The Motion of a Particle under Gravity on the Smooth Surface of a Vertical Paraboloid of Revolution.

A Thesis

by

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Table of Contents.

The Motion of a Particle under Gravity on the Smooth Surface of a Vertical Paraboloid of Revolution.

Sec. 1 - Introductory.

It is proposed to consider the motion of a heavy particle constrained to move on the smooth inner surface of a paraboloid of revolution, symmetrical with respect to the z-axis which is vertical.

The equations of motion will first be set up. From these will be derived the corresponding "Vis Viva Integral" and the equation of angular momentum about the z-axis.

By considering the "Vis Viva" and the momentum equations simultaneously we shall obtain the differential equations for the path and the time. In each of these fundamental equations will appear two constants. By arbitrarily choosing these constants several different cases arise, for each of which the path and the time will be determined. Diagrams will be used to illustrate the shapes of these paths.

Sec. 2 - General Differential Equations of Motion.

The general equations of motion for a particle constrained to move on any surface $\phi\,(x,y,z) = 0$ are:

$$m\,\frac{d^2x}{dt^2} = F_x \quad N_x \equiv X \qquad\text{-----1}$$

$$m\,\frac{d^2y}{dt^2} = F_y \quad N_y \equiv Y \qquad\text{-----2}$$

$$m\,\frac{d^2z}{dt^2} = F_z \quad N_z \equiv Z \qquad\text{-----3}$$

where $F_x \equiv$ Sum of the Components of all the forces in the direction of the x-axis

$N_x \equiv$ Sum of the Components of all the reactions in the direction of the x-axis, etc.

$m =$ mass of particle.

Now in the case of a smooth surface $\phi\,(x,y,z) = 0$ the reaction of the surface, N, is at every point normal to the surface and therefore

$$\frac{N_x}{\frac{\partial\phi}{\partial x}} = \frac{N_y}{\frac{\partial\phi}{\partial y}} = \frac{N_z}{\frac{\partial\phi}{\partial z}} = \lambda \qquad\text{-----4}$$

Further the total reaction

$$N = \sqrt{N_x^2 + N_y^2 + N_z^2} = \lambda\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2} \qquad\text{-----5}$$

Sec. 3 - To Derive the Equations of Motion for the Given Paraboloid.

Here $\phi(xyz) = 0$, becomes $x^2 + y^2 = 2z$

$$\text{or}\quad \tfrac{1}{2}\,(x^2 + y^2 - 2z) = 0 \qquad\text{-----6}$$

We choose here the "latus rectum" as unity. This merely fixes the unit of length and does not affect the generality of the results, but only the numerical values in a particular case.

Sec. 3 - continued.

We have then ------------7

Also since gravity is the only force acting,

$$F_x = 0, \quad F_y = 0, \quad F_z = -g \qquad \text{--------------------8}$$

From (4) etc. when we take the mass equal to unity, the equations (1), (2), (3), take the following forms respectively:

$$\frac{d^2x}{dt^2} = \lambda x \qquad \text{------------9}$$

$$\frac{d^2y}{dt^2} = \lambda y \qquad \text{------------10}$$

$$\frac{d^2z}{dt^2} = -\lambda - g \qquad \text{------------11}$$

We proceed to find the value of and N. For the general equation $\phi\,(x,y,z) = 0$

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\cdot\frac{dz}{dt} = 0$$

Differentiating again

$$\frac{d^2\phi}{dt^2} = \frac{\partial\phi}{\partial x}\cdot\frac{d^2x}{dt^2} + \frac{\partial\phi}{\partial y}\cdot\frac{d^2y}{dt^2} + \frac{\partial\phi}{\partial z}\cdot\frac{d^2z}{dt^2}$$

$$+ \frac{\partial^2\phi}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2\phi}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial^2\phi}{\partial z^2}\left(\frac{dz}{dt}\right)^2$$

$$+ 2\frac{\partial^2\phi}{\partial x\,\partial y}\cdot\frac{dx}{dt}\cdot\frac{dy}{dt} + 2\frac{\partial^2\phi}{\partial y\,\partial z}\cdot\frac{dy}{dt}\cdot\frac{dz}{dt} + 2\frac{\partial^2\phi}{\partial x\,\partial z}\cdot\frac{dx}{dt}\cdot\frac{dz}{dt}$$

$$+ 2\frac{\partial^2\phi}{\partial x\,\partial t}\cdot\frac{dx}{dt} + 2\frac{\partial^2\phi}{\partial y\,\partial t}\cdot\frac{dy}{dt} + 2\frac{\partial^2\phi}{\partial z\,\partial t}\cdot\frac{dz}{dt} = 0 \qquad \text{-------12}$$

Sec. 3 - continued.

In case of paraboloid $\dfrac{\partial \phi}{\partial x} = x$, $\dfrac{\partial \phi}{\partial y} = y$, $\dfrac{\partial \phi}{\partial z} = -1$

as in (7)

hence $\dfrac{\partial^2 \phi}{\partial x^2} = 1$, $\dfrac{\partial^2 \phi}{\partial y^2} = 1$, $\dfrac{\partial^2 \phi}{\partial z^2} = 0$

$\dfrac{\partial^2 \phi}{\partial x . \partial y}$ etc. & $\dfrac{\partial^2 \phi}{\partial x . \partial t}$ etc. all vanish.

Hence Equation (13) becomes

$$0 = x\,(\lambda x) + y\,(\lambda y) + (-1)\,(-\lambda - g) + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= \lambda\,(x^2 + y^2 + 1) + g + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

Whence, on putting $x^2 + y^2 = 2z$, we obtain

$$\lambda = -\frac{1}{2z + 1}\left[g + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] \qquad \text{--------------------13}$$

To find N we have

$$N = \lambda\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]^{\frac{1}{2}}$$

$$= \lambda\,(x^2 + y^2 + 1)^{\frac{1}{2}} \qquad \text{from (7)}$$

$$= \lambda\,(2z + 1)^2 \qquad \text{Since } x^2 + y^2 = 2z$$

$$= -\frac{1}{(2z + 1)^{\frac{1}{2}}}\left[g + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] \qquad \text{-----------------14}$$

by (13)

Substitute this value of λ in (9), (10), (11) and the equations of motion of the particle on the paraboloid become

Sec. 3 - continued.

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{-\,x}{2z + 1}\left[g + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] \\
\frac{d^2y}{dt^2} &= \frac{-\,y}{2z + 1}\left[g + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] \\
\frac{d^2z}{dt^2} &= -\frac{1}{2z + 1}\left[g + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] - g
\end{aligned}\right\} \text{------------15.}$$

Sec. 4. Vis Viva Integral for the Given Surface.

The Vis Viva integral, which gives the kinet energy of the particle, is given by the general relation

$$\tfrac{1}{2}\, m\, V^2 = \int (Xdx + Ydy + Zdz)$$

In the case of the paraboloid under discussion this becomes

$$\begin{aligned}
\frac{V^2}{2} &= \int (\lambda xdx + \lambda ydy - (\lambda + g)\, dz) \\
&= \frac{\lambda}{2}\,(x^2 + y^2) - (\lambda + g)\, z + k' \\
&= \frac{\lambda}{2}\,(2x) - (\lambda + g)\, z + k' \\
&= -\,gz + k'
\end{aligned}$$

, where k' is the constant of integration.

i.e. $V^2 = -\,2g\,z + k$, where $k = 2k'$

Sec. 4 - continued.

If therefore we suppose $V = 0$ when $z = h$ we see that $k = 2gh$ and hence

$$V^2 = 2g\ (h - z) \qquad \text{---------------------16}$$

$h - Z_0$ is the height to which the body would rise if projected vertically from point (X_0, Y_0, Z_0) with a velocity V_0.

Sec. 5. Angular Momentum about the z-axis.

Multiplying the first of equations (15) by y and the second by x and subtracting we have

$$x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} = 0$$

Integrating this equation we get

$$x \frac{dy}{dt} - y \frac{dx}{dt} = c \qquad \text{----------------------17}$$

where c is the constant of integration.

From this equation we see that the angular momentum about the z-axis is constant, or that the area passed over by the projection of the radius vector on the xy-plane is proportional to the time.

Sec. 5 - continued.

If we put $x = r \cos\theta$, $y = r \sin\theta$ where r is the distance of the particle from the z-axis and is the angle between the xz-plane and the plane defined by the position of the particle and the z-axis, equation (17) becomes

$$r \cos\theta . \left(\frac{dr}{dt} \sin\theta + r \cos\theta . \frac{d\theta}{dt}\right) - r \sin\theta\left(\frac{dr}{dt} \cos\theta - r\sin\theta\frac{d\theta}{dt}\right) = c$$

i. e. $$r^2 \frac{d\theta}{dt} = C \qquad \text{------------------17a}$$

which also expresses the fact that the projection of the areal velocity on the xy-plane is constant.

Equations (17) and (17a) may be called the equations of angular momentum.

Sec. 6. Regions of Real Motion.

From equation (16)

$$V^2 = 2g\ (h-z)$$

it is evident that real motion exists only between the planes $z=0$ and $z=h$; since if $z > h$, V is imaginary, and z cannot be negative for any position of the particle on the paraboloid.

Now in using cylindrical coordinates as indicated in sec. 5, the distance r from the z-axis always projects into r on the xy-plane θ projects into θ , and a section of the paraboloid formed by a plane parallel to the xy-plane projects into a circle on the xy-plane any arc of which, S, is given by

$$S = r\ \theta.$$

and also $x^2 + y^2 = r^2$

Sec. 6 - continued.

On the xy-plane

$$V = r \frac{d\theta}{dt}$$

i.e. $$V^2 = r^2 \left(\frac{d\theta}{dt}\right)^2 = \frac{\left(r^2 \cdot \frac{d\theta}{dt}\right)^2}{r^2} = \frac{c^2}{r^2} = \frac{c^2}{2z}$$

where c is the angular momentum, see equation (17) and $r^2 = x^2 + y^2 = 2z$ for any point on the paraboloid.

Now these two values for V^2 may be equated when the vertical component of V^2 in equation (16) has vanished. i.e. when the total velocity has become the tangential velocity in a plane parallel to the xy-plane. This evidently occurs at the turning points in the motion i.e. at the highest and lowest points in any path traced on the surface of the paraboloid.

Hence at points where the vertical component of the velocity vanishes, we have,

$$2g\,(h - z) = \frac{c^2}{2z}$$

or $$4gz^2 - 4ghz + c^2 = 0$$

whence $$z = \frac{4gh + \sqrt{4g^2h^2 - 16gc^2}}{8g}$$

$$= \frac{h}{2} + \frac{1}{2}\sqrt{h^2 - \frac{c^2}{g}} \qquad \text{-----------------18}$$

This gives two parallel planes

$$z = \frac{h}{2} + \frac{1}{2}\sqrt{h^2 - \frac{c^2}{g}}$$

and $$z = \frac{h}{2} - \frac{1}{2}\sqrt{h^2 - \frac{c^2}{g}}$$

between which the motion of the particle must take place.

Sec. 6 - continued.

The constant h may have any value depending on the initial values of V_o and Z_o since $h = \frac{V_o^2}{2g} + Z_o$

The constant c has many positive values.

1) If $\frac{c^2}{g} > h^2$ the values of z as given by (18) are imaginary, and no real motion exists. This case need not be considered further.

2) If $0 \leq \frac{c^2}{g} \leq h^2$ we get two real values for z. Hence we must consider values of c within this range.

Now positive and negative values of c indicate rotation in opposite directions about the z-axis. It is therefore necessary to consider only positive values of c.

Besides the general case (2) presents two special limiting cases as follows:

1) If $c = 0$ i.e. if the angular momentum is zero,

then $r^2 \frac{d\theta}{dt} = c = 0$

i.e. $\frac{d\theta}{dt} = 0$

i.e. θ = a constant.

The motion is then in a plane through the z-axis. Such a plane cuts the paraboloid in a parabolic section and hence the path is along a parabola.

Also $c = 0$ gives $z = 0$ and $z = h$ in equation (8) i.e. motion in this parabolic path is limited between the planes $z = 0$ and $z = h$. The particle will fall along this path, through the origin, and then up along a continuation of the path on the opposite side of the paraboloid.

Sec. 6 - continued.

11) If $\frac{c^2}{g} = h^2 \qquad c \neq 0$ then the two values of z are the same, namely $\frac{h}{2}$ i.e. the two parallel limiting planes come into coincidence and hence the path of the particle is a circle of radius

$$r = \sqrt{2z} \qquad = \sqrt{h}$$

Further since

$$r^2 = 2z \qquad = h \pm \sqrt{h^2 - \frac{c^2}{g}}$$

it is evident that the projection on the xy-plane of the path traced on the paraboloid by the particle will lie between two concentric circles with a common centre at the origin and radii

$$r_1 = \sqrt{h - \sqrt{h^2 - \frac{c^2}{g}}} \qquad\qquad r_2 = \sqrt{h + \sqrt{h^2 - \frac{c^2}{g}}}$$

In the case where $\frac{c^2}{g} = h^2$ these circles become coincident and the radius of each is $r = \sqrt{h}$

Sec. 7. Differential Equations of the Path Projected on the xy-plane

1. The velocity of a particle moving in space is given by

$$v^2 = \left\{\frac{dx}{dt}\right\}^2 + \left\{\frac{dy}{dt}\right\}^2 + \left\{\frac{dz}{dt}\right\}^2$$

Transforming to cylindrical coordinates, (see Sec. 6) by putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

this becomes

Sec. 7 - continued.

$$V^2 = \left\{\frac{dr}{dt} \cos\theta - r \sin\theta \frac{d\theta}{dt}\right\}^2 + \left\{\frac{dr}{dt} \sin + r \cos. \frac{d\theta}{dt}\right\}^2 + \left\{\frac{dz}{dt}\right\}^2$$

$$= r^2 \left\{\frac{d\theta}{dt}\right\}^2 + \left\{\frac{dr}{dt}\right\}^2 + \left\{\frac{dz}{dt}\right\}^2 \qquad \text{------------------19.}$$

Whence

$$V.\ dt = \left[r^2 (d\theta)^2 + (dr)^2 + (dz)^2\right]^{\frac{1}{2}}$$

and since $V = \left[2g\ (h-z)\right]^{\frac{1}{2}}$ ------------------16.

then
$$\frac{Vdt}{V} = dt = \frac{\left[r^2 (d\theta)^2 + (dr)^2 + (dz)^2\right]^{\frac{1}{2}}}{\left[2g\ (h - z)\right]^{\frac{1}{2}}} \qquad \text{-----20.}$$

Since $r^2 \frac{d\theta}{dt} = c$ ------------------17

$$\text{i.e.}\quad dt = \frac{r^2\, d\theta}{c} = 2z\ \frac{d\theta}{c}$$

and moreover

$$r^2 = 2z$$

$$rdr = dz$$

$$(dr)^2 = \frac{(dz)^2}{r^2} = \frac{(dz)^2}{2z}$$

We have, on substituting in (20),

$$4\ z^2\ \frac{(d\theta)^2}{c^2} = \frac{\left[2z\ (d\theta)^2 + \frac{(dz)^2}{2z} + (dz)^2\right]^{\frac{1}{2}}}{\left[2g\ (h - z)\right]^{\frac{1}{2}}}$$

Sec. 7 - continued.

i.e. $$\frac{8\,z^2 g\ (h - z)}{c^2}\,(d\,\theta)^2 = 2z\,(d\,\theta)^2 + \frac{1}{2z}\,(dz)^2(1 + 2z)$$

i.e. $$\frac{8\,z^2 g\ (h - z)}{c^2} = 2z + \frac{1}{2z}\left\{\frac{dz}{d\theta}\right\}^2 (1 + 2z)$$

i.e. $$(2z)^{\frac{1}{2}}\left(\frac{d\,\theta}{dz}\right) = \frac{(1 + 2z)^{\frac{1}{2}}}{\left[\frac{8gz^2}{c^2}\,(h - z) - 2z\right]^{\frac{1}{2}}}$$

i.e. $$d\,\theta = \frac{(1 - 2x)^{\frac{1}{2}}\,dz}{2z\left(\frac{4gz}{c^2}\,(h - z) - 1\right)^{\frac{1}{2}}}$$

i.e. $$\frac{g^{\frac{1}{2}}}{c}\,.\,d\theta = \frac{(1 + 2z)\,dz}{2z\left[\left(4z\,(h - z) - \frac{c^2}{g}\right)(2z + 1)\right]^{\frac{1}{2}}} \quad ---21$$

Integrating (21)

$$\frac{g^{\frac{1}{2}}\theta}{c} = \int \frac{(2z + 1)\,dz}{2z\left[-(4z^2 - 4zh + \frac{c^2}{g})\,(2z + 1\right]^{\frac{1}{2}}}$$

$$= \int \frac{(2z + 1)\,dz}{2z\left[-(2z - a)\,(2z - b)\,(2z + 1)\right]^{\frac{1}{2}}}$$

where $a = h + \sqrt{h^2 - \frac{c^2}{g}}$, $b = h - \sqrt{h^2 - \frac{c^2}{g}}$

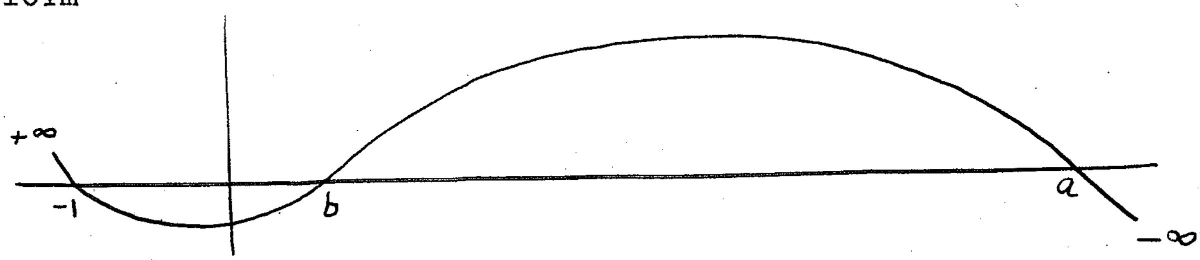
It is now necessary to consider the regions of integrability.

Sec. 7 - continued.

For the expression $\left[-(2z - a)(2z - b)(2z + 1)\right]^{\frac{1}{2}}$, occurring in the integral of (22), to be real, it is evident that 2z must lie between a and b i.e. z must lie between the values $\frac{h}{2} - \frac{1}{2}\sqrt{h^2 - \frac{c^2}{g}}$ and $\frac{h}{2} + \frac{1}{2}\sqrt{h^2 - \frac{c^2}{g}}$ which agrees with the region of real motion of the particle as found in Sec. 7.

Hence the expression is integrable only over the region of real motion of the particle.

The graph of the equation $(2z - a)(2z - b)(2z + 1) = 0$ has the form



And again for a real integral z must lie between a and b

Since $\frac{c^2}{g} \leq h$

$z \geq 0$, $b \nless 0$, . . the order of the roots is

$$-1 \leq b \leq a \quad (0 \leq b)$$

Sec. 8. Determination of the Path. (Special Cases)

i) As we saw in Sec. (6, 2i) we have, for $c = 0$

$$\theta = \theta_0 \qquad \text{a constant.}$$

The particle moved along a POZ section i.e. in a parabolic path determined by

$$r^2 = 2z$$

$$\theta = \theta_0$$

Sec. 8 - continued.

ii) When $\frac{c^2}{g} = h^2 \qquad c \neq 0 \qquad a = b = h$

In this case as we saw in Sec. (6, 2 ii) the two limiting planes coincide and the path of the particle is given by

$$r^2 = 2z \quad \text{and} \quad 2z = h$$

$$\text{i.e.} \quad r^2 = h$$

a circle parallel to the xy-plane of radius $\sqrt{h}$

There follows from this an interesting result from a consideration of the velocity in this path.

From $V^2 = 2g\ (h - z)$ --------------------16.

if $V = V_o$ when $Z = Z_o$

then $V_o^2 = 2g\ (h - z)$

$$h = \frac{V_o^2}{2g} \neq Z_o$$

If $Z = Z_o$ then $h = \frac{V_o^2}{2g}$ which is height body would rise if projected vertically from (X_o, Y_o, Z_o) with velocity V_o.

Now since $r^2 = h = 2Z_o$

$$\frac{V_o^2}{2g} \neq Z_o = 2Z_o$$

$$\therefore \quad V_o^2 = 2gZ_o$$

i.e. if the body is moving in the horizontal circle with a tangential velocity equal to the initial velocity required to raise it to this plane from the xy-plane, then it will continue to rotate in a circle in this plane.

Sec. 9. Determination of the Path in the General Cases.

In the case $h^2 > \frac{c^2}{g}$, $c \neq 0$

$$b \leq 2z \leq a$$

$$\frac{g^{\frac{1}{2}}}{c}\theta = \int \frac{(2z + 1)\,dz}{2z\left[-(2z-a)(2z-b)(2z+1)\right]^{\frac{1}{2}}} \qquad \text{--------22.}$$

$$= \int \frac{dz}{\left[-(2z-a)(2z-b)(2z+1)\right]^{\frac{1}{2}}} + \int \frac{dz}{2z\left[-(2z-a)(2z-b)(2z+1)\right]^{\frac{1}{2}}}$$

To reduce this integral to Legendre's standard form

put $X^2 = \dfrac{2z - a}{b - a}$

or $\quad -(a-b)\,x^2 = 2z - a$

whence $\quad dz = -(a-b)\,xdx$

$$2z - a = -(a-b)\,x^2$$

$$2z - b = (a-b) - (a-b)\,x^2$$

Then the above integral becomes

$$\frac{g^{\frac{1}{2}}}{c}\theta = \int \frac{-(a-b)\,xdx}{\sqrt{-\left(-(a-b)\,x^2\right)\left((a-b)-(a-b)\,x^2\right)\left((1+a)-(a-b)x^2\right)}}$$

$$+ \int \frac{-(a-b)\,x^2\,dx}{\left(a-(a-b)\,x^2\right)\sqrt{-\left(-(a-b)\,x^2\right)\left((a-b)-(a-b)\,x^2\right)\left((1+a)-(a-b)x^2\right)}}$$

Sec. 9 - continued

$$\text{i.e. } \frac{g^{\frac{1}{2}}}{c} = -\frac{1}{\sqrt{1+a}}\left[\int\frac{dx}{\sqrt{(1-x^2)\left(1-\frac{a-b}{1+a}x^2\right)}}\right.$$

$$\left.+\frac{1}{a}\int\frac{dx}{\left(1-\frac{a-b}{a}x^2\right)\sqrt{(1-x^2)\left(1-\frac{a-b}{1+a}x^2\right)}}\right]$$

where x decreases as θ increases, and lies within $1 \geq x^2 \geq 0$

$$\therefore \frac{g^{\frac{1}{2}}}{c}\theta = -\frac{1}{\sqrt{1+a}}\left[F\ (k,x) + \frac{1}{a}\prod(n,k,x)\right] \quad \text{-------23a}$$

$$\text{i.e. } \theta = \frac{c}{\sqrt{g\ (1+a)}}\left[F\ (k\phi) + \frac{1}{a}\prod\ (n,k,\phi)\right] \quad \text{-------23b}$$

in the usual notation for Elliptic Integrals

where $k^2 = \dfrac{a-b}{1+a}$, $n = -\dfrac{a-b}{a}$

$x = \text{Sin}\ \phi$ $\qquad \Delta\ \phi = \sqrt{1 - k^2\ \text{Sin}^2\phi}$

Sec. 10. Differential Equations for the Time.

$$\text{From} \quad V^2 = r^2\left(\frac{d\ \theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \quad \text{----------19}$$

$$\text{and} \quad V^2 = 2g\ (h - z) \quad \text{---------16}$$

We have

$$r^2\left(\frac{d\ \theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2g\ (h - z)$$

$$\therefore (dt)^2 = \frac{\left[r^2\ (d\ \theta)^2 + (dr)^2 + (dz)^2\right]}{\left(2g\ (g - z)\right)}$$

Sec. 10. - continued.

Since $r^2 \dfrac{d\theta}{dt} = c$

$$d\theta = \frac{c\,dt}{r^2} = \frac{c\,dt}{2z}$$

and since $r^2 = 2z$

$$r\,dr = dz$$

and $\therefore$ $(dr)^2 = \dfrac{(dz)^2}{r^2} = \dfrac{(dz)^2}{2z}$

$$\therefore \quad (dt)^2 = \frac{\dfrac{c^2\,(dt)^2}{2z} + \dfrac{(dz)^2}{2z} + (dz)^2}{2g\,(h - z)}$$

i.e. $(dt)^2 \left[4gz\,(h - z) - c^2 \right] = (2x + 1)\,dz$

$$dt = \frac{(2x + 1)^{\frac{1}{2}}\,dz}{\left[4gz\,(h - z) - c^2\right]^{\frac{1}{2}}} \qquad \text{-----------------24}$$

Integrating

$$g^{\frac{1}{2}}t = \int \frac{(2x + 1)\,dz}{\sqrt{-\left(4z^2 - 4zh + \dfrac{c^2}{g}\right)(2x + 1)}}$$

$$= \int \frac{(2x + 1)\,dz}{\sqrt{-\left((2z - h)^2 - \left(h^2 - \dfrac{c^2}{g}\right)\right)\left(2z + 1\right)}}$$

$$= \int \frac{(2z + 1)\,dz}{\sqrt{-\left(2z-h - \sqrt{h^2 - \dfrac{c^2}{g}}\right)\left(2z-h + \sqrt{h^2 - \dfrac{c^2}{g}}\right)(2z + 1)}}$$

$$= \int \frac{(2z + 1)\,dz}{\sqrt{-(2x - a)(2x - b)(2x + 1)}}$$

Sec. 10 - continued.

where $a = h + \sqrt{h^2 - \frac{c^2}{g}}$, $b = h - \sqrt{h^2 - \frac{c^2}{g}}$

$$= \int \sqrt{\frac{2z + 1}{-(2x - a)(2z - b)}}\, dz \qquad \text{---------------25.}$$

Sec. 11. Determination of Time in the General Case.

In

$$g^{\frac{1}{2}}t = \int_{o}^{Z_o} \sqrt{\frac{2z + 1}{-(2z - a)(2z - b)}}\, dz \qquad \text{---------25.}$$

put $x^2 = \frac{2z - a}{b - a}$

or $-(a - b)\,x^2 = (2z - a)$

whence $dz = -(a - b)\,xdx$

$2z + 1 = (1 + a) - (a - b)\,x^2$

$2z - a = -(a - b)\,x^2$

$2x - b = (a - b) - (a - b)\,x^2$

Then we get

$$g^{\frac{1}{2}}t = \int_{X_o}^{o} \sqrt{\frac{(1 + a) - (a - b)\,x^2}{-\left(-(a-b)x^2\right)(a-b)-(a-b)x^2}} \cdot \left(-(a-b)\right) xdx$$

Sec. 11 - continued.

$$= -\int_{X_o}^{0} \sqrt{\frac{(1 + a) - (a - b)\,x^2}{1 - x^2}} \cdot dx$$

$$= \sqrt{1 + a} \int_{0}^{X_o} \sqrt{\frac{1 - \frac{a-b}{1+a}\,x^2}{1 - x^2}} \cdot dx$$

$$= \sqrt{1 + a} \int_{0}^{\phi_o} \cos\phi \sqrt{\frac{1 - k^2 \sin^2\phi}{1 - \sin^2\phi}} \cdot d\phi \text{ where } k^2 = \frac{a-b}{1+a}, \quad x = \sin\phi$$

$$= \sqrt{1 + a} \int_{0}^{\phi_o} \sqrt{1 - k^2 \sin^2\phi} \cdot d\phi$$

$$t = \sqrt{\frac{1 + a}{g}}\; E(k, \phi)$$

For a real integral

$b \leq 2z \leq a$

$\therefore$ Limits of integration from

$2z = a$ to $2z = b$

give $x = 0$ to $x = 1$

and $\phi = 0$ to $\phi = \frac{\pi}{2}$

For the period of one complete cycle we have

$$T = 2\,\frac{(1 + a)^{\frac{1}{2}}}{g^{\frac{1}{2}}}\; E\left(k, \frac{\pi}{2}\right)$$

Sec. 12. Determination of Time in the Special Cases.

i) If in the above the angular momentum $c = 0$ then, as in sections (6, 2 i) and (8 i) the path is parabolic.

Here $a = h + \sqrt{h^2 - \frac{c^2}{g}} = 2h$

$$b = h - \sqrt{h^2 - \frac{c^2}{g}} = 0$$

whence $t = \sqrt{\frac{1 + a}{g}}\, E\,(k, \phi) = \sqrt{\frac{1 + 2h}{g}}\, E\,(k\ \phi)$

$$k^2 = \frac{a - b}{1 + a} = \frac{a}{1 + a} = \frac{2h}{1 + 2h}$$

To find the limits of integration

Since $x^2 = \frac{2x - a}{b - a} = \frac{2z - 2h}{0 - 2h}$

when $z = 0$, $x = 1$, $\phi = \frac{\pi}{2}$

and when $z = h$, $x = 0$, $\phi = 0$

Hence for half the path $\frac{T}{2} = \sqrt{\frac{1 + a}{g}}\, E\,(k, \frac{\pi}{2})$

and for the whole path $T = 2\sqrt{\frac{1 + a}{g}}\ (k, \frac{\pi}{2})$

ii) When $\frac{c^2}{g} = h^2$, $c \neq 0$ then $a = b = h$

As in sections (6, 2ii) and (8,ii) the path is a circle, parallel to xy-plane, of radius $\sqrt{h}$

Sec. 12 - continued.

The velocity as in Sec. (8ii) is given by

$$V_o^2 = 2gZ_o = 2gh$$

and thus $V_o = \sqrt{2gh}$

Hence the periodic time T is given by

$$= \frac{2\pi\sqrt{h}}{\sqrt{2gh}} = \pi\sqrt{\frac{2}{g}}$$

a result independent of h.

This means that the orbit is completed in same time in circles at <u>any</u> height above the xy-plane; i.e. the speed in circular orbits increases with height. This agrees with result in (10,ii) where it was shown that tangential velocity in each of these circular orbits must be equal to the initial velocity required to raise the particle to the level of the orbit in question.

Sec. 13. Summary.

The motion of the particle is bounded by two planes parallel to the xy-plane. These planes are given by the roots of the equation $z^2 - hz + \frac{c^2}{4g} = 0$ ---------------18

If one root is zero, i.e. if c = 0, the motion is parabolic. Sec. (7, 2i)

If two roots are equal, i.e. if the two limiting planes coincide, then the motion is in a circle. Sec. (6, 2ii) and the period of revolution is independent of the height of the plane.

Sec. 13 - continued.

In the general case where the roots are unequal we may consider two cases. 1st where b is small, 2nd where b is nearly equal to a.

Sec. 14. Equation of Path in a Form Adapted to Computation.

From Sec. 11.

$$\frac{g^{\frac{1}{2}}}{c}\theta = -\frac{1}{\sqrt{1 + a}}\left[F\,(k,x) + \frac{1}{a}\Pi(n,k,x)\right] \text{------23a}$$

$$= \frac{c}{\sqrt{g\,(1 + a)}}\left[\int_0^x \frac{dx}{\Delta x} + \frac{1}{a}\int_0^x (1 - nx^2)^{-1}\frac{dx}{\Delta x}\right]$$

where $k^2 = \frac{a - b}{1 + a}$, $n = \frac{a - b}{a}$, $\Delta x = \sqrt{(1-x^2)\,(1-k^2x^2\,)}$

This may be written

$$\frac{\theta}{c} = \int_0^x \frac{dx}{\Delta x} + \frac{1}{a}\int_0^x (1 - nx^2)^{-1}\frac{dx}{\Delta x}$$

where $c_1 = \frac{c}{\sqrt{g\,(1 + a)}}$

Now $(1 - nx^2)^{-1} = 1 + nx + n^2x^4 +$

and this series is convergent for x on the interval we are considering. $x = \operatorname{Sin}\phi$, $\operatorname{Sin}\phi$ being less than unity.

Sec. 14 - continued.

Retaining two terms we have

$$\frac{\theta}{c_1} = \left(1 + \frac{1}{a}\right) F\ (k,x) + \frac{n}{a} \int_0^x \frac{x^2\,dx}{\Delta x}$$

$$= \left(1 + \frac{1}{a}\right) F\ (k,x) + \frac{n}{k^2 a} \int_0^x \frac{k^2 x^2\,dx}{\Delta x}$$

$$= \left(1 + \frac{1}{a}\right) F\ (k,x) + \frac{n}{a} \int_0^x \frac{dx}{\Delta x} - \frac{n}{k^2 a} \int_0^x \frac{(1-k^2x^2)\,dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}$$

$$= \left(1 + \frac{1}{a} + \frac{n}{k^2 a}\right) F\ (k,x) - \frac{n}{k^2 a} \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

i.e.

$$\frac{\theta}{c_1} = \left(1 + \frac{1}{a} + \frac{n}{k^2 a}\right) F\ (k,x) - \frac{n}{k^2 a} E\ (k,x)$$

$$= \left(1 + \frac{1}{a} + \frac{n}{k^2 a}\right) F\ (k,\phi) - \frac{n}{k^2 a} E\ (k,\phi)$$

where $x = \operatorname{Sin} \phi$

From which Θ may be calculated.

Sec. 15. Calculation of Constants.

i) Case where b is small

Let us take $a = 15$, $b = 1$

$$k^2 \equiv \frac{a - b}{1 + a} = \frac{15 - 1}{1 + 15} = \frac{7}{8} \quad \text{and} \; \therefore \; k = .9$$

$$n \equiv \frac{a - b}{a} = \frac{15 - 1}{15} = \frac{14}{15} \quad \text{or} \quad n = .9\dot{3}$$

$$c_1 \equiv \frac{c}{\sqrt{g\,(1 + a)}} = \frac{21.97}{\sqrt{(32.2)\,(16)}} = .9676$$

To evaluate c we have

$$a = h + \sqrt{h^2 - \frac{c^2}{g}}$$

$$b = h - \sqrt{h^2 - \frac{c^2}{g}}$$

Adding, we have $h = \dfrac{a + b}{2}$

Subtracting $(a - b) = 2\sqrt{h^2 - \dfrac{c^2}{g}}$

Whence
$$c^2 = \frac{g}{4}\left(4h^2 - (a - b)^2\right)$$

$$= \frac{32.2}{4}\left(d\,(64) - (14)^2\right) = 483$$

$$\therefore \quad c = 21.97$$

Sec. 15 - continued.

ii) Take $a = 25, \quad b = 10$

$$k^2 = \frac{a - b}{1 + a} = \frac{15}{26} \qquad k = .7$$

$$n = \frac{a - b}{a} = \frac{3}{5} = .6$$

$$c_1 \equiv \frac{c}{\sqrt{g\,(1 + a)}} = \frac{89.1}{\sqrt{(32.2)\,(26)}} = 3.0783$$

As in i) above

$$c^2 = \frac{g}{4}\left(4h^2 - (a - b)^2\right)$$

$$= \frac{32.2}{4}\left(4\left(\frac{35}{2}\right)^2 - (15)^2\right) = 8050$$

$\therefore \quad c = 89.1$

Sec. 16. Calculation of θ and r.

i) To find θ

For $a = 15, \quad b = 1$ using the values of constants found in Sec. (15,i)

$$\frac{\theta}{.9676} = \left(1 + \frac{1}{15} + \frac{\frac{14}{15}}{\frac{7}{8}\,(15)}\right) F\,(k,\phi) - \frac{\frac{14}{15}}{\frac{7}{8}\,(15)} E\,(k,\phi)$$

Sec. 16 - continued.

$$= .9676 \left[1.1377 \; F(k,\phi) - .0711 \; E(k,\phi) \right]$$

$$= 1.1008 \; F(k,\phi) - .06879 \; E(k,\phi)$$

For a = 25 b = 0

$$\frac{\theta}{3.0783} = \left(1 + \frac{1}{25} + \frac{\frac{3}{5}}{\frac{15}{26}(25)} \right) F(k,\phi) - \frac{\frac{3}{5}}{\frac{15}{26}(25)} E(k,\phi)$$

$$= 3.0783 \left[1.0816 \; F(k,\phi) - .04016 \; E(k,\phi) \right]$$

$$= 3.3294 \; F(k,\phi) - .1236 \; E(k,\phi)$$

ii) To find r

From $x^2 = \dfrac{2z - a}{b - a}$

$$2z = a - (a - b)\, x^2$$

But $r^2 = 2z$

$\therefore \; r = \sqrt{a - (a - b)\, x^2}$ where $x = \sin\phi$

For a = 15, b = 1

$$r = \sqrt{15 - 14\sin^2\phi}$$

For a = 25, b = 10

$$r = \sqrt{25 - 15\sin^2\phi}$$

Sec. 17. Tables of Values of r and for Different Values of ϕ

i) For a = 15, b = 1, k = .9, $r = \sqrt{15 - 14\sin^2\phi}$

ϕ	$F(k,\phi)$	$E(k,\phi)$	$1.1008\ F(k,\phi)$	$.06879\ E(k,\phi)$	radians	degrees	r
0	0	0	0	0	0	0	3.873
15°	.264	.259	.2906	.01781	.272	$15^\circ38'$	3.749
30°	.544	.505	.5988	.03473	.564	$32^\circ19'$	3.391
45°	.858	.723	.9444	.04973	.894	$51^\circ16'$	2.828
60°	1.233	.907	1.3573	.06239	1.294	$74^\circ9'$	2.121
75°	1.703	1.053	1.8747	.07095	1.803	$103^\circ31'$	1.392
90°	2.275	1.173	2.5043	.08069	2.423	$138^\circ52'$	1.0

ii) For a = 25, b = 10, k = .7 $r = \sqrt{25 - 15\sin^2\phi}$

ϕ	$F(k,\phi)$	$E(k,\phi)$	$3.3294\ F(k,\phi)$	$.1236\ E(k,\phi)$	radians	degrees	r
0	0	0	0	0	0	0	5
15°	.263	.260	.8756	.03213	.843	$48^\circ20'$	4.898
30°	.536	.512	1.7846	.06326	1.721	$98^\circ44'$	4.609
45°	.826	.748	2.7501	.09245	2.657	$142^\circ17'$	4.183
60°	1.142	.965	3.8022	.11927	3.682	$211^\circ2'$	3.708
75°	1.488	1.163	4.9480	.14375	4.804	$275^\circ17'$	3.317
90°	1.854	1.351	6.1727	.16698	6.005	$344^\circ8'$	3.162

Sec. 18. Conclusion.

The return branch of curve is symmetrical with that indicated above and similar cycles will be traced out in succession.

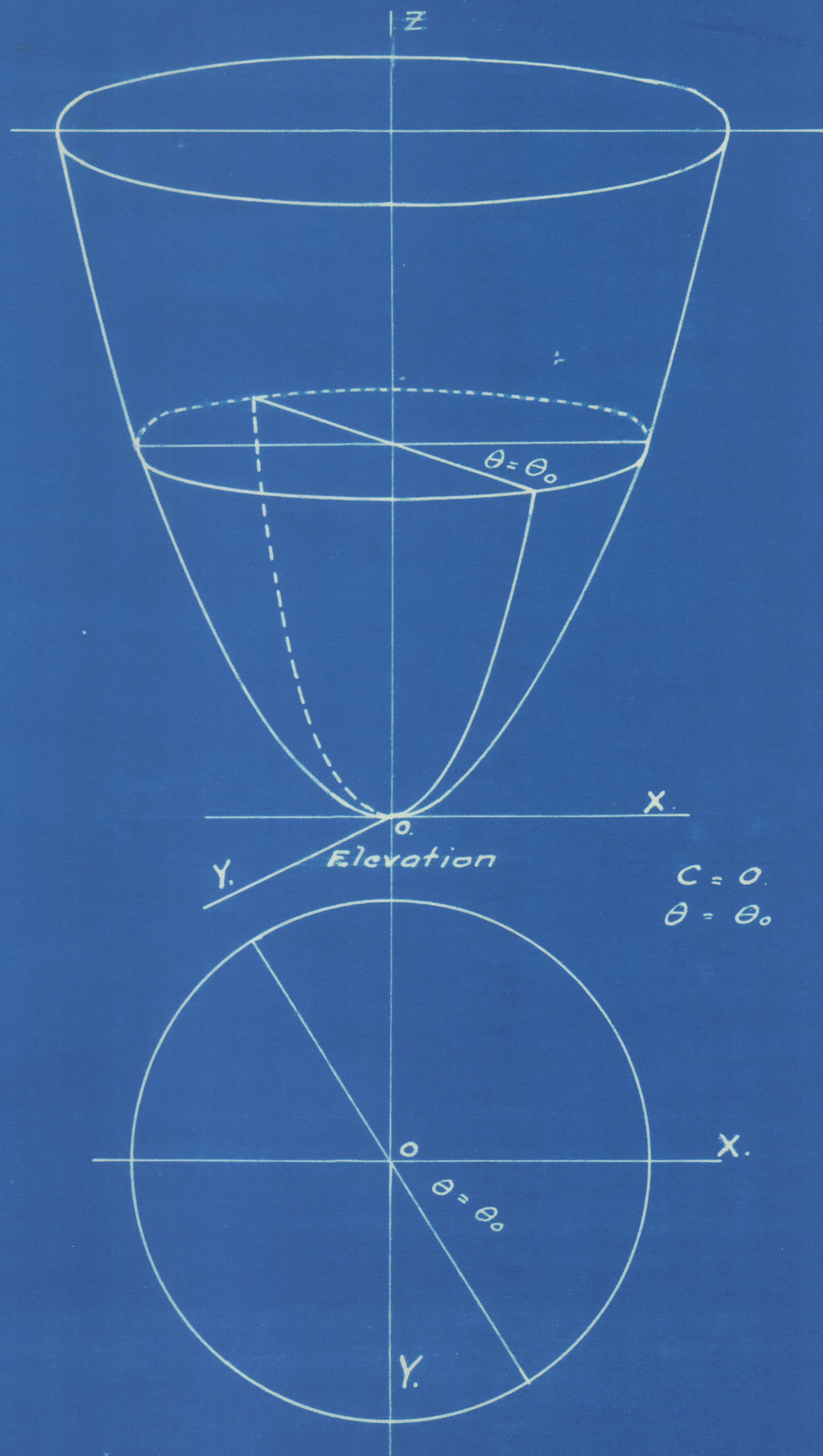
The distance from one apse to the next on the same plane is twice the value of θ corresponding to the complete elliptic integral that is for $\phi = 90^\circ$

Our result for the case a = 15, b = 1 gives the apsidal angles as 2x2.423 = 4.846 radians. Since this result is less than 2π the path is regressing.

Our result for the case a = 25, b = 10 gives the apsidal angle as 2x6.005 = 12.01 radians. Since this is greater than 2π the path is precessing.

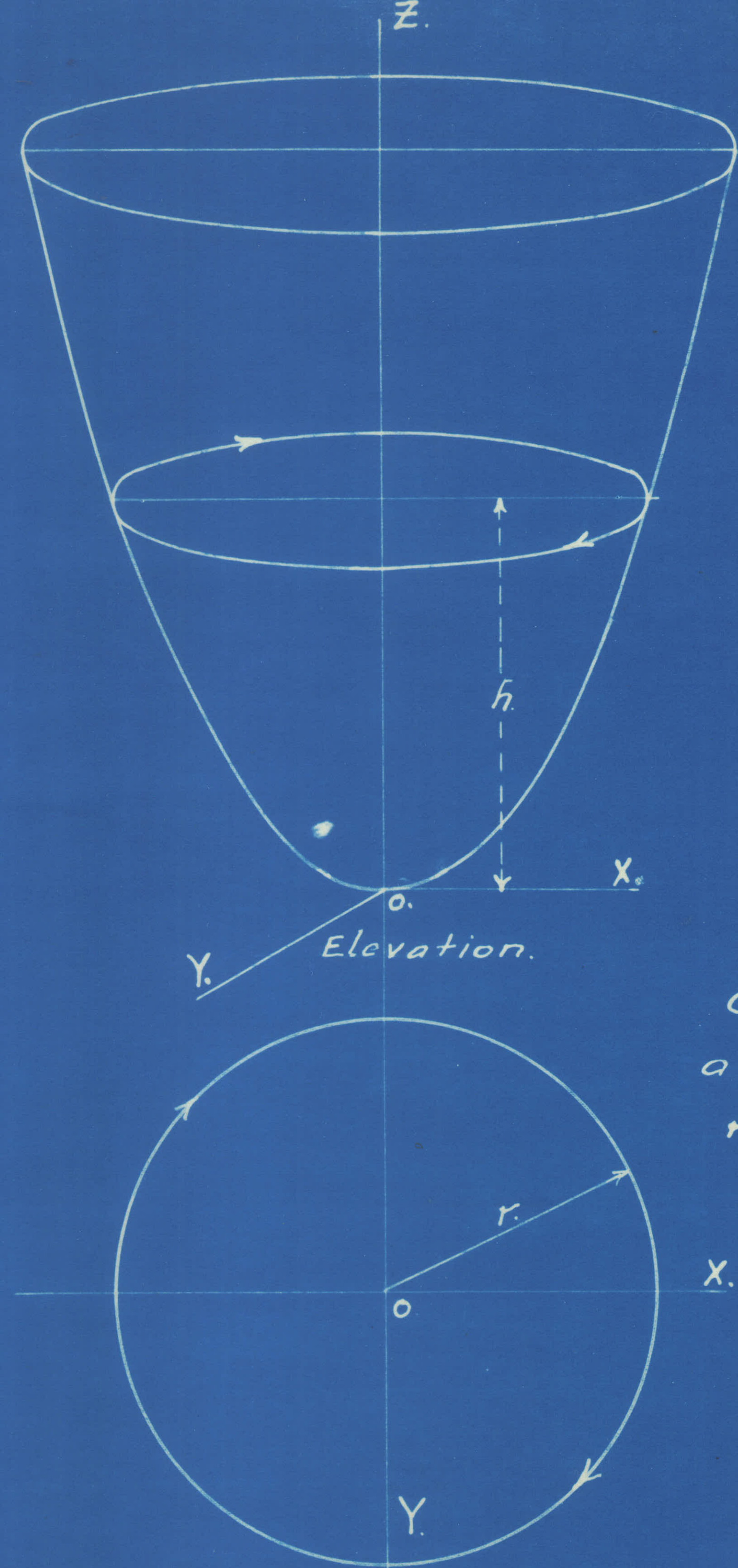
On the following pages are shown drawings of the actual paths and their projections on the xy-plane for the four cases considered.

PLATE, I.



Projection on Plane XY.

PLATE II.



$C \neq 0.$

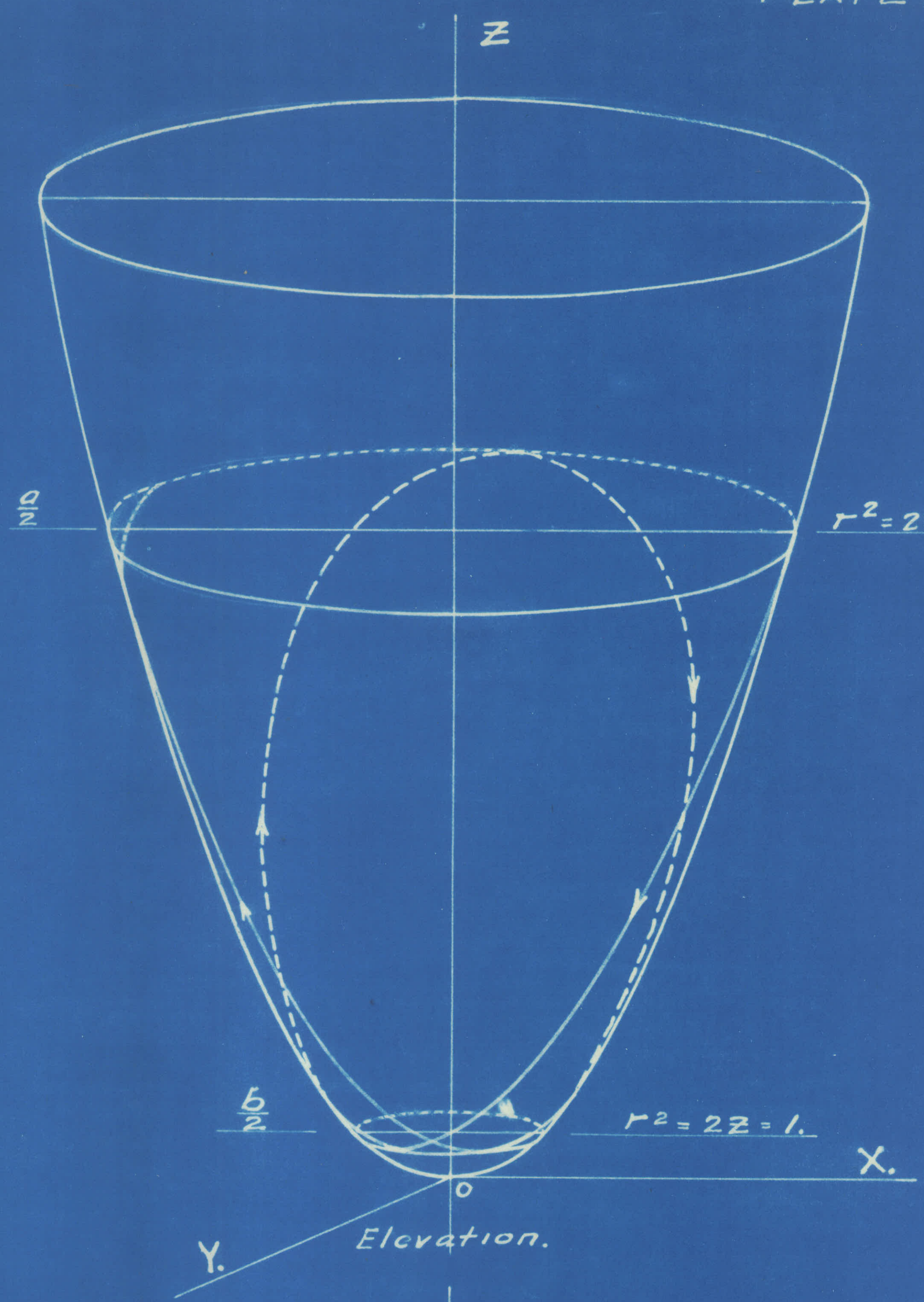
$a = b. = h.$

$r = \sqrt{h}.$

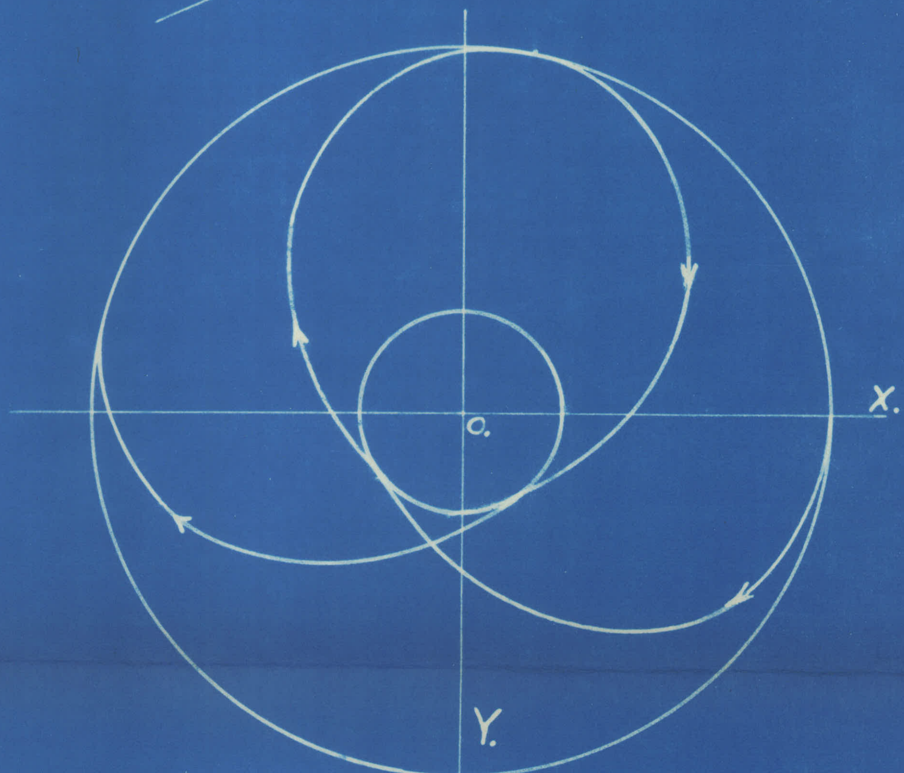
Projection on XY Plane

Circular Orbit.

PLATE III.



Elevation.



Projection on Plane XY.

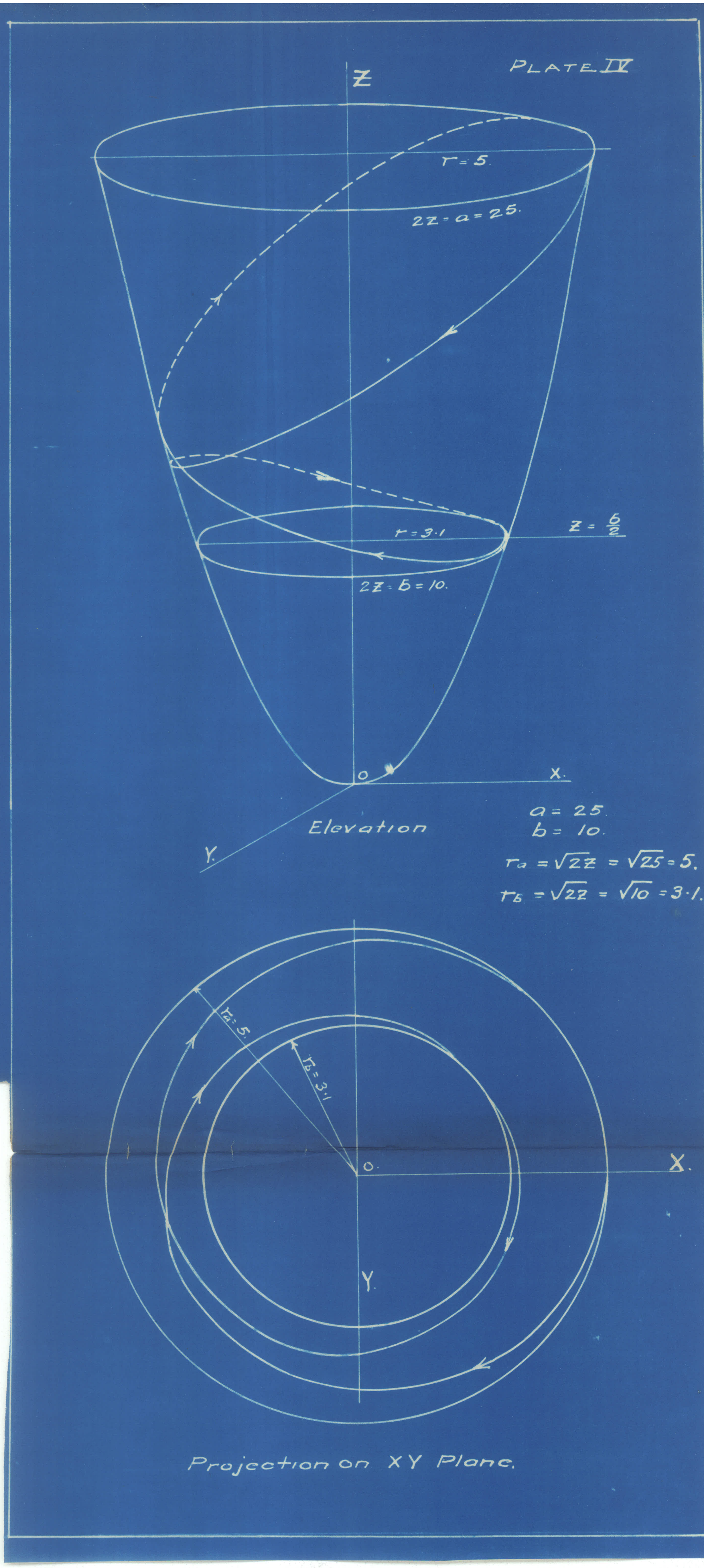


PLATE IV
Z
r = 5.
2z = a = 25.
z = b/2
r = 3.1
2z = b = 10.
O
X.
Y.
Elevation
a = 25.
b = 10.
$r_a = \sqrt{2z} = \sqrt{25} = 5.$
$r_b = \sqrt{2z} = \sqrt{10} = 3.1.$
$r_a = 5$
$r_b = 3.1$
O.
X.
Y.
Projection on XY Plane.