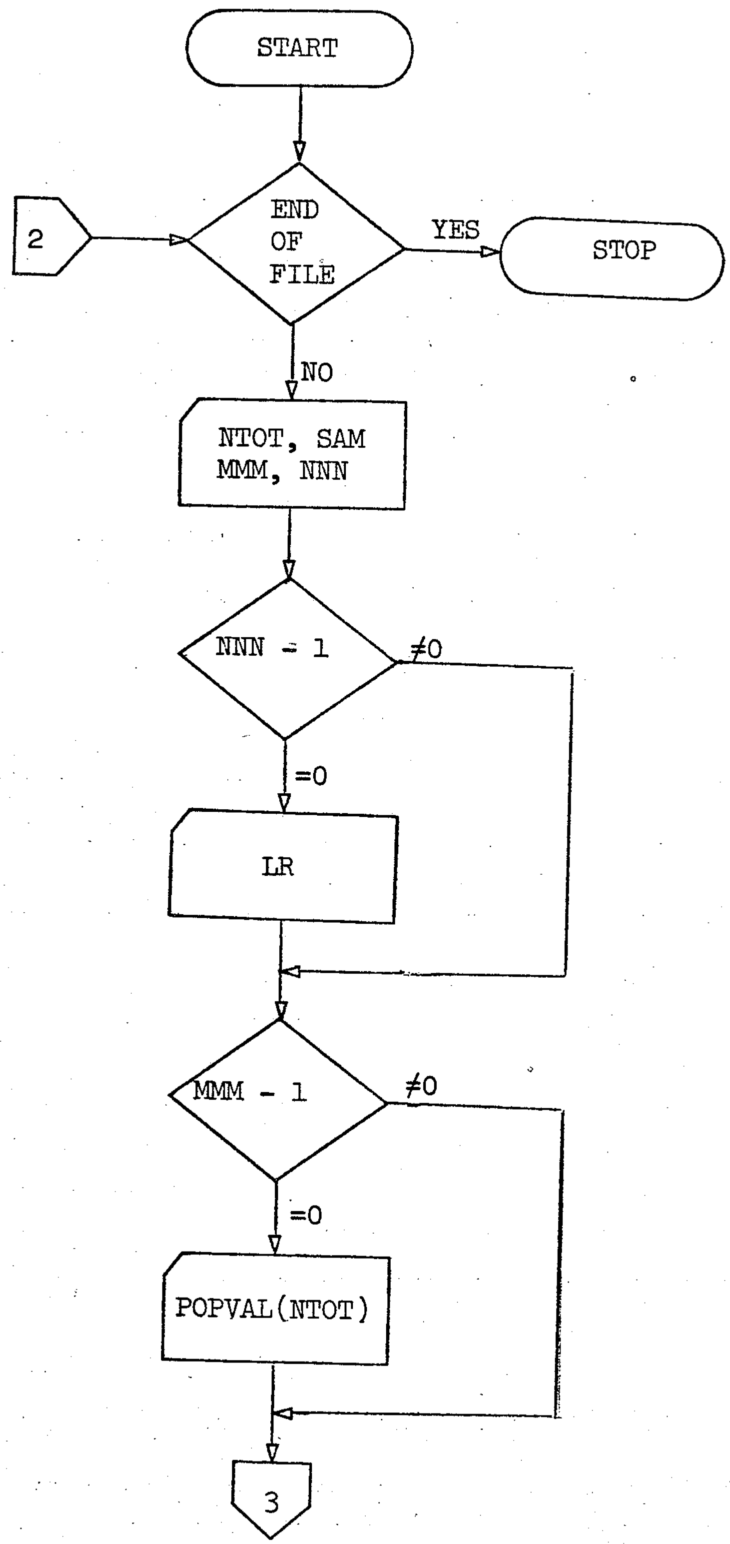
COMPUTER PROGRAM

The computer program given here consists of a main program and five subroutines. The main program generates 2,000 random samples each of a given sample size n for any given population. The random sample generator is the IBM Subroutine RANDU which generates pseudo-random numbers from a uniform distribution. The pseudo-random numbers begin to repeat themselves after approximately 50,000 selections. To overcome this drawback, a new starting position LR is read into the generator for each new population (after 6,000 selections).

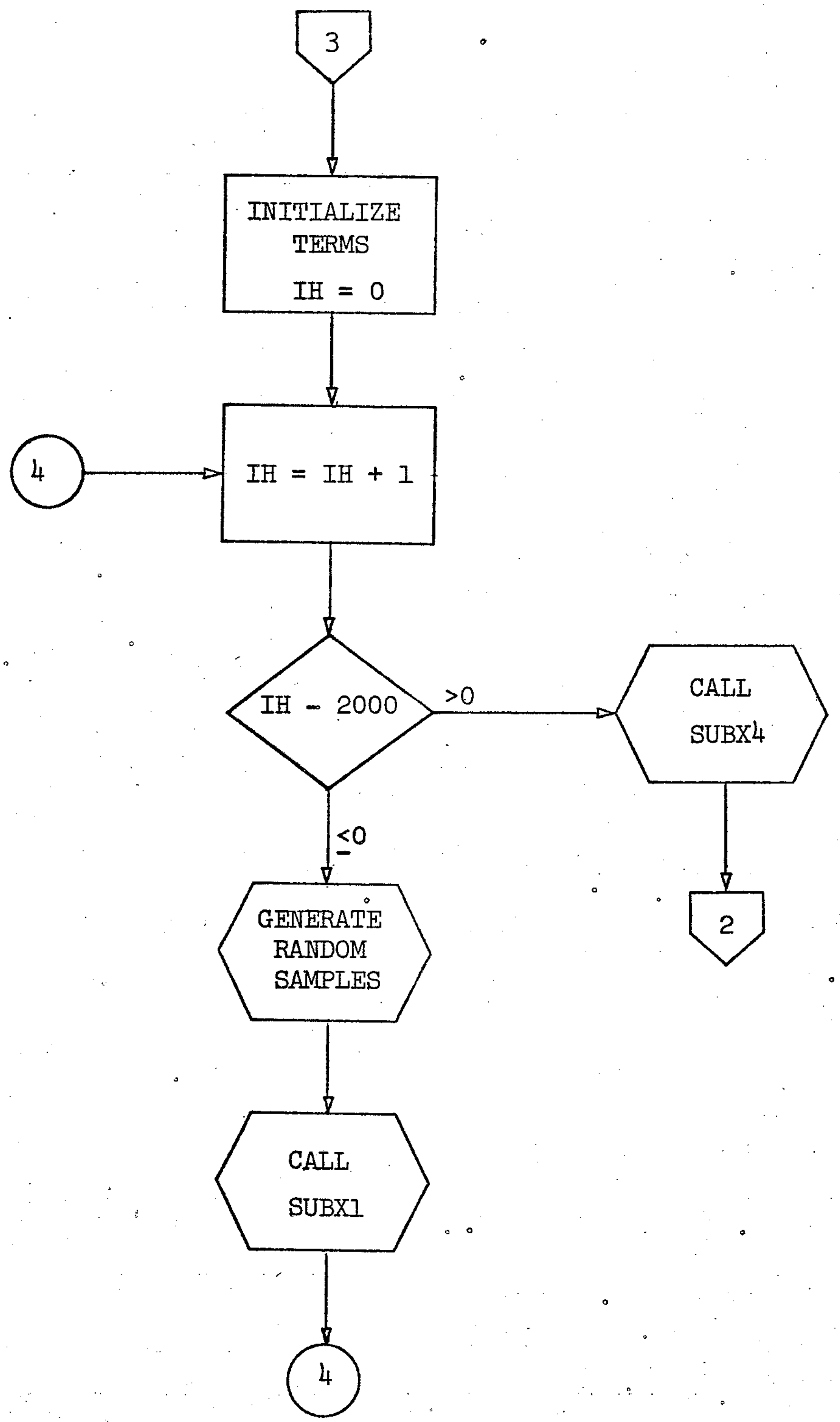
The flowchart for subroutine SUBX1 shows the calculation of the vector A which contains the different terms used in the three estimators explained in the program. The vector A(IJ, IP) was partitioned into two parts as it was found that round-off errors were large for certain terms. TERMS(1) had no round-off error for $n < 3$ and only required a minor adjustment for $n = 3$ (where STOT(1) = $(\Sigma x)_s$ was divided into all terms). TERMS(2) showed a higher degree of discrepancy, so STOT(2) = $(\Sigma x^2)_s$ was needed as a stabilizing factor. The proper factor STOT(I), I = 1, 2 was re-entered after the round-off error had been overcome.

The flowcharts for subroutines SUBX2, SUBX3, SUBX4, and SUBX5 are not shown, as their logic needs no external explanation.

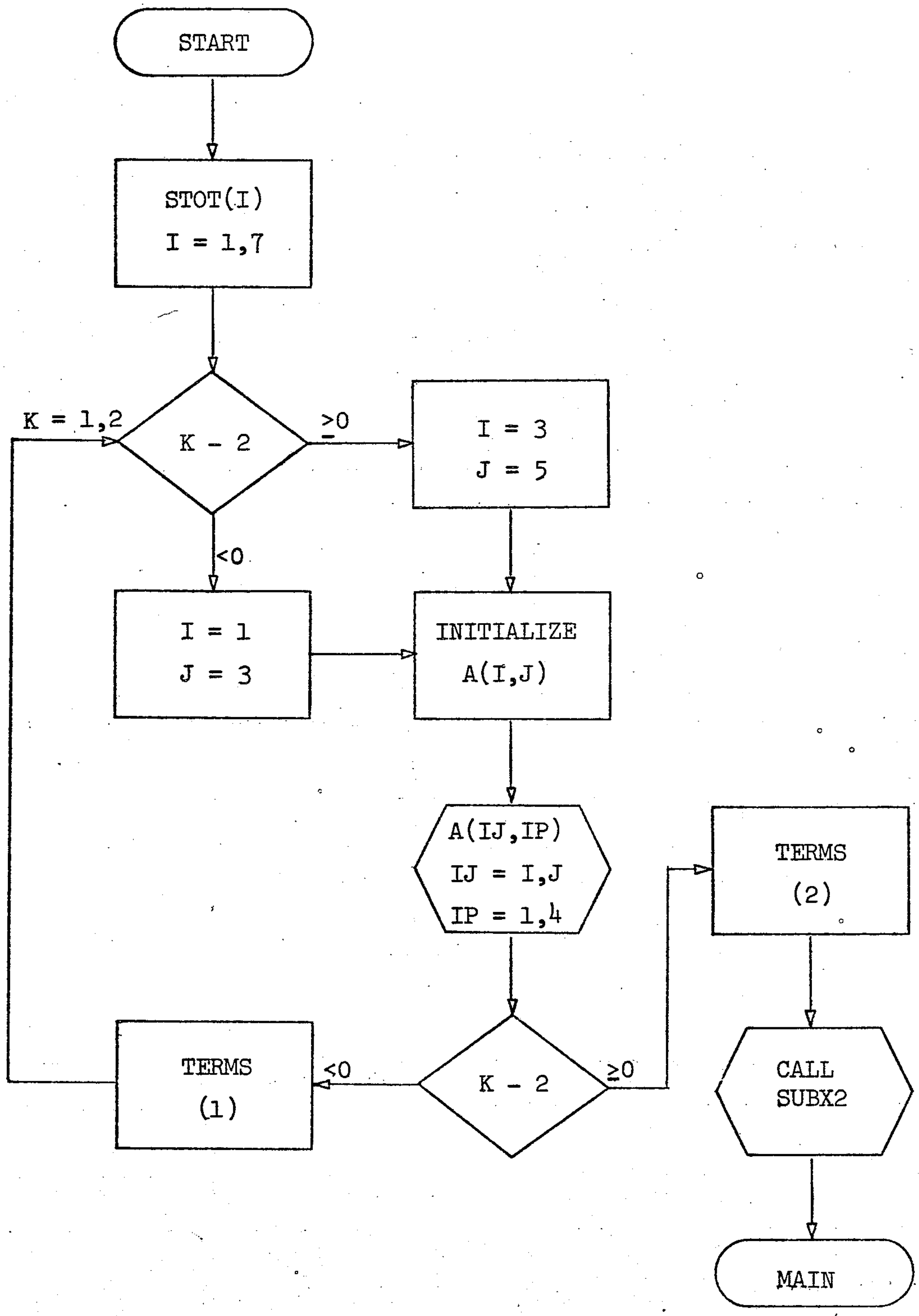
FLOWCHART FOR MAIN PROGRAM



FLOWCHART FOR MAIN PROGRAM
(continued)



FLOWCHART FOR SUBROUTINE SUBX1



```
      COMMON/AAAA/ XXX(12)
      COMMON/BBBB/ SAM, SAMSIZ, SAMSAM, S1, S2, SS1
      COMMON/CCCC/ BLUE, BLACK
      COMMON/DDDD/ TITLE(18), NTOT, CV
      COMMON/EEEE/ ALFA1, ALFA2, ALFA3, ALFA4, ALFA5, ALFA6, ALFA1B,
     1  BCOEF1(3), BCOEF2(3), BCOEF3(3), BCOEF4(3), BCOEF5(3), BCOEF6(3)
      COMMON/FFFF/ BCOEAV(3), ALF1X4, ALF2X4, ALF3X4, ALSQ1, ALSQ2,
     1  ALSQ3, ASQD1(3), ASQD2(3), ASQD3(3), DELSQ1(3), DELSQ2(3),
     2  DELSQ3(3), DELTA1(3), DELTA2(3), DELTA3(3)
      COMMON/GGGG/ PTOT(2), PTOT1, PTOT2, PPTOT, ANTOT, AANT, A1, AT1,
     1  VARPOP, POPVAR
C
      DIMENSION  POPVAL(270), IRN(12), JJ(12)
C
      INTEGER  SAM
C
C                                   ---------------
C
C             NTOT - POPULATION SIZE
C             SAM  - SAMPLE SIZE
C
    1 READ (5,2,END=20) NTOT, SAM, NNN, MMM
    2 FORMAT (4I5)
C
C             LR   - ODD NUMBER USED TO GENERATE RANDOM NUMBER GENERATOR
C
      IF (NNN.NE.1) GO TO 4
      READ (5,3) LR
    3 FORMAT (I9)
C
C             TITLE - VECTOR CONTAINING HEADINGS FOR EACH NEW POPULATION
C
    4 IF (MMM.NE.1) GO TO 10
      READ (5,5) (TITLE(I),I=1,18)
    5 FORMAT (18A4)
C
C             POPVAL - VECTOR CONTAINING THE POPULATION VALUES
C
      READ (5,6) (POPVAL(I),I=1,NTOT)
    6 FORMAT (16F5.0)
C
C             PTOT - VECTOR OF LENGTH 2 CONTAINING POPULATION CALCULATIONS
C                    PTOT(1) - SUM (XI)
C                    PTOT(2) - SUM (XI**2)
C
      DO 7 I = 1,2
    7 PTOT(I) = 0.0
      DO 9 I = 1,NTOT
      PI = POPVAL(I)
      PA = PI
      DO 8 J = 1,2
      PTOT(J) = PTOT(J) + PI
    8 PI = PI*PA
    9 CONTINUE
      PTOT1 = PTOT(1)
      PTOT2 = PTOT(2)
```

```
      PPTOT = PTOT1*PTOT1
      ANTOT = NTOT
      AANT  = ANTOT*ANTOT
      A1    = NTOT - 1
C
C           VARPOP - POPULATION VARIANCE
C           SDPOP  - POPULATION STANDARD DEVIATION
C           CV     - COEFFICIENT OF VARIATION
C
      VARPOP = (ANTOT*PTOT2 - PPTOT)/(A1*ANTOT)
      SDPOP  = SQRT(VARPOP)
      CV     = ANTOT*SDPOP/PTOT1
      AT1    = A1*PTOT1
      AAT    = AT1*AT1
      POPVAR = ANTOT*VARPOP
C
   10 SAMSIZ = SAM
      SAMSAM = SAMSIZ*SAMSIZ
      S1  = SAM - 1
      S2  = SAM - 2
      SS1 = S1*S1
C
      BLUE  = 1.0/SAMSIZ - 1.0/ANTOT
      BLACK = BLUE*BLUE
C
C           INITIALIZE ALL TERMS
C
      ALFA1 = 0.0
      ALFA2 = 0.0
      ALFA3 = 0.0
      ALFA4 = 0.0
      ALFA5 = 0.0
      ALFA6 = 0.0
C
      ALSQ1 = 0.0
      ALSQ2 = 0.0
      ALSQ3 = 0.0
C
      ALF1X4 = 0.0
      ALF2X4 = 0.0
      ALF3X4 = 0.0
C
      DO 11 I = 1,3
      BCOEF1(I) = 0.0
      BCOEF2(I) = 0.0
      BCOEF3(I) = 0.0
      BCOEF4(I) = 0.0
      BCOEF5(I) = 0.0
      BCOEF6(I) = 0.0
C
      DELSQ1(I) = 0.0
      DELSQ2(I) = 0.0
      DELSQ3(I) = 0.0
C
      DELTA1(I) = 0.0
      DELTA2(I) = 0.0
```

```
      DELTA3(I) = 0.0
C
      ASQD1(I)  = 0.0
      ASQD2(I)  = 0.0
   11 ASQD3(I)  = 0.0
C
C            THIS SECTION GENERATES 2000 RANDOM SAMPLES
C            TAKEN WITHOUT REPLACEMENT
C            XXX - VECTOR OF LENGTH 'SAM' CONTAINING THE SAMPLE VALUES
C
      IH = 0
C
   12 IH = IH + 1
      IF (IH.GT.2000) GO TO 19
      DO 18 I = 1,SAM
   13 LR = LR*65539
      IF (LR) 14,15,15
   14 LR = LR + 2147483647 + 1
   15 RL = LR
      RL = RL*.4656613E-9
      IRN(I) = RL*ANTOT + 1.0
      IF (IRN(I).EQ.(NTOT+1)) GO TO 13
      IF (I.EQ.1) GO TO 17
      I1 = I - 1
      DO 16 K = 1,I1
      IF (IRN(I).EQ.IRN(K)) GO TO 13
   16 CONTINUE
   17 II = IRN(I)
      XXX(I) = POPVAL(II)
   18 CONTINUE
C
      CALL SUBX1
C
      GO TO 12
   19 CALL SUBX4
C
      GO TO 1
C
   20 STOP
      END
```

```
      SUBROUTINE  SUBX1
C
C
      COMMON/AAAA/ XXX(12)
      COMMON/BBBB/ SAM, SAMSIZ, SAMSAM, S1, S2, SS1
      COMMON/HHHH/ QV1, QV2, QV3, QV4, QV5, QV6, QV7, QV8, QV9, QV10,
     1 QV11, QV12, QV13, QV14, QV15, QV16, QV17, QV18, QV19, QV20,
     2 QV21, QV22, QV23, QA1(2), QA2(2), QA3(2), QA4(2), QA5(2),
     3 QA6(2), QA7(2), QA8(2), QA9(2), QA10(2), QA11(2), QA12(2),
     4 QA13(2), QA14(2), QA15(2), QA16(2), QA17(2), QA18(2), QA19(2),
     5 QA20(2), QA21(2)
      COMMON/IIII/ STOT(7), SSTOT, ST1, SST, S2TOT, SVAR, SAMVAR, SSVAR
      COMMON/JJJJ/ Q1, Q2, Q3, Q4, Q5, Q6, Q7
C
      DIMENSION  A(5,4), X(5)
C
      EQUIVALENCE  (B1,A(1,1)),(B2,A(2,1)),(B3,A(3,1)),(B5,A(5,1)),
     1             (C1,A(1,2)),(C2,A(2,2)),(C3,A(3,2)),(C5,A(5,2)),
     2             (D1,A(1,3)),(D2,A(2,3)),(D3,A(3,3)),(D5,A(5,3)),
     3             (E1,A(1,4)),(E2,A(2,4)),(E3,A(3,4)),(E5,A(5,4))
      EQUIVALENCE  (STOT1,STOT(1)),(STOT2,STOT(2))
C
      INTEGER  SAM
C
C                              ---------------
C
C         STOT - VECTOR OF LENGTH 7 CONTAINING SAMPLE CALCULATIONS
C                STOT(1) - SUM (XXX(I))
C                STOT(2) - SUM (XXX(I)**2)
C                STOT(3) - SUM (XXX(I)**3)
C                STOT(4) - SUM (XXX(I)**4)
C                STOT(5) - SUM (XXX(I)**5)
C                STOT(6) - SUM (1.0/XXX(I))
C                STOT(7) - SUM (1.0/(XXX(I)**2))
C
      DO 1 I = 1,7
    1 STOT(I) = 0.0
      DO 2 I = 1,SAM
      XT = XXX(I)
      XH = XT
      STOT(1) = STOT(1) + XT
      STOT(6) = STOT(6) + 1.0/XT
      XT = XT*XH
      STOT(2) = STOT(2) + XT
      STOT(7) = STOT(7) + 1.0/XT
      XT = XT*XH
      STOT(3) = STOT(3) + XT
      XT = XT*XH
      STOT(4) = STOT(4) + XT
      XT = XT*XH
    2 STOT(5) = STOT(5) + XT
C
C         SVAR - SAMPLE VARIANCE
C
      SSTOT   = STOT1*STOT1
      ST1     = S1*STOT1
```

```
      SST    = ST1*ST1
      S2TOT  = STOT2*STOT2
      SVAR   =(SAMSIZ*STOT2 - SSTOT)/(S1*SAMSIZ)
      SAMVAR = SAMSIZ*SVAR
      SSVAR  =(SSTOT - SAMVAR)/SSTOT
C
C            A - VECTOR CONTAINING THE DIFFERENT TERMS USED IN
C                MICKEY'S, QUENOUILLE'S AND TUKEY'S ESTIMATORS.
C                THE EQUATIONS ARE SET UP ALONG THE LINES SHOWN
C                IN APPENDIX III.
C
      DO 13 K = 1,2
      IF (K - 2) 3,4,4
    3 MAR = 1
      MOP = 3
      GO TO 5
    4 MAR = 3
      MOP = 5
    5 DO 7 I = MAR,MOP
      DO 6 J = 1,4
    6 A(I,J) = 0.0
    7 CONTINUE
      DO 10 IJ = MAR,MOP
      IF (IJ.EQ.4) GO TO 10
      J = IJ - 1
      DO 9 I = 1,SAM
      XI  = XXX(I)
      SXI = STOT1 - XI
      SXJ = SXI
      IF (IJ.GE.3) SXI = (STOT1 - XI)*STOT(K)
      X(IJ) = XI**J
      DO 8 IP = 1,4
      A(IJ,IP) = X(IJ)/SXI + A(IJ,IP)
    8 SXI = SXI*SXJ
    9 CONTINUE
   10 CONTINUE
C
      IF (K - 2) 11,12,12
C
   11 WO  = B1*B1 - C1
      WO1 = B1*B2 - C2
      QV1 = SS1*C1
      QV3 = S1*C1
      QV5 = SS1*WO
      QV8 = S2*WO
      QA2(1) = STOT1*C1 - C2
      QA6(1) = STOT1*WO - 2.0*WO1
      Q1 = B1
      Q2 = C1
      Q4 = WO
      Q5 = STOT1*WO - 2.0*WO1
      IF (SAM.LT.4) GO TO 13
C
      W1  = C1*C1 - E1
      W11 = C1*C2 - E2
      W12 = C1*C3 - E3
```

```
      W14 = C2*C2/STOT1 - E3
      W2  = B1*D1 - E1
      W21 = B1*D2 - E2
      W22 = B1*D3 - E3
      W24 = B2*D1 - E2
      W26 = B2*D2/STOT1 - E3
      W3  = B1*B1*C1 - C1*C1 - 2.0*B1*D1 + 2.0*E1
      W31 = B1*B1*C2 - C1*C2 - 2.0*B1*D2 + 2.0*E2
      W32 = B1*B1*C3 - C1*C3 - 2.0*B1*D3 + 2.0*E3
      W34 = B1*B2*C1 - C1*C2 - B2*D1 - B1*D2 + 2.0*E2
      W36 =(B1*B2*C2 - C2*C2 - B2*D2)/STOT1 - B1*D3 + 2.0*E3
      W38 =(B2*B2*C1 - 2.0*B2*D2)/STOT1 - C1*C3 + 2.0*E3
      W4  = B1*B1*B1*B1 + 8.0*B1*D1 - 6.0*B1*B1*C1 + 3.0*C1*C1 - 6.0*E1
      W41 = B1*B1*B1*B2 + 2.0*B2*D1 + 6.0*B1*D2 - 3.0*B1*B2*C1 -
     1  3.0*B1*B1*C2 + 3.0*C1*C2 - 6.0*E2
      W43 =(B1*B1*B2*B2 + 4.0*B2*D2 - B2*B2*C1 - 4.0*B1*B2*C2 +
     1     2.0*C2*C2)/STOT1 + 4.0*B1*D3 - B1*B1*C3 + C1*C3 - 6.0*E3
C
      QV2 = (3.0*SAMSAM - 6.0*SAMSIZ + 3.0)*E1
      QV4 = S1*E1
      QV6 = (3.0*SAMSAM - 10.0*SAMSIZ + 9.0)*W1
      QV7 = (3.0*SAMSAM -  9.0*SAMSIZ + 6.0)*W2
      QV9 = (4.0*SAMSIZ -  6.0)*W1
      QV10 = (2.0*SAMSIZ - 3.0)*W2
      QV11 = S1*W1
      QV12 = S2*W2
      QV13 = S1*W2
      QV14 = S2*W1
      QV15 = (3.0*SAMSAM - 11.0*SAMSIZ + 10.0)*W3
      QV16 = (4.0*SAMSIZ - 7.0)*W3
      QV17 = (2.0*SAMSIZ - 4.0)*W3
      QV18 = S2*W3
      QV19 = S1*W3
      QV20 = S2*W3
      QV21 = (3.0*SAMSAM - 12.0*SAMSIZ + 12.0)*W4
      QV22 = (4.0*SAMSIZ -  8.0)*W4
      QV23 = S2*W4
C
      QA1(1) = 3.0*STOT1*(E3 - 2.0*E2 + STOT1*E1)
      QA3(1) = STOT1*E1 - E2
      QA4(1) = STOT1*(4.0*W12 + 5.0*W14 - 10.0*W11 + 3.0*STOT1*W1)
      QA5(1) = 3.0*STOT1*(W22 + W26 - W24 - 2.0*W21 + STOT1*W2)
      QA7(1) = 4.0*STOT1*W1 - 6.0*W11
      QA8(1) = 2.0*STOT1*W2 - 2.0*W21 - W24
      QA9(1) = STOT1*W1 - W11
      QA10(1) = STOT1*W2 - W21 - W24
      QA11(1) = STOT1*W2 - W21
      QA12(1) = STOT1*W1 - 2.0*W11
      QA13(1) = STOT1*(2.0*(W32 + W38 + 3.0*W36) - 5.0*W31 - 6.0*W34 +
     1  3.0*STOT1*W3)
      QA14(1) = 4.0*STOT1*W3 - 3.0*W31 - 4.0*W34
      QA15(1) = 2.0*STOT1*W3 - 2.0*W31 - 2.0*W34
      QA16(1) = STOT1*W3 - 2.0*W34
      QA17(1) = STOT1*W3 - W31
      QA18(1) = STOT1*W3 - W31 - W34
      QA19(1) = 3.0*STOT1*(4.0*(W43 - W41) + STOT1*W4)
```

```
      QA20(1) = 4.0*STOT1*W4 - 8.0*W41
      QA21(1) = STOT1*W4 - 2.0*W41
      GO TO 13
C
   12 W02 = B1*B3 - C3
      QA2(2) = STOT2*(C1 - C3)
      QA6(2) = STOT2*(W0 - 2.0*W02)
      Q3 = STOT2*(C1 - C3)
      Q6 = STOT2*(W0 - 2.0*W02)
      Q7 = STOT2*(B1 - B3)
      IF (SAM.LT.4) GO TO 13
C
      W12 = C1*C3 - E3
      W13 = C1*C5 - E5
      W15 = C3*C3*STOT2 - E5
      W22 = B1*D3 - E3
      W23 = B1*D5 - E5
      W25 = B3*D1 - E3
      W27 = B3*D3*STOT2 - E5
      W32 = B1*B1*C3 - C1*C3 - 2.0*B1*D3 + 2.0*E3
      W33 = B1*B1*C5 - C1*C5 - 2.0*B1*D5 + 2.0*E5
      W35 = B1*B3*C1 - C1*C3 - B3*D1 - B1*D3 + 2.0*E3
      W37 = 2.0*E5 - B1*D5 + STOT2*(B1*B3*C3 - C3*C3 - B3*D3)
      W39 = 2.0*E5 - C1*C5 + STOT2*(B3*B3*C1 - 2.0*B3*D3)
      W42 = B1*B1*B1*B3 + 2.0*B3*D1 + 6.0*B1*D3 - 3.0*B1*B3*C1 -
     1  3.0*B1*B1*C3 + 3.0*C1*C3 - 6.0*E3
      W44 = 4.0*B1*D5 - B1*B1*C5 + C1*C5 - 6.0*E5 + STOT2*(B1*B1*B3*B3 -
     1      B3*B3*C1 + 2.0*(C3*C3 + 2.0*(B3*D3 - B1*B3*C3)))
C
      QA1(2) = 3.0*STOT2*(E5 + STOT2*(E1 - 2.0*E3))
      QA3(2) = STOT2*(E1 - E3)
      QA4(2) = STOT2*(4.0*W13 + 5.0*W15 + STOT2*(3.0*W1 - 10.0*W12))
      QA5(2) = 3.0*STOT2*(W23 + W27 + STOT2*(W2 - 2.0*W22 - W25))
      QA7(2) = 2.0*STOT2*(2.0*W1 - 3.0*W12)
      QA8(2) = STOT2*(2.0*(W2 - W22) - W25)
      QA9(2) = STOT2*(W1 - W12)
      QA10(2) = STOT2*(W2 - W22 - W25)
      QA11(2) = STOT2*(W2 - W22)
      QA12(2) = STOT2*(W1 - 2.0*W12)
      QA13(2) = STOT2*(2.0*(W33 + 3.0*W37 + W39) + STOT2*(3.0*(W3 -
     1          2.0*W35) - 5.0*W32))
      QA14(2) = STOT2*(4.0*(W3 - W35) - 3.0*W32)
      QA15(2) = 2.0*STOT2*(W3 - W32 - W35)
      QA16(2) = STOT2*(W3 - 2.0*W35)
      QA17(2) = STOT2*(W3 - W32)
      QA18(2) = STOT2*(W3 - W32 - W35)
      QA19(2) = 3.0*STOT2*(4.0*W44 + STOT2*(W4 - 4.0*W42))
      QA20(2) = 4.0*STOT2*(W4 - 2.0*W42)
      QA21(2) = STOT2*(W4 - 2.0*W42)
C
   13 CONTINUE
C
      CALL SUBX2
C
      RETURN
      END
```

```
      SUBROUTINE  SUBX2
C
C
      COMMON/BBBB/ SAM, SAMSIZ, SAMSAM, S1, S2, SS1
      COMMON/FFFF/ BCOEAV(3), ALF1X4, ALF2X4, ALF3X4, ALSQ1, ALSQ2,
     1  ALSQ3, ASQD1(3), ASQD2(3), ASQD3(3), DELSQ1(3), DELSQ2(3),
     2  DELSQ3(3), DELTA1(3), DELTA2(3), DELTA3(3)
      COMMON/GGGG/ PTOT(2), PTOT1, PTOT2, PPTOT, ANTOT, AANT, A1, AT1,
     1  VARPOP, POPVAR
      COMMON/HHHH/ QV1, QV2, QV3, QV4, QV5, QV6, QV7, QV8, QV9, QV10,
     1  QV11, QV12, QV13, QV14, QV15, QV16, QV17, QV18, QV19, QV20,
     2  QV21, QV22, QV23, QA1(2), QA2(2), QA3(2), QA4(2), QA5(2),
     3  QA6(2), QA7(2), QA8(2), QA9(2), QA10(2), QA11(2), QA12(2),
     4  QA13(2), QA14(2), QA15(2), QA16(2), QA17(2), QA18(2), QA19(2),
     5  QA20(2), QA21(2)
      COMMON/IIII/ STOT(7), SSTOT, ST1, SST, S2TOT, SVAR, SAMVAR, SSVAR
      COMMON/KKKK/ AMOTHR
      EQUIVALENCE  (STOT1,STOT(1)),(STOT2,STOT(2)),(STOT3,STOT(3)),
     1             (STOT4,STOT(4)),(STOT5,STOT(5))
C
      DIMENSION  DELINT(3), ASQINT(3), DELSQA(3)
C
      INTEGER  SAM
C
C                     ---------------
C
C          FIRST AND SECOND VARIANCES (PRELIMINARY CALCULATIONS)
C
      ALINT  = SAMSAM*SAMVAR*SAMVAR/(SSTOT*SSTOT)
C
      AMOTHR = SAMSAM*PPTOT/(AANT*SSTOT)
      FATHER = AMOTHR*AMOTHR
      CARPET = 2.0/SS1 + SAMVAR/SSTOT*(3.0*SAMVAR/SSTOT + 6.0/S1 + 1.0)
      RUGG   = 2.0/S1 + 3.0*SAMVAR/SSTOT
      AMOM = 3.0*SVAR*SVAR/(SSTOT*SSTOT) + (SAMSAM + 2.0*SAMSIZ + 3.0)/
     1      (SAMSAM*SS1) + 2.0*(SAMSIZ + 3.0)*SVAR/(SAMSIZ*SSTOT*S1)
      DAD  = 2.0*(SAMSIZ + 2.0)/(SAMSIZ*SS1) + 4.0*SVAR/(ST1*STOT1)
C
      ASQINT(1) = 4.0*SAMSAM*STOT2/SST - 4.0*SAMSIZ/S1*RUGG +
     1           2.0*SAMSIZ*CARPET
      ASQINT(2) = 4.0*SAMSAM*STOT3/SST - 4.0*SAMSIZ*STOT2/ST1*RUGG +
     1           2.0*STOT1*CARPET
      ASQINT(3) = 4.0*SAMSAM*STOT4/SST - 4.0*SAMSIZ*STOT3/ST1*RUGG +
     1           2.0*STOT2*CARPET
C
      DELINT(1) = 4.0*SAMSIZ*STOT2/SST - 8.0*STOT1/(ST1*S1) +
     1            SAMSAM*AMOM
      DELINT(2) = SAMSIZ*DAD + STOT1*((8.0*STOT1 - 4.0*SAMSIZ*STOT1)/
     1            SST - 12.0/(S1*ST1) - 12.0*SAMVAR/(ST1*SSTOT))
      DELSQA(1) = DELINT(1) + DELINT(2)
C
      DELINT(1) = 4.0*STOT1*STOT3/SST - 8.0*STOT3/(ST1*S1) + SSTOT*AMOM
      DELINT(2) = STOT2*DAD + STOT2*((8.0*STOT2 - 4.0* SSTOT) /SST -
     1            12.0*STOT1/(SAMSIZ*S1*ST1) - 12.0*STOT1*SVAR/
     2            (ST1*SSTOT))
      DELSQA(2) = DELINT(1) + DELINT(2)
```

```
C
      DELINT(1) = 4.0*STOT2*STOT4/SST - 8.0*STOT5/(ST1*S1) + S2TOT*AMOM
      DELINT(2) = STOT4*DAD + STOT3*((8.0*STOT3 - 4.0*STOT2*STOT1)/SST -
     1            12.0*STOT2/(SAMSIZ*S1*ST1) - 12.0*STOT2*SVAR/
     2            (ST1*SSTOT))
      DELSQA(3) = DELINT(1) + DELINT(2)
C
      ALF1X4 = ALINT + ALF1X4
      ALF2X4 = ALINT*FATHER + ALF2X4
      DO 1 I = 1,3
      ASQD1(I)  = ASQINT(I) + ASQD1(I)
      ASQD2(I)  = ASQINT(I)*FATHER + ASQD2(I)
      DELSQ1(I) = DELSQA(I) + DELSQ1(I)
    1 DELSQ2(I) = DELSQA(I)*FATHER + DELSQ2(I)
C
C          THIRD VARIANCE (PRELIMINARY CALCULATIONS)
C
      IF (SAM.LT.4) GO TO 3
C
      BOOK = 1.0/SAMSIZ*(S1*QV1 - QV5)
C
      ALF3X4    = BOOK*BOOK + ALF3X4
      ASQD3(1)  = SS1/SAMSAM*(2.0*QV9 + 4.0*QV16 + QV22 + 4.0*QV14 +
     1            8.0*QV20 + 2.0*QV23 - S1*(4.0*QV12 + 2.0*QV18 +
     2            4.0*QV13 + 2.0*QV19 + 8.0*QV10 + 4.0*QV17 - S1*
     3           (6.0*QV4 + 2.0*QV11 + 4.0*QV14))) + ASQD3(1)
      DELSQ3(1) = (SS1*QV2 + (SS1 + 2.0)*QV6 + (4.0 - 2.0*S1)*QV15 +
     1             QV21 - 4.0*S1*QV7)/SAMSAM + DELSQ3(1)
      DO 2 I = 2,3
      J = I - 1
      ASQD3(I)  = SS1/SAMSAM*(2.0*QA7(J) + 4.0*QA14(J) + QA20(J) +
     1            4.0*QA12(J) + 8.0*QA18(J) + 2.0*QA21(J) - S1*
     2           (4.0*QA10(J) + 2.0*QA16(J) + 4.0*QA11(J) + 2.0*QA17(J)
     3          + 8.0*QA8(J) + 4.0*QA15(J) - S1*(6.0*QA3(J) + 2.0*QA9(J)
     4          + 4.0*QA12(J)))) + ASQD3(I)
    2 DELSQ3(I) =(SS1*QA1(J) + (SS1 + 2.0)*QA4(J) + (4.0 - 2.0*S1)*
     1            QA13(J) + QA19(J) - 4.0*S1*QA5(J))/SAMSAM + DELSQ3(I)
C
C          FIRST AND SECOND BIASES (PRELIMINARY CALCUALTIONS)
C
    3 HOME = (STOT2 + SSTOT)/(ST1*STOT1)
C
      ALINT     = SAMSIZ/S1*(SAMSIZ*STOT2 - SSTOT)/SSTOT
      DELINT(1) = SAMSIZ*HOME - 2.0/S1
      DO 4 I = 2,3
      J = I - 1
    4 DELINT(I) = STOT(J)*HOME - 2.0*STOT(I)/ST1
C
      ALSQ1 = ALINT + ALSQ1
      ALSQ2 = ALINT*AMOTHR + ALSQ2
      DO 5 I = 1,3
      DELTA1(I) = DELINT(I) + DELTA1(I)
    5 DELTA2(I) = DELINT(I)*AMOTHR + DELTA2(I)
C
C          THIRD BIAS (PRELIMINARY CALCULATIONS)
C
```

```
      DEAD   = PPTOT*S1/(AANT*SAMSIZ)
C
      ALSQ3     = DEAD*(S1*QV1 - QV5) + ALSQ3
      DELTA3(1) = DEAD*(S1*QV3 - QV8) + DELTA3(1)
      DO 6 I = 2,3
      J = I - 1
    6 DELTA3(I) = DEAD*(S1*QA2(J) - QA6(J)) + DELTA3(I)
C
      CALL SUBX3
C
      RETURN
      END
```

```
      SUBROUTINE  SUBX3
C
C
      COMMON/BBBB/ SAM, SAMSIZ, SAMSAM, S1, S2, SS1
      COMMON/CCCC/ BLUE, BLACK
      COMMON/EEEE/ ALFA1, ALFA2, ALFA3, ALFA4, ALFA5, ALFA6, ALFA1B,
     1  BCOEF1(3), BCOEF2(3), BCOEF3(3), BCOEF4(3), BCOEF5(3), BCOEF6(3)
      COMMON/GGGG/ PTOT(2), PTOT1, PTOT2, PPTOT, ANTOT, AANT, A1, AT1,
     1  VARPOP, POPVAR
      COMMON/IIII/ STOT(7), SSTOT, ST1, SST, S2TOT, SVAR, SAMVAR, SSVAR
      COMMON/JJJJ/ Q1, Q2, Q3, Q4, Q5, Q6, Q7
      COMMON/KKKK/ AMOTHR
C
      EQUIVALENCE  (STOT1,STOT(1)),(STOT2,STOT(2)),(STOT3,STOT(3)),
     1             (STOT4,STOT(4)),(STOT5,STOT(5)),(STOT6,STOT(6)),
     2             (STOT7,STOT(7))
C
      REAL  MB1, MB2
C
      INTEGER  SAM
C
C                             ---------------
C
C           CLASSICAL RATIO ESTIMATOR
C           ALFA1B - BIAS ACLCULATIONS
C
      BED = SAMSIZ*PTOT1
      DEB = ANTOT*STOT1
      BEDDEB = BED - DEB
      BBEDD  = BEDDEB*BEDDEB
C
      ALFA1B = BEDDEB/DEB + ALFA1B
      ALFA1 = BBEDD/(DEB*DEB)+ ALFA1
      BCOEF1(1) = BED/(ANTOT*DEB)*(PTOT1/STOT1-2.0) + 1.0/ANTOT +
     1            BCOEF1(1)
      BCOEF1(2) = PTOT1/AANT*(PTOT1-STOT1)/STOT1 + BCOEF1(2)
      BCOEF1(3) = STOT2*PTOT1*(PTOT1-2.0*STOT1)/(DEB*DEB) +
     1            PTOT2/AANT + BCOEF1(3)
C
C           BEALE
C
      SPTOT = STOT1*PTOT1
      PPA = ANTOT - SAMSIZ
      PPB = PPA*SAMVAR
      PPC = PPA*PPB
      PPD = DEB*STOT1 + PPB
      PPE = (STOT1*BEDDEB - PPB)/PPD
      PPF = SAMSIZ*PPA*PTOT1/(ANTOT*S1*PPD)
      PPG = (S2 + SAMSIZ/ANTOT)/S1
C
      ALFA2 = PPE*PPE + ALFA2
      BCOEF2(1) = BED/(ANTOT*PPD)*(PTOT1*(ST1*STOT1*AANT + PPC)/
     1           (ANTOT*S1*PPD) - 2.0*STOT1) + 1.0/ANTOT + BCOEF2(1)
      DO 1 J = 1,2
      J1 = J + 1
      J2 = J + 2
```

```
      COEF1 = STOT(J2)*PPF*PPF + STOT(J1)*2.0*BED*PPA/(AANT*S1*PPD)*
     1        (SPTOT*(ANTOT*S2 + SAMSIZ)/(S1*PPD) - 1.0)
      COEF2 = STOT(J)*PTOT1/(AANT*PPD)*(SPTOT*AANT*PPG*PPG*STOT1/PPD -
     1        2.0*STOT1*(ANTOT*S2 + SAMSIZ)/S1) + PTOT(J)/AANT
    1 BCOEF2(J1) = COEF1 + COEF2 + BCOEF2(J1)
C
C          MODIFIED BEALE
C
      TALL  = BED/DEB
      SHORT = TALL*SSVAR - 1.0
      PDEB  = PTOT1/DEB
      TBS1  = TALL*BLUE/S1
      MB1 = 1.0 - SAMSIZ*SAMVAR*BLUE/SSTOT
      MB2 = MB1*TALL - 1.0
C
      ALFA3      = MB2*MB2 + ALFA3
      BCOEF3(1) = (MB2 + 1.0)*(PDEB*MB1 - 2.0/ANTOT) + SAMSAM*AMOTHR*
     1             SVAR*BLACK/(ST1*STOT1) + 1.0/ANTOT + BCOEF3(1)
      DO 2 J = 1,2
      J1 = J + 1
      J2 = J + 2
      COEF1 = STOT(J2)*SAMSAM*AMOTHR*BLACK/SST + STOT(J1)*(2.0*SAMSIZ*
     1        TBS1/STOT1*(PDEB*MB1 - TBS1 - 1.0/ANTOT))
      COEF2 = STOT(J)*(PDEB*MB1*(PDEB*MB1 - 2.0*TBS1 - 2.0/ANTOT) +
     1        TBS1*(TBS1 + 2.0/ANTOT)) + PTOT(J)/AANT
    2 BCOEF3(J1) = COEF1 + COEF2 + BCOEF3(J1)
C
C          HARTLEY - ROSS
C
      USSR = (ANTOT - SAMSIZ)/(ANTOT*S1)
      PINK = PTOT1/(ANTOT*SAMSIZ) - (A1*STOT1/(ANTOT*SAMSIZ*S1))
      PPINK  = PINK*PINK
      RUBBER = PINK*STOT6 + USSR
      BLAST = 2.0*PINK*USSR
      BEER  =(ANTOT - 2.0*SAMSIZ+1.0)*A1/(AANT*SS1)
C
      ALFA4 = RUBBER*RUBBER + ALFA4
      BCOEF4(1) = PPINK*STOT7 + BLAST*STOT6 + (A1*A1*SAMSIZ - S1*(2.0*A1
     1             *SAMSIZ - ANTOT*S1))/(AANT*SS1) + BCOEF4(1)
      BCOEF4(2) = PPINK*STOT6 + BLAST*SAMSIZ + STOT1*BEER +
     1             PTOT1/AANT + BCOEF4(2)
      BCOEF4(3) = PPINK*SAMSIZ + BLAST*STOT1 + STOT2*BEER +
     1             PTOT2/AANT + BCOEF4(3)
C
C          MICKEY
C
      HAS   = ANTOT - SAMSIZ + 1.0
      HHASS = HAS*(HAS - 2.0)
      OPEN  = (PTOT1 - HAS*STOT1)/(SAMSIZ*ANTOT)
      CLOSE = OPEN*OPEN
      PARTY = OPEN*(ANTOT-SAMSIZ)/ANTOT
      HEX   = Q1*S1*OPEN + S1*(ANTOT-SAMSIZ)/ANTOT
C
      ALFA5 = HEX*HEX + ALFA5
      BCOEF5(1) = (SAMSIZ*HHASS+ANTOT)/AANT + (S1*Q2+S2*Q4)*CLOSE +
     1            2.0*S1*Q1*PARTY + BCOEF5(1)
```

```
      BCOEF5(2) = STOT1*HHASS/AANT + (Q1+Q5)*CLOSE + 2.0*SAMSIZ*PARTY +
     1            PTOT1/AANT + BCOEF5(2)
      BCOEF5(3) = STOT2*HHASS/AANT + (Q3+Q6)*CLOSE + 2.0*Q7*PARTY +
     1            PTOT2/AANT + BCOEF5(3)
C
C           QUENOUILLE
C
      PLATE = BED*HAS
      FORK  = DEB*ANTOT
      SPOON = PLATE/FORK
      ANIFE = (ANTOT - SAMSIZ)*S1*PTOT1
      CUP   = ANIFE/(SAMSIZ*AANT)
      DISH  = CUP*CUP
      ALOVE = (SAMSIZ*PLATE - FORK)/FORK - CUP*Q1*S1
      WAR   = ANIFE*(PLATE - DEB)/(SAMSIZ*STOT1*AANT*AANT)
      SPOT  = SPOON*(SPOON - 2.0/ANTOT)
      SATOT = SAMSIZ*ANTOT
C
      ALFA6 = ALOVE*ALOVE + ALFA6
      BCOEF6(1) = SAMSIZ*SPOT + DISH*(S1*Q2 + S2*Q4) - 2.0*S1*Q1*WAR +
     1         1.0/ANTOT + BCOEF6(1)
      BCOEF6(2) = STOT1*SPOT + DISH*(Q1+Q5) - 2.0*SAMSIZ*WAR +
     1            PTOT1/AANT + BCOEF6(2)
      BCOEF6(3) = STOT2*SPOT + DISH*(Q3+Q6) - 2.0*Q7*WAR +
     1            PTOT2/AANT + BCOEF6(3)
C
      RETURN
      END
```

```
      SUBROUTINE  SUBX4
C
C
      COMMON/BBBB/ SAM, SAMSIZ, SAMSAM, S1, S2, SS1
      COMMON/CCCC/ BLUE, BLACK
      COMMON/DDDD/ TITLE(18), NTOT, CV
      COMMON/EEEE/ ALFA1, ALFA2, ALFA3, ALFA4, ALFA5, ALFA6, ALFA18,
     1  BCOEF1(3), BCOEF2(3), BCOEF3(3), BCOEF4(3), BCOEF5(3), BCOEF6(3)
      COMMON/FFFF/ BCOEAV(3), ALF1X4, ALF2X4, ALF3X4, ALSQ1, ALSQ2,
     1  ALSQ3, ASQD1(3), ASQD2(3), ASQD3(3), DELSQ1(3), DELSQ2(3),
     2  DELSQ3(3), DELTA1(3), DELTA2(3), DELTA3(3)
      COMMON/GGGG/ PTOT(2), PTOT1, PTOT2, PPTOT, ANTOT, AANT, A1, AT1,
     1  VARPOP, POPVAR
      COMMON/KKKK/ AMOTHR
      COMMON/LLLL/COMB, ALFAV
C
      DIMENSION  RATIO1(3)
C
      INTEGER  SAM
C
C                                 ---------------
C
      ANC2 = 2000
C
      RED    = BLACK/ANC2
      WHITE  = BLUE/ANC2
      OFFICE = PPTOT*PPTOT*SS1/(AANT*AANT)
      REDOF  = RED*OFFICE
C
      ALF1X4 = ALF1X4*RED
      ALF2X4 = ALF2X4*RED
      ALSQ1  = ALSQ1*WHITE
      ALSQ2  = ALSQ2*WHITE
      ALSQ3  = ALSQ3*WHITE
C
      IF (SAM.LT.4) GO TO 1
      ALF3X4 = ALF3X4*REDOF
C
    1 DO 2 I = 1,3
      DELTA1(I) = DELTA1(I)*WHITE
      DELTA2(I) = DELTA2(I)*WHITE
      DELTA3(I) = DELTA3(I)*WHITE
      ASQD1(I)  = ASQD1(I)*RED
      ASQD2(I)  = ASQD2(I)*RED
      DELSQ1(I) = DELSQ1(I)*RED
      DELSQ2(I) = DELSQ2(I)*RED
C
      IF (SAM.LT.4) GO TO 2
      ASQD3(I)  = ASQD3(I)*REDOF
      DELSQ3(I) = DELSQ3(I)*REDOF
    2 CONTINUE
C
C          AVERAGE THE COEFFICIENTS FOR THE MSE (OR VAR)
C          FOR THE DIFFERENT ESTIMATORS
C
```

```
      ALFAV = ALFA1/ANC2
      ALFA2 = ALFA2/ANC2
      ALFA3 = ALFA3/ANC2
      ALFA4 = ALFA4/ANC2
      ALFA5 = ALFA5/ANC2
      ALFA6 = ALFA6/ANC2
C
      ALFA1B = ALFA1B/ANC2
      ALFA1B = ALFA1B*ALFA1B
C
      DO 3 I = 1,3
      BCOEAV(I) = BCOEF1(I)/ANC2
      BCOEF2(I) = BCOEF2(I)/ANC2
      BCOEF3(I) = BCOEF3(I)/ANC2
      BCOEF4(I) = BCOEF4(I)/ANC2
      BCOEF5(I) = BCOEF5(I)/ANC2
    3 BCOEF6(I) = BCOEF6(I)/ANC2
C
      WRITE (6,4)  (TITLE(I),I=1,18), SAM, NTOT, CV
    4 FORMAT (1H1, 18A4 // 25X, '2000 INDEPENDENT SAMPLES GENERATED -',
     1        'EACH OF SIZE', I3, 1X, 'FROM A POPN OF SIZE', I4 / T53,
     2        'COEF OF VAR =', E15.8 /)
C
      WRITE (6,5)  ALFA1B
    5 FORMAT (15X, 'CLASSICAL RATIO ESTIMATOR', T75, 'RATIO - ',
     1        'ESTIMATORS/MOD-BEALE' / T16, 'B*B = E(ALFA/2000)**2 =',
     2         E15.8)
      RATIO = ALFAV/ALFA3
      DO 11 K = 1,3
      M = K - 1
      RATIO1(K) = BCOEAV(K)/BCOEF3(K)
      IF (K.EQ.1) GO TO 6
      GO TO 8
    6 WRITE (6,7)  M, ALFAV, RATIO, BCOEAV(1), RATIO1(1)
    7 FORMAT (1X, 'T =', I2, 12X, 'AVERAGE ALPHA-SQ =', E15.8, T75,
     1         E15.8 / 21X, 'AVERAGE DCOEF =', E15.8, T75, E15.8)
      GO TO 11
    8 WRITE (6,9)  M, BCOEAV(K), RATIO1(K)
    9 FORMAT (1X, 'T =', I2, 15X, 'AVERAGE DCOEF =' E15.8, T75, E15.8)
      IF (K.NE.3) GO TO 11
      WRITE (6,10)
   10 FORMAT (1X /)
   11 CONTINUE
C
      WRITE (6,12)
   12 FORMAT (31X, 'BEALE')
      RATIO = ALFA2/ALFA3
      DO 15 K = 1,3
      M = K - 1
      RATIO1(K) = BCOEF2(K)/BCOEF3(K)
      IF (K.EQ.1) GO TO 13
      GO TO 14
   13 WRITE (6,7)  M, ALFA2, RATIO, BCOEF2(1), RATIO1(1)
      GO TO 15
   14 WRITE (6,9)  M, BCOEF2(K), RATIO1(K)
      IF (K.NE.3) GO TO 15
```

```
      WRITE (6,10)
   15 CONTINUE
C
      WRITE (6,16)
   16 FORMAT (31X, 'MODIFIED BEALE')
      DO 21 K = 1,3
      M = K - 1
      IF (K.EQ.1) GO TO 17
      GO TO 19
   17 WRITE (6,18)  M, ALFA3, BCOEF3(1)
   18 FORMAT (1X, 'T =', I2, 12X, 'AVERAGE ALPHA-SQ =', E15.8)
      GO TO 21
   19 WRITE (6,20)  M, BCOEF3(K)
   20 FORMAT (1X, 'T =', I2, 12X, 'AVERAGE DCOEF =', E15.8)
      IF (K.NE.3) GO TO 21
      WRITE (6,10)
   21 CONTINUE
C
      WRITE (6,22)
   22 FORMAT (31X, 'HARTLEY - ROSS')
      RATIO = ALFA4/ALFA3
      DO 25 K = 1,3
      M = K - 1
      RATIO1(K) = BCOEF4(K)/BCOEF3(K)
      IF (K.EQ.1) GO TO 23
      GO TO 24
   23 WRITE (6,7)  M, ALFA4, RATIO, BCOEF4(1), RATIO1(1)
      GO TO 25
   24 WRITE (6,9)  M, BCOEF4(K), RATIO1(K)
      IF (K.NE.3) GO TO 25
      WRITE (6,10)
   25 CONTINUE
C
      WRITE (6,26)
   26 FORMAT (31X, 'MICKEY')
      RATIO = ALFA5/ALFA3
      DO 29 K = 1,3
      M = K - 1
      RATIO1(K) = BCOEF5(K)/BCOEF3(K)
      IF (K.EQ.1) GO TO 27
      GO TO 28
   27 WRITE (6,7)  M, ALFA5, RATIO, BCOEF5(1), RATIO1(1)
      GO TO 29
   28 WRITE (6,9)  M, BCOEF5(K), RATIO1(K)
      IF (K.NE.3) GO TO 29
      WRITE (6,10)
   29 CONTINUE
      WRITE (6,30)
   30 FORMAT (31X, 'QUENOUILLE')
      RATIO = ALFA6/ALFA3
      DO 33 K = 1,3
      M = K - 1
      RATIO1(K) = BCOEF6(K)/BCOEF3(K)
      IF (K.EQ.1) GO TO 31
      GO TO 32
   31 WRITE (6,7)  M, ALFA6, RATIO, BCOEF6(1), RATIO1(1)
```

```
      GO TO 33
   32 WRITE (6,9)  M, BCOEF6(K), RATIO1(K)
      IF (K.NE.3) GO TO 33
      WRITE (6,10)
   33 CONTINUE
C
C            COMB - ALPHA-SQ TERM OF THE VARIANCE OF THE
C                   CLASSICAL RATIO ESTIMATOR
C
      COMB  = ALFAV - ALFA1B
      RATIO = ALFA4/ALFA5
      DO 38 K = 1,3
      RATIO1(K) = BCOEF4(K)/BCOEF5(K)
      IF (K.EQ.1) GO TO 34
      GO TO 36
   34 WRITE (6,35)  COMB,  RATIO, RATIO1(1)
   35 FORMAT (1X /// 26X, 'VAR OF CRE (V)', T75, 'RATIO - HARTLEY-',
     1       'ROSS/MICKEY' / T21, '(ALPHA-SQ)-B*B =', E15.8, T75,
     2        E15.8 / T75, E15.8)
      GO TO 38
   36 WRITE (6,37)  RATIO1(K)
   37 FORMAT (T75, E15.8)
   38 CONTINUE
C
      CALL SUBX5
C
      RETURN
      END
```

```
      SUBROUTINE  SUBX5
C
C
      COMMON/BBBB/ SAM, SAMSIZ, SAMSAM, S1, S2, SS1
      COMMON/DDDD/ TITLE(18), NTOT, CV
      COMMON/FFFF/ BCOEAV(3), ALF1X4, ALF2X4, ALF3X4, ALSQ1, ALSQ2,
     1  ALSQ3, ASQD1(3), ASQD2(3), ASQD3(3), DELSQ1(3), DELSQ2(3),
     2  DELSQ3(3), DELTA1(3), DELTA2(3), DELTA3(3)
      COMMON/LLLL/ COMB, ALFAV
C
      DIMENSION  EB1D(3), EB2D(3), EB3D(3), V1VASD(3), V2VASD(3),
     1  V3VASD(3), VVASD(3), EE1ASD(3), EE2ASD(3), EE3ASD(3), V1VDX2(3),
     2  V2VDX2(3), V3VDX2(3), VVDX2(3), EE1DX2(3), EE2DX2(3), EE3DX2(3),
     3  EE1DSQ(3), EE2DSQ(3), EE3DSQ(3)
C
      INTEGER  SAM
C
C                                   ---------------
C
C             E BIAS(VI) = EBI = (VI - V)
C                                  WHERE VI ARE THE 3 PRELIMINARY
C                                  BIAS CALCULATIONS
C
C
C                ALFA-SQ TERM
C
      EB1ASQ = ALSQ1 - COMB
      EB2ASQ = ALSQ2 - COMB
      EB3ASQ = ALSQ3 - COMB
C
C                DELTA TERM
C
      DO 1 I = 1,3
      EB1D(I) = DELTA1(I) - BCOEAV(I)
      EB2D(I) = DELTA2(I) - BCOEAV(I)
    1 EB3D(I) = DELTA3(I) - BCOEAV(I)
C
C             E MSE(VI) = EEI = (VI - V)**2
C                             =  VVI - 2.0*VIV + VV
C                                     WHERE VI ARE THE 3 PRELIMINARY
C                                     VARIANCE CALCULATIONS
C
C
C                ALFA**4 TERM
C
      V1VAX4 = ALSQ1*COMB
      V2VAX4 = ALSQ2*COMB
C
      VVAX4  = COMB*COMB
C
      EE1AX4 = ALF1X4 - 2.0*V1VAX4 + VVAX4
      EE2AX4 = ALF2X4 - 2.0*V2VAX4 + VVAX4
      IF (SAM.LT.4) GO TO 2
      V3VAX4 = ALSQ3*COMB
      EE3AX4 = ALF3X4 - 2.0*V3VAX4 + VVAX4
C
C                ALFA-SQ-DELTA TERM
C
```

```
    2 DO 3 I = 1,3
      V1VASD(I) = ALSQ1*BCOEAV(I) + COMB*DELTA1(I)
      V2VASD(I) = ALSQ2*BCOEAV(I) + COMB*DELTA2(I)
C
      VVASD(I) = 2.0*COMB*BCOEAV(I)
C
      EE1ASD(I) = ASQD1(I) - 2.0*V1VASD(I) + VVASD(I)
      EE2ASD(I) = ASQD2(I) - 2.0*V2VASD(I) + VVASD(I)
C
C             DELTA-SQ TERM
C
      V1VDX2(I) = BCOEAV(I)*DELTA1(I)
      V2VDX2(I) = BCOEAV(I)*DELTA2(I)
C
      VVDX2(I) = BCOEAV(I)*BCOEAV(I)
C
      EE1DSQ(I) = DELSQ1(I) - 2.0*V1VDX2(I) + VVDX2(I)
      EE2DSQ(I) = DELSQ2(I) - 2.0*V2VDX2(I) + VVDX2(I)
C
      IF (SAM.LT.4) GO TO 3
C
      V3VASD(I) = ALSQ3*BCOEAV(I) + COMB*DELTA3(I)
      EE3ASD(I) = ASQD3(I) - 2.0*V3VASD(I) + VVASD(I)
      V3VDX2(I) = BCOEAV(I)*DELTA3(I)
      EE3DSQ(I) = DELSQ3(I) - 2.0*V3VDX2(I) + VVDX2(I)
    3 CONTINUE
C
      WRITE (6,4)  (TITLE(I),I=1,18), SAM, NTOT, CV
    4 FORMAT (1H1, 18A4 // 25X, '2000 INDEPENDENT SAMPLES GENERATED -',
     1        'EACH OF SIZE', I3, 1X, 'FROM A POPN OF SIZE', I4 / T53,
     2        'COEF OF VAR =', E15.8 /)
      WRITE (6,5)
    5 FORMAT (51X, 'BIASES OF VARIANCE ESTIMATORS' / T40, 'V1-V' /)
      DO 9 M = 1,3
      K = M - 1
      IF (M.GT.1) GO TO 7
      WRITE (6,6)  K, EB1ASQ, EB1D(1)
    6 FORMAT (1X, 'T =', I2, 10X, 'ALFA-SQ =', E15.8, T43,
     1  'DELTA =', E15.8)
      GO TO 9
    7 WRITE (6,8)  K, EB1D(M)
    8 FORMAT (1X, 'T =', I2, T43, 'DELTA =', E15.8)
    9 CONTINUE
C
      WRITE (6,10)
   10 FORMAT (T40, 'V2-V')
      DO 12 M = 1,3
      K = M -1
      IF (M.GT.1) GO TO 11
      WRITE (6,6)  K, EB2ASQ, EB2D(1)
      GO TO 12
   11 WRITE (6,8)  K, EB2D(M)
   12 CONTINUE
C
      WRITE (6,13)
   13 FORMAT (T40, 'V3-V')
```

```
      DO 15 M = 1,3
      K = M - 1
      IF (M.GT.1) GO TO 14
      WRITE (6,6)  K, EB3ASQ, EB3D(1)
      GO TO 15
   14 WRITE (6,8)  K, EB3D(M)
   15 CONTINUE
C
C
      WRITE (6,16)  EE1AX4
   16 FORMAT (1X // 30X, 'MSE OF VARIANCE ESTIMATORS' / T56,
     1        'ALFA**4 =', E15.8)
      DO 17 K = 1,3
      M = K - 1
   17 WRITE (6,18)  M, EE1ASD(K), EE1DX2(K)
   18 FORMAT (T5, 'T =', I2, T15, 'ALFA-SQ-DELTA =', E15.8, 10X,
     1        'DELTA-SQ =', E15.8 /)
C
      WRITE (6,19) EE2AX4
   19 FORMAT (T56, 'ALFA**4 =', E15.8)
      DO 20 K = 1,3
      M = K -1
   20 WRITE (6,18)  M, EE2ASD(K), EE2DX2(K)
C
      IF (SAM.LT.4) GO TO 22
C
      WRITE (6,19) EE3AX4
      DO 21 K = 1,3
      M = K - 1
      WRITE (6,18)  M, EE3ASD(K), EE3DX2(K)
C
   21 CONTINUE
C
   22 RETURN
      END
```

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