THE UNIVERSITY OF MANITOBA

FREELY VIBRATING BEAMS

by

Daqing Chang

A Thesis

Submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

of

DOCTOR OF PHILOSOPHY

To Dr. C. S. Chang

and

To the memory of my late father

ABSTRACT

An operator based formulation is used to show the completeness of the eigenvectors of a non-uniform, axially loaded, transversely vibrating Euler-Bernoulli beam having eccentric masses and supported by off-set linear springs. This result generalizes the classical expansion theorem for a beam having conventional end conditions. Furthermore, the effect of truncating a series approximation of the initial deflection is investigated for the first time. New asymptotic forms of the eigenvalues and eigenvectors are determined which are themselves often sufficiently accurate for high frequency calculations.

A numerical procedure normally needs to be used for a transversely vibrating Euler-Bernoulli beam having complicated interior and end conditions because closed form solutions (including their asymptotic forms) are mostly beyond reach. The Rayleigh-Ritz approximate procedure has been applied widely to self-adjoint problems in structural dynamics. However, the numerical convergence of the Rayleigh-Ritz procedure deteriorates significantly if Gibbs phenomenon occurs. In this thesis, generalized force mode functions are suggested as one means of avoiding this effect. The convergence rate of the eigenvalue approximations resulting from the use of such functions is determined for a discontinuous, freely vibrating Euler-Bernoulli beam. Moreover, the pointwise convergence of the derivatives that correspond to the practically important bending moment and shear force is examined for the first time. Then, a numerical example is given to corroborate the new theory.

Non-self-adjoint systems are encountered when viscous damping forces or a gyroscopic effect exists. The generalized force mode functions method is extended to accommodate a spinning Timoshenko beam having a stepped cross-section. Numerical data suggests that this approach can very accurately approximate the backward and forward precession frequencies, bending moment and shear force.

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LIST OF SYMBOLS

Beam's Geometrical Properties and Deformation

L	length
$\hat{t}$	time
A or $A(x)$	cross-sectional area (at distance x from left end)
A_1, A_2	cross-sectional areas of the first and second segments of a stepped Timoshenko beam
I or $I(x)$	moment of inertia of a cross-section
I_x	polar moment of inertia of a cross-section
O_x	the centroid of the cross-sectional area
O_1, O_2	the centre of gravity of the lumped masses, M_0 and M_1, respectively
$w(x, \hat{t})$	translational deformation of the Euler-Bernoulli beam
$\theta(x, \hat{t})$	rotation of the cross-section of the Euler-Bernoulli beam

Mechanical Properties

E	Young's modulus
G	shear modulus
k	shear coefficient
ρ	mass density

External Force, Internal Force and Supports

$A_f(x)$	external, distributed longitudinal load
P_0, P_1	external, time-independent, concentrated longitudinal load at $x = 0$ and $x = L$, respectively
$M_f(x, \hat{t})$, $V_f(x, \hat{t})$	bending moment and shear force of the Euler-Bernoulli beam, respectively
p or $p(x)$	axial force
$p_i \equiv p(x_i)$	axial force at $x = x_i$
p^m, p^M	uniformly distributed axial forces
k_e	stiffness of elastic foundation
N	a (given) positive integer
K_i, $i = 0, ..., N$	external rectilinear spring stiffness
η_0, η_1	longitudinal off-set of the tips of linear springs
β_i, $i = 0, ..., N$	torsional spring constants
M_i, $i = 0, ..., N$	lumped, eccentric masses
J_i, $i = 0, ..., N$	rotary inertia
e_0, e_1	eccentricity of lumped masses M_0 and M_1, respectively

Admissible Functions

$\{\psi_m(x)\}$	eigenvectors of a uniform Euler-Bernoulli beam
Ω_m	m th characteristic value associated with $\psi_m(x)$

$\alpha_{rm}^{(i)}$	asymptotic form of the ith derivative of the mth eigenvector of a uniform Euler-Bernoulli beam at $x = x_r$
Λ_{rm}	higher order component of the m th eigenvector, $\psi_m(x)$, of a uniform Euler-Bernoulli beam at $x = x_r$
$\bar{\Lambda}_{rm}$	higher order component of the first order derivative of $(L/\Omega_m)\psi_m(x)$ at $x = x_r$
$\{\zeta_{ir}(x)\}$	generalized force mode (GFM) functions
$\varphi_{11}^t(x)$, $\varphi_{12}^t(x)$	GFM functions for the deflection of a Timoshenko beam
$\varphi_{21}^t(x)$, $\varphi_{22}^t(x)$	GFM functions for the slope of a Timoshenko beam

Set and Function Spaces

$\mathbf{C}$	set of complex numbers
$C^\infty(0, L)$	a set whose elements have continuous derivatives upto any arbitrarily high order
$\mathfrak{I}_i^q$, $\mathscr{G}_i^q$, $\mathscr{F}_i^q$	three sets of functions by which q-GFM functions are defined with respect to $\{\psi_m\}$
S_n	n-dimensional subspace
$W^{(2)}(0, L)$	Sobolev space in which every element and its first derivative are absolutely continuous whilst the second derivative is square integrable in $0 < x < L$
D, B	a general Hilbert space and energy space, respectively
$<\cdot, \cdot>_D$	inner product of space D

$\langle \cdot, \cdot \rangle_B$	inner product of space B
$\mathcal{L}^2(0, L)$	Hilbert space of square integral functions
$\mathcal{L}^2(\rho A, 0, L)$	Hilbert space of square integral functions having weight ρA
$H^{(i)}$, $E^{(i)}$	a Hilbert space and an energy space, respectively, of a vector having given integer i components
$\langle \cdot, \cdot \rangle_{H^{(i)}}$	inner product of space $H^{(i)}$
$\langle \cdot, \cdot \rangle_{E^{(i)}}$	inner product of space $E^{(i)}$
$\\|\cdot\\|_{H^{(i)}}$	norm of space $H^{(i)}$
$\\|\cdot\\|_{E^{(i)}}$	norm of space $E^{(i)}$
R(F), R(F^*)	Rayleigh-quotients for $F \in H^{(5)}$ and $F^* \in H^{(4)}$, respectively

Linear Mappings and Operators

τ_i, $i = 1, \ldots, 5$	linear mappings
Π, Π^{-1}	linear vector operator and its inverse
Π^*, Π^{*-1}	linear vector operator corresponding to $J_1 = 0$
$Dom(\Pi)$, $Dom(\Pi^*)$	domain of operators Π and Π^*, respectively
I	identity operator
P	an orthogonal projection of space, B, on an n-dimensional subspace
P_n	an orthogonal projection of a Hilbert space, H, on a (n-1)-dimensional subspace
U_{ir}, $i = 1, 2$, $r = 0, 1, \ldots, N$	linear operators defined at a beam's end

U_{ir}^{+}, U_{ir}^{-}, $\mathfrak{C}_{ir}^{+}$, $\mathfrak{C}_{ir}^{-}$, $i = 1, 2$, $r = 0, 1, ..., N$	linear operators defined at the interior points of a beam

Eigenvalues and Eigenvectors

(1) Euler-Bernoulli Beam

$\boldsymbol{W}_j$, $\boldsymbol{W}_j^*$, $j = 1, 2, ...$	jth eigenvector of operators Π and Π^*, respectively
$\boldsymbol{W}_{j,J_1}^*$	a four-component vector obtained by eliminating the fifth component of $\boldsymbol{W}_j$
w_{1j}, $j = 1, 2, ...$	jth eigenvector of a Euler-Bernoulli beam
$w_{1j}^{(1)}$, $w_{1j}^{(2)}$	the first and second order asymptotic forms of w_{1j}, respectively
w_j^n	jth eigenvector approximation in an n-dimensional subspace
x	co-ordinate of a beam that starts from its left end
$z_j\sigma$, $j = 1, 2, ...$	jth characteristic value
$(z_j)_1\sigma$, $(z_j)_2\sigma$	the first and second asymptotic estimates of the jth characteristic value, respectively
ϑ_m	the phase of the mth eigenvector of a uniform beam having standard end conditions
λ_j, $j = 1, 2, ...$	jth eigenvalue of a beam

λ_j^*, $j = 1, 2, \ldots$	jth eigenvalue of a modified beam by adding mass or rotatory inertia to or by removing rotatory inertia from one end of the beam.
λ_j^m, λ_j^M $j = 1, 2, \ldots$	jth eigenvalue corresponding to the constants p^m and p^M, respectively
λ_j^n, $j = 1, 2, \ldots$	jth eigenvalue approximation in an n-dimensional subspace
μ_j, μ_j^n $j = 1, 2, \ldots$	jth frequency parameter and its approximation in an n-dimensional subspace, respectively
ω_j, $j = 1, 2, \ldots$	jth (circular) natural frequency
ω_{0j}, $j = 1, 2, \ldots$	jth (circular) natural frequency for a lumped mass with no off-set
$\omega_j^2 \Upsilon_j$ $j = 1, 2, \ldots$	kinetic energy of a uniform Euler-Bernoulli beam corresponding to the jth natural frequency

(2) **Timoshenko Beam**

ω_j^t, $j = \pm 1, \pm 2, \ldots$	jth ($j > 0$) forward or backward natural frequency
$u_{1j}^t(x)$, $u_{2j}^t(x)$ $j = \pm 1, \pm 2, \ldots$	translations of a spinning Timoshenko beam corresponding to the jth natural frequency in the χ_1 and χ_2 directions, respectively
$\Phi_{1j}^t(x)$, $\Phi_{2j}^t(x)$ $j = \pm 1, \pm 2, \ldots$	slopes, in a fixed frame, of a spinning Timoshenko beam due to bending at the jth natural frequency
Θ	angular speed of a spinning Timoshenko beam

Linear Ordinary Differential (LOD) Equation

$G(x, \xi)$	Green's function of a multiple-point boundary value problem
$\phi_{ir}(x)$, $i = 1, 2, 3, 4$, $r = 1, 2, \ldots, N$	four independent solutions of a LOD equation in $x_{r-1} \le x \le x_r$

Miscellaneous

$y_0(x)$	initial deflection of a Euler-Bemoulli beam
$O(\ )$	Landau's notation in the asymptotic analysis
( ), { }	row and column vector, respectively
$(\ )^T$, $\{\ \}^T$	transpose of a row and column vector, respectively
[ ]	square matrix
OA_r	a ray running from the origin, O, to an arbitrary point, A_r
$\theta_0(x_r)$	initial angle of OA_r with respect to the positive χ_0 axis
$\theta_m(x_r)$	angle of OA_r with respect to the positive χ_0 axis after m stepped increment counterclockwise from $\theta_0(x_r)$

CHAPTER 1

INTRODUCTION

1.1 Introduction

Both the rotary inertia and shear deformation are neglected in the Euler-Bernoulli model of a free, transversely vibrating beam. Although the model is limited to a beam having a small flexural wavelength to length ratio [1], it is a simple and widely used approximation for beam-like structures which may have additional mass or rotary inertia at their ends (e.g. a mast supporting an antenna [2], or a single joint robot carrying an end payload [3, 4]) or which may be loaded axially (an accelerating missile [5]). In the analysis and control of beam vibrations, it is extremely important to understand the eigenvalue distributions and eigenvectors as well as the influence of parameters, such as the off-set of a lumped mass and axial force, on the dynamic behaviour [6, 7]. In most cases, it is impossible to obtain an analytical solution for a beam having a complicated cross-section, except for several particular cases, e.g. [8]. On the other hand, first order asymptotic estimates of natural frequencies have been presented in [9] for a non-uniform beam having conventional end conditions such as simple supports, fixed, sliding or pinned ends. However, no rigorous justification was given. A similar procedure has been employed in [10] for a beam having arbitrary elastic displacement and rotation constraints at its ends. However, reference [10] failed to explicitly identify the data included in the estimates. By clarifying the asymptotic solution of a cantilevered beam, this thesis derives, rigorously and explicitly for the first time, the first and second order asymptotic forms of the eigenvalues and eigenvectors of a non-uniform Euler-Bernoulli beam having non-conventional end conditions, i.e. eccentric masses supported by off-set linear springs.

In addition to the asymptotic estimates of the natural frequencies, a practically important problem concerns how a perturbation of the beam's end conditions influences its natural frequencies. This question may arise from the dynamic analysis and control of a robotic manipulator. Numerical results [3, 4] illustrate that such behaviour is particularly difficult to analyze when the centre of gravity of the manipulator's payload does not coincide with the manipulator's end or alters as a task changes. Furthermore, no theoretical analysis has been derived to indicate whether the classical inclusion principle can be employed to estimate the natural frequencies. To clarify this important point, a detailed theoretical analysis as well as numerical data are presented in this thesis. It is concluded that the classical inclusion principle is invalid for an off-set mass. Furthermore, the off-set of a mass mainly influences the positioning accuracy.

An important issue in the numerical simulation of the dynamic response of an externally excited beam concerns the completeness of the eigenvectors. Completeness is needed in a Hilbert space and corresponding energy space to ensure that (i) a beam's initial conditions or an external force can be truly expanded in terms of the eigenvectors, and (ii) the bending moment can be predicted reliably [11]. Furthermore, completeness is a fundamental requirement when eigenvectors are used in the Rayleigh-Ritz method for self-adjoint problems or in the Galerkin method for non-self-adjoint problems. For a non-uniform beam having conventional ends, completeness is a direct result of the well-known Sturm-Liouville theorem [12]. On the other hand, this theorem cannot be applied straightforwardly when a mass or rotary inertia is connected with a longitudinal off-set to an end of a beam. Then the integral kernel of the eigenvalue problem depends upon the eigenvalues themselves [13]. A formal statement of the completeness of the eigenvectors has been given in [14, 15] by observing the orthogonality of the eigenvectors and employing the delta function. However, it has been shown in [16] that the orthogonality of the eigenvectors is neither a necessary condition nor a sufficient condition for their completeness. A rigorous proof of completeness may employ a Hilbert space theory [13, 17 - 19] or a S-Hermitian boundary value approach [20]. The main idea behind these two methods is to transfer the original eigenvalue problem to an integral equation with a kernel function independent of the eigenvalue. For example, a Hilbert space formalism was used in [21] to prove the completeness of the eigenvectors of a transversely vibrating, non-uniform rotating beam having one end fixed and a mass located precisely at the other end. Completeness was also shown in a Hilbert space in [22] by using a perturbation theory for a non-uniform cantilever beam having an axial force and the centre of a (eccentric) mass off-set from the free end. In this thesis, the completeness of eigenvectors in both a Hilbert space and an energy space is confirmed by employing operator theory for a beam having more than one eccentric masses and supported by off-set springs.

In addition to completeness, another fundamental problem is to detect how closely eigenvectors can approximate a known function, like an initial deflection, when a transient response is formulated in terms of such eigenvectors. In other words, it is important to determine how rapidly numerical errors decrease as the number of eigenvectors increases.

Suppose, for example, that a function has continuous derivatives upto order three and also possesses a piecewise continuous fourth order derivative. The classical expansion theorem for a beam having conventional ends [23] states that, if this function satisfies all the beam's end conditions, a series expansion as well as each series obtained by differentiating it upto three times converge uniformly and absolutely at each point of the beam. However, the work presented in this thesis demonstrates that the classical expansion theorem still applies when the function is expanded in terms of the eigenvectors of a Euler-Bernoulli beam having an eccentric mass and possibly springs off-set from both ends - even when the function does not satisfy a single end condition.

Analytical solutions, including asymptotic forms, are clearly important because of the insight they provide into a structure's behaviour. Unfortunately, they cannot be found for most real structures so that numerical methods have to be employed. The Rayleigh-Ritz procedure is a well established numerical method. It traditionally employs continuously differentiable functions to approximate the eigenvalues and eigenvectors of, say, a freely vibrating Euler-Bernoulli beam [24]. These functions may be a set of independent polynomial functions or the eigenvectors of a uniform Euler-Bernoulli beam having conventional end conditions. However, such functions can produce significant numerical oscillations in the practically important second and third deflection derivatives near discontinuities or the beam's boundaries. This is called Gibbs' phenomenon [25]. Moreover, the eigenvalues are approximated poorly [25 - 27]. To avoid this phenomenon, a mixed Rayleigh variational approach [26], in which the deflection and stress are considered simultaneously, can be used for beams having a continuous stress distribution despite material discontinuities. However, a larger eigenvalue problem is generated and the eigenvalue estimates are not necessarily upper bounds. Another approach reported in [26] approximates merely the stress. Although the co-ordinate transformation given in [28] can be applied, a complicated second order differential equation is produced. A non-standard finite element approach has also been proposed [29, 30] in which solely the deflection is approximated by using a suitable average for the varying cross-section or material characteristics of an element [30]. However, this procedure again does not necessarily produce upper bound estimates for the eigenvalues. Moreover, these methods cannot avoid Gibbs' phenomenon at the discontinuities of the bending moment and shear force of a beam having interior linear and torsional spring supports or lumped masses.

Force mode functions and quasi-comparison functions have been used, with a uniform beam's eigenvectors, to accommodate interior springs [27] and natural end conditions [31]. The force mode functions are associated with the static deflection of the beam. In particular, the first order force mode function is the deflection found by replacing an intermediate spring with an analogous concentrated force. However, an outstanding issue concerns appropriate functions when a rigid body motion occurs after a spring has been removed. Furthermore, it may be difficult to derive an analytical form when a non-uniformity is not piecewise constant [32]. On the other hand, quasi-comparison functions involve at least two sets of eigenvectors of a uniform beam corresponding to different natural boundary conditions. When a beam has discontinuities, the beam has to be divided into different pieces at the discontinuities. Then a set of quasi-comparison functions have to be defined on each component. Finally, the approximate solution is resolved by using a component mode synthesis. It can be expected that, with more discontinuities, a larger eigenvalue problem is created again. Moreover, regardless of approach, no pointwise error estimates have been derived yet for the bending moment and shear force. To overcome these problems, a unified approach, called the generalized force mode (GFM) function method, is proposed. The pointwise convergence of the practically important bending moment and shear is derived and confirmed numerically for a freely vibrating, Euler-Bernoulli beam. Finally, the GFM function method is applied, in conjunction with the Galerkin method, to solve the free vibrations of a non-self-adjoint, spinning Timoshenko beam having a discontinuous cross-section. An easy way of constructing the GFM functions is proposed and numerical data demonstrates that Gibbs phenomenon does not happen at the discontinuity of a bending moment or shear force.

1.2 Objectives of Thesis

The main objectives of this thesis are stated next.

A. To present a detailed procedure which generalizes and significantly extends previous expansion theorem [22]. The extension enables a non-uniform beam to have more than one eccentric mass and be supported by springs that are off-set from one or both its ends. Furthermore, the first and second order asymptotic estimates of the natural frequencies are derived explicitly for the first time. The error from truncating a series approximation of the initial deflection is also investigated. This investigation uses an extended inclusion principle to formulate new and easily enumerated, asymptotic forms of the eigenvalues and eigenvectors when an eccentric mass is added to a beam. Moreover, the principle affirms the numerical data given in [3] and disproves the paradoxical observation stated in [4] that a larger mass, at a given eccentricity, can increase a particular natural frequency. Finally, the mathematical formula demonstrating the influence of the off-set of a lumped mass is derived for the first time.

B. To develop a unified procedure for selecting admissible functions in order to handle, in the Rayleigh-Ritz method, a complex Euler-Bernoulli beam having complicated interior as well as end conditions. These functions involve the eigenvectors and generalized force mode functions of a uniform beam having conventional ends. Generalized force mode (GFM) functions may be constructed by finding the static deflection of a uniform beam arising from either a concentrated moment or force acting at the location of a discontinuity. Thus, discontinuous deflection derivatives are approximated by discontinuous functions. A rigorous treatment of GFM functions is also needed to guarantee that approximate solutions have a high convergence rate with an increasing number of admissible functions. This important aspect is presented in this thesis along with new error estimates for the eigenvalues and eigenvectors. Furthermore, sufficient conditions are proposed for the pointwise convergence of the second and third deflection derivatives. These conditions are proved for a beam having an arbitrarily located discontinuity. They are suggested numerically for more than one discontinuity. A numerical example is also given to confirm the theory and demonstrate that Gibbs phenomenon is avoided when GFM functions are employed in conjunction with the eigenvectors of a uniform Euler-Bernoulli beam having standard end conditions.

C. To extend the GFM method in order to analyze a non-self-adjoint problem involving a stepped, spinning Timoshenko beam. To achieve this end, a general method which employs Hermite polynomial interpolation is proposed for the construction of the GFM functions. This approach advantageously avoids the need to solve a boundary value problem in order to find the static deflection. Furthermore, it may provide simpler forms of the GFM functions. Accurate numerical data suggests that the approach has great potential.

1.3 Thesis Layout

This thesis has five chapters and thirteen Appendices. The results needed to achieve objective A are presented and discussed in Chapter 2. Chapter 3 deals with objective B whilst Chapter 4 considers objective C. Finally, conclusions and recommendations are presented in Chapter 5. Detailed proofs are given more conveniently in Appendices.

CHAPTER 2

FREE VIBRATIONS OF A NON-UNIFORM EULER-BERNOULLI BEAM

2.1 Introduction

The completeness of a beam's eigenvectors is fundamentally important in a generalized Fourier's series expansion. This is because completeness ensures convergence when the eigenvectors are employed to approximate, for example, an initial deflection. Moreover, completeness is a primary requirement in the successful application of the Rayleigh-Ritz or Galerkin methods. Reference [22] has shown the completeness of the eigenvectors of a non-uniform, axially loaded, cantilever Euler-Bernoulli beam having a mass off-set from its free end. This chapter generalizes and significantly extends this work to a beam having two eccentrically located masses as well as off-set springs. For simplicity, the static deflection caused by the total weight is assumed negligible as in [2, 3, 4, 5, 22].

Completeness guarantees convergence only when the number of eigenvectors tends to infinity. In practical computations, however, only a finite number of eigenvectors can be employed. This limitation leads inevitably to a so-called truncation error. Consequently, a question arises as to how the truncation error decreases with the use of an increasing number of eigenvectors. This chapter clarifies this important issue by deriving the convergence rate of a generalized Fourier's series based upon the use of asymptotic forms of the eigenvalues and eigenvectors. These asymptotic forms are derived by employing an extended inclusion principle. They are useful not only in a convergence analysis but also in the approximation of the exact higher valued eigenvalues and corresponding eigenvectors. Furthermore, the influence of an off-set mass on the eigenvalues is clarified. This work is motivated by the need to assess the positioning accuracy of a robotic arm when the payload's centre of gravity changes with different tasks or does not coincide with the arm's end.

2.2 Euler-Bernoulli Beam Theory

A non-uniform beam having length L is illustrated in Figure 2.1. The M_r, J_r, K_r and β_r ($r = 0, 1$) shown indicate (positive) lumped masses and rotary inertia as well as rectilinear and torsional spring constants. The non-negative e_r and η_r, on the other hand, represent respectively the distances (i.e. longitudinal off-sets) of the centres of gravity of the lumped masses and the tips of the linear springs outside the ends of the beam. $A(x)$ and ρ are, respectively, the area of cross-section and density of the beam whilst $E(x)$ and $I(x)$ are the Young's modulus and moment of inertia of a cross-section, respectively. Furthermore, $EI(x)$ is positive and assumed to be twice differentiable whilst $\rho A(x)$ is positive and continuous. That is, there exist two positive constants c_1 and c_2 such that $EI(x) \geq c_1 > 0$ and $\rho(x) \geq c_2 > 0$ for $0 \leq x \leq L$. Then the free vibrations can be found from (M.25) and (M.31) of Appendix M to be

$$\tau_1 w_{1j} \equiv (\rho A)^{-1} [(EI\, w_{1j}'')'' - (p\, w_{1j}')'] = \lambda_j w_{1j}, \quad 0 < x < L \tag{2.2.1}$$

where $p(x)$ is a continuous axial force such that buckling is avoided [5]. Note that an equation number with a letter prefix is given in the corresponding Appendix. Moreover, λ_j and $w_{1j}(x)$ are the jth eigenvalue and corresponding eigenvector, respectively, and a prime superscript indicates a differentiation with respect to x. If the explicit function of x is omitted for brevity, it can be found from (M.26) through (M.30) and (M.32) through (M.35) that the end conditions can be written as

$$\tau_{2+k}\, w_{(2+k)j} \equiv M_r^{-1}\,[K_r(w_{1j}-(-1)^r\eta_r w_{1j}')-(-1)^r p w_{1j}'+ \\ +(-1)^r(EIw_{1j}'')']\big|_{x=x_r} = \lambda_j w_{(2+k)j} \tag{2.2.2}$$

and

$$\tau_{3+k}\, w_{(3+k)j} \equiv J_r^{-1}\,[\,e_r(EIw_{1j}'')'+(-1)^r K_r(e_r-\eta_r)(w_{1j}-(-1)^r\eta_r w_{1j}')- \\ -(-1)^r EIw_{1j}''+(\beta_r-pe_r)w_{1j}']\,\big|_{x=x_r} = \lambda_j w_{(3+k)j} \tag{2.2.3}$$

where $k = 0$ or 2, $r = 0$ if $k = 0$ otherwise $r = 1$, whilst $x_0 = 0$, $x_1 = L$. The τ_n, $n = 1, 2, \ldots, 5$ define a mapping in which

$$\left.\begin{aligned} & w_{2j} \equiv w_{1j}(0)-e_0 w_{1j}'(0), \qquad w_{3j} \equiv w_{1j}'(0), \\ \textit{and} \quad & \\ & w_{4j} \equiv w_{1j}(L)+e_1 w_{1j}'(L), \qquad w_{5j} \equiv w_{1j}'(L). \end{aligned}\right\} \tag{2.2.4}$$

Completeness of the eigenfunctions will be considered next.

2.3. Completeness of Eigenvectors

By reformulating (2.2.1) through (2.2.3) in an operator form, the eigenvectors' completeness can be determined in a Hilbert space and an energy space. Following the proposal of Freidman [13], define $H^{(5)}$ as a Hilbert space having five-component vectors such that

$$H^{(5)} = \mathscr{L}^2(\rho A,\ 0,\ L) \oplus \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \tag{2.3.1}$$

with the inner product given by

$$<F,\ G>_{H^{(5)}} = \left(\int_0^L \rho A f_1\, \overline{g_1}\, dx\right) + M_0 f_2\, \overline{g_2} + J_0 f_3\, \overline{g_3} + M_1 f_4\, \overline{g_4} + J_1 f_5\, \overline{g_5} \tag{2.3.2}$$

for two arbitrary vectors $F = (f_1,\ \ldots,\ f_5)$ and $G = (g_1,\ \ldots,\ g_5) \in H^{(5)}$. Here $\mathscr{L}^2(\rho A,\ 0,\ L)$ designates a Hilbert space of square-integrable functions with weight $\rho A(x)$, $\mathbf{C}$ is the set of complex numbers and the overhead bar denotes the complex conjugate. Furthermore, $\|F\|_{H^{(5)}} = (<F,\ F>_{H^{(5)}})^{1/2}$, $F \in H^{(5)}$, defines the norm of $H^{(5)}$.

Define a linear vector operator Π: $Dom(\Pi) \rightarrow H^{(5)}$ such that

$$\Pi\, Y = (\tau_1 y_1,\ \tau_2 y_2,\ \tau_3 y_3,\ \tau_4 y_4,\ \tau_5 y_5) \tag{2.3.3}$$

for every $Y = (y_1,\ y_2,\ \ldots,\ y_5) \in Dom(\Pi)$ where the y_i, $i \geq 2$, are defined in terms of $y_1(x)$ and its derivative at $x_0 = 0$ and $x_1 = L$ by (2.2.4). The $Dom(\Pi)$ describes a domain of Π that is dense in $H^{(5)}$. The proof of the density of $Dom(\Pi)$ in $H^{(5)}$ is given in Appendix A. Moreover, the range of Π is in $H^{(5)}$ so that the $y_1''''(x)$ needed in (2.2.1) lies in $\mathscr{L}^2(\rho A,\ 0,\ L)$. Furthermore, (2.2.1) through (2.2.3) may be rewritten succinctly for the

jth eigenvalue, λ_j, and corresponding eigenvector, $w_{1j}(x)$, as

$$\Pi\, W_j = \lambda_j W_j \tag{2.3.4}$$

where

$$W_j = (w_{1j}, w_{2j}, w_{3j}, w_{4j}, w_{5j}) \tag{2.3.5}$$

is the jth eigenvector of operator Π. Relation (2.3.5) shows that the completeness of $w_{1j}(x)$ follows from the completeness of eigenvector W_j which is stated formally next but proved in Appendix B.

Theorem 2.3.1. A positive constant c exists such that an energy space $E^{(5)}$ can be formed by completing a space having the inner product

$$<F, G>_{E(5)} = <\Pi F, G>_{H(5)} + c<F, G>_{H(5)} \tag{2.3.6}$$

whilst $\|F\|_{E(5)} = (<F, F>_{E(5)})^{1/2}$ for $F, G \in Dom(\Pi)$. Moreover, the eigenvectors, W_j, $j = 1, 2, \ldots$ form a complete orthogonal system in $H^{(5)}$ and $E^{(5)}$. That is,
$<W_i, W_j>_{H(5)} = 0 = <W_i, W_j>_{E(5)}$, $i \neq j$ and

$$\lim_{n\to\infty} \| F - \sum_{j=1}^{n} \frac{a_j W_j}{\|W_j\|_{H(5)}} \|_{H(5)} = 0 \tag{2.3.7}$$

for an arbitrary vector $F \in H^{(5)}$ whilst

$$\lim_{n\to\infty} \| F - \sum_{j=1}^{n} \frac{a_j W_j}{\|W_j\|_{H(5)}} \|_{E(5)} = 0 \tag{2.3.8}$$

for an arbitrary $F \in E^{(5)}$ where

$$a_j = \langle F, W_j \rangle_{H(S)} / \| W_j \|_{H(S)} . \tag{2.3.9}$$

To estimate the error introduced by truncating (2.3.7), the order of the a_j given by (2.3.4) is important. This point is considered in section 2.5 after first presenting the required asymptotic estimates of the eigenvalues and eigenfunctions.

2.4. Eigenvalue Properties

The fundamental properties of the eigenvalues are investigated next. Then the first and second order asymptotic estimates are presented. Finally, the eigenvalues of a flexible manipulator are considered as a practical application of the theory.

2.4.1 Theoretical Analysis

The first order asymptotic estimate of the eigenvalue, λ_j, of a non-uniform cantilever beam may be employed as a convenient base for other end conditions. First, however, the general second order asymptotic form, $w_{1j}^{(2)}(x)$, of the jth eigenvector, $w_{1j}(x)$, is needed as $j \to \infty$. It has been shown in Appendix C (based upon [9]) to take the form

$$\begin{aligned} w_{1j}^{(2)}(x) = \alpha(x)(A_j \cos \xi_1(x) + B_j \sin \xi_1(x) + C_j \exp(-\xi_2(x)) \\ + D_j \exp(-(\sigma z_j + \sigma^* / z_j - \xi_2(x))) \end{aligned} \tag{2.4.1}$$

with $0 \le x \le L$ as well as

$$z_j^4 = \lambda_j, \quad \xi_1(x) = \hat{x}(x) z_j - \chi(x)/z_j, \quad \xi_2 = \hat{x}(x) z_j + \chi(x)/z_j \tag{2.4.2}$$

whilst

$$\hat{x}(x) = \int_0^x \hat{b}(x)dx \quad and \quad \alpha(x) = (\hat{b}(x))^{-3/2}(EI(x))^{-1/2} \tag{2.4.3}$$

whereas

$$\chi(x) = \int_0^x [(\frac{5\hat{b}''}{4\hat{b}^2} - \frac{15(\hat{b}')^2}{8\hat{b}^3}) - \frac{3((EI)')^2}{8\hat{b}(EI)^2} + \frac{(EI)''}{2EI\hat{b}} + \frac{p(x)}{4EI\hat{b}}]dx, \tag{2.4.4}$$

with

$$\hat{b}(x) = (\frac{\rho A(x)}{EI(x)})^{1/4}, \quad \sigma = \hat{x}(L) \quad and \quad \sigma^* = \chi(L). \tag{2.4.5}$$

The A_j, B_j, C_j and D_j used in (2.4.1) are constants which depend upon the order, j, as well as the particular end conditions of a beam. By substituting (2.4.1) into (2.2.2) and (2.2.3),

$$[\Xi_{ij}](A_j\ B_j\ C_j\ D_j)^T = (0\ 0\ 0\ 0)^T \tag{2.4.6}$$

is obtained where the Ξ_{ij}, i and j = 1, 2, 3, 4 are detailed in Appendix D. The A_j, B_j, C_j and D_j are not generally zero simultaneously so that the determinant of $[\Xi_{ij}]$, $\det(\Xi_{ij})$, will normally be

$$\det(\Xi_{ij}) = 0. \tag{2.4.7}$$

This condition, of course, provides the frequency equation from which λ_j can be estimated. By employing (2.4.1), (D.28) and (D.33), the following result can be shown.

Theorem 2.4.1. The first order asymptotic estimate, $(z_j)_1\,\sigma$, of $z_j\sigma$, defined within (2.4.2) and (2.4.5), is $(2j - 1)\pi/2$ for a non-uniform beam which is cantilevered.

Proof

Let

$$(z_j)_1\,\sigma = (j+j_0+1/2)\pi + \nabla_j \tag{2.4.8}$$

where j_0 and ∇_j need to be determined for $|\nabla_j| \leq \pi/2$. It can be shown, essentially from (D.28) and (D.33), that, for $K_0 = \infty$, $\beta_0 = \infty$, $K_1 = 0$, $\beta_1 = 0$, $M_1 = 0$ and $J_1 = 0$, the asymptotic frequency equation and the corresponding A_j, B_j, C_j and D_j are given, as $j \to \infty$, by

$$\left.\begin{aligned} \cos z_j\sigma = O(z_j^{-1}),\ \ A_j = -B_j = -C_j = 1, \\ D_j = -(\sin z_j\sigma + \cos z_j\sigma) \end{aligned}\right\} \tag{2.4.9}$$

in which $f(j) = O(g(j))$ means that there exists a positive constant, c, which is independent of j and such that $|f(j)| \leq c|g(j)|$ as $j \to \infty$. Combining (2.4.8) with (2.4.1) and (2.4.9) yields $\nabla_j \to 0$ as $j \to \infty$ as well as the first order asymptotic form

$$\begin{aligned} w_{1j}^{(1)}(x) = \alpha(x)[\cos\,\Omega_n\hat{x}/\sigma - \sin\,\Omega_n\hat{x}/\sigma - \exp(-\Omega_n\hat{x}/\sigma) \\ +(-1)^j\exp(-(1-\hat{x}/\sigma)\Omega_n)] + O(j^{-1}) \end{aligned} \tag{2.4.10}$$

where

$$\Omega_n = (2n-1)\pi/2 \quad and \quad n = j+j_0+1. \tag{2.4.11}$$

Now the function in the square parentheses of (2.4.10) constitutes the nth eigenvector of a *uniform* cantilever beam [33]. Thus, $w_{1j}^{(1)}(x)$ has $(n - 1)$ nodes in $0 < x \leq L$ [34].

However, it is known [34] that $w_{1j}^{(1)}(x)$ has $(j - 1)$ nodes so that j must equal n. Consequently, it can be seen from (2.4.11) that j_0 must equal -1 and, from (2.4.8), that $z_j\sigma$ has the required first order asymptotic value of $(2j - 1)\pi/2$ because $\nabla_j \to 0$ as $j \to \infty$.

(*Remark.* Although the asymptotic estimate of λ_j has been stated previously for a non-uniform cantilever beam, e.g. [9], no rigorous proof has been presented.)

An inclusion theorem is given next as a means of finding the approximate eigenvalues of a non-uniform beam due to a change in either the axial force or the beam's end conditions.

Theorem 2.4.2 (i) If an eccentric mass or rotary inertia is added to one end of a non-uniform beam, which has the j th eigenvalue λ_j^*, the modified beam's corresponding eigenvalue, λ_j, will satisfy $\lambda_{j-1}^* \leq \lambda_j \leq \lambda_j^*$.

(ii) Let λ_j^m and λ_j^M be the j th eigenvalues corresponding to a beam having a constant axial force p^m or p^M, respectively, where $p^m \leq p(x) \leq p^M$ for $0 \leq x \leq L$. Then the j th eigenvalue, λ_j, corresponding to the spatially varying axial force $p(x)$ satisfies $\lambda_j^m \leq \lambda_j \leq \lambda_j^M$.

Theorem 2.4.2 is an extension of the classical inclusion principle (e.g. [35]) in which a mass without eccentricity is considered. Its proof (using min-max and max-min principles) is very similar to the classical one. Details can be found in Appendix E.

The following lemma and theorem are needed to determine more precise asymptotic estimates of the eigenvalues.

Lemma 2.4.1. The eigenvector $w_{1j}(x)$ and its first spatial derivative, which correspond to a simple eigenvalue, depend continuously upon (finite) K_r, β_r, M_r, J_r, $r = 0$, 1 and $p(x)$.

This lemma can be obtained directly, except for $M_r = 0 = J_r$, $r = 0$ and 1, by using the classical Rellich's perturbation theorem on operator Π [36]. For $M_r = 0 = J_r$, Π needs to be modified so that it can be defined in a Hilbert space whose elements correspond to a vector having fewer than five components. In this case, the proof of the lemma is analogous to that given in [37] for the numerical stability of the round-off error introduced by different admissible functions in the Rayleigh-Ritz approach. The proof is provided in Appendix G.

Theorem 2.4.3. If a non-zero λ_j is not a repeated eigenvalue, then

$$\left.\begin{aligned}
\frac{\partial z_j}{\partial M_r} &= -\frac{z_j}{4\| W_j \|^2_{H^{(5)}}}(w_{1j}-(-1)^r e_r w'_{1j})^2 \,|_{x=x_r} && 0 \le M_r < \infty, \\
\frac{\partial z_j}{\partial J_r} &= -\frac{z_j}{4\| W_j \|^2_{H^{(5)}}}(w'_{1j})^2 \,|_{x=x_r} && 0 \le M_r < \infty, \\
\frac{\partial z_j}{\partial K_r} &= \frac{1}{4z_j^3\| W_j \|^2_{H^{(5)}}}(w_{1j}-(-1)^r \eta_r w'_{1j})^2 \,|_{x=x_r} && 0 \le K_r < \infty, \\
\frac{\partial z_j}{\partial \beta_r} &= \frac{1}{4z_j^3\| W_j \|^2_{H^{(5)}}}(w'_{1j})^2 \,|_{x=x_r} && 0 \le \beta_r < \infty \\
\frac{\partial z_j}{\partial e_r} &= \frac{(-1)^r M_r z_j}{2\| W_j \|^2_{H^{(5)}}}(w_{1j}-(-1)^r e_r w'_{1j}) w'_{1j} \,|_{x=x_r} && 0 \le e_r < \infty
\end{aligned}\right\} \qquad (2.4.12)$$

for a M_r with eccentricity e_r and a K_r with off-set η_r, $r = 0$ and 1. Furthermore, when $p(x)$ is constant,

$$\frac{\partial z_j}{\partial p} = \frac{1}{4z_j^3 \| W_j \|^2_{H^{(5)}}} \int_0^L (w'_{1j}(x))^2 \, dx . \tag{2.4.13}$$

Proof

Suppose $r = 1$ and that $\infty > M_1 \geq 0$ is changed to M_1^* due to the augmentation or loss of a mass at $x = L$ whilst all other parameters (like e_0, e_1 etc.) remain fixed. Denote the corresponding Hilbert space, the jth eigenvalue and eigenvector by $H^{*(5)}$, λ_j^* and W_j^*, respectively. Then it can be shown straightforwardly that

$$<(AW_j^* - \lambda_j^* W_j^*),\ W_j>_{H^{*(5)}} + \lambda_j^* <W_j^*,\ W_j>_{H^{*(5)}} - \lambda_j <W_j,\ W_j^*>_{H^{(5)}} = 0. \tag{2.4.14}$$

Consequently, the partial derivative of λ_j with respect to M_1 can be found to be

$$\frac{\partial \lambda_j}{\partial M_1} = \lim_{M_1^* \to M_1} \frac{\lambda_j^* \left(<W_j^*,\ W_j>_{H^{*(5)}} - <W_j,\ W_j^*>_{H^{(5)}} \right)}{(M_1 - M_1^*) <W_j,\ W_j^*>_{H^{(5)}}} . \tag{2.4.15}$$

Substituting $z_j = \lambda_j^{1/4} > 0$ from (2.4.2) into the last equality and using Lemma 2.4.1 yields

$$\frac{\partial z_j}{\partial M_1} = -\frac{z_j}{4 \| W_j \|^2_{H^{(5)}}} (w_{1j} + e_1 w'_{1j})^2 \, |_{x = x_1} . \tag{2.4.16}$$

The last equation is simply (2.4.12) with $i = 1$. Similar proofs can also be demonstrated for the other derivatives.

Two practical applications are presented in the following sections.

2.4.2 Asymptotic Estimates

It can be shown from Theorem 2.4.3 that $\partial z_j/\partial K_r$, $\partial z_j/\partial\beta_r$ and $\partial z_j/\partial p$ tend to zero as $j \to \infty$ for a beam having a constant axial force. Then Theorem 2.4.2 (ii) indicates that $(z_j)_1\,\sigma$ is independent of finite K_r, β_r and $p(x)$. Equation (2.4.12), on the other hand, can be used in conjunction with Table 2.1 to find the analogous effect of adding M_r or J_r at $x = x_r$, $r = 0$ and 1. For example if $r = 1$, (2.2.2) and (2.2.4) lead to

$$(w_{1j}(L)+e_1 w_{1j}'(L)) = (1/z_j^4 M_1)[K_1(w_{1j}+\eta_1 w_{1j}') - (EIw_{1j}'')' + pw_{1j}']\,|_{x\equiv L}\,. \tag{2.4.17}$$

The last equation does not depend upon $w_{1j}(x)$ and its derivatives at $x = 0$ so that, by using (2.4.1), it can be demonstrated that $(w_{1j}(L)+e_1 w'_{1j}(L)) = O(j^{-1})$ as $j \to \infty$ regardless of the conditions at $x = 0$. (Moreover, it is shown later that all z_j have the same asymptotic order of j whilst $\|W_j\|_{H^{(5)}}$ is demonstrated in Appendix F to be bounded below and above by constants.) Therefore the asymptotic order of $\partial z_j/\partial M_1$ can be found from (2.4.12) to be $j \times j^{-2} = j^{-1}$. Consequently, adding M_1 at $x = x_1$ does not change the asymptotic eigenvalue. The asymptotic order of $w_{1j}(x)$ and its derivatives at $x = x_r$, $r = 0$ or 1, can be determined similarly. Only the results are summarized in Table 2.1.

The first order asymptotic eigenvalue estimates of the beam shown in Figure 2.1 can be derived now by using Theorems 2.4.1 through 2.4.3 and Table 2.1. To illustrate the procedure, suppose a non-uniform beam has a constant axial force and the end conditions $K_0 = \beta_0 = \infty$ and $M_1 \neq 0 \neq J_1$, $\beta_1 < \infty = K_1$ with $\eta_1 = 0 \neq e_1$. Consider first, however, the same beam but without the axial force, eccentric mass, M_1, and rotary inertia, J_1. The classical inclusion principle [12] and (D.28) indicate that $(z_j)_1\,\sigma$ then respectively satisfies

$$(2j-1)\pi/2 \le (z_j)_1\sigma \le (2j+1)\pi/2 \quad and \quad \cos(z_j)_1\sigma - \sin(z_j)_1\sigma = 0 \tag{2.4.18}$$

where the upper and lower bounds correspond to the jth and $(j + 1)$ th values of $(z_j)_1\,\sigma$ for a cantilever beam. Thus $(z_j)_1\,\sigma$ must equal $(4j + 1)\pi/4$. Now, add the mass M_1 eccentrically at $x = x_1$. Theorem 2.4.2 (i) and (D.28) show that

$$(4j-3)\pi/4 \le (z_j)_1\sigma \le (4j+1)\pi/4 \quad and \quad \cos(z_j)_1\sigma = 0 \tag{2.4.19}$$

so that $(z_j)_1\,\sigma = (2j - 1)\pi/2$. Finally, add the rotary inertia, J_1, at $x = x_I$. Theorem 2.4.2 (i) and (D.28) indicate that $(z_j)_1\sigma$ then satisfies

$$(2j-3)\pi/2 \le (z_j)_1\sigma \le (2j-1)\pi/2 \quad and \quad \cos(z_j)_1\sigma = 0. \tag{2.4.20}$$

Now $(z_j)_1\,\sigma$ cannot be determined uniquely from (2.4.20) so that Theorem 2.4.3 and Table 2.1 are needed. First, the asymptotic order of $w'_{1j}(L)$, corresponding to the previous end conditions of $J_1 = 0$ with $e_1 \neq 0 \neq M_1$, $\beta_1 < \infty = K_1$ and $\eta_1 = 0$, can be found to be j^{-2} from the intersection of the fourth column from the right and the fifth row from the bottom of Table 2.1. Moreover, Theorem 2.4.3 indicates that $\partial z_j/\partial J_1$ tends to zero as $j \to \infty$. That is, there is no change in $(z_j)_1\,\sigma$ when $j \to \infty$ due to the addition of J_1. Furthermore, it can be shown from (2.4.13) and Theorem 2.4.2 (ii) that a constant axial force does not influence the first order asymptotic estimate so that the previous $(z_j)_1\,\sigma = (2j - 1)\pi/2$ still applies and it is the final result.

Other end conditions can be treated similarly. The results are summarized in Table 2.2. This table confirms that, for given conditions at $x = x_{r_1}$, $(z_j)_1\,\sigma$ i.e. $(z_j)_1$ is independent of K_{r_2}, β_{r_2} as well as η_{r_2} if $K_{r_2} < \infty$ and $\beta_{r_2} < \infty$, where r_1, $r_2 = 0$ or 1 but $r_1 \neq r_2$. On the other hand, if $J_{r_2} \neq 0$, Table 2.2 indicates that $(z_j)_1$ is independent of e_{r_2}. Furthermore, if $K_{r_2} = \infty$ in addition to $J_{r_2} \neq 0$, $(z_j)_1$ is also independent of η_{r_2}.

As a matter of interest, the second order asymptotic estimate, $(z_j)_2\sigma$, can be found from (D.28) and (2.4.5) to be

$$(z_j)_2\sigma = [(z_j)_1\sigma + \frac{\sigma^*}{(z_j)_1}] - \frac{\Delta_1}{\Delta_2} \qquad for \ \frac{\Delta_1}{\Delta_2} \to 0, \tag{2.4.21}$$

$$(z_j)_2\sigma = [(z_j)_1\sigma + \frac{\sigma^*}{(z_j)_1}] + (\frac{\Delta_1}{\Delta_2})^{-1} \qquad for \ \frac{\Delta_1}{\Delta_2} \to \infty, \tag{2.4.22}$$

and

$$(z_j)_2\sigma = [(z_j)_1\sigma + \frac{\sigma^*}{(z_j)_1}] - [\tan((z_j)_1\sigma) + \frac{\Delta_1}{\Delta_2}]/2 \quad for \ \left|\frac{\Delta_1}{\Delta_2}\right| \to 1. \tag{2.4.23}$$

Δ_1 and Δ_2 are given by (D.29) and $|\Delta_1/\Delta_2|$ can tend only to the indicated limits as $j \to \infty$. The Δ_2/Δ_1 in (2.4.22) was inadvertently neglected in [8] for the particular example of a non-uniform cantilever beam.

By substituting the first (or second) order asymptotic estimates of $z_j\sigma$ into (D.31) and (D.33), the corresponding A_j, B_j, C_j and D_j can be found and the first (second) order asymptotic estimates of the eigenfunctions can be obtained straightforwardly. These results are used to determine the order of the a_j defined by (2.3.9).

2.4.3 Influence of an Off-Set, Lumped Mass

Theorem 2.4.3 is employed in this section to investigate the practical effect of an off-set payload on the positioning accuracy of a flexible manipulator. For simplicity, only a uniform beam is considered for which $K_1 = \beta_1 = 0$. Then it can be seen from equation

(2.4.12) that

$$\frac{\partial \omega_j}{\partial e_1} = -\frac{1}{2\,\omega_j \Upsilon_j}[\,\omega_j^2 M_1 (w_{1j}(L) + e_1 w_{1j}'(L)) w_{1j}'(L)\,] \tag{2.4.24}$$

where ω_j is the j th natural frequency satisfying $\omega_j = (\lambda_j)^{1/2}$ whilst a prime superscript indicates a differentiation with respect to the spatial co-ordinate x and

$$\Upsilon_j = 1/2\,[(\int_0^L \rho A\, w_{1j}^2(x)dx) + M_0(w_{1j}(0) - e_0\, w_{1j}'(0))^2 + J_0\,(w_{1j}'(0))^2 + \\ + M_1(w_{1j}(L) + e_1\, w_{1j}'(L))^2 + J_1\,(w_{1j}'(L))^2\,]\,. \tag{2.4.25}$$

The ρA in the last equation is the beam's mass per unit length so that $\omega_j^2 \Upsilon_j$ is the kinetic energy of the jth mode. Equation (2.4.25) leads to the observation that a unit, independent change in e_1 modifies $\omega_j^2 \Upsilon_j$ by $\omega_j^2 M_1(w_{1j}(L) + e_1 w'_{1j}(L))w'_{1j}(L)$ - the term contained in the square parentheses of equation (2.4.24). Consequently Rayleigh's principle [12] may be used to straightforwardly validate equation (2.4.24) given that, to first order, the strain energy is unaffected by e_1 being modified. Hence, as suggested by equation (2.4.24), the frequency variation depends solely upon the change in the kinetic energy.

When, as done here, $w_{1j}'(L)$ is set invariably to unity, the inequality

$$\frac{\partial \omega_j}{\partial e_1} \lesseqgtr 0 \quad if \quad w_{1j}(L) + e_1 \gtreqless 0 \tag{2.4.26}$$

can be found immediately from equation (2.4.24). Therefore ω_j will increase (decrease) when e_1 is changed alone if $(w_{1j}(L) + e_1)$ or, in non-dimensional form, $(w_{1j}(L) + e_1)/L$ is negative (positive). Although such trends are apparently simple to predict, the application of inequality (2.4.26) may be limited by the often tedious computation of $w_{1j}(L)$ (and $w'_{1j}(L)$) [3]. However, a numerical example is given next to demonstrate that the inequality provides a useful check when variations in ω_j are complex.

Suppose $\rho A(x) \equiv$ constant and $EI(x) \equiv$ constant such that

$$\frac{10e_0}{L} = \frac{100\rho AL}{M_1} = \frac{10J_1}{\rho AL^3} = 1 \tag{2.4.27}$$

and

$$\frac{EIM_0}{\rho AL^4K_0} = \frac{\rho AL^3}{J_0} = \frac{\beta_0 L}{EI} = 5. \tag{2.4.28}$$

These values are identical to those used in [3]. The ratio $\omega_j/\ \omega_{0j}$, where ω_{0j} is the j th natural frequency for an identical payload with no off-set, was computed by using double precision arithmetic on a SUN/4 - 280 workstation [38, 39]. The results for different values of e_1/L are presented in Figures 2.2(a) and 2.3. Good agreement is demonstrated with limited previous data [3]. On the other hand, Figure 2.2(b) gives, for the first time, the values of $(w_{1j}(L) + e_1)/L$, $w'_{1j}(L) - 1$, corresponding to the frequency ratios of Figure 2.2(a).

Figures 2.2(a) and 2.3 reveal that the fundamental natural frequency is affected most by a given variation in a payload's off-set. Indeed, these figures demonstrate that $\omega_j/\ \omega_{0j}$ deviates more from one as the mode number, j, generally decreases and e_1/L increases. Not unexpectedly, therefore, a particular payload has a detrimentally greater influence on the overall dynamics as the off-set grows. If an off-set is neglected, then the lowest frequency modes - particularly the fundamental mode - should be controlled to achieve more accurate positioning of a heavy payload [7].

A careful comparison of Figures 2.2(a) and 2.2(b) corroborates inequality (2.4.26). These figures show that $\omega_j/\ \omega_{0j}$ decreases for mode $j = 1$ but increases for $j = 5$ because the corresponding $(w_{1j}(L) + e_1)/L$ is always positive or negative, respectively. Mode $j = 3$ is more interesting. The $(w_{1j}(L) + e_1)/L$ is negative upto about $e_1/L = 0.0345$ but is positive otherwise. Thus, the corresponding ω_j/ω_{0j} grows and then diminishes with increasing e_1/L in a way that is consistent with inequality (2.4.26). This dichotomous behaviour, moreover, confirms that an inclusion principle cannot hold for the third natural frequency because its value may either increase or decrease when e_1/L is perturbed around 0.03.

2.5. Convergence Rate Estimates

The objective of this section is to investigate the convergence rate, j^{-k}, of $|a_j|$ as $j \to \infty$ for a non-uniform beam's initial deflection, $y_0(x)$, in order to determine the difference $|y_0(x) - y_n(x)|$. Here $y_n(x)$ is the summation of the first n terms of the infinite series

$$\sum_{j=1}^{\infty} a_j w_{1j}(x)$$

constituting $y_0(x)$. The next result is needed to achieve the objective.

Theorem 2.5.1. Let φ_j represent any one of the four functions $w_{1j}(x)$, $(1/z_j\sigma)w_{1j}'(x)$, $(1/z_j\sigma)^2 w_{1j}''(x)$ and $(1/z_j\sigma)^3 w_{1j}'''(x)$ as $j \to \infty$. Then the inequality

$$|\int_0^L y(x)\varphi_j(x)dx| \le \frac{c}{j}, \tag{2.5.1}$$

holds for any function, $y(x)$, that is piecewise continuous in $0 \le x \le L$ [40]. Here c and subsequently c_i, $i = 0, 1, 2, \ldots$, are generic positive constants.

Proof

Consider initially the case of $\varphi_j(x) \equiv w_{1j}(x)$. Set the $\chi(x)$ of (2.4.2) to zero so that the resulting $w_{1j}^{(1)}(x)$ from (2.4.1) represents the first order approximation of $w_{1j}(x)$. Hence, it is known from [34] that there exists a positive constant, c_1, which is independent of j and such that

$$|w_{1j} - w_{1j}^{(1)}| \le \frac{c_1}{j} \tag{2.5.2}$$

as $j \to \infty$. Equation (2.4.1), combined with the generic inequality $|ab| \le |a(b-c)| + |ac|$, leads to

$$\begin{aligned} \left|\int_0^L y(x)\, w_{1j}(x)\, dx\right| \le\ & c_1 \left|\int_0^L y(x) dx\right| / j + c_2 \left\{ \left|\int_0^L y(x) \sin (z_j)_1 \hat{x}\, dx\right| \right. \\ & + \left|\int_0^L y(x) \cos (z_j)_1 \hat{x}\, dx\right| + \left|\int_0^L y(x) \exp(-(z_j)_1 \hat{x})\, dx\right| \qquad (2.5.3) \\ & \left. + \left|\int_0^L y(x) \exp(-(z_j)_1 (\sigma - \hat{x}))\, dx\right| \right\} \end{aligned}$$

where $c_2 = max(\alpha(x)A_j, \alpha(x)B_j, \alpha(x)C_j, \alpha(x)D_n)$. It can be shown straightforwardly [41] that

$$\left| \int_0^L y(x) \sin(z_j)_1 \hat{x}\, dx \right| \le c_3/j, \quad \left| \int_0^L y(x) \cos(z_j)_1 \hat{x}\, dx \right| \le c_3/j \tag{2.5.4}$$

and

$$\left| \int_0^L y(x) \exp(-(z_j)_1 \hat{x})\, dx \right| \le c_3/j \tag{2.5.5}$$

where c_3 is a positive constant which is independent of j. Consequently,

$$\left| \int_0^L y(x) w_{1j}(x) dx \right| \le (c_1 c_4 + 4 c_2 c_3)/j \quad and \quad c_4 = \left| \int_0^L y(x) dx \right|. \tag{2.5.6}$$

Taking $c = c_1 c_4 + 4c_2 c_3$ produces (2.5.1). Similar proofs can also be formulated for the remaining functions.

By using Table 2.1 and Theorem 2.5.1, the convergence rate, j^{-k}, of $|a_j|$ can be found as $j \to \infty$. To illustrate the procedure, suppose the initial deflection, $y_0(x)$, of the non-uniform beam of Figure 2.1 has continuous derivatives up to fifth order. Let the sixth derivative be piecewise continuous. Integrating (2.3.9) by parts and using (2.2.2) and (2.2.3) leads to

$$\begin{aligned} a_j = \{ z_j^{-8} \int_0^L (EIR_a'' w_{1j}'' - pR_a')' w_{1j} dx + z_j^{-8} \sum_{r=0}^{1} (EIR_a' w_{1j}'' - pR_a' w_{1j}) \big|_{x=x_r} \\ + z_j^{-4} \sum_{r=0}^{1} [R_{br} w_{1j} + R_{cr} w_{1j}' - z_j^{-4} R_a ((EIw_{1j}'')' - pw_{1j}')] \big|_{x=x_r} \} \| W_j \|_{H^{(5)}}^{-1} \end{aligned} \tag{2.5.7}$$

where the function $R_a(x)$ as well as R_{ar}, R_{br} and R_{cr}, $r = 0, 1$, are detailed in the footnote of Table 2.3. It can be seen from Table 2.2 that, as suggested previously, the z_j all have the same asymptotic order of n. Furthermore, it is shown in Appendix F that two positive constants, c_1 and c_2, exist such that $c_1 < \|W_j\|_{H(S)} < c_2$ for all j. On the other hand if, for example, $y(x) = EI(x)R_a''(x)$ and $z(x) = -(p(x)R_a'(x))'$, Theorem 2.5.1 indicates that the integral multiplied by z_j^{-8} in (2.5.7) behaves like $j^{-8} \times j = j^{-7}$ as $j \to \infty$. Moreover, the asymptotic behaviour of $w_{1j}(x_r)$, $w_{1j}'(x_r)$, $w_{1j}''(x_r)$ and $(EIw_{1j}''(x_r))'$ can be found for this particular beam from the twelfth rightmost row of Table 2.1 to be j^{-1}, j^{-1}, j^{2} and j^{3}, respectively. Therefore, the second and third terms in (2.5.7), which correspond to the summations, have the asymptotic order j^{-6} and j^{-5}, respectively. Consequently, the third asymptotic term dominates and $|a_j| \leq cj^{-5}$ as $j \to \infty$, where c is a positive constant which is independent of j. However, if the initial deflection, $y_0(x)$, satisfies the further conditions $R_{ar} = R_{br} = R_{cr} = 0$, $r = 0$ and 1, the third term becomes zero and, hence, $|a_j| \leq cj^{-6}$ as $j \to \infty$. If, in addition, $R_{ar}' = 0$, $r = 0$ and 1, then the second term is also zero so that $|a_j| \leq cj^{-7}$ as $j \to \infty$. Now the footnote of Table 2.3 suggests that the conditions $R_{br} = R_{cr} = 0$ correspond to the static end conditions when M_r and J_r, $r = 0$ and 1, do not exist. Thus, an initial deflection caused by a static external force automatically satisfies $R_{cr} = R_{dr} = 0$ but not necessarily $R_{ar} = R_{ar}' = 0$, $r = 0, 1$.

By using Table 2.3, the error produced by truncating $y_0(x)$ can be estimated from the inequality [40]

$$\sum_{j=n}^{\infty} j^{-k} \leq n^{-(k-1)}/(k-1) \quad for\ 1 < k. \tag{2.5.8}$$

For example, if $|a_j| \leq cj^{-k}$ as $j \to \infty$, the last inequality, when introduced into the truncated series, gives

$$|y_0(x)-\sum_{j=1}^{n} a_j w_{1j}(x)| = |\sum_{j=n+1}^{\infty} a_j w_{1j}| \leq c\sum_{j=n+1}^{\infty} j^{-k} \leq \frac{c}{(k-1)}(n+1)^{-(k-1)}. \quad (2.5.9)$$

Therefore, an appropriate number of terms can be chosen, a priori, for the series with knowledge of the asymptotic behaviour of $|a_j|$. Table 2.3 indicates that this behaviour depends strongly upon a beam's end conditions.

(*Remark.* Table 2.3 can also be applied to a series expansion of an initial velocity as well as to the situation involving an external force.)

2.6. Conclusions

Completeness has been shown in a Hilbert space and an energy space by employing an operator theory for the eigenvectors of a non-uniform, axially loaded, Euler-Bernoulli beam having eccentric masses and supported by springs. Consequently, the general solution to the forced and free vibrations of the beam can be obtained in terms of a summation of these eigenvectors. On the other hand, a truncation error is inevitable in practical computations. Therefore, in order to determine how the truncation error decreases with the use of an increasing number of eigenvectors, the convergence rate has been analyzed and tabulated. The results demonstrate, for the first time, that a series expansion in terms of the eigenvectors, as well as each series obtained by differentiating it upto three times, converge uniformly and absolutely. This result implies that the eigenvectors of a non-uniform beam having eccentric masses and springs should produce a high convergence rate when used as the Rayleigh-Ritz base functions in the component mode synthesis. This conjecture is substantiated numerically in [43] through illustrative examples.

In addition to the completeness and convergence rate, asymptotic estimates of the eigenvalues and eigenvectors have been derived to approximate the higher order, exact eigenvalues and corresponding eigenvectors. These estimates are simple in form so that they can be applied, for example, to the design of distributed feedback by using independent modal-space control [6]. Then the optimal distributed control force is a summation of modes whose weighting coefficients can be approximated easily and accurately at high frequencies by employing the asymptotic estimates. Moreover, the effect of an off-set lumped mass on the lower natural frequencies of a beam has also been investigated. It is demonstrated that, for a given mass, an off-set influences mainly the fundamental natural frequency. Practically, this means that a robot's positioning accuracy, say, can be influenced by the off-set of the payload's gravity centre.

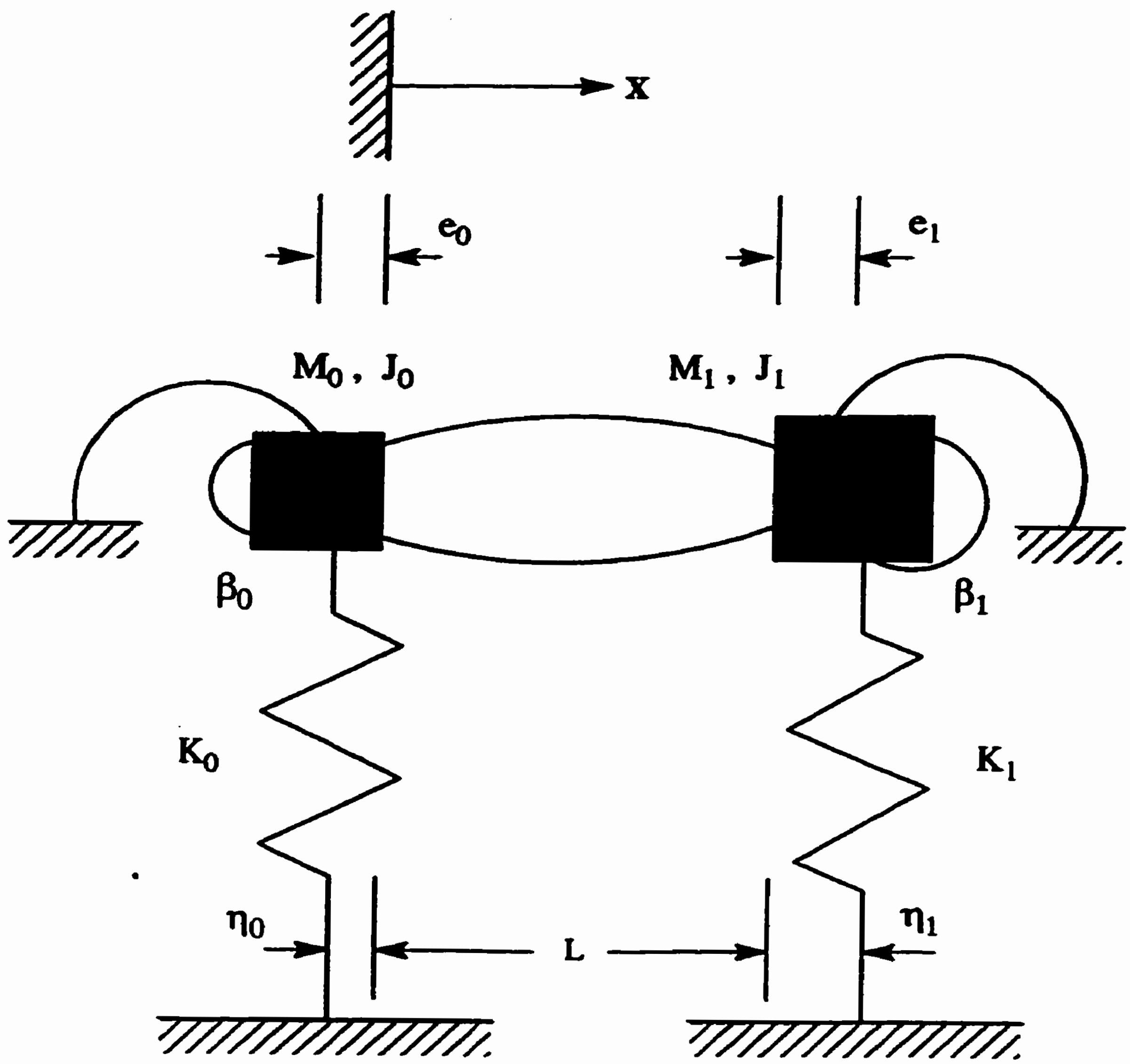

Figure 2.1. A non-uniform beam having general end conditions.

Table 2.1. Asymptotic estimates of $w_{1j}(x)$ and its derivatives at $x = x_r$ with $p_r \equiv p(x_r)$, $r = 0$ and 1.

Conditions at $x = x_r$						$w_{1j}(x_r)$	$w_{1j}'(x_r)$	$w_{1j}''(x_r)$	$(EIw_{1j}''(x_r))'$	$w_{1j}(x_r)-(-1)^r e_r w_{1j}'(x_r)$
$e_r = 0$	$K_r \neq \infty$	$M_r \neq 0$				$O(j^{-1})$			$O(j^3)$	$O(j^{-1})$
		$M_r = 0$	$\eta_r = p_r = 0$			$O(1)$			$O(1)$	$O(1)$
			$\eta_r \neq 0$ or $p_r \neq 0$			$O(1)$			$O(j)$	$O(1)$
	$\beta_r \neq \infty$	$J_r \neq 0$	$K_r \neq \infty$				$O(j^{-2})$	$O(j^2)$		
			$K_r = \infty$	$\eta_r = 0$		0	$O(j^{-2})$	$O(j^2)$	$O(j^3)$	0
				$\eta_r \neq 0$		$O(j^{-1})$	$O(j^{-1})$	$O(j^2)$	$O(j^3)$	$O(j^{-1})$
		$J_r = 0$	$K_r \neq \infty$				$O(j)$	$O(j)$		
			$K_r = \infty$	$\eta_r = 0$		0	$O(j)$	$O(j)$	$O(j^3)$	0
				$\eta_r \neq 0$	$M_r = 0$	$O(1)$	$O(1)$	$O(j^2)$	$O(j^2)$	$O(1)$
					$M_r \neq 0$	$O(j^{-1})$	$O(j^{-1})$	$O(j^2)$	$O(j^3)$	$O(j^{-1})$
$e_r \neq 0$ $M_r \neq 0$	$\beta_r \neq \infty$	$J_r \neq 0$	$K_r \neq \infty$			$O(j^{-1})$	$O(j^{-1})$	$O(j^2)$	$O(j^3)$	$O(j^{-1})$
			$K_r = \infty$	$\eta_r = 0$		0	$O(j^{-2})$	$O(j^2)$	$O(j^3)$	$O(j^{-2})$
				$\eta_r = e_r \neq 0$		$O(j^{-1})$	$O(j^{-1})$	$O(j^2)$	$O(j^3)$	0
				$0 \neq \eta_r \neq e_r$		$O(j^{-1})$	$O(j^{-1})$	$O(j^2)$	$O(j^3)$	$O(j^{-1})$
		$J_r = 0$	$K_r \neq \infty$			$O(1)$	$O(1)$	$O(j^2)$	$O(j^2)$	$O(j^{-2})$
			$K_r = \infty$	$\eta_r = 0$		0	$O(j^{-2})$	$O(j^2)$	$O(j^3)$	$O(j^{-2})$
				$\eta_r = e_r \neq 0$		$O(1)$	$O(1)$	$O(j^2)$	$O(j^2)$	0
				$0 \neq \eta_r \neq e_r$		$O(j^{-1})$	$O(j^{-1})$	$O(j^2)$	$O(j^3)$	$O(j^{-1})$
	$\beta_r = \infty$		$K_r \neq \infty$			$O(j^{-1})$	0	$O(j^2)$	$O(j^3)$	$O(j^{-1})$
			$K_r = \infty$			0	0	$O(j^2)$	$O(j^3)$	0

Table 2.2. First order asymptotic estimate $(z_j)_1\,\sigma$.
(For $|p(x)| < \infty$ and $z_1 > 0$ whilst $e_{r_1} \neq 0 \neq M_{r_1}$, $0 \leq \beta_{r_1} < \infty$, $r_1, r_2 = 0, 1$ but $r_1 \neq r_2$.)

End Conditions					$(z_j)_1\,\sigma$
$0 \leq K_{r_1} < \infty$, $0 \leq K_{r_2} < \infty$, $0 \leq \beta_{r_2} < \infty$	$J_{r_1} \neq 0$	$M_{r_2} = 0, J_{r_2} = 0$			$(2j-5)\pi/2$
		$M_{r_2} = 0, J_{r_2} \neq 0$			$(4j-13)\pi/4$
		$M_{r_2} \neq 0$, $J_{r_2} = 0$	$e_{r_2} = 0$		$(4j-11)\pi/4$
			$e_{r_2} \neq 0$		$(4j-13)\pi/4$
		$M_{r_2} \neq 0$	$J_{r_2} \neq 0$		$(2j-7)\pi/2$
	$J_{r_1} = 0$	$M_{r_2} = 0, J_{r_2} = 0$			$(4j-9)\pi/4$
		$M_{r_2} = 0, J_{r_2} \neq 0$			$(j-3)\pi$
		$M_{r_2} \neq 0$, $J_{r_2} = 0$	$e_{r_2} = 0$		$(2j-5)\pi/2$
			$e_{r_2} \neq 0$		$(j-3)\pi$
$0 \leq K_{r_1} < \infty$, $K_{r_2} = \infty$, $0 \leq \beta_{r_2} < \infty$	$J_{r_1} \neq 0$	$J_{r_2} = 0$	$M_{r_2} = 0$	$\eta_{r_2} = 0$	$(4j-7)\pi/4$
				$\eta_{r_2} \neq 0$	$(4j-9)\pi/4$
			$M_{r_2} \neq 0$	$\eta_{r_2} = e_{r_2} \neq 0$	$(4j-9)\pi/4$
				$\eta_{r_2} \neq e_{r_2}$	$(2j-5)\pi/2$
		$J_{r_2} \neq 0$	$M_{r_2} = 0$ or $M_{r_2} \neq 0$		$(2j-5)\pi/2$
	$J_{r_1} = 0$	$J_{r_2} = 0$	$M_{r_2} = 0$	$\eta_{r_2} = 0$	$(2j-3)\pi/2$
				$\eta_{r_2} \neq 0$	$(j-2)\pi$
			$M_{r_2} \neq 0$	$\eta_{r_2} = e_{r_2} \neq 0$	$(j-2)\pi$
				$\eta_{r_2} \neq e_{r_2}$	$(4j-9)\pi/4$
		$J_{r_2} \neq 0$	$M_{r_2} = 0$ or $M_{r_2} \neq 0$		$(4j-9)\pi/4$
$0 \leq K_{r_1} < \infty$, $0 \leq K_{r_2} < \infty$, $\beta_{r_2} = \infty$	$J_{r_1} \neq 0$	$M_{r_2} = 0$			$(4j-9)\pi/4$
		$M_{r_2} \neq 0$			$(2j-5)\pi/2$
	$J_{r_1} = 0$	$M_{r_2} = 0$			$(j-2)\pi$
		$M_{r_2} \neq 0$			$(4j-9)\pi/4$
$0 \leq K_{r_1} < \infty$, $K_{r_2} = \infty = \beta_{r_2}$	$J_{r_1} \neq 0$				$(2j-3)\pi/2$
	$J_{r_1} = 0$				$(4j-5)\pi/4$
$K_{r_1} = K_{r_2} = \infty$, $\beta_{r_2} = \infty$	$J_{r_1} \neq 0$				$(2j-1)\pi/2$
	$J_{r_1} = 0$			$\eta_{r_1} = e_{r_1}$	$(4j-1)\pi/4$
				$\eta_{r_1} \neq e_{r_1}$	$(2j-1)\pi/2$

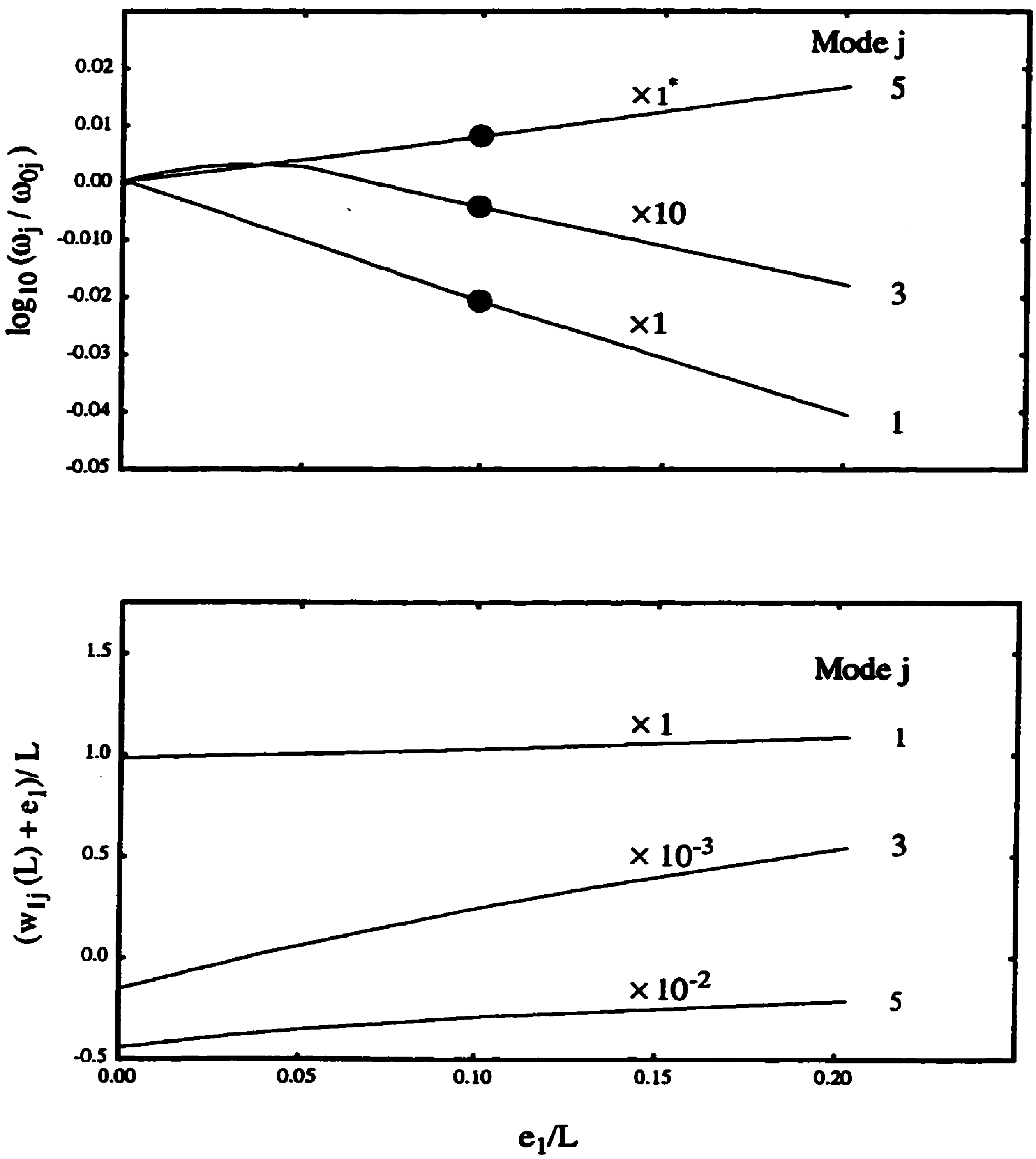

Figure 2.2 . (a) The natural frequencies and (b) corresponding criterion values. ●, Data of reference [3]; *, ordinate scales have to be multiplied by given factors.

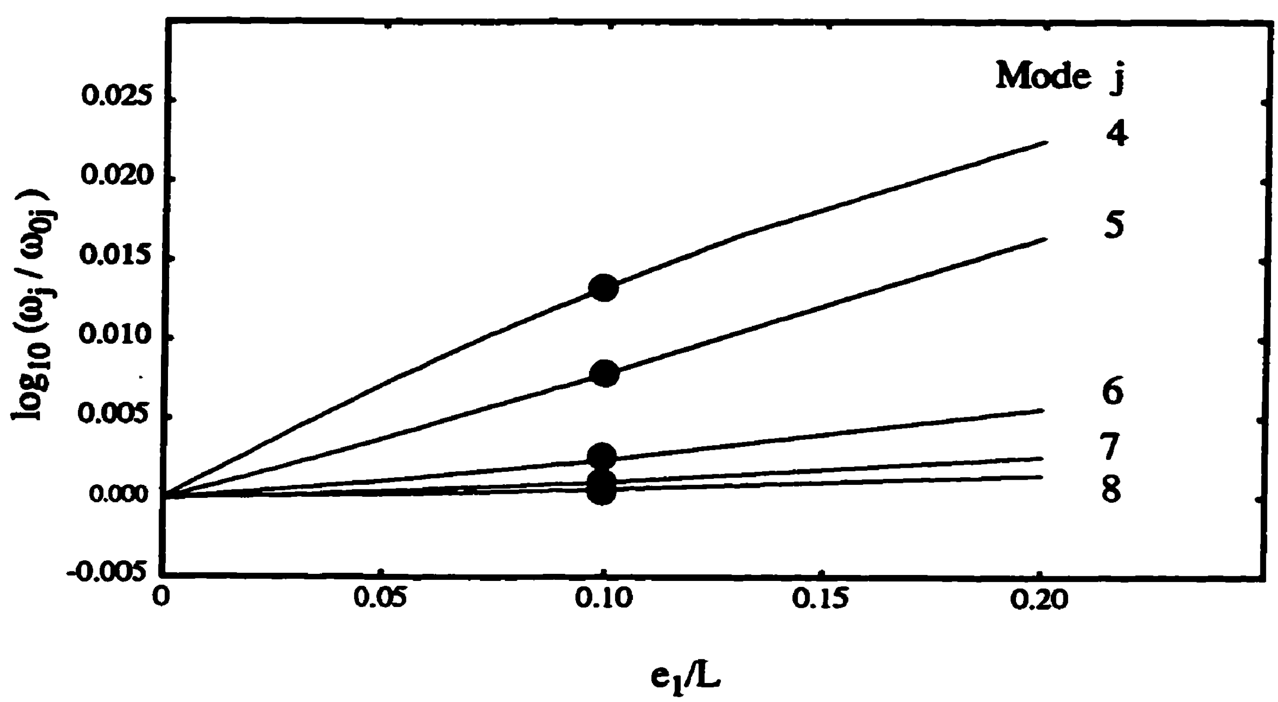

Figure 2.3. Variation of higher frequencies with an off-set mass.

(● data of [3])

Table 2.3. The order, j^{-k}, of $|a_j|$ as $j \to \infty$ for a beam's initial deflection, $y_0(x)$, whose $(k-1)$th derivative is piecewise continuous in $0 \le x \le L$.

End Conditions, $r = 0$ or 1					$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$e_r = 0$, $\beta_r \ne \infty$	$J_r \ne 0$, $K_r \ne \infty$	$M_r = 0$	$\eta_r \ne 0$ or $p_r \ne 0$					$R_{br} = 0$	$R_{ar} = 0$	$R_{ar}' = R_{cr} = 0$
			$\eta_r = 0 = p_r$					$R_{br} = 0$		$R_{ar}' = R_{cr} = 0$
		$M_r \ne 0$							$R_{br} = R_{ar} = 0$	$R_{ar}' = R_{cr} = 0$
	$J_r \ne 0$	$K_r = \infty$	$\eta_r = 0$		$R_{dr} = 0$				$R_{ar} = 0$	$R_{ar}' = R_{cr} = 0$
			$\eta_r \ne 0$		$R_{dr} = 0$				$R_{cr} = R_{ar} = 0$	$R_{ar}' = 0$
	$J_r = 0$	$K_r \ne \infty$	$M_r = 0$				$R_{cr} = 0$	$R_{br} = 0$		
			$M_r \ne 0$				$R_{cr} = 0$		$R_{br} = R_{ar} = 0$	
		$K_r = \infty$	$\eta_r = 0$		$R_{dr} = 0$		$R_{cr} = 0$		$R_{ar} = 0$	
			$\eta_r \ne 0$	$M_r = 0$		$R_{dr} = 0$		$R_{cr} = 0$		$R_{ar} = R_{ar}' = 0$
				$M_r \ne 0$	$R_{dr} = 0$				$R_{cr} = R_{ar} = 0$	$R_{ar}' = 0$
$M_r \ne 0$, $e_r \ne 0$, $\beta_r \ne \infty$	$J_r \ne 0$	$K_r \ne \infty$							$R_{br} = U_l = R_{ar} = 0$	$R_{ar}' = 0$
		$K_r = \infty$	$\eta_r = 0$		$R_{dr} = 0$				$R_{ar} = 0$	$R_{ac} = R_{ar}' = 0$
			$\eta_r \ne 0$		$R_{dr} = 0$				$R_{cr} = R_{ar} = 0$	$R_{ar}' = 0$
	$J_r = 0$	$K_r \ne \infty$						$R_{br} = R_{cr} = 0$		$R_{ar} = R_{ar}' = 0$
		$K_r = \infty$	$\eta_r = 0$		$R_{dr} = 0$				$R_{ar} = 0$	$R_{ac} = R_{ar}' = 0$
			$\eta_r = e_r \ne 0$			$R_{dr} = 0$		$R_{cr} = 0$		$R_{ar} = R_{ar}' = 0$
			$0 \ne \eta_r \ne e_r$		$R_{dr} = 0$				$R_{cr} = R_{ar} = 0$	$R_{ar}' = 0$
$M_r \ne 0$	$e_r \ne 0$	$\beta_r = \infty$	$K_r \ne \infty$			$y_0'(x_r) = 0$			$R_{br} = R_{ar} = 0$	$R_{ar}' = 0$
$\beta_r = \infty$		$K_r = \infty$			$y_0(x_r) = 0$	$y_0'(x_r) = 0$			$R_{ar} = 0$	$R_{ar}' = 0$

Note: $R_a = [(EIy_0'')'' - (py_0')']/m$, $R_{ar} = (-1)^r R_a|_{x=x_r}$, $R_{ar}' = (-1)^r R_a'|_{x=x_r}$, $R_{br} = (-1)^r (EIy_0'')' + K_r(y_0 - (-1)^r \eta_r y_0') - (-1)^r py_0'|_{x=x_r}$, $R_{cr} = (-1)^{r+1} EIy_0'' + \beta_r y_0' - (-1)^r K_r \eta_r (y_0 - (-1)^r \eta_r y_0')|_{x=x_r}$ if $K_r \ne \infty$, otherwise $U_r = (-1)^{r+1} EIy_0'' + \beta_r y_0'|_{x=x_r}$, $p_r \equiv p(x_r)$ where $r = 0$ or 1. $R_{dr} = y_0 - (-1)^r \eta_r y_0'|_{x=x_r}$, $R_{er} = (-1)^{r+1} EIy_0'' + (EIy_0'')' \eta_r + \beta_r y_0' - py_0' \eta_r|_{x=x_r}$.

CHAPTER 3

A UNIFIED APPROACH AND ITS NUMERICAL APPLICATION

3.1 Introduction

Asymptotic solutions were derived in the previous chapter for a single-span, non-uniform Euler-Bernoulli beam having complex end conditions. Unfortunately, most real beams have discontinuous cross-sections or materials so that asymptotic solutions may not be found so easily or even may not exist. Furthemore, low frequency information is often needed in the practical design of a beam. An exact solution approach, however, becomes less tractable as the variation in a beam's cross-section gets more complicated. Then a Rayleigh-Ritz or Galerkin procedure that employs the eigenvectors of a uniform beam having a standard fixed, free or simple end support is generally preferred. However, such an approach can produce poor approximations due to Gibbs phenomenon [25, 27] when a beam has discontinuous material properties, interior masses and spring supports as well as non-conventional end conditions. In this chapter, a generalized force mode (GFM) method is introduced to avoid the Gibbs phenomenon and speed the convergence rate of an approximation. To present this approach, the Rayleigh-Ritz procedure is reviewed first for a Euler-Bernoulli beam having general interior and end conditions. Then the concept of GFM functions is defined. Subsequently, error estimates of the eigenvalues and eigenvectors are derived when GFM functions are used in conjunction with the eigenvectors of a uniform Euler-Bernoulli beam having conventional end conditions. Furthermore, pointwise error estimates of the second and third deflection derivatives are derived under specified conditions. Finally, a numerical example is given to confirm the theory and verify that Gibbs phenomenon is truly avoided. An extension of this method to Galerkin's procedure is given in the next chapter.

3.2 Rayligh-Ritz Approach

Consider a freely vibrating Euler-Bernoulli beam having length L. Unlike the previous chapter, the flexural rigidity $EI(x)$, mass per unit length $\rho A(x)$ and an axial force $p(x)$, where x indicates a typical distance from the beam's left end, may not continuous. Furthermore, the beam is supported by an elastic foundation having stiffness $k_e(x)$. Let

$$0 = x_0 < x_1 \ldots < x_N = L \tag{3.2.1}$$

denote a partition of the interval $0 \le x \le L$ in which the knots x_r, $1 \le r \le N\text{-}1$, correspond to the locations of discontinuities which may involve $EI(x)$, $\rho A(x)$, $p(x)$ and $k_e(x)$. Furthermore, rectilinear and torsional springs, K_r and β_r, as well as lumped masses and rotary inertia, M_r and J_r, may be located at x_r, $r = 0, 1, \ldots, N$.

Suppose that λ_j is the j th exact and distinct, free vibration eigenvalue having multiplicity φ_j. Let $w_j(x)$ be an arbitrary eigenvector in a subspace, $M(\lambda_j)$, that is spanned by all the eigenvectors corresponding to λ_j. It can be shown similarly to (M.20) through (M.24) that the λ_j and $w_j(x)$ are governed by

$$\begin{gathered}(EI(x)w_j''(x))'' - (p(x)w_j'(x))' + k_e(x)w_j(x) = \lambda_j \rho A(x) w_j(x), \\ x_{r-1} < x < x_r,\ r = 1, \ldots, N\end{gathered} \tag{3.2.2}$$

where a prime superscript indicates differentiation with respect to x. If K_r, β_r, M_r and

J_r, $r = 0, 1, \ldots, N$, are all positive and finite, $w_j(x)$ satisfies the end conditions

$$\left.\begin{aligned} K_0 w_j(0) - p(0) w_j'(0) + (EI w_j''(0))' &= \lambda_j M_0 w_j(0) \\ -EI w_j''(0) + \beta_0 w_j'(0) &= \lambda_j J_0 w_j'(0) \end{aligned}\right\} \tag{3.2.3}$$

and

$$\left.\begin{aligned} K_N w_j(L) + p(L) w_j'(L) - (EI w_j''(L))' &= \lambda_j M_N w_j(L) \\ EI w_j''(L) + \beta_N w_j'(L) &= \lambda_j J_N w_j'(L) \end{aligned}\right\} \tag{3.2.4}$$

as well as the following interior conditions at $x = x_r$, $r = 1, \ldots, (N - 1)$,

$$\left.\begin{aligned} & w_j(x_r^+) = w_j(x_r^-), \quad w_j'(x_r^+) = w_j'(x_r^-) \\ & -EI w_j''\big|_{x=x_r^+} + EI w_j''\big|_{x=x_r^-} + \beta_r w_j'(x_r) = \lambda_j J_r w_j'(x_r) \\ & ((EI w_j'')' - p w_j')\big|_{x=x_r^+} - ((EI w_j'')' - p w_j')\big|_{x=x_r^-} \\ & \qquad + K_r w_j(x_r) = \lambda_j M_r w_j(x_r) \end{aligned}\right\} \tag{3.2.5}$$

Negative and positive superscripts indicate limiting values as x approaches x_r from the left and right, respectively. The variational form of equations (3.2.2) through (3.2.5) can be written as [44]

$$B(w_j, y) = \lambda_j D(w_j, y) \tag{3.2.6}$$

for any $y \in W^{(2)}(0, L)$, a Sobolev space in which every element and its first derivative are absolutely continuous whilst the corresponding second derivative is square integrable in $0 \le x \le L$ [45]. Now

$$B(w_j, y) = \int_0^L (EIw_j'' y'' + p w_j' y' + k_e w_j y)\,dx + \sum_{r=0}^{N} (K_r w_j y + \beta_r w_j' y')\,|_{x=x_r} \tag{3.2.7}$$

and

$$D(w_j, y) = \int_0^L \rho A w_j y\,dx + \sum_{r=0}^{N} (M_r w_j y + J_r w_j' y')\,|_{x=x_r} \tag{3.2.8}$$

where a function's dependence upon x is omitted for convenience.

Suppose that there exist positive constants c_i, $i = 0, 1$ (subsequently $i = 2, 3, \ldots$ and c also denote a positive constant) such that $B(u, w)$ and $D(u, w)$ satisfy [46]

$$|B(u, y)| \le c_0 \| u \| \| y \|, \quad B(u, u) \ge c_1 \| u \|^2 \; \textit{and} \; D(u, u) > 0 \tag{3.2.9}$$

for arbitrary non-zero $u(x) \in W^{(2)}(0, L)$ and $y(x) \in W^{(2)}(0, L)$. The $\|\bullet\|$ represents the norm of $W^{(2)}(0, L)$ whilst $D(u, u)^{1/2}$ is assumed to be compact with respect to $\|\bullet\|$ [46]. Then $\|u\|_B = B(u, u)^{1/2}$ defines a norm for an energy space, B, which is equivalent to $W^{(2)}(0, L)$ whilst $\|u\|_D = D(u, u)^{1/2}$ introduces a norm for a Hilbert space, D. These assumptions are required to ensure that the eigenvalue problem is self-adjoint so that the Rayleigh-Ritz procedure can be employed.

(*Remark* 3.2.1. Should $K_r = \infty$ or $\beta_r = \infty$, $r = 0, N$, the end conditions (3.2.3) and (3.2.4) take the simpler form $w_j(x_r) = 0$ or $w_j'(x_r) = 0$. Then it can be shown that equation (3.2.6) still holds if terms involving $w_j(x_r)$ or $w_j'(x_r)$ in equations (3.2.7) and (3.2.8) are omitted and $y(x_r) = 0$ or $y'(x_r) = 0$ for any $y(x) \in B$, $r = 0, N$. This last constraint is a so-called geometric end condition that should be satisfied by any admissible function in the Rayleigh-Ritz procedure [44].)

In order to estimate the errors produced by the Rayleigh-Ritz procedure, it is known [43, 44] that the nature of the discontinuities (i.e. the regularity [47, 48]) of the $w_j(x)$ as well a the solution, $w(x)$, of the equation

$$B(w,u) = D(f,u), \ \textit{for a given } f(x) \in B \textit{ but any } u \in B, \tag{3.2.10}$$

needs to be clarified. The last equation describes the static deflection, $z(x)$, of the complex Euler-Bernoulli beam under consideration when subjected to a temporally independent, distributed force, $\rho A(x)f(x)$, in each interval: $x_{r-1} < x < x_r$ as well as a concentrated force, $M_r f(x_r)$, and bending moment, $J_r f'(x_r)$, located at $x = x_r$, $r = 0, 1, ..., N$. Eigenvalue problem (3.2.6) can be considered a special case of equation (3.2.10) in which $f(x)$ is replaced by $\lambda_j w_j(x)$.

Theorem 3.2.1. Suppose that $EI(x)$, $\rho A(x)$, $k_e(x)$ and $p(x)$ are all differentiable to an arbitrarily high order in each sub-interval V_r: $x_{r-1} \le x \le x_r$, $1 \le r \le N$. Then all order spatial derivatives of the eigenvectors, $w_j(x)$, are continuous in each V_r if (3.2.9) holds for any $u(x) \in B$ and $w(x) \in B$. In addition, $w(x)$ has continuous derivatives upto order five whilst $d^6w(x)/dx^6$ is square integrable in each V_r if $f''(x)$ is square integrable. Furthermore, the $w_j(x)$ and $w(x)$ satisfy conditions (3.2.3) through (3.2.5). (It is worth noting that distributional derivatives of functions having finite jumps at the knots are circumvented by considering these functions only in each individual sub-interval V_r.)

The proof of Theorem 3.2.1 is similar to that given in [49] where a Green's function is employed for differential equations of motion that involve continuously differentiable coefficients but no interior conditions. Details can be found in Appendix H.

Suppose S_n is an n-dimensional subspace of B. Then the n th Rayleigh-Ritz approximation, λ_j^n and $w_j^n(x)$, of λ_j and $w_j(x)$, $j \leq n$, are found from [29]

$$B(w_j^n, u) = \lambda_j^n D(w_j^n, u), \quad \textit{for } w_j^n(x) \in S_n \textit{ and any } u \in S_n. \qquad (3.2.11)$$

However, a solution's convergence rate depends significantly upon the base chosen for S_n. This aspect is considered in the next section. There S_n is spanned by GFM functions and a simple uniform beam's eigenvectors, $\{\psi_m(x)\}$, whose analytical form is given generally by [33]

$$\begin{aligned}\psi_m(x) = {} & Q_{1m}\cos(\Omega_m x/L + \vartheta_m(x)) + Q_{2m}\exp(-\Omega_m x/L) \\ & + Q_{3m}\exp(-(L-x)\Omega_m/L).\end{aligned} \qquad (3.2.12)$$

Phase $\vartheta_m(x)$, the m th characteristic value, Ω_m, and the coefficients Q_{1m}, Q_{2m} and Q_{3m} are determined by the beam's standard end conditions [50]:

$$\frac{d^{\gamma_{rv}+4j}\psi_m(x_r)}{dx^{\gamma_{rv}+4j}} \equiv U_{rv}^j(\psi_m) = 0, \quad r = 0, N, \quad v = 1, 2, \quad j = 0, 1, \ldots . \qquad (3.2.13)$$

Here $\gamma_{r\,v}$ is an integer that depends upon the uniform beam's specific end conditions at $x_0 = 0$ and $x_N = L$. Moreover, $\gamma_{r2} > \gamma_{r1} \geq 0$ and the notation $d^0\psi_m(x)/dx^0 \equiv \psi_m(x)$ is used.

3.3. Asymptotic Error Estimates

Three sets, $\mathfrak{I}_i^q$, $\mathscr{F}_i^q$ and $\mathcal{F}_i^q$, in which i and q are positive integers satisfying $q \geq i + 1$, are described next. They are needed to estimate piecewise asymptotic errors for the higher order deflection derivatives.

Definition 3.3.1. Set $\Im_i^q$ contains a finite number of functions, $\zeta_{ir}(x)$, $r = 1, \ldots, (N - 1)$, that individually satisfy $U_{0\nu}^{\,j}(\zeta_{ir}) = U_{N\nu}^{\,j}(\zeta_{ir}) = 0$ ($\nu = 1, 2$ and $j = 0, 1, \ldots,$ $j_0 \geq [q - \gamma_{0\nu} - 1]/4$). The $\zeta_{ir}(x)$ have uniformly continuous derivatives upto order $(i - 1)$ in $0 \leq x \leq L$ and upto order $(q + 1)$ in both $0 \leq x < x_r$ and $x_r < x \leq L$, $1 \leq r \leq (N - 1)$. Furthermore, $d^i\zeta_{ir}(x_r^+)/dx^i \neq d^i\zeta_{ir}(x_r^-)/dx^i$.

Definition 3.3.2. Set $\mathscr{E}_i^q$ ($\mathscr{F}_i^q$) $\subset C^\infty(0, L)$ contains just one function $\zeta_{i0}(x)$ ($\zeta_{iN}(x)$). $\zeta_{i0}(x) \neq 0$ ($\zeta_{iN}(x) \neq 0$) if, for a given positive integer i, there exists a positive integer, j_1, such that $j_1 = [i - \gamma_{0\nu_0}]/4$ ($j_1 = [i - \gamma_{N\nu_0}]/4$) when $\nu_0 = 1$ or 2. Moreover, $\zeta_{i0}(x)$ ($\zeta_{iN}(x)$) satisfies

(a) $U_{0\nu_0}^{\,j_1}(\zeta_{i0}) \neq 0$ ($U_{N\nu_0}^{\,j_1}(\zeta_{ir}) \neq 0$) and $U_{N\nu}^{\,j}(\zeta_{i0}) = 0$ ($U_{0\nu}^{\,j}(\zeta_{iN}) = 0$), $\nu = 1, 2$ and $j = 0, 1, \ldots, j_2$, where j_2 is an integer satisfying $j_2 \geq [i - \gamma_{N1}]/4$ ($j_2 \geq [i - \gamma_{01}]/4$). In addition,

(b) $U_{01}^{\,j}(\zeta_{i0}) = 0$ ($U_{N1}^{\,j}(\zeta_{iN}) = 0$), $j = 0, 1, \ldots, j_3$, and $U_{02}^{\,j}(\zeta_{i0}) = 0$ ($U_{N2}^{\,j}(\zeta_{iN}) = 0$), $j = 0, 1, \ldots, j_4$. The j_3 and j_4 are positive integers satisfying $j_3 < [i - \gamma_{01}]/4 \leq j_3 + 1$ ($j_3 < [i - \gamma_{N1}]/4 \leq j_3 + 1$) and $j_4 < [i - \gamma_{02}]/4 \leq j_4 + 1$ ($j_4 < [i - \gamma_{N2}]/4 \leq j_4 + 1$).

By using these definitions, q-GFM functions can be defined concisely for an arbitrary function, $w(x) \in B$, that has continuous derivatives upto order q in each sub-interval V_r.

Definition 3.3.3. Let i_0 and i_1 be two positive integers satisfying $2 \leq i_0 \leq i_1 \leq (q - 1)$. Suppose that r_{i0} and r_i are two positive integers for a given positive integer i that satisfy $r_{i0} \leq r_i \leq N$ whilst i satisfies $i_0 \leq i \leq i_1$. The set of non-zero functions

$$\{\, \zeta_{ir}(x) : \zeta_{ir} \in \bigcup \Im_i^q \bigcup \mathscr{E}_i^q \bigcup \mathscr{F}_i^q \,,\ i_0 \leq i \leq i_1,\ r_{i0} \leq r \leq r_i \} \qquad (3.3.1)$$

corresponds to q-GFM functions with respect to $\{\psi_m(x)\}$ and the function $w(x)$ if there exists a set of real constants, h_{ir}, such that the function $g(x)$ defined by

$$g(x) = w(x) - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir} \zeta_{ir}(x), \quad x \neq x_k, \quad k = 1, \ldots, N-1, \tag{3.3.2}$$

and

$$\frac{d^j g(x_k)}{dx^j} = \frac{d^j w(x_k^{\pm})}{dx^j} - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir} \frac{d^j \zeta_{ir}(x_k^{\pm})}{dx^j}, \quad j = 2, \ldots, q-1, \tag{3.3.3}$$

$$k = 1, \ldots, N-1,$$

has a series expansion with respect to $\{\psi_m(x)\}$ whose derivatives can be taken, term by term, up to order (q - 1) without loss of uniform convergence in $0 \leq x \leq L$. Furthermore, the qth derivative of $g(x)$ is fully or piecewise continuous. Moreover, if q is independent of $w(x)$ when $w(x)$ is a solution of equation (3.2.10) for a $f(x) \in B$, the $\{\zeta_{ir}(x)\}$ are said to be q-GFM functions with respect to $\{\psi_m(x)\}$ and equation (3.2.10). Then the following result can be obtained immediately but its proof is given more conveniently in Appendix I.

Lemma 3.3.1. The q-GFM functions with respect to $\{\psi_m(x)\}$ and equation (3.2.10) satisfy $2 \leq q \leq 5$.

Two main concerns arise. One concern is how to construct the sets $\Im_i^q$, $\mathcal{B}_i^q$ and $\mathcal{F}_i^q$. It is easily found that the conditions needed by these sets involve the end conditions (3.2.13) as well as the left and right derivatives of a function at $x = x_r$, $1 \leq r \leq N$ - 1. Consequently, the functions $\{\zeta_{ir}(x)\}$ can be obtained from standard references, e.g. [51], by employing the static deflection of a uniform beam having no rigid body motion. The second concern relates to the existence of the constants, h_{ir}. It can be demonstrated directly that they can be found from

$$h_{ir} = \begin{cases} \left[\dfrac{d^i w(x_r^+)}{dx^i} - \dfrac{d^i w(x_r^-)}{dx^i} - \displaystyle\sum_{k=i_0}^{i-1} h_{kr}\left(\dfrac{d^i \zeta_{kr}(x_r^+)}{dx^i} \right.\right. \\ \qquad \left.\left. - \dfrac{d^i \zeta_{kr}(x_r^-)}{dx^i}\right)\right] \Big/ \left(\dfrac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \dfrac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right), \\ \qquad\qquad i_0 < i \le i_1,\ 0 < r < N \\ \left(\dfrac{d^i w(x_r^+)}{dx^i} - \dfrac{d^i w(x_r^-)}{dx^i}\right) \Big/ \left(\dfrac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \dfrac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right), \\ \qquad\qquad i = i_0,\ 0 < r < N \\ \left[\dfrac{d^i w(x_r)}{dx^i} - \displaystyle\sum_{k=i_0}^{i-1} \dfrac{d^i \zeta_{kr}(x_r)}{dx^i}\right] \Big/ \dfrac{d^i \zeta_{ir}(x_r)}{dx^i}, \qquad i_0 < i \le i_1, \\ \qquad\qquad r = 0, N,\ \zeta_{ir}(x) \neq 0 \\ \dfrac{d^i w(x_r)}{dx^i} \Big/ \dfrac{d^i \zeta_{ir}(x_r)}{dx^i}, \qquad i = i_0,\ r = 0, N,\ \zeta_{ir}(x) \neq 0. \end{cases} \tag{3.3.4}$$

Asymptotic lower errors are determined next for the convergence rates of the eigenvalue and eigenvector errors.

Theorem 3.3.1. Suppose that the conditions used in Theorem 3.2.1 hold and S_n is spanned by n functions consisting of the m_1 linearly independent functions $\{\zeta_{ir}(x)\}$ of set (3.3.1) in addition to $\{\psi_m(x),\ m = 1, \ldots, n - m_1\}$. If the $\{\zeta_{ir}(x)\}$ form q_1-GFM functions with respect to $\{\psi_m(x)\}$ and $w_j(x)$, the asymptotic errors arising from the nth Rayleigh-Ritz eigenvalue and eigenvector approximations, λ_j^n and $w_j^n(x)$, to their true counterparts, λ_j

and $w_j(x)$, are bounded by

$$\lambda_j^n - \lambda_j \le c_1 n^{-2q_2}(\lceil w_j \rceil_{q_1})^2 \quad and \quad \| w_j - w_j^n \|_B \le c_2 n^{-q_2} \lceil w_j \rceil_{q_1} \tag{3.3.5}$$

where

$$q_2 = (2q_1 - 3)/2 \quad and \quad \lceil w_j \rceil_{q_1} = \sum_{i=0}^{q_1+1} \sum_{r=1}^{N} \left(\int_{x_{r-1}}^{x_r} \left(\frac{d^i w_j}{dx^i}\right)^2 dx \right)^{1/2}. \tag{3.3.6}$$

Moreover, if $\{\zeta_{ir}(x)\}$ also forms a set of q_3-GFM functions with respect to $\{\psi_m(x)\}$ and equation (3.2.10), then

$$\| w_j - w_j^n \|_D \le c_3 n^{-(q_2+q_4)} \lceil w_j \rceil_{q_1} \tag{3.3.7}$$

and

$$| w_j(x) - w_j^n(x) | \le c_4 n^{-q_{30}} \lceil w_j \rceil_{q_1}, \quad | w_j'(x) - w_j^{n\prime}(x) | \le c_5 n^{-q_{31}} \lceil w_j \rceil_{q_1}. \tag{3.3.8}$$

The c_i, $i = 1, \ldots, 5$, are not only positive constants but they are independent of n and w_j. Also,

$$q_{3i} = (q_2 + q_4)(3 - 2i)/4 + q_2(1 + 2i)/4, \quad i = 0, 1 \tag{3.3.9}$$

and

$$q_4 = \begin{cases} 1/2, & q_3 = 2 \\ 3/2, & q_3 = 3, 4 \\ 7/4, & q_3 = 5. \end{cases} \tag{3.3.10}$$

Proof

Inequalities (3.3.5) and (3.3.7) can be obtained straightforwardly by using inequalities (I.59) and (I.61) of Lemma I.1 as well as inequalities (I.85) and (I.86). (They can all be found in Appendix I.) Then it can be seen from inequalities (3.3.5) and (3.3.7) that

$$A_n = \left(\int_0^L |w_j - w_j^n|^2 dx\right)^{1/2} \le c n^{-(q_2+q_4)} \lceil w_j \rceil_{q_1} \tag{3.3.11}$$

and

$$B_n = \left(\int_0^L |w_j'' - w_j^{n\,\prime\prime}|^2 dx\right)^{1/2} \le c n^{-q_2} \lceil w_j \rceil_{q_1}. \tag{3.3.12}$$

Inequalities (3.3.11) and (3.3.12), on the other hand, produce

$$A_n \to 0 \quad and \quad A_n^{-k+\frac{3}{2}} B_n^{k+\frac{1}{2}} \to 0 \quad as\ n \to \infty,\ k = 0,\ 1. \tag{3.3.13}$$

Furthermore, it can be found from inequalities (3.3.11) and (3.3.12) that $A_n / B_n \to 0$ as $n \to \infty$ so that

$$E_n = (A_n^2 + B_n^2)^{1/2} = B_n(1 + A_n^2/B_n^2)^{1/2} \le \sqrt{2}\, c n^{-q_2} \lceil w_j \rceil_{q_1}. \tag{3.3.14}$$

Thus, the conditions required for Theorem I.2 of Appendix I are satisfied. Consequently, the inequalities labelled (3.3.8) can be derived by substituting the A_n and E_n, defined by relations (3.3.11) and (3.3.14), into inequality (I.89).

Inequalities (3.3.8) provide pointwise asymptotic estimates over $0 \le x \le L$ but only for the error of the deflection and its first derivative. The practically important bending moment and shear force are considered next.

3.4. Pointwise Convergence of the Higher Derivatives

Sufficient conditions for the pointwise convergence of the second and third deflection derivatives are determined first.

Theorem 3.4.1. Suppose that the conditions employed in Theorem 3.3.1 and Lemma J.1 hold. Let the i and r of the GFM function $\zeta_{ir}(x)$, which is employed in Theorem 3.3.1, satisfy $2 \le i_0 \le i \le i_1 \le 3$ and $0 \le r_{i0} \le r \le r_i \le N$. Then there exist four positive constants, c_1 and c_2, that are independent of n and such that, at the continuous points of $w_j''(x)$ and $w_j'''(x)$, the inequalities

$$|w_j''(x) - w_j^{n}{}''(x)| \begin{cases} \le c_1 n^{-1/2} \lceil w_j \rceil_{q_1}, & q_1 = 3 \\ \le c_1 n^{-[q_2(q_1 - \frac{7}{2}) + \frac{1}{4}]/(q_1 - 3)} \lceil w_j \rceil_{q_1}, & q_1 > 3 \end{cases} \tag{3.4.1}$$

and

$$|w_j'''(x) - w_j^{n}{}'''(x)| \begin{cases} \le c_2 n^{-1/2} \lceil w_j \rceil_{q_1}, & q_1 = 4 \\ \le c_2 n^{-[q_2(q_1 - \frac{9}{2}) + \frac{3}{4}]/(q_1 - 3)} \lceil w_j \rceil_{q_1}, & q_1 > 4 \end{cases} \tag{3.4.2}$$

hold for their approximate counterparts, $w_j^{n}{}''(x)$ and $w_j^{n}{}'''(x)$. The q_2 and $\lceil w_j \rceil_{q_1}$ are defined by equations (3.3.6).

Proof

The nth Rayleigh-Ritz approximation, $w_j^n(x)$, of $w_j(x)$ may be expressed as

$$w_j^n(x) = \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} b_{ir}\zeta_{ir}(x) + \sum_{m=1}^{n-m_1} a_m\psi_m(x) \tag{3.4.3}$$

where b_{ir} and a_m are determined from equation (3.2.11). On the other hand, the m_1 functions $\{\zeta_{ir}''(x)\}$ constitute a set of q_1 - GFM functions with respect to the $(n - m_1)$ functions $\{\psi_m(x)\}$ and the $w_j(x)$. Hence, it can be shown similarly to the proof of Lemma I.2 of Appendix I that there exist constants, h_{ir}, such that the series expansion

$$w_j(x) = \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x) + \sum_{m=1}^{\infty} d_m\psi_m(x) \tag{3.4.4}$$

has the error estimate

$$\| w_j''(x) - f_n''(x) \|_H = \| \sum_{m=(n-m_1+1)}^{\infty} d_m\psi_m''(x) \|_H < c_3 n^{(2q_1-3)/2} \overline{\| w_j \|}_{q_1}. \tag{3.4.5}$$

The c_3 is a positive constant, $\|\bullet\|_H$ is a norm of a Hilbert space, H, given by equation (J.2) and

$$f_n(x) = \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x) + \sum_{m=1}^{n-m_1} d_m\psi_m(x). \tag{3.4.6}$$

Furthermore, d_m is defined by equation (J.6) in which $g(x)$ is given by

$$g(x) = w_j(x) - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x). \tag{3.4.7}$$

Subtracting equation (3.4.3) from equation (3.4.4) produces

$$w_j(x) - w_j^n(x) = \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} (h_{ir} - b_{ir}) \zeta_{ir}(x) + \sum_{m=1}^{n-m_1} (d_m - a_m) \psi_m(x) + \sum_{m=(n-m_1+1)}^{\infty} d_m \psi_m(x) . \tag{3.4.8}$$

Suppose P_{st} is an orthogonal projection of the Hilbert space, H, on a $(n - 1)$ dimensional subspace, B_{n-1}, that is spanned by $\{\psi_m''(x),\ m = 1, \dots, (n - m_1)\}$ and the $(m_1 - 1)$ GFM functions $\{\zeta_{ir}''(x)\}$ in which $\zeta_{st}''(x)$ is omitted. Here s and t are two (given) positive integers satisfying $2 \le s \le 4$ and $0 \le t \le N$. Then it can be shown, by using Lemma J.1, that there exists a positive constant, c_4, such that

$$\begin{aligned} |h_{st} - b_{st}| &\le \frac{\| w_j''(x) - w_j^{n\,\prime\prime}(x) \|_H + \| \sum_{m=(n-m_1+1)}^{\infty} d_m \psi_m''(x) \|_H}{\| \zeta_{st}''(x) - P_{st} \zeta_{st}''(x) \|_H} \\ &\le c_4 n^{-(q_2-s+1)} \overline{|w_j|}_{q_1} . \end{aligned} \tag{3.4.9}$$

On the other hand, by multiplying equation (3.4.8) by $\psi_m(x)$, the resulting term involving $(d_m - a_m)$ can be determined from

$$(d_m - a_m) \int_0^L (\psi_m'')^2 dx = \int_0^L (w_j'' - w_j^{n\,\prime\prime}) \psi_m'' dx - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} (h_{ir} - b_{ir}) \int_0^L \zeta_{ir}'' \psi_m'' dx .$$

It can be shown straightforwardly that

$$|d_m - a_m| \le \left(|\hat{t}_m| + \sum_{i=i_0}^{i_1} |\hat{w}_{im}| n^{-(q_2-i+1)} \overline{|w_j|}_{q_1} \right) / \int_0^L (\psi_m'')^2 dx \tag{3.4.10}$$

where

$$\hat{t}_m = \int_0^L (w_j'' - w_j^{n\,''})\psi_m'' dx \quad and \quad \hat{w}_{im} = \int_0^L \sum_{r=r_{i0}}^{r_i} \zeta_{ir}''\psi_m'' dx. \tag{3.4.11}$$

By using equations (3.4.6) and (3.4.8), the Rayleigh-Ritz approximation, $d^{q_1-1}w_j^n(x)/dx^{q_1-1}$, can be found to be

$$\begin{aligned} \frac{d^{q_1-1}w_j^n(x)}{dx^{q_1-1}} = \frac{d^{q_1-1}f_n(x)}{dx^{q_1-1}} - \sum_{i=i_0}^{i_1}\sum_{r=r_{i0}}^{r_i}(h_{ir}-b_{ir})\frac{d^{q_1-1}\zeta_{ir}(x)}{dx^{q_1-1}} \\ - \sum_{m=1}^{n-m_1}(d_m-a_m)\frac{d^{q_1-1}\psi_m(x)}{dx^{q_1-1}}. \end{aligned} \tag{3.4.12}$$

It can be demonstrated, by using Definition 3.3.3 and inequality (3.4.9), that the first term on the right of the last equation converges absolutely and uniformly to the true derivative $d^{q_1-1}w_j(x)/dx^{q_1-1}$ whilst the second term converges to zero. Thus, the last summation on the right determines whether $d^{q_1-1}w_j^n(x)/dx^{q_1-1}$ converges absolutely or uniformly. Now relations (3.4.10) and (3.4.11) lead to

$$\begin{aligned} \left|\sum_{m=1}^{n-m_1}(d_m-a_m)\frac{d^{q_1-1}\psi_m(x)}{dx^{q_1-1}}\right| \le c_5\Big(\sum_{i=i_0}^{i_1} n^{-(q_2-i+1)}\sum_{m=1}^{n-m_1} m^{-[i-(q_1-2)]} \\ + n^{-[q_2-(q_1-2)]}\Big)\|w_j\|_{q_1} \end{aligned} \tag{3.4.13}$$

where c_5 is a positive constant. Combining the last inequality with relations (3.3.6) and (3.4.9) results in $|(d^{q_1-1}/dx^{q_1-1})(w_j^n(x) - w_j(x))| \to 0$ like $n^{-1/2}$ as $n \to \infty$. Consequently, the required inequalities (3.4.1) and (3.4.2) can be obtained by using Theorems 3.3.1 and I.2.

The next result, whose proof is given in Appendix J, can be considered a corollary.

Corollary 3.4.1. Suppose that the conditions employed in Theorem 3.3.1 are valid and $N \leq 2$. If the i and r of the $\zeta_{ir}(x)$ employed in Theorem 3.3.1 satisfy $2 \leq i_0 \leq i \leq i_1 \leq 3$ and $0 \leq r_{i0} \leq r \leq r_i \leq N$, inequalities (3.4.1) and (3.4.2) hold.

A numerical example is given next.

3.5. Numerical Example

The cantilever beam shown in Figure 3.1 has a torsional spring located at $x = x_1 \equiv L/4$ and a stepped cross-section at $x = x_2 \equiv L/2$. It is used solely for illustration because an exact solution is available. In this example, $K_0 = \infty = \beta_0$, $K_1 = K_2 = K_3 = \beta_2 = \beta_3 = 0$ but $\beta_1 L/EI(0) = 40$. Moreover, $M_r = J_r = 0$, $r = 0, 1, 2, 3$ and $p(x) = 0 = k_e(x)$ for $0 \leq x \leq L$. The $EI(x)$ and $\rho(x)$ are constant in each sub-interval V_r ($r = 1, 2, 3$). They satisfy $EI(x_1^+) = EI(x_1^-)$, $\rho(x_1^+) = \rho(x_1^-)$, $EI(x_2^+)/EI(x_2^-) = 10$ and $\rho(x_2^+)/\rho(x_2^-) = 10^{1/2}$. It follows from Remark 3.2.1 that $w(0) = w'(0) = 0$ for any $w(x) \in B$. Furthermore, it can be shown straightforwardly that conditions (2.9) are satisfied for any $u(x) \in B$ and $w(x) \in B$. Thus, Theorem 3.2.1 holds.

Equations (3.2.5) suggest that the example problem's second deflection derivative is discontinuous at $x = x_1$ and $x = x_2$ whilst the left third deflection derivative equals the right third deflection derivative at $x = x_1$ but not at $x = x_2$. Therefore, the three GFM functions presented in Table 3.2 are used in the Rayleigh-Ritz procedure. They correspond to the static deflection of a uniform cantilever beam caused by a moment positioned at either $x = x_1$ or $x = x_2$ and a transverse force located at $x = x_2$ [51]. Thus, the n-dimensional subspace, S_n, in Theorems 3.3.1 and 3.4.1 is formed by these three functions in addition to the $(n - 3)$ eigenvectors $\{\psi_m(x), m = 1, ..., (n - 3)\}$ of a uniform cantilevered beam. Obviously, the dimensionality, n, grows as the number of eigenvectors is enlarged.

Definition 3.3.3 indicates that the functions of Table 3.2 form a set of 4-GFM functions with respect to $\{\psi_m(x)\}$ and $w_j(x)$. Therefore, from Theorem 3.3.1, the n th approximation, $\mu_j^n = (\lambda_j^n\, \rho(0)L^4/EI(0))^{1/2}$, should converge ultimately like at least n^{-5} as n is increased. Furthermore, it is demonstrated in Appendix L that the conditions of Theorem 3.4.1 are satisfied. This last theorem indicates that the convergence of the corresponding $w_j^{n\,\prime\prime}(x)$ and $w_j^{n\,\prime\prime\prime}(x)$ should be at least $n^{-3/2}$ and $n^{-1/2}$, respectively.

To confirm the above predictions, an analytical expression was derived for the exact j th eigenvector. It takes the form

$$w_j(x) = \begin{cases} \varsigma_{1j}(\sin\mu_j^{1/2}x/L - \sinh\mu_j^{1/2}x/L) \\ \quad + \varsigma_{2j}(\cos\mu_j^{1/2}x/L - \cosh\mu_j^{1/2}x/L) & 0 \le x \le x_1 \\[2ex] \left.\begin{aligned} &\varsigma_{3j}\sin\mu_j^{1/2}(x/L - 1/4) + \varsigma_{4j}\sinh\mu_j^{1/2}(x/L - 1/4) + \\ &\varsigma_{5j}\cos\mu_j^{1/2}(x/L - 1/4) + \varsigma_{6j}\cosh\mu_j^{1/2}(x/L - 1/4) \end{aligned}\right\} & x_1 \le x \le x_2 \\[2ex] \left.\begin{aligned} &\varsigma_{7j}(\sin a\mu_j^{1/2}(1 - x/L) + \sinh a\mu_j^{1/2}(1 - x/L)) + \\ &\quad (\cos a\mu_j^{1/2}(1 - x/L) + \cosh a\mu_j^{1/2}(1 - x/L))/2 \end{aligned}\right\} & x_2 \le x \le L \end{cases} \tag{3.5.1}$$

where $a = 0.1^{1/8}$ and $\mu_j = (\lambda_j\rho(0)L^4/EI(0))^{1/2}$. The lowest three values of μ_j are presented in Table 3.3 whilst the corresponding coefficients ς_{kj}, $k = 1, 2, \ldots, 7$ and $j = 1, 2, 3$, are detailed in Table 3.4 when both $w_j(L)$ and $w_j''(L)$ are taken as an arbitrary 1 m. Furthermore, equation (3.2.11) was solved numerically by using double precision arithmetic and the IMSL eigenvalue subroutine, DGVCSP [38], running on a SUN/4-280 workstation. The resulting errors $(\mu_j^n - \mu_j)$, $j = 1, 2, 3$ are given in Figure 3.2 for increasing n. They tend to zero like the n^{-5} predicted bound. The corresponding comparisons of the approximate and analytical second and third derivatives of the fundamental eigenvector are presented in Figures 3.3 and 3.4. These figures show that the overall errors are generally close to about n^{-2} and n^{-1}. Therefore, they tend to zero somewhat faster than the predicted lower rates of $n^{-3/2}$ and $n^{-1/2}$. In addition, there is no evidence of Gibbs phenomenon.

3.6. Conclusions

A generalized force mode (GFM) function approach has been introduced in this chapter for a self-adjoint eigenvalue problem corresponding to a Euler-Bernoulli beam having material or cross-sectional discontinuities, interior spring supports, or non-classical end conditions. Generalized force mode (GFM) functions are defined for the first time. A priori error estimates are derived and corroborated numerically for the eigenvalues of a Euler-Bernoulli beam having complicated constraints when GFM functions are used in the Rayleigh-Ritz approach. These estimates not only implicitly indicate a fast convergence, even when each approximation function does not satisfy non-standard boundary conditions, but they also demonstrate that Gibbs phenomenon is avoided. Consequently, the bending moment and shear force can be advantageously analyzed in a pointwise fashion even in the neighbourhood of a discontinuity.

The next chapter extends the present method to a non-self-adjoint eigenvalue problem by considering a simply supported, spinning Timoshenko beam.

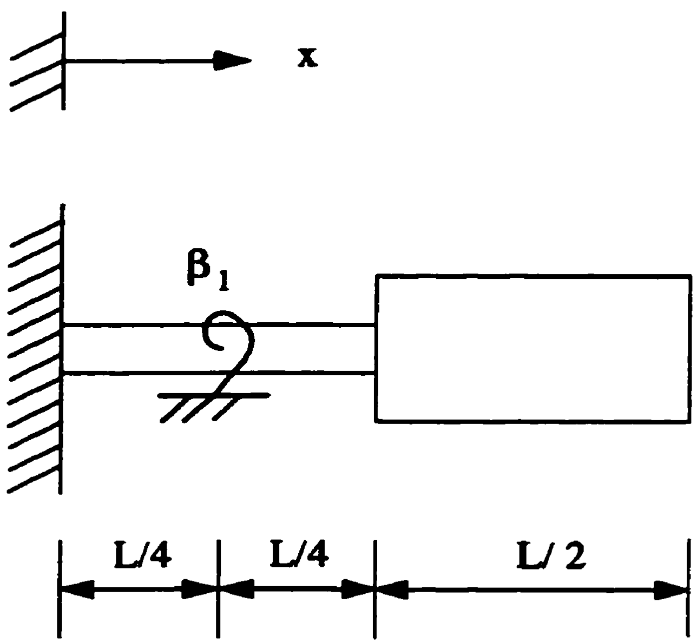

Figure 3.1. A stepped beam having an interior spring support.

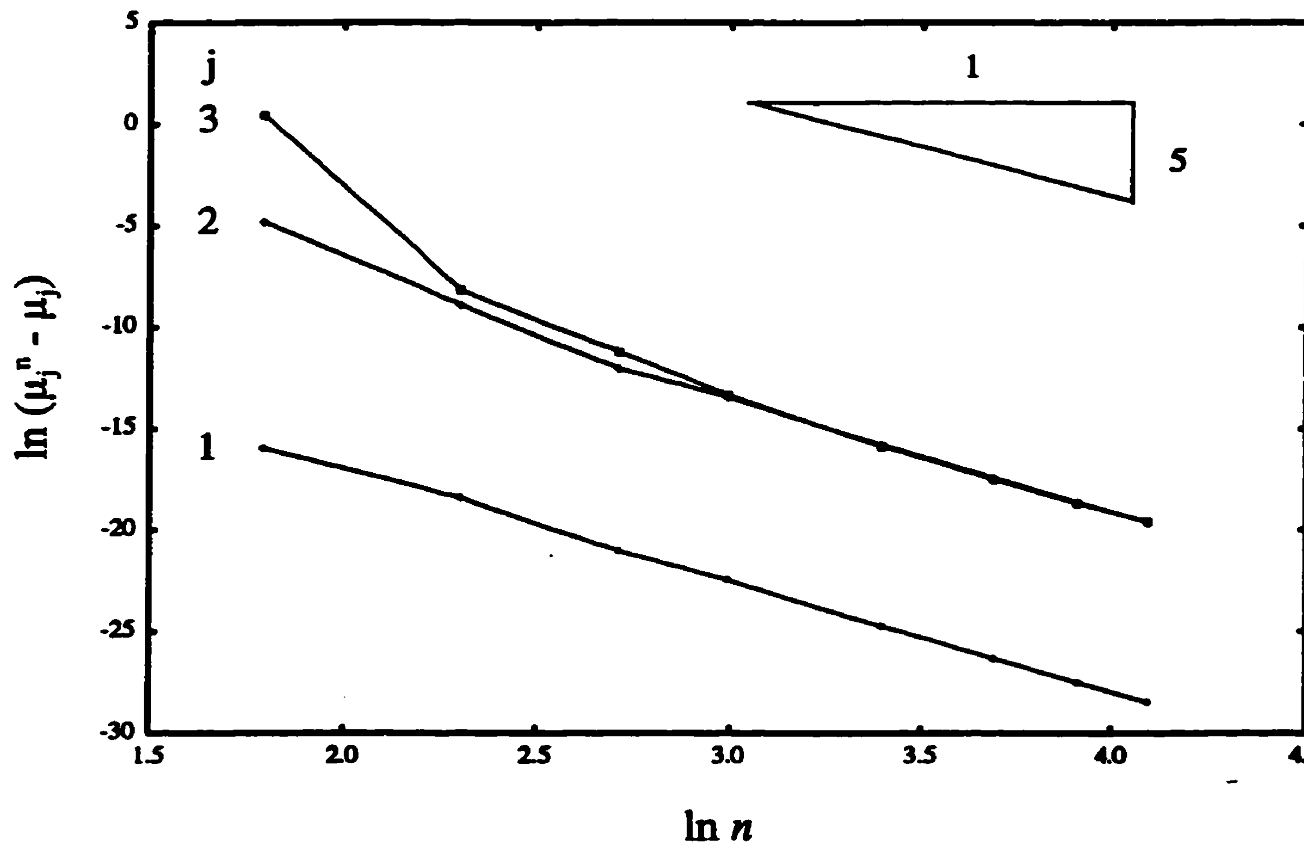

Figure 3.2. Lowest three frequency errors.

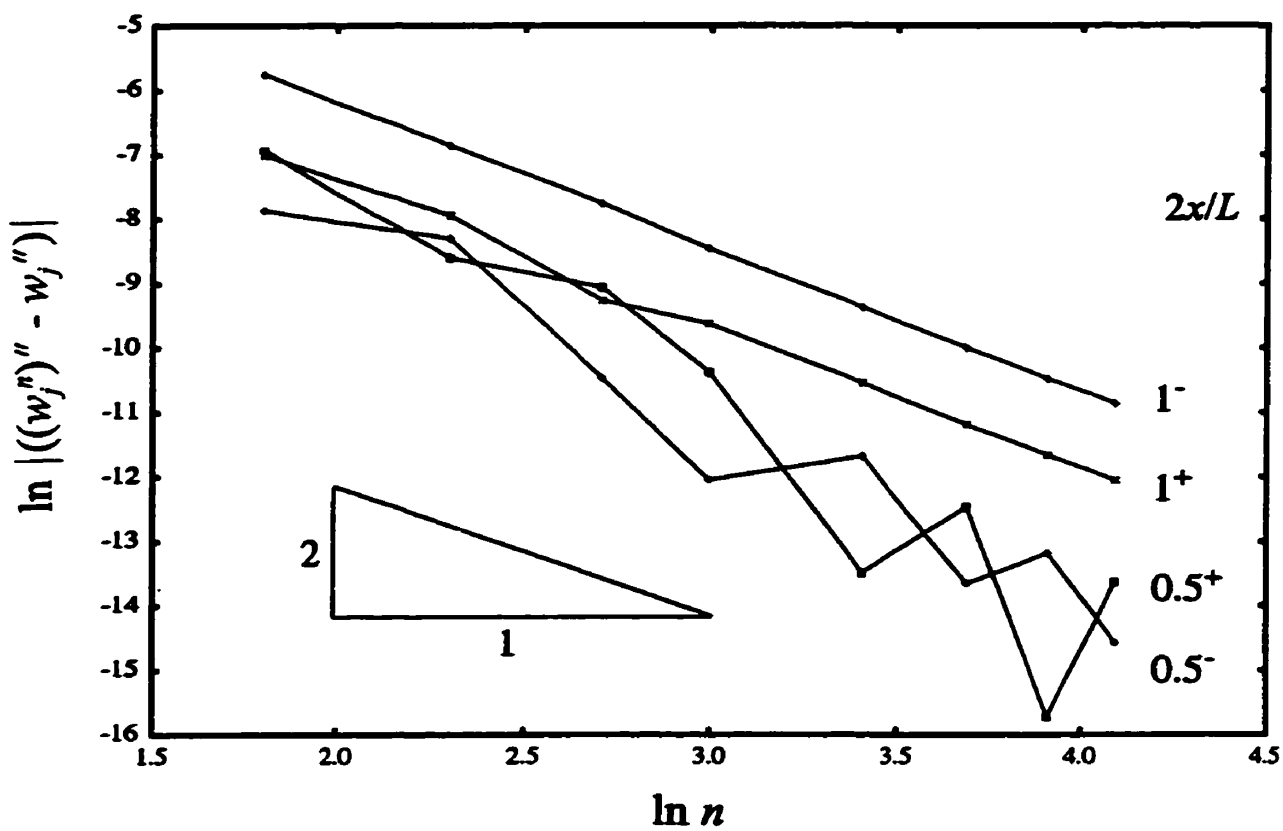

Figure 3.3. Absolute second derivative errors for the fundamental eigenvector.

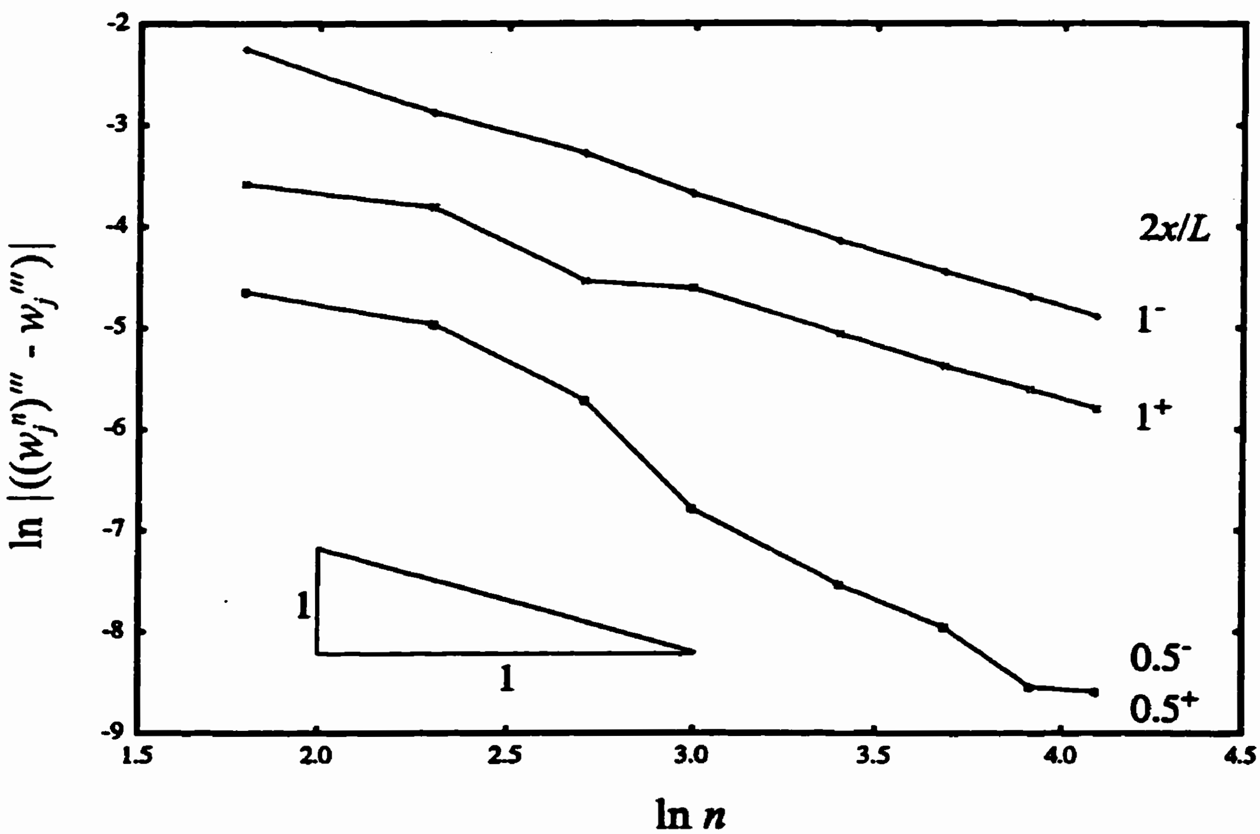

Figure 3.4. Absolute third derivative errors for the fundamental eigenvector.

Table 3.1. Values of γ_{rv}.

Standard end conditions at $x = x_r$, $r = 0, N$		γ_{r1}	γ_{r2}
clamped	$\psi_m(x_r) = d\psi_m(x_r)/dx = 0$	0	1
pinned	$\psi_m(x_r) = d^2\psi_m(x_r)/dx^2 = 0$	0	2
sliding	$d\psi_m(x_r)/dx = d^3\psi_m(x_r)/dx^3 = 0$	1	3
free	$d^2\psi_m(x_r)/dx^2 = d^3\psi_m(x_r)/dx^3 = 0$	2	3

Table 3.2. GFM functions.

$\zeta_{21} = x^2/2,\ 0 \le x \le L/4,$	$\zeta_{21} = L(2x - L/4)/2,$	$L/4 \le x \le L$
$\zeta_{22} = x^2/2,$	$\zeta_{32} = x^2(3L/2 - x)/6,$	$0 \le x \le L/2$
$\zeta_{22} = L(2x - L/2)/2,$	$\zeta_{32} = L^2(3x - L/2)/6,$	$L/2 \le x \le L$

Table 3.3. Lowest three analytical values of μ_j.

μ_1	μ_2	μ_3
3.44940	27.63121	85.64654

Table 3.4. Eigenvector coefficients ς_{ij}, $i = 1, 2, ..., 7$ corresponding to Table 3.3.

j	ς_{1j}	ς_{2j}	ς_{3j}	ς_{4j}	ς_{5j}	ς_{6j}	ς_{7j}
1	0.95119	-0.32684	0.99685	-0.89830	-0.92746	0.96620	-0.53938
2	-1.77976	1.28946	-1.69902	1.32469	0.03013	-0.92429	-0.63676
3	1.39006	-1.40385	0.09379	-0.06846	1.91865	0.26260	-0.49583

CHAPTER 4

FREE VIBRATIONS OF A STEPPED, SPINNING TIMOSHENKO BEAM

4.1. Introduction

A generalized force mode (GFM) method was introduced in the last chapter for a free, transversely vibrating Euler-Bernoulli beam. It was demonstrated that the beam's static deflection can be employed as GFM functions. On the other hand, it is well-known that Timoshenko beam theory can provide more accurate eigenvalues and eigenvectors as the beam's depth increases and as the wavelength of vibration decreases [24]. This is because the Timoshenko theory considers the effects of rotary inertia and the transverse shear deformation. Thus, from a structural engineering viewpoint, an extension of the GFM method to a Timoshenko beam is desirable. However, the static deflection of a Euler-Bernoulli beam that arises from a concentrated force may not be used as a GFM function for a Timoshenko beam. This is because the first derivative of such a static deflection is always continuous. Conversely, a static deflection caused by a concentrated force acting on a uniform Timoshenko beam (having standard end conditions) may be employed directly as a GFM function because its first derivative is discontinuous. However, from a computational viewpoint, a question arises as to the simplicity of its analytical form. It can be shown [51] that the latter static deflection can be expressed as a polynomial of degree three on each side of a concentrated force. On the other hand, it can be seen from Definition 3.3.1 that a polynomial function having only degree one on each side of a discontinuous cross-section can also be used as a GFM function. Thus a more general approach, called Hermite polynomial interpolation, is suggested in this chapter for the construction of the GFM functions. This procedure involves two steps. First, polynomials are found on each side of a discontinuity which satisfy the conditions at the contiguous end. Second, the polynomials must be chosen so that the transverse deflection and the slope due to bending are continuous at the location of a discontinuity. The approach not only provides simple analytical forms for the GFM functions but also avoids the need to solve a boundary value problem. This latter advantage may be even more important for a two dimensional problem in which the static deflection may have to be found numerically.

A self-adjoint eigenvalue problem was considered in the previous chapter. However, non-self-adjoint problems are often encountered in practice due to viscous damping and gyroscopic effects. It is known [11] that a non-self-adjoint problem can be approximated by the Galerkin approach. In this chapter, the simply supported, stepped spinning Timoshenko beam shown in Figure 4.1 is employed to demonstrate the extension of the GFM method needed to apply Galerkin's approach. A numerical example illustrates the usefulness of this procedure.

4.2. Outline of Analysis

Consider a Timoshenko beam having length L and a circular cross-section which is discontinuous at $x = L/2$. Suppose that the beam spins at a constant angular speed, Θ, about the x axis which coincides with the beam's geometric centre in the fixed (inertial) coordinate frame of Figure 4.1. The beam has mass density, ρ, Young's modulus, E, shear modulus, G, and shear coefficient κ. Let $A(x)$, $I(x)$ and J_x be the area, moment and polar moment of inertia of a cross-section that is distance x from the left end. The transverse deflections corresponding to the jth natural frequency of the beam are designated u_{1j}^t and u_{2j}^t, in the $O\chi_1$ and $O\chi_2$ directions, respectively, whilst Φ_{1j}^t and Φ_{2j}^t represent the analogous bending angles. The free vibrations of the spinning beam are governed by [48]

$$(\kappa AG(\Phi_{1j}^t + \frac{du_{1j}^t}{dx}))' + \varsigma_j^2 \rho A u_{1j}^t = 0 \tag{4.2.1}$$

$$(\kappa AG(\Phi_{2j}^t + \frac{du_{2j}^t}{dx}))' + \varsigma_j^2 \rho A u_{2j}^t = 0 \tag{4.2.2}$$

$$-(EI\frac{d\Phi_{1j}^t}{dx})' + \kappa AG(\Phi_{1j}^t + \frac{du_{1j}^t}{dx}) + \varsigma_j^2 \rho I \Phi_{1j}^t + \varsigma_j \Theta J_p \Phi_{2j}^t = 0 \tag{4.2.3}$$

and

$$-(EI\frac{d\Phi_{2j}^t}{dx})' + \kappa AG(\Phi_{2j}^t + \frac{du_{2j}^t}{dx}) + \varsigma_j^2 \rho I \Phi_{2j}^t - \varsigma_j \Theta J_p \Phi_{1j}^t = 0 \tag{4.2.4}$$

where a prime superscript indicates differentiation with respect to x whilst $\varsigma_j = \omega_j^t i'$, $i' = (-1)^{1/2}$. Here ω_j^t represents the jth forward natural frequency when $j > 0$. However, ω_j^t depicts the jth backward natural frequency when $j < 0$. The simply supported ends are denoted by

$$u_{1j}^t(0) = u_{1j}^t(L) = u_{2j}^t(0) = u_{2j}^t(L) = 0 \tag{4.2.5}$$

and

$$\Phi_{1j}^{t}{}'(0) = \Phi_{1j}^{t}{}'(L) = \Phi_{2j}^{t}{}'(0) = \Phi_{2j}^{t}{}'(L) = 0 . \tag{4.2.6}$$

On the other hand, the force compatibility conditions at $x = L/2 \equiv x_0 = L/2$ are

$$\frac{d\Phi_{1j}^{t}(x_0^-)}{dx} = \frac{EI(x_0^+)}{EI(x_0^-)}\frac{d\Phi_{1j}^{t}(x_0^+)}{dx}, \qquad \frac{d\Phi_{2j}^{t}(x_0^-)}{dx} = \frac{EI(x_0^+)}{EI(x_0^-)}\frac{d\Phi_{2j}^{t}(x_0^+)}{dx} \tag{4.2.7}$$

$$(\Phi_{1j}^{t}(x_0^-) + \frac{du_{1j}^{t}(x_0^-)}{dx}) = \frac{\kappa GA(x_0^+)}{\kappa GA(x_0^-)}(\Phi_{1j}^{t}(x_0^+) + \frac{du_{1j}^{t}(x_0^+)}{dx}) \tag{4.2.8}$$

and

$$(\Phi_{2j}^{t}(x_0^-) + \frac{du_{2j}^{t}(x_0^-)}{dx}) = \frac{\kappa GA(x_0^+)}{\kappa GA(x_0^-)}(\Phi_{2j}^{t}(x_0^+) + \frac{du_{2j}^{t}(x_0^+)}{dx}) . \tag{4.2.9}$$

Assume approximate solutions have the form

$$u_{1j}^{tn}(x) = \sum_{\ell=1}^{n} \delta_{1\ell}^{tn}\varphi_{1\ell}^{t}(x), \quad u_{2j}^{tn}(x) = \sum_{\ell=1}^{n} \delta_{2\ell}^{tn}\varphi_{1\ell}^{t}(x) \tag{4.2.10}$$

and

$$\Phi_{1j}^{tn} = \sum_{\ell=1}^{n} \delta_{3\ell}^{tn}\varphi_{2\ell}^{t}(x), \quad \Phi_{2\ell}^{tn} = \sum_{\ell=1}^{n} \delta_{4\ell}^{tn}\varphi_{2\ell}^{t}(x) \tag{4.2.11}$$

where $\varphi_{i\ell}^{t}(x)$ ($i = 1, 2$ and $\ell = 1, 2, ..., n$) are admissible functions whilst $\delta_{i\ell}^{m}$ are undetermined coefficients. Substituting these forms into the left sides of (4.2.1) through

(4.2.4) leads to the residual errors

$$\varepsilon_{1n}^{t} = (\kappa AG(\Phi_{1j}^{tn}+\frac{du_{1j}^{tn}}{dx}))' + \varsigma_j^2\rho A u_{1j}^{tn} \tag{4.2.12}$$

$$\varepsilon_{2n}^{t} = (\kappa AG(\Phi_{2j}^{tn}+\frac{du_{2j}^{tn}}{dx}))' + \varsigma_j^2\rho A u_{2j}^{tn} \tag{4.2.13}$$

$$\varepsilon_{3n}^{t} = -(EI\frac{d\Phi_{1j}^{tn}}{dx})'+\kappa AG(\Phi_{1j}^{tn}+\frac{du_{1j}^{tn}}{dx}) + \varsigma_j^2\rho I\Phi_{1j}^{tn}+\varsigma_j\Theta J_p\Phi_{2j}^{tn} \tag{4.2.14}$$

and

$$\varepsilon_{4n}^{t} = -(EI\frac{d\Phi_{2j}^{tn}}{dx})'+\kappa AG(\Phi_{2j}^{tn}+\frac{d\Phi_{2j}^{tn}}{dx}) + \varsigma_j^2\rho I\Phi_{2j}^{tn}-\varsigma_j\Theta J_p\Phi_{1j}^{tn} \tag{4.2.15}$$

Coefficients $\delta_{i\ell}^{m}$ are determined from the requirement that [11]

$$\left.\begin{aligned} &\int_0^L \varphi_{1\ell}^{t}(x)\varepsilon_1^{n}dx = 0, \quad \int_0^L \varphi_{1\ell}^{t}(x)\varepsilon_2^{n}dx = 0, \\ \textit{and} \quad & \\ &\int_0^L \varphi_{2\ell}^{t}(x)\varepsilon_3^{n}dx = 0, \quad \int_0^L \varphi_{2\ell}^{t}(x)\varepsilon_4^{n}dx = 0 \end{aligned}\right\} \tag{4.2.16}$$

which leads, in matrix notation, to

$$\varsigma_j^2[M^t]\{\delta^{m}\}+\varsigma_j[C^t]\{\delta^{m}\}+[K^t]\{\delta^{m}\} = 0 \tag{4.2.17}$$

where

$$\{\delta^{tn}\} = (\delta^{tn}_{11} \ \dots \ \delta^{tn}_{1n}, \ \dots \ \delta^{tn}_{41} \ \dots \ \delta^{tn}_{4n})^T. \tag{4.2.18}$$

The $[M^t]$ and $[K^t]$ are symmetric mass and stiffness matrix, respectively, whilst $[C^t]$ is a skew-symmetric gyroscopic matrix. These matrices are given by

$$[M^t] = \begin{bmatrix} [M_1^t] & [0] \\ [0] & [M_1^t] \end{bmatrix}, \quad [K^t] = \begin{bmatrix} [K_1^t] & [0] \\ [0] & [K_1^t] \end{bmatrix} \tag{4.2.19}$$

and

$$[C^t] = \Theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [I_1^t]^T \\ 0 & 0 & 0 & 0 \\ 0 & -[I_1^t] & 0 & 0 \end{bmatrix}. \tag{4.2.20}$$

with

$$[M_1^t] = \begin{bmatrix} [I_2^t] & [0] \\ [0] & [I_3^t] \end{bmatrix} \quad and \quad [K_1^t] = \begin{bmatrix} [0] & [0] \\ [0] & [I_4^t] \end{bmatrix} + \kappa G \begin{bmatrix} [I_5^t] & [I_7^t]^T \\ [I_7^t] & [I_6^t] \end{bmatrix}. \tag{4.2.21}$$

Here

$$[I_1^t] = [\int_0^L \rho J_x \{\overline{\varphi_2^{tn}}\}^T \{\overline{\varphi_2^{tn}}\} dx] \tag{4.2.22}$$

$$[I_2^t] = [\int_0^L \rho A\{\overline{\varphi_1^{tn}}\}^T\{\overline{\varphi_1^{tn}}\}dx], \quad [I_3^t] = [\int_0^L \rho I\{\overline{\varphi_2^{tn}}\}^T\{\overline{\varphi_2^{tn}}\}dx] \tag{4.2.23}$$

$$[I_4^t] = [\int_0^L EI\{\overline{\varphi_2^{tn}}\}'^T\{\overline{\varphi_2^{tn}}\}'dx], \quad [I_5^t] = [\int_0^L GI\{\overline{\varphi_1^{tn}}\}'^T\{\overline{\varphi_1^{tn}}\}'dx] \tag{4.2.24}$$

$$[I_6^t] = [\int_0^L GA\{\overline{\varphi_2^{tn}}\}^T\{\overline{\varphi_2^{tn}}\}dx], \quad [I_7^t] = [\int_0^L GA\{\overline{\varphi_2^{tn}}\}^T\{\overline{\varphi_1^{tn}}\}'dx]. \tag{4.2.25}$$

$$\{\overline{\varphi_1^{tn}}\} = (\varphi_{11}^t(x), \ldots, \varphi_{1n}^t(x))^T, \quad \{\overline{\varphi_2^{tn}}\}' = ((\varphi_{21}^t(x))', \ldots, (\varphi_{2n}^t(x))')^T \tag{4.2.26}$$

and

$$\{\overline{\varphi_2^{tn}}\} = (\varphi_{21}^t(x), \ldots, \varphi_{2n}^t(x))^T, \quad \{\overline{\varphi_2^{tn}}\}' = ((\varphi_{21}^t(x))', \ldots, (\varphi_{2n}^t(x))')^T. \tag{4.2.27}$$

Equation (4.2.17) represents a system that is not self-adjoint. It can be rewritten as

$$\begin{bmatrix} 0 & [I] \\ -[M^t]^{-1}[K^t] & -[M^t]^{-1}[C^t] \end{bmatrix} \begin{Bmatrix} \{\delta^{tn}\} \\ \varsigma_j\{\delta^{tn}\} \end{Bmatrix} = \varsigma_j \begin{Bmatrix} \{\delta^{tn}\} \\ \varsigma_j\{\delta^{tn}\} \end{Bmatrix} \tag{4.2.28}$$

in order to employ a standard eigenvalue solver. A specific beam that has the material and dimensional properties given in Table 4.1 is considered next.

4.3. Numerical Results

The first and second order deflection derivatives as well as the slope due to bending of the example beam are discontinuous at its stepped midpoint, $x = 0.5$ m. Consequently, the corresponding derivatives of the GFM functions must also be discontinuous at this location. They should also satisfy the contiguous end conditions. These GFM functions are designated arbitrarily in (4.2.10) and (4.2.11) by $\varphi^t_{11}(x)$ and $\varphi^t_{12}(x)$ for the deflection, and by $\varphi^t_{21}(x)$ and $\varphi^t_{22}(x)$ for the slope due to bending. Their piecewise polynomial forms, obtained by following the procedure outlined in the section 4.1, are summarized in Table 4.2 for the situation when $L = 1$ m. They possess the properties

$$\varphi^{t\,\prime}_{12}(x) = \varphi^t_{22}(x), \quad \varphi^{t\,\prime}_{21}(x) = 8\varphi^t_{11}(x), \quad x \neq 0.5\ m, \tag{4.3.1}$$

and

$$\varphi^{t\,\prime}_{22}(x) = \begin{cases} 24\varphi^t_{11}(x), & x \leq 0.5\ m \\ -24\varphi^t_{11}(x), & x \geq 0.5\ m, \end{cases} \tag{4.3.2}$$

that make the calculation of $[K]$ in (4.2.19) easier. The remaining admissible functions are taken to be the eigenfunctions of a uniform, non-spinning Euler-Bernoulli beam having simply supported ends for the deflection and sliding-sliding ends for the slope due to the bending. i.e. $\varphi^t_{1\ell}(x) = sin(\ell - 2)\pi x/L$ and $\varphi^t_{2\ell}(x) = cos(\ell - 3)\pi x/L$ for $\ell = 3, \ldots, n$. It can be shown directly that all the $\varphi^t_{1\ell}(x)$ and $\varphi^t_{2\ell}(x)$, $\ell \geq 1$, satisfy the end conditions (4.2.5) and (4.2.6), respectively. The resulting numerical data for the first four forward and backward precession frequencies, computed with $n = 10$ in (4.2.10) and (4.2.11), are presented in Table 4.3 alongside the exact values obtained by using standard method. Data calculated without the generalized force mode functions are also given. In this case, $\varphi^{t}_{1\ell}(x) = sin\, \ell\pi x/L$ and $\varphi^{t}_{2\ell}(x) = cos(\ell - 1)\pi x/L$ are employed in (4.2.10) and(4.2.11) for $\ell = 1, ..., n$. It can be seen that the GFM functions certainly improve the accuracy of the natural frequencies.

To ascertain if Gibbs phenomenon [25] occurs in the bending moment and shear force due to the stepped cross-section, the $d\Phi^{t}_{11}(x)/dx$ and $du^{t}_{11}(x)/dx + \Phi^{t}_{11}(x)$ for the first forward precession frequency are compared with their exact values in Figures 4.2 and 4.3, respectively. Corresponding results computed without the GFM functions are also presented again. For convenience, $u^{t}_{11}(x)$ is taken as 1 m at the beam's midpoint. Figure 4.2 demonstrates that the exact results and those obtained with the inclusion of the GFM functions overlap, despite the discontinuous nature of the derivatives. However, the data obtained without these functions oscillate around the midpoint. A similar oscillation can also be found in Figure 4.3. Furthermore, this last figure indicates that the numerical data obtained with the GFM functions converge to the exact results with an increasing n.

4.4. Conclusions

The numerical results presented in this chapter demonstrate that Hermite polynomial interpolation is a simple way of constructing GFM functions. Furthermore, GFM functions enable the free vibrations of a non-self-adjoint Timoshenko beam to be found without Gibbs phenomenon occurring.

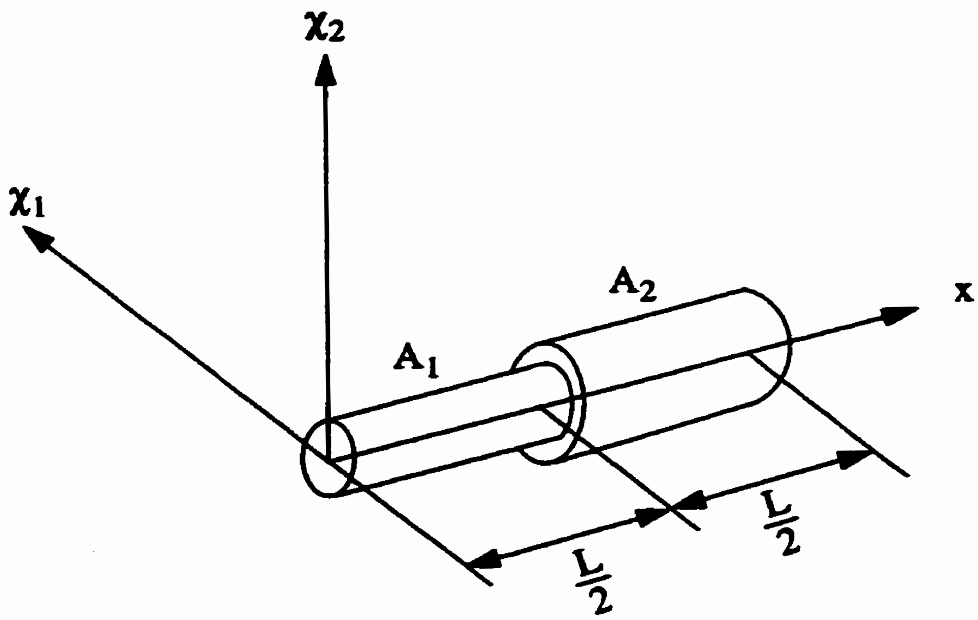

Figure 4.1. The inertial co-ordinates χ_i, $i = 1, 2$.

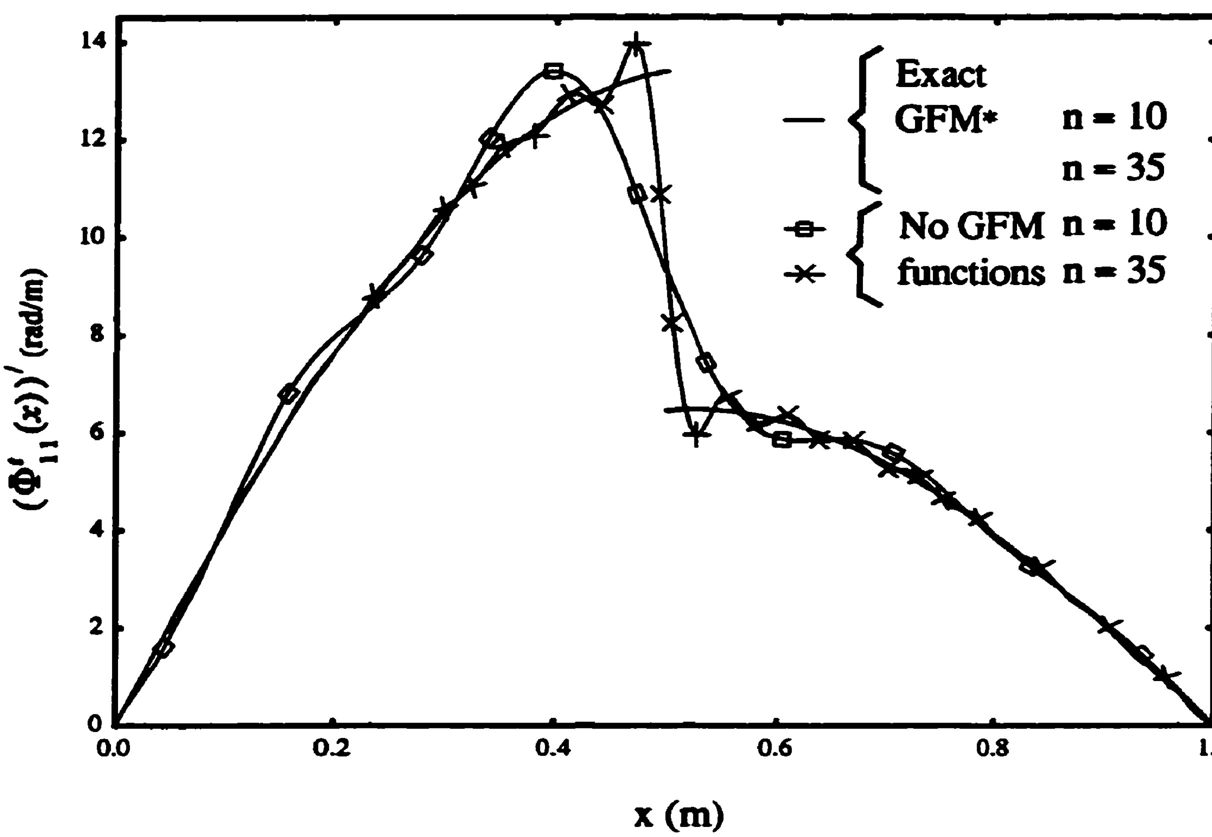

Figure 4.2. Exact and numerical values of $(\Phi'_{11}(x))'$.

* Includes the GFM functions.

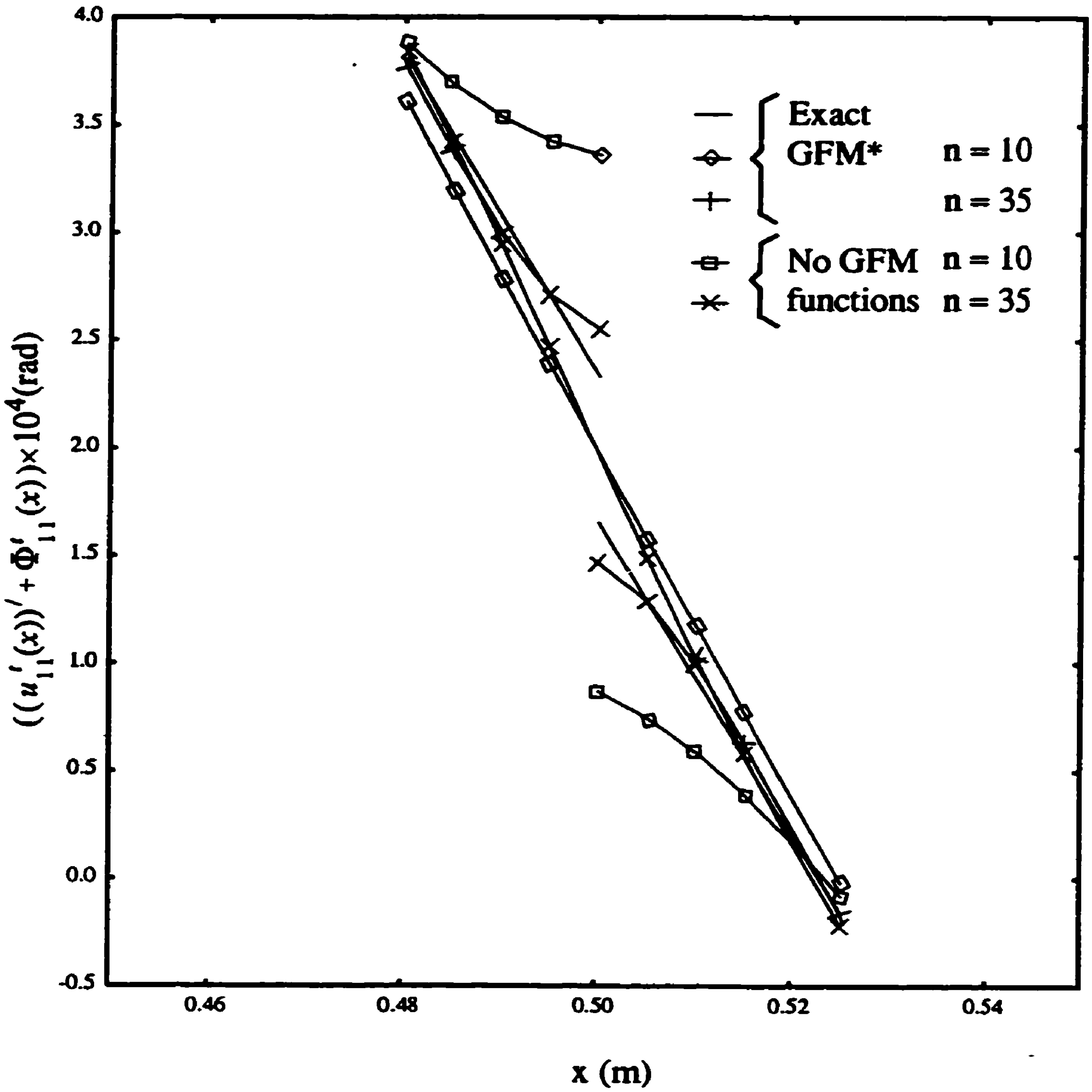

Figure 4.3. Exact and numerical values of $(u'_{11}(x))' + \Phi'_{11}(x)$.

* Includes the GFM functions.

Table 4.1. Properties of the spinning beam.

$L = 1$ m	$k = 0.9$
$A_1 = 3.14159\times10^{-4}$ m^2	$E = 200$ GPa
$A_2 = 4.52389\times10^{-4}$ m^2	$G = 83$ GPa
$\rho = 7833.5$ kg/m^3	$\Theta = 200$ rad/s

Table 4.2. GFM functions in the inertial co-ordinate frame.

GFM functions for the deflection	GFM for the bending angle
$\varphi^t_{11} = x, \quad \varphi^t_{12} = x(4x^2 - 1)$ $0.0 < x < 0.5$	$\varphi^t_{21} = 4x^2 - 1, \quad \varphi^t_{22} = 12x^2 - 1$ $0.0 < x < 0.5$
$\varphi^t_{11} = 1 - x, \ \varphi^t_{12} = 4x^3 - 12x^2 + 11x - 3$ $0.5 < x < 1.0$	$\varphi^t_{21} = -4x^2 + 8x - 3, \ \varphi^t_{22} = 12x^2 - 24x + 11$ $0.5 < x < 1.0$

Table 4.3. Values of $\omega_j{}'$ for a stepped, simply supported, rotating beam.

j	Present Method	No GFM functions	Exact Results
1	264.53	266.04	264.53
-1	-264.41	-265.92	-264.41
2	1104.84	1105.23	1104.70
-2	-1104.37	-1104.76	-1104.23
3	2415.35	2428.57	2415.32
-3	-2414.27	-2427.49	-2414.25
4	4364.84	4367.21	4361.92
-4	-4362.74	-4365.38	-4360.10

- indicates a backward natural frequency.

Table 4.4. $u_{11}^{t}(x)$ and $\Phi_{11}^{t}(x)$ corresponding to the first forward natural frequency of a spinning Timoshenko beam having a stepped, circular cross-section.

$$u_{11}^{t}(x) = \begin{cases} -0.057898\, \sinh(3.235359\, x) + 1.141479\, \sin(3.236267\, x) & 0 < x < 0.5 \\ 0.062353 \sinh[2.953382(1-x)] + 0.874441\, \sin[2.954386(1-x)] & 0.5 < x < 1 \end{cases}$$

$$\Phi_{11}^{t}(x) = \begin{cases} 0.187452\, \cosh(3.235359\, x) - 3.691554\, \cos(3.236267\, x) & 0 < x < 0.5 \\ 0.184308\, \cosh[2.953382(1-x)] + 2.581273\, \cos[2.954386(1-x)] & 0.5 < x < 1 \end{cases}$$

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

5.1 Conclusions

An operator approach has been used to show the completeness of the eigenvectors of a non-uniform, axially loaded Euler-Bernoulli beam having eccentric masses and supported by off-set springs at both ends. The motivation is to verify the validity of using these eigenvectors in a generalized Fourier series expansion or in the Rayleigh-Ritz or Galerkin methods. This generalization extends the work presented in [22] for a cantilever beam having solely an eccentric mass at its free tip. Furthermore, the order of the j th coefficient of a series approximation of a continuous initial deflection, $y_0(x)$, has been determined, for the first time, as $j \to \infty$. Consequently, the error caused by truncating such a series can be found straightforwardly. An important conclusion which arises is that, for any three times differentiable function whose fourth order derivative is piecewise continuous, a series expansion in terms of these eigenvectors, as well as each series obtained by differentiating it upto three times, converge uniformly and absolutely. This result significantly extends a classical expansion theorem [23] in which a function is required to satisfy all the beam's end conditions. Moreover, it can be expected from this generalization that the eigenvectors should produce a higher convergence rate when used as the Ritz base functions in the component mode synthesis. This conjecture is substantiated numerically in [43] through illustrative examples.

In addition to the generalized expansion theorem, asymptotic estimates of the eigenvalues and eigenvectors have been derived for the first time. These estimates can be applied, for example, to the design of distributed feedback by using independent modal-space control [6]. Then the optimal distributed control force is a summation of modes whose weighting coefficients can be approximated easily and accurately at high frequencies by employing the asymptotic estimates. Furthermore, the effect of an off-set lumped mass on the lower natural frequencies of a beam has also been investigated. This work is motivated by the recent growth in the use of industrial robots. A new criterion is proposed for predicting how the natural frequencies of a beam (i.e. a flexible robot having a single arm) vary with a payload's off-set. It is demonstrated that, for a given payload, an off-set influences mainly the fundamental natural frequency and, hence, the robot's positioning accuracy. Moreover, a numerical example confirms that an inclusion principle cannot be generally used to estimate the natural frequencies. To obtain low frequency data, a numerical procedure like the Rayleigh-Ritz method or the finite element method is possibly the best way to obtain such information. Of course, such procedures are not restricted to low frequencies but experience [11] suggests that they can produce inaccurate high frequency modes even with the additional penalty of severe computational effort.

When a beam has complex interior conditions such as discontinuous cross-sections, spring supports or lumped masses, or non-conventional end conditions, an outstanding question concerns possible extraneous numerical oscillations around the discontinuities as well as at the beam's ends. This so called Gibbs phenomenon can lead to a slowly converging approximate solution. To avoid this difficulty, a unified procedure for selecting admissible functions has been developed in this thesis for the Rayleigh-Ritz method. In this approach, generalized force mode (GFM) functions are employed as admissible functions together with the eigenvectors of a uniform Euler-Bernoulli beam having conventional end conditions. The key idea is that discontinuous deflection derivatives can be approximated more efficiently in a pointwise fashion only by using discontinuous functions. Based upon this viewpoint, GFM functions are defined rigorously for the first time. To justify the practical usefulness of the unified approach, a priori error estimates are derived and corroborated numerically for the eigenvalues of a Euler-Bernoulli beam having complicated constraints. These estimates not only implicitly indicate a fast convergence, even when each approximation function does not satisfy non-standard boundary conditions, but they also demonstrate that the Gibbs phenomenon is avoided. Consequently, the practically important bending moment and shear force can be approximated accurately.

It is known [11] that the Rayleigh-Ritz approach can be used only for a self-adjoint eigenvalue problem. However, non-self-adjoint problems are often encountered in practical structures. A Galerkin procedure is needed for such problems. In this thesis, a spinning Timoshenko shaft having a stepped cross-section is employed as an example to demonstrate how the unified approach can be extended to non-self-adjoint problems. Moreover, a Hermite polynomial interpolation method has been proposed as an easier way of constructing the GFM functions. Furthermore, numerical data again show no Gibbs phenomenon in the bending moment and shear force. Consequently, an extension of the unified approach to non-self-adjoint problems appears to be possible.

5.2 Recommendations

This thesis studies the eigenvalues and eigenvectors of a single span Euler-Bernoulli beam having general end conditions. Furthermore, a novel numerical approach has been proposed to approximate the eigenvalues and eigenvectors of a beam having complex interior and end conditions. Several other interesting topics could be developed in future research. They may include:

- a rigorous asymptotic analysis of the eigenvalues and eigenvectors of a Timoshenko beam having general end conditions;
- extension of GFM functions to handle a steady, transient or random dynamic response;
- extension of GFM functions to accommodate vibrating plates and shells; and an
- extension of the approach to a finite element analysis of structures having discontinuous cross-sections, interior spring supports and lumped masses.

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APPENDIX A

This appendix justifies the assertion that the domain of operator Π is dense in the Hilbert space $\boldsymbol{H}^{(5)}$. The following theorem, proved in [45], is needed to accomplish this task.

Theorem A.1 The function space $C^{\infty}(0, L)$ with $f(0) = f(L) = 0$ for all $f(x) \in C^{\infty}(0, L)$ is dense in the Hilbert space $\mathcal{L}^2(0, L)$.

$C^{\infty}(0, L)$ consists of functions that are infinitely differentiable. Thus, it is known from Theorem A.1 that, for an arbitrary vector $\boldsymbol{F} = (f_1, f_2, f_3, f_4, f_5)$ in $\boldsymbol{H}^{(5)}$ and an arbitrary $\varepsilon > 0$, there exists a function $f(x) \in C^{\infty}(0, L)$ with $f(0) = f(L) = 0$ such that

$$\int_0^L |f_1(x) - f(x)|^2 dx \leq \varepsilon^2 . \tag{A.1}$$

Furthermore, it is easily shown that there exists a cubic polynomial, denoted by $f_A(x)$, such that

$$f_A(x) = c_1 + c_2(x/L) + c_3(x/L)^2 + c_4(x/L)^3 \tag{A.2}$$

where

$$c_1 = f_2 + e_0 f_3, \quad c_2 = f_3 L, \quad c_3 = -(L + 3e_1)f_5 + 3f_4 - 2c_2 - 3c_1 \tag{A.3}$$

and

$$c_4 = (L+2e_1)f_5 - 2f_4 + c_2 + 2c_1 \tag{A.4}$$

whilst

$$f_2 - f_A(0) + e_0 f_A'(0) = f_3 - f_A'(0) = f_4 - f_A(L) - e_1 f_A'(L) = f_5 - f_A'(L) = 0. \tag{A.5}$$

For $f_A(x)$ given by (A.1) and an arbitrary $\varepsilon > 0$, it is known from Theorem A.1 that there exists a function, denoted by $f_B(x) \in C^{\infty}(0, L)$ with $f_B(0) = f_B(L) = 0$, such that

$$\int_0^L |f_A(x) - f_B(x)|^2 dx \le \varepsilon^2 . \tag{A.6}$$

Define a vector $F^* \in E^{(5)}$ as

$$F^* = (f_1^*, f_2^*, f_3^*, f_4^*, f_5^*) \tag{A.7}$$

where

$$f_1^*(x) = f(x) + f_A(x) - f_B(x). \tag{A.8}$$

$$f_2^* \equiv f_1^*(0) - e_0 f_1^{*\prime}(0), \quad f_3^* \equiv f_1^{*\prime}(0) \tag{A.9}$$

and

$$f_4^* \equiv f_1^*(L) + e_1 f_1^{*\prime}(L), \quad f_5^* \equiv f_1^{*\prime}(L). \tag{A.10}$$

It is clear that F^* is in the domain of operator Π. By using (A.1) and (A.6) as well as the generic inequality $(a + b)^2 \le 2(a^2 + b^2)$, it can be shown straightforwardly that

$$\| F^{*} - F \|^2_{H^{(5)}} \le 2 \int_0^L [(f_1(x) - f(x))^2 + (f_A(x) - f_B(x))^2]\, dx \le 4\varepsilon^2. \qquad \text{(A.11)}$$

The last inequality indicates that the domain of operator Π is, indeed, dense in $H^{(5)}$.

APPENDIX B

To simplify the proof of Theorem 2.3.1, operator $\Pi + c\,I$ is considered instead of Π where I is the identity operator from $Dom(\Pi)$ to $H^{(5)}$ whilst, c is a positive constant. By rewriting (2.3.4) as

$$(\Pi + c\,I)\,W_j = (\lambda_j + c)\,W_j \tag{B.1}$$

it is easily seen that the eigenvalue problem of operator $(\Pi + c\,I)$ is equivalent to that of operator Π . The following theorems are needed to prove Theorem 2.3.1.

Theorem B.1 A positive constant c exists such that, for any vector $F \in Dom(\Pi)$,

$$<(\Pi + cI)F, F>_{H^{(5)}} \;\geq\; p_M \, \|F\|^2_{H^{(5)}} \tag{B.2}$$

where $p_M = max\,|p(x)\,|$. That is, operator $\Pi + c\,I$ is positive-bounded-below.

Proof

Integrating the left side of (B.2) and using (2.2.4) leads to

$$\begin{aligned}
<(\Pi + c\,I)F,\; F>_{H^{(5)}} &= \;<\Pi F,\; F>_{H^{(5)}} + c<F,\; F>_{H^{(5)}} \\
&= \left(\int_0^L EI\,|f_1''|^2\,dx\right) + \int_0^L p\,|f_1'|^2\,dx \\
&\quad + \sum_{r=0}^{1} K_r\,|f_1 - (-1)^r \eta_r f_1'|^2\,|_{x=x_r} \qquad\qquad (B.3)\\
&\quad + \sum_{r=0}^{1} \beta_r\,|f_1'|^2\,|_{x=x_r} + c\,\|F\|^2_{H^{(5)}}.
\end{aligned}$$

there exists one subsequence, $\{f_{1n_k}\}$, which converges in $\mathcal{L}^2(\rho A, 0, L)$. On the other hand, the subsequence $\{f_{2n_k}\}$, $\{f_{3n_k}\}$, $\{f_{4n_k}\}$ and $\{f_{5n_k}\}$ must lie in a bounded and closed set due to the boundedness of these sequences in the complex space, **C**. It is known [40] from the compactness of a bounded closed set that there exists at least one convergent subsequence $\{f_{2n_{k_j}}\}$, $\{f_{3n_{k_j}}\}$, $\{f_{4n_{k_j}}\}$ and $\{f_{5n_{k_j}}\}$. Consequently, the subsequence $\{F_j\} = \{(f_{1n_{k_j}}, f_{2n_{k_j}}, f_{3n_{k_j}}, f_{4n_{k_j}}, f_{5n_{k_j}})\}$ converges in $\boldsymbol{H}^{(5)}$ and Theorem B.2 is proved.

Theorem B.3 (See [44].) Let a positive, bounded from below operator be such that any set of functions (vectors), whose energy norms are all bounded, have at least one convergent sequence in a Hilbert space. Then the corresponding eigenfunctions (or eigenvectors) form a complete orthogonal system in both a Hilbert space and an energy space.

Theorem 2.3.1 can be proved now. First, Theorem B.1 reveals that $(\Pi + c\,\boldsymbol{I})$ is a positive, bounded from below operator. Then Theorem B.2 shows that, an arbitrary sequence $\{F_j\}$, whose energy norm in $\boldsymbol{E}^{(5)}$ is bounded, has at least one convergent sequence. Thus, Theorem A.3 demonstrates that the eigenvectors of $(\Pi + c\,\boldsymbol{I})$ or Π are complete in $\boldsymbol{H}^{(5)}$ and $\boldsymbol{E}^{(5)}$, i.e. Theorem 2.3.1 holds.

APPENDIX C

The second order asymptotic expression, $w_{1j}^{(2)}(x)$, is derived in this appendix for the eigenvector, $w_{1j}(x)$, of a non-uniform beam having the general end conditions shown in Figure 2.1. It is known from [9] that the second order expression for $w_{1j}^{(2)}(x)$ takes the general form

$$w_{1j}^{(2)}(x) = \exp(\omega_j^{1/2} \sum_{m=0}^{2} u_m(x)\ \omega_j^{-m/2}) \tag{C.1}$$

where $\omega_j = z_j^2 = \lambda_j^{1/2}$. By substituting the above expression into (2.2.1) and comparing the resulting coefficients of ω_j^2, $\omega_j^{3/2}$ and ω, the following ordinary differential equations are obtained:

$$(u_0')^4 EI(x)/\rho(x) = 1 \tag{C.2}$$

$$EI(x)(4(u_0')^3 u_1' + 6(u_0')^2 u_0'') + 2(EI(x))'(u_0')^3 = 0 \tag{C.3}$$

and

$$\begin{aligned} &EI(x)(4(u_0')^3 u_2' + 6(u_0')^2 (u_1')^2 + 3(u_0'')^2 + 4u_0''' u_0' + 6(u_0')^2 u_1'' + \\ &12 u_0' u_0'' u_1' + 2(EI(x))'(3(u_0')^2 u_1' + 3u_0'' u_0') + ((EI(x))'' - p)(u_0')^2 = 0. \end{aligned} \tag{C.4}$$

It is known from [9] that the solutions of equations (C.2) through (C.4) are given by

$$u_0(x) = \pm\hat{x}(x),\quad u_1(x) = \ln\alpha(x) \quad and \quad u_2(x) = \pm\chi(x) \tag{C.5}$$

as well as

$$u_0(x) = \pm i'\hat{x}(x),\quad u_1(x) = \ln\alpha(x) \quad and \quad u_2(x) = \mp i'\chi(x)\,. \tag{C.6}$$

Here, $i' = (-1)^{1/2}$ and

$$\hat{x}(x) = \int_0^x \hat{b}(x)dx,\quad \alpha(x) = (\hat{b}(x))^{-3/2}(EI(x))^{-1/2} \quad and \quad \hat{b}(x) = (\frac{\rho A(x)}{EI(x)})^{1/4} \tag{C.7}$$

whilst

$$\chi(x) = \int_0^x [(\frac{5\hat{b}''}{4\hat{b}^2} - \frac{15(\hat{b}')^2}{8\hat{b}^3}) - \frac{3((EI)')^2}{8\hat{b}(EI)^2} + \frac{(EI)''}{2EI\hat{b}} + \frac{p(x)}{4EI\hat{b}}]dx\,. \tag{C.8}$$

By substituting (C.5) and (C.6) into (C.1), $w_{1j}^{(2)}(x)$ can be expressed as

$$\begin{aligned} w_{1j}^{(2)}(x) = {} & \alpha(x)(A_j\cos\xi_1(x) + B_j\sin\xi_1(x) + C_j\exp(-\xi_2(x)) \\ & + D_j\exp(-(\sigma z_j + \sigma^*/z_j - \xi_2(x))). \end{aligned} \tag{C.9}$$

where

$$\xi_1(x) = \hat{x}(x)z_j - \chi(x)/z_j,\quad \xi_2 = \hat{x}(x)z_j + \chi(x)/z_j,\quad z_j^4 = \lambda_j \tag{C.10}$$

whilst

$$\sigma = \hat{x}(L) \quad and \quad \sigma^{*} = \chi(L). \tag{C.11}$$

By substituting (C.11) into the end conditions (2.2.2) and (2.2.3), coefficients A_j, B_j, C_j and D_j can be obtained by solving the equation

$$[\Xi_{ij}]\{A_j \quad B_j \quad C_j \quad D_j\}^T = 0. \tag{C.12}$$

Elements Ξ_{ij}, $i, j = 1, 2, 3, 4$ and the coefficients A_j, B_j, C_j and D_j are detailed in the next appendix.

APPENDIX D

This appendix presents explicit expressions for all the coefficients as well as details of the frequency equation appearing in section 2.4.1 and Appendix C.

1. Coefficients Ξ_{ij}

$$\Xi_{11}(\hat{x}=0) = [M_0 e_0 z_j^4 - (K_0\eta_0 + p)]\hat{b}\alpha' - d_4\alpha(\xi_1')^2 - 3d_5\alpha'(\xi_1')^2 + (K_0 - M_0 z_j^4)\alpha + O(1) \tag{D.1}$$

$$\Xi_{12}(\hat{x}=0) = [d_3 + M_0 e_0 \hat{b} z_j^4 - (K_0\eta_0 + p)\hat{b}]\alpha\xi_1' + 2d_4\alpha'\xi_1' - d_5 \times (3\alpha''\xi_1' - \alpha(\xi_1')^3) + O(z_j^{-1}) \tag{D.2}$$

$$\Xi_{13}(\hat{x}=0) = [d_3 + M_0 e_0 \hat{b} z_j^4 - (K_0\eta_0 + p)\hat{b}](\alpha' - \alpha\xi_2') + d_4(-2\alpha'\xi_2' + \alpha(\xi_2')^2) + d_5(-3\alpha''\xi_2' + 3\alpha'(\xi_2')^2 - \alpha(\xi_2')^3) + (K_0 - M_0 z_j^4)\alpha + O(1) \tag{D.3}$$

$$\Xi_{14}(\hat{x}=\sigma) = O(\exp(-z_j\sigma)) \tag{D.4}$$

$$\Xi_{21}(\hat{x}=\sigma) = h_1\cos\xi_1 + h_2\sin\xi_1 \tag{D.5}$$

$$\Xi_{22}(\hat{x}=\sigma) = h_1 \sin\xi_1 - h_2 \cos\xi_1 \tag{D.6}$$

$$\Xi_{23}(\hat{x}=\sigma) = O(\exp(-z_j\sigma)) \tag{D.7}$$

$$\begin{aligned}\Xi_{24}(\hat{x}=\sigma) = {} & [d_3 + M_1 e_1 \hat{b} z_j^4 - (K_1\eta_1 + p)\hat{b}](\alpha' + \alpha\xi_2') + d_4(2\alpha'\xi_2' + \\ & + \alpha(\xi_2')^2) + d_5(3\alpha''\xi_2' + 3\alpha'(\xi_2')^2 - \alpha(\xi_2')^3) - \\ & - (K_1 - M_1 z_j^4)\alpha + O(1)\end{aligned} \tag{D.8}$$

$$\begin{aligned}\Xi_{31}(\hat{x}=0) = {} & d_2\alpha(\xi_1')^2 + [\hat{b}((\beta_0 - p\eta_0) - (J_0 + M_0(e_0 - \eta_0)e_0)z_j^4) - d_1]\alpha' + \\ & + M_0(e_0 - \eta_0)\alpha z_j^4 - \eta_0[d_4\alpha(\xi_1')^2 + 3d_5\alpha'(\xi_1')^2] + O(1)\end{aligned} \tag{D.9}$$

$$\begin{aligned}\Xi_{32}(\hat{x}=0) = {} & -2d_2\alpha'\xi_1' + [\hat{b}((\beta_0 - p\eta_0) - (J_0 + M_0(e_0 - \eta_0)e_0)z_j^4) - d_1]\alpha\xi_1' + \\ & + \eta_0[d_3\alpha\xi_1' + 2d_4\alpha'\xi_1' - d_5(3\alpha''\xi_1' - \alpha(\xi_1')^3)] + O(1)\end{aligned} \tag{D.10}$$

$$\begin{aligned}\Xi_{33}(\hat{x}=0) = {} & d_2(2\alpha'\xi_2' - \alpha(\xi_2')^2) + [\hat{b}(\beta_0 - p\eta_0) - (J_0 + M_0(e_0 - \eta_0)e_0)z_j^4) - \\ & - d_1](\alpha' - \alpha\xi_2') + M_0(e_0 - \eta_0)\alpha z_j^4 + \eta_0[-d_3\alpha\xi_2' + d_4(-2\alpha'\xi_2' + \\ & + \alpha(\xi_2')^2) + d_5(-3\alpha''\xi_2' + 3\alpha'(\xi_2')^2 - \alpha(\xi_2')^3)] + O(1)\end{aligned} \tag{D.11}$$

$$\Xi_{34}(\hat{x} = \sigma) = O(\exp(-z_j\sigma)) \tag{D.12}$$

$$\Xi_{41}(\hat{x} = \sigma) = h_3\cos\xi_1 + h_4\sin\xi_1 \tag{D.13}$$

$$\Xi_{42}(\hat{x} = \sigma) = h_3\sin\xi_1 - h_4\cos\xi_1 \tag{D.14}$$

$$\Xi_{43}(\hat{x} = \sigma) = O(\exp(-z_j\sigma)) \tag{D.15}$$

$$\begin{aligned}\Xi_{44}(\hat{x} = \sigma) = {} & d_2(2\alpha'\xi_2' + \alpha(\xi_2')^2) + [\hat{b}(\beta_1 - p\eta_1) - (J_1 + M_1(e_1 - \eta_1)e_1)z_j^4) + \\ & + d_1](\alpha' + \alpha\xi_2') - M_1(e_1 - \eta_1)\alpha z_j^4 + \eta_1[d_3\alpha\xi_2' + d_4(2\alpha'\xi_2' + \\ & + \alpha(\xi_2')^2) + d_5(3\alpha''\xi_2' + 3\alpha'(\xi_2')^2 + \alpha(\xi_2')^3)] + O(1)\end{aligned} \tag{D.16}$$

where

$$d_1(x) = \vartheta^2(x)\hat{b}'(x),\quad d_2(x) = \vartheta^2(x)\hat{b}(x),\quad d_3(x) = d_1'(x)\hat{b}(x) \tag{D.17}$$

$$d_4(x) = \hat{b}(x)(d_1(x) + d_2'(x)),\quad d_5(x) = \vartheta^2(x)\hat{b}^2(x) \tag{D.18}$$

$$\vartheta^2(\hat{x}) = (EI(\hat{x}))^{3/4}(\rho A(\hat{x}))^{1/4}. \tag{D.19}$$

$$z_j^4 = \lambda_j,\quad \xi_1(x) = \hat{x}(x)z_j - \chi(x)/z_j,\quad \xi_2 = \hat{x}(x)z_j + \chi(x)/z_j \tag{D.20}$$

$$\hat{x}(x) = \int_0^x \hat{b}(x)\,dx, \quad \alpha(x) = (\hat{b}(x))^{-3/2}(EI(x))^{-1/2} \tag{D.21}$$

$$\chi(x) = \int_0^x [(\frac{5\hat{b}''}{4\hat{b}^2} - \frac{15(\hat{b}')^2}{8\hat{b}^3}) - \frac{3((EI)')^2}{8\hat{b}(EI)^2} + \frac{(EI)''}{2EI\hat{b}} + \frac{p(x)}{4EI\hat{b}}]\,dx, \tag{D.22}$$

and

$$\hat{b}(x) = (\frac{\rho A(x)}{EI(x)})^{1/4}, \quad \sigma = \hat{x}(L). \tag{D.23}$$

Moreover,

$$\begin{aligned} h_1(\hat{x} = \sigma) = {} & [M_1 e_1 z_j^4 - (K_1\eta_1 + p)]\hat{b}\alpha' - d_4\alpha(\xi_1')^2 - 3d_5\alpha'(\xi_1')^2 - \\ & -(K_1 - M_1 z_j^4)\alpha + O(1) \end{aligned} \tag{D.24}$$

$$\begin{aligned} h_2(\hat{x} = \sigma) = {} & -[(d_3 + M_1 e_1 \hat{b} z_j^4 - (K_1\eta_1 + p)\hat{b})\alpha + 2d_4\alpha' + 3d_5\alpha'']\xi_1' + \\ & + d_5\alpha(\xi_1')^3 + O(1) \end{aligned} \tag{D.25}$$

$$\begin{aligned} h_3(\hat{x} = \sigma) = {} & -d_2\alpha(\xi_1')^2 + [\hat{b}(\beta_1 - p\eta_1) - (J_1 + M_1(e_1 - \eta_1)e_1)z_j^4) + d_1]\times \\ & \times\alpha' - M_1(e_1 - \eta_1)\alpha z_j^4 - \eta_1(d_4\alpha(\xi_1')^2 + 3d_5\alpha'(\xi_1')^2) \end{aligned} \tag{D.26}$$

$$h_4(\hat{x} = \sigma) = -\{2d_2\alpha' + [\hat{b}(\beta_1 - p\eta_1) - (J_1 + M_1(e_1 - \eta_1)e_1)z_j^4) + d_1]\alpha\} \times$$
$$\times \xi_1' + \eta_1[-(d_3\alpha + 2d_4\alpha' + 3d_5\alpha'')\xi_1' + d_5\alpha(\xi_1')^3 + O(1) \tag{D.27}$$

Except for (D.22), a prime superscript in this appendix indicates a differentiation with respect to $\hat{x}$.

2. Frequency equation

By expanding $\mathrm{Det}(\Xi_{ij})$, the frequency equation can be shown to be

$$\Delta_1 \cos\xi_1(z_n) + \Delta_2 \sin\xi_1(z_n) = 0 \tag{D.28}$$

where

$$\left.\begin{aligned} \Delta_1 &= \Xi_{24}(\upsilon_1 h_3 - \upsilon_2 h_4) + \Xi_{44}(\upsilon_2 h_2 - \upsilon_1 h_1) \\ \Delta_2 &= \Xi_{24}(\upsilon_1 h_4 + \upsilon_2 h_3) - \Xi_{44}(\upsilon_1 h_2 + \upsilon_2 h_1) \end{aligned}\right\} \tag{D.29}$$

and

$$\upsilon_1 = (\Xi_{12}\Xi_{33} - \Xi_{13}\Xi_{32}), \qquad \upsilon_2 = (\Xi_{13}\Xi_{31} - \Xi_{11}\Xi_{33}). \tag{D.30}$$

3. A_j, B_j, C_j and D_j

A_j, B_j, C_j and D_j can be derived from equation (2.4.6) as

$$\left.\begin{aligned} A_j = \upsilon_1/\upsilon_2, \quad B_j = 1, \quad C_j = -(\Xi_{12}\Xi_{31} - \Xi_{11}\Xi_{32})/\upsilon_2 \\ D_j = -(\Xi_{22} + \Xi_{21}\upsilon_1/\upsilon_2)/\Xi_{24} \end{aligned}\right\} \tag{D.31}$$

for a beam whose left end conditions are either

$$K_0=\infty,\ \ J_0=0,\ \ \beta_0<\infty \quad or \quad M_0\neq 0,\ \ J_0=0,\ \ \beta_0<\infty. \tag{D.32}$$

Moreover,

$$\left.\begin{aligned} A_j = 1,\ \ B_j = \upsilon_2/\upsilon_1,\ \ C_j &= -(\Xi_{12}\Xi_{31}-\Xi_{11}\Xi_{32})/\upsilon_1 \\ D_j &= -(\Xi_{21}+\Xi_{22}\upsilon_2/\upsilon_1)/\Xi_{24} \end{aligned}\right\} \tag{D.33}$$

for a beam having the other end conditions specified at the leftmost side of Table 2.1.

APPENDIX E

The proof of Theorem 2.4.2 is presented here. First, the min-max principle is needed which states that the jth eigenvalue of a completely continuous, self-adjoint, positive operator, $\boldsymbol{Q}$, is given by [12]

$$\lambda_j = \min_{V_j \in \boldsymbol{H},\ \dim V_j = j} \ \max_{Y \in V_j} \ R(Y), \quad j = 1, 2, \ldots \tag{E.1}$$

where $R(Y)$ is the Rayleigh-quotient defined by

$$R(Y) = \frac{\langle Y, Y \rangle_E}{\langle Y, Y \rangle_H} \tag{E.2}$$

for $Y \in \boldsymbol{H}$. The $\boldsymbol{H}$ and $\boldsymbol{E}$ are the Hilbert space and energy space corresponding to operator $\boldsymbol{Q}$, respectively, whilst $\boldsymbol{Q}$ is merely the operator Π^{-1} for a beam. Consider, without loss of generality, the beam shown in Figure 2.1 except that $M_1 = 0$. This beam can be regarded as a beam that has been modified by an additional M_1 having eccentricity e_1 at $x = L$. Designate the eigenvalues and eigenvectors of the unmodified beam by λ_j^* and $w_{1j}^*(x)$, respectively, where $w_{1j}^*(x)$ is normalized in the corresponding Hilbert space. That is,

$$\int_0^L \rho A (w_{1j}^*(x))^2 dx + M_0(w_{1j}^*(0) - e_0 w_{1j}^{*\prime}(0))^2 + J_0(w_{1j}^{*\prime}(0))^2 + J_1(w_{1j}^{*\prime}(L))^2 = 1. \tag{E.3}$$

Consider an arbitrary linear combination of the vectors X_n, $n = 1, 2, \ldots, j$, i.e

$$U_j = c_1 X_1 + c_2 X_2 + \ldots + c_j X_j \neq 0 \tag{E.4}$$

where

$$X_n = (w^*_{1n}(x),\ w^*_{1n}(0) - e_0 w^{*\prime}_{1n}(0),\ w^{*\prime}_{1n}(0),\ w^*_{1n}(L) + e_1 w^{*\prime}_{1n}(L),\ w^{*\prime}_{1n}(L))\,. \tag{E.5}$$

Substituting U_j, corresponding to the modified beam, into the Rayleigh quotient (E.2) results in

$$R(U_j) = \frac{c_1^2\lambda_1^* + c_2\lambda_2^* + \ldots + c_j^2\lambda_j^*}{c_1^2 + c_2^2 + \ldots + c_j^2 + M_1(\sum_{n=1}^{j}(w^*_{1n}(L) - e_1 w^{*\prime}_{1n}(L)))^2} \le \lambda_j^*\,. \tag{E.6}$$

On the other hand, assume that at least one of $(w_{1n}(L) + e_1 w_{1n}'(L))$ $(n = 1, 2, \ldots, j - 1)$ is non-zero. Here $w_{1n}(x)$ is the nth eigenvector of the modified beam. Without loss of generality, let $(w_{11}(L) + e_1 w_{11}'(L)) \neq 0$. Consider the vector $V_{j-1} \neq 0$ given by

$$V_{j-1} = c_2 Y_2 + c_3 Y_3 + \ldots + c_j Y_j \tag{E.7}$$

where

$$Y_n = h_n Z_1 + Z_n,\ h_n = -\frac{w_{1n}(L) + e_1 w_{1n}'(L)}{w_{11}(L) + e_1 w_{11}'(L)},\ n > 1 \tag{E.8}$$

$$Z_1 = (w_{11}(x), w_{11}(0) - e_0 w_{11}'(0), w_{11}'(0), w_{11}'(L)) \tag{E.9}$$

and

$$Z_n = (w_{1n}(x), w_{1n}(0) - e_0 w_{1n}'(0), w_{1n}'(0), w_{1n}'(L)), \quad n = 2, 3, \ldots, j. \qquad \text{(E.10)}$$

By substituting V_{j-1} corresponding to the unmodified beam into the quotient (E.2), it is easily shown that

$$R(V_{j-1}) = \frac{(\sum_{n=2}^{j} h_n c_n)^2 \lambda_1 + c_2^2 \lambda_2 + c_3^2 \lambda_3 + \ldots + c_j^2 \lambda_j}{(\sum_{n=2}^{j} h_n c_n)^2 + c_2^2 + c_3^2 + \ldots + c_j^2} \leq \lambda_j \qquad \text{(E.11)}$$

for any $V_{j-1} \neq 0 \in \text{span}\ \{Y_2, \ldots, Y_j\}$. From the min-max principle (E.1), inequalities (E.6) and (E.11) imply that Theorem 2.4.2 (i) holds when at least one of $(w_{1n}(L) + e_1 w_{1n}'(L)) \neq 0$ ($n = 1, 2, \ldots, j - 1$) is non-zero.

If $(w_{1n}(L) + e_1 w_{1n}'(L)) = 0$ for $n = 1, 2, \ldots, j - 1$, the eigenvectors $w_{1n}(x)$, $n = 1, 2, \ldots, j - 1$, also satisfy the unmodified beam's conditions at $x = L$. According to the completeness of the eigenvectors of the unmodified system, it can be seen that the first $(j - 1)$ eigenvalues are the same as those of the unmodified beam. Therefore, Theorem 2.4.2 (i) still holds.

Theorem 2.4.2 (ii) can be shown analogously.

APPENDIX F

This appendix demonstrates that there exists two positive constants, c_1 and c_2, that are independent of the eigenvector $\boldsymbol{W}_j$ and such that

$$c_1 < \| \boldsymbol{W}_j \|_{H^{(5)}} < c_2 \tag{F.1}$$

is valid for a sufficiently large j. Here

$$\| \boldsymbol{W}_j \|^2_{H^{(5)}} = \int_0^L \rho A (w_{1j})^2 dx + M_0 (w_{2j})^2 + J_0 (w_{3j})^2 + M_1 (w_{4j})^2 + J_1 (w_{5j})^2 \tag{F.2}$$

whilst $w_{1j}(x)$ is the first component of $\boldsymbol{W}_j$. By setting the $\chi(x)$ of (2.4.2) to zero, the first order, asymptotic form, $w_{1j}^{(1)}(x)$, of $w_{1j}(x)$ can be found from (2.4.1) to be

$$\begin{aligned} w_{1j}^{(1)}(x) = \alpha(x)((A_j^2 + B_j^2)^{1/2} \cos((z_j)_1 \hat{x} - \hat{\vartheta}_j) + C_j \exp(-(z_j)_1 \hat{x}) \\ + D_j \exp(-(\sigma - \hat{x})(z_j)_1)) \end{aligned} \tag{F.3}$$

where

$$\hat{\vartheta}_j = \sin^{-1} \frac{B_j}{(A_j^2 + B_j^2)^{1/2}}, \quad \sigma = \hat{x}(L) \tag{F.4}$$

and

$$\hat{x}(x) = \int_0^x \hat{b}(x) dx, \quad \alpha(x) = (\hat{b}(x))^{-3/2} (EI(x))^{-1/2}, \quad \hat{b}(x) = \left(\frac{\rho A(x)}{EI(x)}\right)^{1/4}. \tag{F.5}$$

The A_j, B_j, C_j and D_j are given by (D.31) and (D.33). To show (F.1), the following inequalities

$$-3 < A_j < 3, \quad -3 < B_j < 3, \quad -3 < C_j < 3 \quad and \quad -3 < D_j < 3 \tag{F.6}$$

are needed. To validate (F.6), first consider a simply supported beam, i.e.

$$K_0 = \infty = K_1 \tag{F.7}$$

and

$$e_0 = \eta_0 = e_1 = \eta_1 = M_0 = J_0 = M_1 = J_1 = \beta_0 = \beta_1 = 0. \tag{F.8}$$

By substituting (F.8) into (D.1) through (D.12), (D.24), (D.25) and (D.31), A_j, B_j, C_j and D_j can be found as follows

$$A_j = \frac{(\Xi_{12}\Xi_{33} - \Xi_{13}\Xi_{32})}{(\Xi_{13}\Xi_{31} - \Xi_{11}\Xi_{33})}, \; B_j = 1 \tag{F.9}$$

and

$$C_j = -\frac{(\Xi_{12}\Xi_{31} - \Xi_{11}\Xi_{32})}{(\Xi_{13}\Xi_{31} - \Xi_{11}\Xi_{33})}, \quad D_j = (\frac{\Xi_{22}}{\Xi_{24}} + \frac{\Xi_{21}}{\Xi_{24}}A_j) \tag{F.10}$$

where

$$\Xi_{11}(\hat{x} = 0) = -p\hat{b}\frac{d\alpha}{d\hat{x}} - d_4\alpha((z_j)_1)^2 - 3d_5\frac{d\alpha}{d\hat{x}}((z_j)_1)^2 + K_0\alpha + O(1) \tag{F.11}$$

$$\Xi_{12}(\hat{x} = 0) = (d_3 - p\hat{b})\alpha(z_j)_1 + 2d_4\frac{d\alpha}{d\hat{x}}(z_j)_1$$

$$+ d_5(3\frac{d^2\alpha}{d\hat{x}^2}(z_j)'_1 - \alpha((z_j)_1)^3) + O(z_j^{-1}) \tag{F.12}$$

$$\Xi_{13}(\hat{x} = 0) = (d_3 - p\hat{b})(\frac{d\alpha}{d\hat{x}} - \alpha(z_j)_1) + d_4(-2\frac{d\alpha}{d\hat{x}}(z_j)_1 + \alpha((z_j)_1)^2)$$

$$+ d_5(-3\frac{d^2\alpha}{d\hat{x}^2}(z_j)_1 + 3\frac{d\alpha}{d\hat{x}}((z_j)_1)^2 - \alpha((z_j)_1)^3) \tag{F.13}$$

$$+ K_0\alpha + O(1)$$

$$\Xi_{21}(\hat{x} = \sigma) = h_1\cos(z_j)_1 + h_2\sin(z_j)_1 \tag{F.14}$$

$$\Xi_{22}(\hat{x} = \sigma) = h_1\sin(z_j)_1 - h_2\cos(z_j)_1 \tag{F.15}$$

$$\Xi_{24}(\hat{x} = \sigma) = (d_3 - p\hat{b}](\frac{d\alpha}{d\hat{x}} + \alpha(z_j)_1) + d_4(2\frac{d\alpha}{d\hat{x}}(z_j)_1 +$$

$$+ \alpha((z_j)_1)^2) + d_5(3\frac{d^2\alpha}{d\hat{x}^2}(z_j)_1 + 3\frac{d\alpha}{d\hat{x}}((z_j)_1)^2 \tag{F.16}$$

$$+ \alpha((z_j)_1)^3) - K_1\alpha + O(1)$$

$$\Xi_{31}(\hat{x} = 0) = d_2\alpha((z_j)_1)^2 - d_1\frac{d\alpha}{d\hat{x}} + O(1) \tag{F.17}$$

$$\Xi_{32}(\hat{x} = 0) = -2d_2\frac{d\alpha}{d\hat{x}}(z_j)_1 - d_1\alpha(z_j)_1 + O(1) \tag{F.18}$$

and

$$\Xi_{33}(\hat{x} = 0) = d_2(2\frac{d\alpha}{d\hat{x}}(z_j)_1 - \alpha((z_j)_1)^2) - d_1(\frac{d\alpha}{d\hat{x}} - \alpha(z_j)_1) + O(1) \tag{F.19}$$

whilst

$$h_1(\hat{x} = \sigma) = -p\hat{b}\frac{d\alpha}{d\hat{x}} - d_4\alpha((z_j)_1)^2 - 3d_5\frac{d\alpha}{d\hat{x}}((z_j)_1)^2 - \\ -K_1\alpha + O(1) \tag{F.20}$$

and

$$h_2(\hat{x} = \sigma) = -[(d_3 - p\hat{b})\alpha + 2d_4\frac{d\alpha}{d\hat{x}} + 3d_5\frac{d^2\alpha}{d\hat{x}^2}](z_j)_1 + \\ + d_5\alpha((z_j)_1)^3 + O(1). \tag{F.21}$$

Here d_i, $i = 1, 2, ..., 5$, are given by (D.17) and (D.18). On the other hand, when $K_0 \to \infty$, it can be found straightforwardly from (F.11) through (F.13), (F.17) and (F.19) that

$$\lim_{K_0\to\infty}\frac{\Xi_{11}}{K_0} = \alpha = \lim_{K_0\to\infty}\frac{\Xi_{13}}{K_0}, \quad \lim_{K_0\to\infty}\frac{\Xi_{12}\Xi_{33}}{K_0} = 0 = \lim_{K_0\to\infty}\frac{\Xi_{12}\Xi_{31}}{K_0} \tag{F.22}$$

Consequently, the A_j and C_j given in (F.9) and (F.10), respectively, can be simplified, for

$K_0 = \infty$, as

$$A_j = \lim_{K_0\to\infty} \frac{\Xi_{12}\Xi_{33}-\Xi_{13}\Xi_{32}}{\Xi_{13}\Xi_{31}-\Xi_{11}\Xi_{33}} = \lim_{K_0\to\infty} \frac{\dfrac{\Xi_{12}\Xi_{33}-\Xi_{13}\Xi_{32}}{K_0}}{\dfrac{\Xi_{13}\Xi_{31}-\Xi_{11}\Xi_{33}}{K_0}}$$

or

$$A_j = \frac{-\alpha\Xi_{32}}{\alpha\Xi_{31}-\alpha\Xi_{33}} \tag{F.23}$$

and, similarly,

$$C_j = \lim_{K_0\to\infty} \frac{\Xi_{12}\Xi_{31}-\Xi_{11}\Xi_{32}}{\Xi_{13}\Xi_{31}-\Xi_{11}\Xi_{33}} = \lim_{K_0\to\infty} \frac{\dfrac{\Xi_{12}\Xi_{31}-\Xi_{11}\Xi_{32}}{K_0}}{\dfrac{\Xi_{13}\Xi_{31}-\Xi_{11}\Xi_{33}}{K_0}}$$

so that

$$C_j = \frac{-\alpha\Xi_{32}}{\alpha\Xi_{31}-\alpha\Xi_{33}} = A_j. \tag{F.24}$$

Now consider D_j. When $K_1 \to \infty$, it can be found from (F.16), (F.20) and (F.21) that

$$\lim_{K_1\to\infty} \frac{h_1}{K_1} = -\alpha, \quad \lim_{K_1\to\infty} \frac{h_2}{K_1} = 0, \quad \lim_{K_1\to\infty} \frac{\Xi_{24}}{K_1} = -\alpha. \tag{F.25}$$

Consequently, the limit

$$\lim_{K_1\to\infty}\frac{\Xi_{22}}{\Xi_{24}} = \lim_{K_1\to\infty}\frac{\frac{\Xi_{22}}{K_1}}{\frac{\Xi_{24}}{K_1}} = \lim_{K_1\to\infty}\frac{\frac{h_1\sin(z_j)_1 - h_2\cos(z_j)_1}{K_1}}{\frac{\Xi_{24}}{K_1}}$$

$$= \frac{(-\alpha\sin(z_j)_1 - 0\times\cos(z_j)_1)}{\alpha}$$

or

$$\lim_{K_1\to\infty}\frac{\Xi_{22}}{\Xi_{24}} = -\sin(z_j)_1 \tag{F.26}$$

can be found from (F.15) and (F.25). Furthermore, $(z_j)_1 = j\pi$. This leads to

$$\lim_{K_1\to\infty}\frac{\Xi_{22}}{\Xi_{24}} = -\sin(j\pi) = 0. \tag{F.27}$$

A similar procedure can be used to obtain

$$\lim_{K_1\to\infty}\frac{\Xi_{21}}{\Xi_{24}} = -\cos(z_j)_1. \tag{F.28}$$

Substituting (F.27) and (F.28) into (F.10) yields

$$D_j = \lim_{K_1\to\infty}\left(\frac{\Xi_{22}}{\Xi_{24}} + \frac{\Xi_{21}}{\Xi_{24}}A_j\right) = (0 - A_j\cos(z_j)_1) = -A_j\cos(z_j)_1. \tag{F.29}$$

On the other hand, the limits

$$\lim_{(z_j)_1\to\infty}\frac{\Xi_{31}}{((z_j)_1)^3}=0=\lim_{(z_j)_1\to\infty}\frac{\Xi_{32}}{((z_j)_1)^3},\quad \lim_{(z_j)_1\to\infty}\frac{\Xi_{33}}{((z_j)_1)^3}=-d_5\alpha\neq 0 \tag{F.30}$$

can be determined from (F.17) through (F.19). Thus, the limits of A_j and C_j, as $j\to\infty$, can be found from (F.23) and (F.24) to be

$$\lim_{j\to\infty}A_j=\lim_{j\to\infty}C_j=\lim_{j\to\infty}\frac{-\alpha\Xi_{32}}{\alpha\Xi_{31}-\alpha\Xi_{33}}=\lim_{j\to\infty}\frac{\dfrac{-\alpha\Xi_{32}}{((z_j)_1)^3}}{\dfrac{\alpha\Xi_{31}-\alpha\Xi_{33}}{((z_j)_1)^3}}=\frac{0}{0-d_5\alpha^2}$$

or, by employing (F.30),

$$\lim_{j\to\infty}A_j=\lim_{j\to\infty}C_j=0. \tag{F.31}$$

Furthermore, it is known [42] that $|\cos(z_j)_1|\le 1$ for any j so that

$$0\le|A_j\cos(z_j)_1|\le|A_j||\cos(z_j)_1|\le|A_j|. \tag{F.32}$$

Consequently, the inequality

$$0\le\lim_{j\to\infty}|A_j\cos(z_j)_1|\le\lim_{j\to\infty}|A_j| \tag{F.33}$$

must hold [42]. This last inequality, when combined with (F.31), leads immediately to

$$\lim_{j\to\infty}|A_j\cos(z_j)_1|=0. \tag{F.34}$$

On the other hand, it is known from [40] that (F.34) is equivalent to

$$\lim_{j\to\infty} (A_j \cos(z_j)_1) = 0. \qquad \text{(F.35)}$$

Substituting (F.35) into (F.29) leads to

$$\lim_{j\to\infty} D_j = \lim_{j\to\infty} (-A_j \cos(z_j)_1) = -\lim_{j\to\infty} (A_j \cos(z_j)_1) = 0. \qquad \text{(F.36)}$$

Thus, (F.31) and (F.36) demonstrate that $A_j \to 0$, $C_j \to 0$ and $D_j \to 0$ as $j \to \infty$ so that there must exist a positive integer, j_0, such that

$$-3 < A_j < 3,\ -3 < C_j < 3 \quad \textit{and} \quad -3 < D_j < 3 \qquad \text{(F.37)}$$

for $j > j_0$. Furthermore, (F.9) indicates $B_j = 1$. Therefore, (F.6), indeed, holds for a simply supported beam. A similar proof can be given for a beam having the other end conditions stated in Table 2.1.

Now (D.31) and (D.33) demonstrate that either $A_j = 1$ or $B_j = 1$ for the end conditions given in Table 2.1. Thus, it can be shown from (F.6) that the inequalities

$$1 \le A_j^2 + B_j^2 < 10,\ 0 \le C_j^2 < 9,\ 0 \le D_j^2 < 9 \qquad \text{(F.38)}$$

$$-24 < 2C_j(A_j^2 + B_j^2)^{1/2} < 24,\ -24 < 2D_j(A_j^2 + B_j^2)^{1/2} < 24, \qquad \text{(F.39)}$$

and

$$-18 < 2C_jD_j < 18 \qquad \text{(F.40)}$$

can be found straightforwardly for any condition given in Table 2.1. Furthermore, it can be shown, by employing (F.3), that

$$\int_0^L \rho A (w_{1j}^{(1)})^2 dx = (A_j^2 + B_j^2) \int_0^L \rho A \alpha^2(x) \cos^2((z_j)_1 \hat{x} - \hat{\vartheta}_j) dx$$

$$+ C_j^2 \int_0^L \rho A \alpha^2(x) \exp(-2(z_j)_1 \hat{x}) dx$$

$$+ D_j^2 \int_0^L \rho A \alpha^2(x) \exp(-2(\sigma - \hat{x})(z_j)_1) dx$$

$$2C_j(A_j^2 + B_j^2)^{1/2} \int_0^L \rho A \alpha^2(x) \cos((z_j)_1 \hat{x} - \hat{\vartheta}_j) \qquad \text{(F.41)}$$

$$\times \exp(-(z_j)_1 \hat{x}) dx$$

$$+ 2D_j(A_j^2 + B_j^2)^{1/2} \int_0^L \rho A \alpha^2(x) \cos((z_j)_1 \hat{x} - \hat{\vartheta}_j) \times$$

$$\exp(-(\sigma - \hat{x})(z_j)_1) dx$$

$$+ 2C_j D_j \int_0^L \rho A \alpha^2(x) \exp(-\sigma (z_j)_1) dx.$$

By employing the standard integral formulae [42]

$$\int_0^\sigma \exp(-i(z_j)_1 \hat{x}) d\hat{x} = \frac{(1 - \exp(-i(z_j)_1 \sigma))}{i(z_j)_1}, \quad i = 1, 2, \qquad \text{(F.42)}$$

and

$$\int_0^{\sigma} \cos^2((z_j)_1 \hat{x} + \hat{\vartheta}_j) d\hat{x} = \frac{\sigma}{2} + \frac{(\sin(2((z_j)_1 \sigma + \hat{\vartheta}_j)) - \sin(2\hat{\vartheta}_j))}{4(z_j)_1}, \tag{F.43}$$

integral (F.41) is shown next to be bounded. First, it can be found from (F.43) that the inequalities

$$\begin{aligned}\int_0^{\sigma} \cos^2((z_j)_1 \hat{x} + \hat{\vartheta}_j) d\hat{x} &\geq \frac{\sigma}{2} - \frac{|(\sin(2((z_j)_1 \sigma + \hat{\vartheta}_j)) - \sin(2\hat{\vartheta}_j))|}{4(z_j)_1} \\ &\geq \frac{\sigma}{2} - \frac{2}{4(z_j)_1}\end{aligned}$$

i.e.

$$\int_0^{\sigma} \cos^2((z_j)_1 \hat{x} + \hat{\vartheta}_j) d\hat{x} \geq \frac{\sigma}{2} - \frac{1}{2(z_j)_1} \tag{F.44}$$

and

$$\begin{aligned}\int_0^{\sigma} \cos^2((z_j)_1 \hat{x} + \hat{\vartheta}_j) d\hat{x} &\leq \frac{\sigma}{2} + \frac{|(\sin(2((z_j)_1 \sigma + \hat{\vartheta}_j))| + |\sin(2\hat{\vartheta}_j))|}{4(z_j)_1} \\ &\leq \frac{\sigma}{2} + \frac{2}{4(z_j)_1}\end{aligned}$$

or

$$\int_0^{\sigma} \cos^2((z_j)_1 \hat{x} + \hat{\vartheta}_j) d\hat{x} \le \frac{\sigma}{2} + \frac{1}{2(z_j)_1} \tag{F.45}$$

are valid. On the other hand, it is known from (F.5) that

$$\hat{x}(x) = \int_0^{x} \hat{b}(x) dx. \tag{F.46}$$

Moreover, it is known from the derivative of a definite integral having a variable upper limit [40] that

$$\frac{d\hat{x}}{dx} = \frac{d}{dx}\int_0^{x} \hat{b}(x) dx = \hat{b}(x). \tag{F.47}$$

Hence, for an arbitrary integrable function, $f(x)$, the identity

$$\int_0^{\sigma} \frac{f(x)}{\hat{b}(x)} d\hat{x} = \int_0^{L} \frac{f(x)}{\hat{b}(x)} \frac{d\hat{x}}{dx} dx = \int_0^{L} \frac{f(x)}{\hat{b}(x)} \hat{b}(x) dx = \int_0^{L} f(x) dx. \tag{F.48}$$

can be found by employing (F.47). The use of (F.38), (F.44) and (F.48) yields

$$(A_j^2+B_j^2)\int_0^L \rho A\alpha^2(x)\cos^2((z_j)_1\hat{x}+\hat{\vartheta}_j)dx = (A_j^2+B_j^2)\int_0^\sigma \frac{\rho A\alpha^2(x)}{\hat{b}(x)}\cos^2((z_j)_1\hat{x}+\hat{\vartheta}_j)d\hat{x}$$

$$\geq \int_0^\sigma \frac{\rho A\alpha^2(x)}{\hat{b}(x)}\cos^2((z_j)_1\hat{x}+\hat{\vartheta}_j)d\hat{x}$$

$$\geq \min(\frac{\rho A\alpha^2(x)}{\hat{b}(x)})\int_0^\sigma \cos^2((z_j)_1\hat{x}+\hat{\vartheta}_j)d\hat{x}$$

or

$$(A_j^2+B_j^2)\int_0^L \rho A\alpha^2(x)\cos^2((z_j)_1\hat{x}+\hat{\vartheta}_j)dx \geq \min(\frac{\rho A\alpha^2(x)}{\hat{b}(x)})\times (\frac{\sigma}{2}-\frac{1}{2(z_j)_1}). \tag{F.49}$$

Furthermore, the inequality

$$(A_j^2+B_j^2)\int_0^L \rho A\alpha^2(x)\cos^2((z_j)_1\hat{x}+\hat{\vartheta}_j)dx \leq 10\max(\frac{\rho A\alpha^2(x)}{\hat{b}(x)})\times (\frac{\sigma}{2}+\frac{1}{2(z_j)_1}). \tag{F.50}$$

can be obtained similarly from the use of (F.38), (F.44) and (F.48). Now

$$c_3 = \max(\rho A\alpha^2(x)/\hat{b}(x)) > 0 \quad \textit{and} \quad c_4 = \min(\rho A\alpha^2(x)/\hat{b}(x)) > 0. \tag{F.51}$$

Hence the inequalities (F.49) and (F.50) can be rewritten succinctly as

$$c_4\left(\frac{\sigma}{2}-\frac{1}{2(z_j)_1}\right) \le \frac{\int_0^L \rho A \alpha^2(x)\cos^2((z_j)_1\hat{x}+\hat{\vartheta}_j)dx}{(A_j^2+B_j^2)^{-1}} \le 10c_3\left(\frac{\sigma}{2}+\frac{1}{2(z_j)_1}\right). \tag{F.52}$$

By employing the inequalities

$$|\cos((z_j)_1\hat{x}+\hat{\vartheta}_j)| \le 1, \quad \rho A > 0 \quad \textit{and} \quad \alpha(x) > 0 \tag{F.53}$$

as well as (F.6) and (F.38) through (F.40), the following inequalities can be shown similarly

$$0 < C_j^2 \int_0^L \rho A(\alpha(x))^2 \exp(-2(z_j)_1\hat{x})dx \le \frac{9c_3}{2(z_j)_1} \tag{F.54}$$

$$0 < D_j^2 \int_0^L \rho A(\alpha(x))^2 \exp(-2(\sigma-\hat{x})(z_j)_1)dx \le \frac{9c_3}{2(z_j)_1} \tag{F.55}$$

$$-\frac{24c_3}{(z_j)_1} \le \frac{2C_j\int_0^L \rho A \alpha^2(x)\cos((z_j)_1\hat{x}-\hat{\vartheta}_j)\exp(-(z_j)_1\hat{x})dx}{(A_j^2+B_j^2)^{-1/2}} \le \frac{24c_3}{(z_j)_1} \tag{F.56}$$

$$-\frac{24c_3}{(z_j)_1} \le \frac{2D_j\int_0^L \rho A \alpha^2(x)\cos((z_j)_1\hat{x}-\hat{\vartheta}_j)\exp(-(\sigma-\hat{x})(z_j)_1)dx}{(A_j^2+B_j^2)^{-1/2}} \le \frac{24c_3}{(z_j)_1} \tag{F.57}$$

and

$$-\frac{18c_3\sigma}{\exp(\sigma(z_j)_1)} \le 2C_jD_j\int_0^L \rho A\alpha^2(x)\exp(-\sigma(z_j)_1)dx \le \frac{18c_3\sigma}{\exp(\sigma(z_j)_1)}. \quad (F.58)$$

Then, by employing (F.52) and the leftmost and rightmost inequalities of (F.54) through (F.58), integral (F.41) can be bounded immediately by

$$\int_0^L \rho A(w_{1j}^{(1)})^2dx \ge c_4(\frac{\sigma}{2}-\frac{1}{2(z_j)_1})-c_3(\frac{18\sigma}{\exp(\sigma(z_j)_1)}+\frac{48}{(z_j)_1}) \quad (F.59)$$

and

$$\int_0^L \rho A(w_{1j}^{(1)})^2dx \le 10c_3(\frac{\sigma}{2}+\frac{1}{2(z_j)_1})+c_3(\frac{18\sigma}{\exp(\sigma(z_j)_1)}+\frac{57}{(z_j)_1}). \quad (F.60)$$

Table 2.2 indicates that $(z_j)_1^{-1} \to 0$ and $\exp(-\sigma(z_j)_1) \to 0$ as $j \to \infty$. Thus, it is known [36] that there must exist a positive integer, j_1, such that,

$$0 < \frac{c_4\sigma}{2(z_j)_1}-c_3(\frac{18\sigma}{\exp(\sigma(z_j)_1)}+\frac{48}{(z_j)_1}) < \frac{c_4\sigma}{100} \quad (F.61)$$

and

$$0 < \frac{10c_3\sigma}{2(z_j)_1}+c_3(\frac{18\sigma}{\exp(\sigma(z_j)_1)}+\frac{57}{(z_j)_1}) < \frac{c_3\sigma}{100} \quad (F.62)$$

for two given c_3 and c_4 and $j > j_1$. By employing (F.61) and (F.62), the inequalities

$$c_4\left(\frac{\sigma}{2}-\frac{1}{2(z_j)_1}\right)-c_3\left(\frac{18\sigma}{\exp(\sigma(z_j)_1)}+\frac{48}{(z_j)_1}\right) > \frac{c_4\sigma}{2}-\frac{c_4\sigma}{100} > \frac{c_4\sigma}{3} \tag{F.63}$$

and

$$10c_3\left(\frac{\sigma}{2}+\frac{1}{2(z_j)_1}\right)+c_3\left(\frac{18\sigma}{\exp(\sigma(z_j)_1)}+\frac{57}{(z_j)_1}\right) < \frac{10c_3\sigma}{2}+\frac{c_3\sigma}{2} = \frac{11c_3\sigma}{2} \tag{F.64}$$

can be obtained because (1/2 - 1/100) > 1/3 . Combining (F.59) and (F.61) with (F.63) and (F.64) produces

$$\frac{c_4\sigma}{3} < \int_0^L \rho A(w_{1j}^{(1)})^2 dx < \frac{11c_3\sigma}{2} \tag{F.65}$$

for a sufficiently large j.

Now, by employing (F.65) and the identity

$$w_{1j}(x) \equiv w_{1j}(x)+w_{1j}^{(1)}(x)-w_{1j}^{(1)}(x) \quad \textit{because} \; w_{1j}^{(1)}(x)-w_{1j}^{(1)}(x) \equiv 0, \tag{F.66}$$

as well as the generic inequality $|a - b|^2 \leq 2(|a|^2 + |b|^2)$, where a and b are any two real values, it can be shown that

$$\begin{aligned}\int_0^L \rho A(w_{1j}(x))^2 dx &= \int_0^L \rho A(w_{1j}^{(1)}+w_{1j}-w_{1j}^{(1)})^2 dx \\ &\leq 2\int_0^L \rho A(w_{1j}^{(1)}(x))^2 dx + 2\int_0^L \rho A(w_{1j}-w_{1j}^{(1)})^2 dx.\end{aligned} \tag{F.67}$$

On the other hand, it is known from [34] that there must exist a positive constant, c, which is independent of j and such that

$$|w_{1j}(x)-w_{1j}^{(1)}(x)| \le \frac{c}{j}. \tag{F.68}$$

By employing (F.68), the inequality

$$\int_0^L \rho A(w_{1j}-w_{1j}^{(1)})^2 dx \le \int_0^L \rho A(\frac{c}{j})^2 dx \le \int_0^L \max(\rho A)(\frac{c}{j})^2 dx$$

or

$$\int_0^L \rho A(w_{1j}-w_{1j}^{(1)})^2 dx \le \max(\rho A)L(\frac{c}{j})^2 \tag{F.69}$$

can be found. When

$$j^2 > \frac{\max(\rho A)Lc^2}{c_3\sigma/100} \tag{F.70}$$

(F.67) can be simplified, by employing (F.65) and (F.69), to

$$\int_0^L \rho A(w_{1j}(x))^2 dx \le 2\frac{11c_3\sigma}{2}+2L\max(\rho A)(\frac{c}{j})^2 < 11c_3\sigma+\frac{c_3\sigma}{50}$$

or

$$\int_0^L \rho A(w_{1j}(x))^2 dx < 12c_3\sigma \tag{F.71}$$

for a sufficiently large j because $1 > 1/50$. On the other hand, by employing (F.66), it can be found that

$$\int_0^L \rho A (w_{1j}(x))^2 dx = \int_0^L \rho A (w_{1j} - w_{1j}^{(1)} + w_{1j}^{(1)})^2 dx$$

$$= \int_0^L \rho A (w_{1j}^{(1)}(x))^2 dx + 2\int_0^L \rho A w_{1j}^{(1)} (w_{1j} - w_{1j}^{(1)}) dx \qquad \text{(F.72)}$$

$$+ \int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2 dx.$$

Employing the generic inequality $|a - b| \geq |a| - |b|$ yields

$$\left| \int_0^L \rho A (w_{1j}(x))^2 dx \right| = \left| \int_0^L \rho A (w_{1j}^{(1)}(x))^2 dx + 2\int_0^L \rho A w_{1j}^{(1)} (w_{1j} - w_{1j}^{(1)}) dx \right.$$

$$\left. + \int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2 dx \right|$$

$$\geq \left| \int_0^L \rho A (w_{1j}^{(1)}(x))^2 dx + 2\int_0^L \rho A w_{1j}^{(1)} (w_{1j} - w_{1j}^{(1)}) dx \right|$$

$$- \left| \int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2 dx \right|$$

or

$$\int_0^L \rho A (w_{1j}(x))^2 dx \geq \int_0^L \rho A (w_{1j}^{(1)}(x))^2 dx - 2 \left| \int_0^L \rho A w_{1j}^{(1)} (w_{1j} - w_{1j}^{(1)}) dx \right| - \int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2 dx. \tag{F.73}$$

Furthermore, from (F.73) and Schwarz's inequality [42], viz

$$\left| \int_0^L f(x) g(x) dx \right| \leq \left(\int_0^L f^2(x)\, dx \,\middle|\right)^{1/2} \left(\int_0^L g^2(x)\, dx \,\middle|\right)^{1/2}, \tag{F.74}$$

the inequality

$$\int_0^L \rho A (w_{1j}(x))^2 dx \geq \int_0^L \rho A (w_{1j}^{(1)}(x))^2 dx - \int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2 dx$$

$$-2 \left(\int_0^L \rho A (w_{1j}^{(1)})^2 dx \right)^{1/2} \left(\int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2\, dx \right)^{1/2}$$

or, from (F.65),

$$\int_0^L \rho A (w_{1j}(x))^2 dx > \frac{c_4 \sigma}{3} - \int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2 dx - 2 \left(\frac{11 c_3 \sigma}{2} \right)^{1/2} \left(\int_0^L \rho A (w_{1j} - w_{1j}^{(1)})^2\, dx \right)^{1/2} \tag{F.75}$$

can be found. In addition to (F.70), if j also satisfies

$$j > \max\left(\frac{(100\max(\rho A)L)^{1/2}c}{(c_4\sigma)^{1/2}},\ 200\left(\frac{11c_3\sigma}{2}\right)^{1/2}\frac{(\max(\rho A)L)^{1/2}c}{c_4\sigma}\right), \quad \text{(F.76)}$$

it can be shown from (F.69) and (F.76) that

$$\int_0^L \rho A(w_{1j}-w_{1j}^{(1)})^2 dx \le \max(\rho A)L\left(\frac{c}{j}\right)^2 < \frac{c_4\sigma}{100} \quad \text{(F.77)}$$

and

$$2\left(\frac{11c_3\sigma}{2}\right)^{1/2}\left(\int_0^L \rho A(w_{1j}-w_{1j}^{(1)})^2 dx\right)^{1/2} \le 2\left(\frac{11c_3\sigma}{2}\right)^{1/2}(\max(\rho A)L)^{1/2}\frac{c}{j}$$

or

$$2\left(\frac{11c_3\sigma}{2}\right)^{1/2}\left(\int_0^L \rho A(w_{1j}-w_{1j}^{(1)})^2 dx\right)^{1/2} < \frac{c_4\sigma}{100}. \quad \text{(F.78)}$$

Thus, by using (F.77) and (F.78), (F.75) can be simplified to

$$\int_0^L \rho A(w_{1j}(x))^2 dx > \frac{c_4\sigma}{3}-\frac{c_4\sigma}{100}-\frac{c_4\sigma}{100} = \frac{c_4\sigma}{3}-\frac{c_4\sigma}{50} = \frac{47c_4\sigma}{150}$$

or

$$\int_0^L \rho A (w_{1j}(x))^2 dx > \frac{c_4 \sigma}{6} \tag{F.79}$$

because 47/150 > 1/6. By employing (F.2) and (F.79), the inequality

$$\begin{aligned}\| W_j \|^2_{H^{(5)}} &= \int_0^L \rho A (w_{1j})^2 dx + M_0 (w_{2j})^2 + J_0 (w_{3j})^2 + M_1 (w_{4j})^2 + J_1 (w_{5j})^2 \\ &\geq \int_0^L \rho A (w_{1j})^2 dx\end{aligned}$$

or

$$\| W_j \|^2_{H^{(5)}} > \frac{c_4 \sigma}{6} \tag{F.80}$$

can be found. On the other hand, Table 2.2 indicates that the four rightmost terms of equation (F.2) tend to zero as $j \to \infty$, i.e. there exists a positive j_2 such that

$$M_0 (w_{2j})^2 < \frac{c_3 \sigma}{4},\ J_0 (w_{3j})^2 < \frac{c_3 \sigma}{4},\ M_1 (w_{4j})^2 < \frac{c_3 \sigma}{4}\ and\ J_1 (w_{5j})^2 < \frac{c_3 \sigma}{4} \tag{F.81}$$

for $j > j_2$. Thus, from (F.71) and (F.81), the inequality

$$\begin{aligned}\| W_j \|^2_{H^{(5)}} &= \int_0^L \rho A (w_{1j})^2 dx + M_0 (w_{2j})^2 + J_0 (w_{3j})^2 + M_1 (w_{4j})^2 + J_1 (w_{5j})^2 \\ &< \int_0^L \rho A (w_{1j})^2 dx + 4 \frac{c_3 \sigma}{4}\end{aligned}$$

or

$$\| W_j \|^2_{H(S)} < 12c_3\sigma + 4\frac{c_3\sigma}{4} = 13c_3\sigma \tag{F.82}$$

can be obtained for a sufficiently large j. By taking

$$c_1 = (c_4\sigma/6)^{1/2} \quad \textit{and} \quad c_2 = (13c_3\sigma)^{1/2}, \tag{F.83}$$

the required inequality (F.1) is proved by combining (F.80) and (F.82).

APPENDIX G

This appendix presents the proof of Lemma 2.4.1.

Suppose $J_1 = 0$ for the beam shown in Figure 2.1. Then a Hilbert space having four-component vectors, $H^{(4)}$, is defined by

$$H^{(4)} = \mathcal{L}^2(\rho A,\ 0,\ L) \oplus \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \tag{G.1}$$

with the inner product given by

$$<F^*,\ G^*>_{H^{(4)}} = \left(\int_0^L \rho A f_1^* \overline{g_1^*}\, dx\right) + M_0 f_2^* \overline{g_2^*} + J_0 f_3^* \overline{g_3^*} + M_1 f_4^* \overline{g_4^*} \tag{G.2}$$

for two arbitrary vectors $F^* = (f_1^*, \ldots, f_4^*)$ and $G^* = (g_1^*, \ldots, g_4^*) \in H^{(4)}$. Furthermore, $\|F^*\|_{H^{(4)}} = (<F^*,\ F^*>_{H^{(4)}})^{1/2}$, $F^* \in H^{(4)}$, represents the norm of $H^{(4)}$. Then, the j th eigenvalue, λ_j^*, and corresponding eigenvector, $W_j^* = (w_{1j}^*, \ldots, w_{4j}^*)$ are determined by

$$\Pi^* W_j^* = \lambda_j^* W_j^* \tag{G.3}$$

where Π^* is a linear vector operator defined by

$$\Pi^* Y^* = (\tau_1 y_1^*, \tau_2 y_2^*, \tau_3 y_3^*, \tau_4 y_4^*) \tag{G.4}$$

with

$$e_1 (EIy_1^{*\prime\prime})' - K_1 (e_1 - \eta_1)(y_1^* + \eta_1 y_1^{*\prime}) + EI y_1^{*\prime\prime} + (\beta_1 - pe_1) y_1^{*\prime} |_{x=L} = 0 \tag{G.5}$$

for every $Y^* = (y_1^*, y_2^*, y_3^*, y_4^*) \in Dom(\Pi^*)$. Moreover, the y_i^*, $i \geq 2$, are defined in terms of $y_1^*(x)$ and its first derivative at $x_0 = 0$ and $x_1 = L$ by the equations labelled (2.2.4). The $Dom(\Pi^*)$ describes a domain of operator Π^*. On the other hand, operator Π^* can be proved, in a similar way to the proof given in Appendix B for operator Π, to be completely continuous, positive and self-adjoint in $H^{(4)}$. Consequently, the jth eigenvalue, λ_j^*, of Π^* can be characterized by the min-max principle (E.1) as

$$\lambda_j^* = \min_{V_j^* \in H^{(4)},\ \dim V_j^* = j} \ \max_{Y^* \in V_j^*} R(Y^*), \quad j = 1, 2, \ldots \tag{G.6}$$

where $R(Y^*)$ is the Rayleigh-quotient defined by

$$R(Y^*) = \frac{<Y^*, Y^*>_{E^{(4)}}}{<Y^*, Y^*>_{H^{(4)}}}. \tag{G.7}$$

Here $E^{(4)}$ is an energy space which is completed by the inner product

$$<F^*, G^*>_{E^{(4)}} = <F^*, \Pi^* G^*>_{H^{(4)}}, \quad F^*, G^* \in D(\Pi^*) \tag{G.8}$$

i.e.

$$<F^*, G^*>_{E^{(4)}} = \int_0^L EI f_1^{*''} \overline{g_1^{*''}} dx + \sum_{i=0}^{1} K_i (f_1^* - (-1)^i \eta_i f_1^{*\prime}) (\overline{g_1^*} - (-1)^i \eta_i \overline{g_1^{*\prime}}) |_{x=x_i}$$

$$+ \sum_{i=0}^{1} \beta_i f_1^{*\prime} \overline{g_1^{*\prime}} |_{x=x_i}. \tag{G.9}$$

It can be found from (2.2.4) that, for an arbitrary vector $F^* = (f_1^*, \ldots, f_4^*) \in E^{(4)}$, there exists a unique vector $F = (f_1, \ldots, f_4, f_5) \in E^{(5)}$ in which $f_1 = f_1^*, f_2 = f_2^*, f_3 = f_3^*, f_4 = f_4^*$ and $f_5 = f_1^{*\prime}(L)$. On the other hand, for an arbitrary $F = (f_1, \ldots, f_4, f_5) \in E^{(5)}$, the vector $F^* = (f_1^*, \ldots, f_4^*) \in E^{(4)}$ can be determined uniquely by $f_1^* = f_1, f_2^* = f_2, f_3^* = f_3, f_4^* = f_4$. Consequently, it can be seen from (E.1) that the jth eigenvalue, λ_j, of a beam having a non-zero J_1 can be rewritten as

$$\lambda_j = \min_{V_j \in H^{(4)},\ \dim V_j = j} \quad \max_{F \in V_j} \quad R(F) \tag{G.10}$$

where R(*F*) is given by (E.2)

$$\lambda_j = \min_{V_j^* \in H^{(4)},\ \dim V_j^* = j} \quad \max_{F^* \in V_*^*} \quad \frac{<F^*,\ \Pi^* F^*>_{H(4)}}{<F^*,\ F^*>_{H(4)} + J_1 (f_5)^2}. \tag{G.11}$$

It can be seen from (G.6), (G.7), (G.10) and (G.11) that the jth eigenvalue, λ_j, is a perturbation of λ_j^* due to a non-zero $(J_1 f_5^2)$. If the term $(J_1 f_5^2)$ is considered equivalent to a numerical error in the Rayleigh-Ritz-Galerkin procedure, the convergence analysis presented in [37] demonstrates that

$$\| W_{j,J_1}^* - W_j^* \|_{E(4)} \to 0, \quad as \quad J_1 \to 0 \tag{G.12}$$

where W_{j,J_1}^* denotes a four-component vector obtained by eliminating the fifth component of $W_j = (w_{1j}, ..., w_{5j})$. W_j is the jth eigenvector of the beam shown in Figure 2.1 and it has a non-zero J_1. Thus, it can be shown from (G.9) and (G.12) that the following limits hold

$$\lim_{J_1 \to 0} \int_0^L \left| \frac{d^k w_{1j}}{dx^k} - \frac{d^k w_{1j}^*}{dx^k} \right|^2 dx = 0, \quad k = 0,\ 1,\ 2 \tag{G.13}$$

and

$$\lim_{J_1 \to 0} | w_{ij} - w_{ij}^* | = 0, \quad i = 2,\ 3,\ 4,\ 5. \tag{G.14}$$

By repeatedly employing Schwarz's inequality i.e.

$$|\int_0^L f(x)g(x)dx| \le (\int_0^L |f(x)|^2 dx)^{1/2} (\int_0^L |g(x)|^2 dx)^{1/2}, \tag{G.15}$$

and

$$w_{1j}(x) - w_{1j}^*(x) = \int_0^x (w'_{1j}(x) - w_{1j}^{*\prime}(x))dx + (w_{1j}(0) - w_{1j}^*(0)) \tag{G.16}$$

the following two inequalities can be obtained straightforwardly

$$\lim_{J_1 \to 0} |w_{1j}(x) - w_{1j}^*(x)| \le \lim_{J_1 \to 0} L^{1/2} \int_0^L (w_{1j}{}'(x) - w_{1j}^*{}'(x))^2 dx)^{1/2}$$

$$+ \lim_{J_1 \to 0} |w_{1j}(0) - w_{1j}^*(0)|$$

i.e.

$$\lim_{J_1 \to 0} |w_{1j}(x) - w_{1j}^*(x)| = 0. \tag{G.17}$$

Similarly, the inequality

$$\lim_{J_1 \to 0} |w'_{1j}(x) - w_{1j}^{*\prime}(x)| \le \lim_{J_1 \to 0} L^{1/2} \int_0^L (w''_{1j}(x) - w_{1j}^{*\prime\prime}(x))^2 dx)^{1/2}$$

$$+ \lim_{J_1 \to 0} |w'_{1j}(0) - w_{1j}^{*\prime}(0)|$$

i.e.

$$\lim_{J_1 \to 0} |w'_{1j}(x) - w_{1j}^{*\prime}(x)| = 0 \tag{G.18}$$

can be obtained. Limits (G.17) and (G.18) illustrate that Lemma 2.4.1 holds as the parameter J_1 tends to zero. A similar procedure can also be used for parameters like M_0, M_1 and J_0.

APPENDIX H

This appendix presents the proof of Theorem 3.2.1. The notation of section 3.2 is also used in this appendix. It is well known that a boundary value problem has two forms [53]. One is called a strong form which consists of a differential equation as well as interior and end conditions. The other is a weak form, i.e. a variational equation. The corresponding solutions are called the classical solution and the weak solution, respectively. These two solutions are identical when strong and weak forms both have unique solutions [53]. Moreover, the regularity, i.e. continuities or discontinuities, of a weak solution is equivalent then to that of the classical solution. On the other hand, the uniqueness of the classical solution can be demonstrated by employing a Green's function. this approach gives the classical solution of the eigenvalue problem, described in (3.2.2) through (3.2.5), in the integral form

$$w_j(x) = \lambda_j\Big(\int_0^L G(x,\xi)\,w_j(x)\,dx + \sum_{r=0}^{N}\big(G_{1r}(x)M_0\,w_j(x_r) + G_{2r}(x)J_1\,w_j'(x_r)\big)\Big). \qquad \text{(H.1)}$$

$G_{ir}(x)$, $i = 1, 2$, is defined later in this section. $G(x,\xi)$, conversely, is the Green's function of the multiple-point boundary value problem

$$\mathcal{L}[y] \equiv \frac{d^2}{dx^2}\Big(EI(x)\frac{d^2y(x)}{dx^2}\Big) - \frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big) + k_e(x)y(x) = f(x), \qquad \text{(H.2)}$$

$$x_{r-1} < x < x_r, \quad r = 1, \ldots, N$$

with the end conditions at $x = 0$ and $x = L$ given, respectively, by

$$
\left.
\begin{aligned}
U_{10}[y] &\equiv K_0 y - p y' + (EIy'')' \,|_{x=0} = 0 \\
U_{20}[y] &\equiv -EIy'' + \beta_0 y' \,|_{x=0} = 0
\end{aligned}
\right\}
\qquad \text{(H.3)}
$$

and

$$
\left.
\begin{aligned}
U_{1N}[y] &\equiv K_N y + p y' - (EIy'')' \,|_{x=L} = 0 \\
U_{2N}[y] &\equiv EIy'' + \beta_N y' \,|_{x=L} = 0\,.
\end{aligned}
\right\}
\qquad \text{(H.4)}
$$

The interior conditions at $x = x_r$ are given by

$$
\left.
\begin{aligned}
&\mathfrak{C}_{1r}^{-}[y] \equiv y(x_r^-) = y(x_r^+) \equiv \mathfrak{C}_{1r}^{+}[y]\,, \quad \mathfrak{C}_{2r}^{-}[y] \equiv y'(x_r^-) = y'(x_r^+) \equiv \mathfrak{C}_{2r}^{+}[y] \\
&U_{1r}^{-}[y] \equiv -(EIy'')' + K_1 y + p y' \,|_{x=x_r^-} = -(EIy'')' + p y' \,|_{x=x_r^+} \equiv U_{1r}^{+} \\
&U_{2r}[y] \equiv EIy'' + \beta_r y' \,|_{x=x_r^-} = EIy'' \,|_{x=x_r^+} \equiv U_{2r}^{+} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r = 1,\, 2,\, \ldots,\, N-1\,.
\end{aligned}
\right\}
\qquad \text{(H.5)}
$$

The negative and positive superscripts indicate limiting values as x approaches x_r from the left and right, respectively. On the other hand, $G_{1r}(x)$ and $G_{2r}(x)$ are the respective solutions of

$$
\left.
\begin{aligned}
&\mathfrak{L}[G_{1r}] = 0\\
&\mathfrak{C}_{1i}^{-}[G_{1r}] - \mathfrak{C}_{1i}^{+}[G_{1r}] = \mathfrak{C}_{2i}^{-}[G_{1r}] - \mathfrak{C}_{2i}^{+}[G_{1r}] = U_{2i}^{-}[G_{1r}]\\
&\qquad\qquad - U_{2i}^{+}[G_{1r}] = 0, \quad i = 1, 2, \ldots, N-1\\
&U_{20}[G_{1r}] = U_{2N}[G_{1r}] = 0,\\
&U_{10}[G(_{1r}] = 1, \quad U_{1N}[G_{1r}] = U_{1i}^{-}[G_{1r}]\\
&\qquad - U_{1i}^{+}[G_{1r}] = 0, \quad \mathit{for} \quad r = 0, \quad i = 1, 2, \ldots, N-1\\
&U_{1N}[G_{1r}] = 1, \quad U_{10}[G_{1r}] = U_{1i}^{-}[G_{1r}]\\
&\qquad - U_{1i}^{+}[G_{1r}] = 0, \quad \mathit{for} \quad r = N, \quad i = 1, 2, \ldots, N-1\\
&U_{1r}^{-}[G_{1r}] - U_{1r}^{+}[G_{1r}] = 1, \; U_{10}[G_{1r}] = U_{1N}[G_{1r}] = U_{1i}^{-}[G_{1r}]\\
&\qquad - U_{1i}^{+}[G_{1r}] = 0, \quad \mathit{for} \quad 1 \le r \le N-1 \quad \mathit{and} \quad \mathit{all} \quad i \ne r
\end{aligned}
\right\} \qquad \text{(H.6)}
$$

and

$$
\left.
\begin{aligned}
&\mathfrak{L}[G_{2r}] = 0\\
&\mathfrak{C}_{1i}^{-}[G_{2r}] - \mathfrak{C}_{1i}^{+}[G_{2r}] = \mathfrak{C}_{2i}^{-}[G_{2r}] - \mathfrak{C}_{2i}^{+}[G_{2r}] = U_{1i}^{-}[G_{2r}]\\
&\qquad\qquad - U_{1i}^{+}[G_{2r}] = 0, \quad i = 1, 2, \ldots, N-1\\
&U_{10}[G_{2r}] = U_{1N}[G_{2r}] = 0,\\
&U_{20}[G_{2r}] = 1, \quad U_{2N}[G_{2r}] = U_{2i}^{-}[G_{2r}] - U_{2i}^{+}[G_{2r}] = 0,\\
&\qquad\qquad \mathit{for} \quad r = 0, \quad i = 1, 2, \ldots, N-1\\
&U_{2N}[G_{2r}] = 1, \quad U_{20}[G_{2r}] = U_{2i}^{-}[G_{2r}] - U_{2i}^{+}[G_{2r}] = 0,\\
&\qquad\qquad \mathit{for} \quad r = N, \quad i = 1, 2, \ldots, N-1\\
&U_{2r}^{-}[G_{2r}] - U_{2r}^{+}[G_{2r}] = 1, \; U_{20}[G_{2r}] = U_{2N}[G_{2r}] = U_{2i}^{-}[G_{2r}]\\
&\qquad - U_{2i}^{+}[G_{2r}] = 0, \quad \mathit{for} \quad 1 \le r \le N-1 \quad \mathit{and} \quad \mathit{all} \quad i \ne r.
\end{aligned}
\right\} \qquad \text{(H.7)}
$$

The Green's function, $G(x, \xi)$, is constructed to satisfy the following requirements.

(1) $G(x,\xi)$ is regarded as a function of x for a fixed ξ. It satisfies the homogeneous differential equation $\mathfrak{L}[G] = 0$ for all x except $x = \xi$ and $x = x_r$, $r = 0, 1, \ldots, N$. Moreover, it also satisfies the end and interior conditions (H.3) through (H.5).

(2) $G(x,\xi)$ and $\partial G(x,\xi)/\partial x$ are continuous in the square defined by $0 \le x, \xi \le L$.

(3) The $\partial^\nu G(x,\xi)/\partial x^\nu$, $\nu = 2, 3, 4$, are continuous in

$$0 \le x \le x_1\,,\ \ldots,\ x_{r_0-1} \le x < \xi \le x_{r_0}\,,\ \ldots,\ x_{N-1} \le x \le x_N \qquad \text{(H.8)}$$

if ξ satisfies $x_{r_0-1} \le \xi \le x_{r_0}$ for a given positive integer, r_0, such that $1 \le r_0 \le N$. The partial derivatives of $G(x,\xi)$ at $x = x_r$, $r = 0, 1, \ldots, N$ should be considered as left partial derivatives when $x < x_r$ or right partial derivatives when $x > x_r$. Furthermore, $\partial^2 G(x,\xi)/\partial x^2$ is continuous in the square defined by $x_{r_0-1} \le x, \xi \le x_{r_0}$.

(4) The following equalities hold for $x_{r-1} < x < x_r$, $r = 1, \ldots, N$:

$$\frac{\partial^3 G(\xi^+,\xi)}{\partial x^3} - \frac{\partial^3 G(\xi^-,\xi)}{\partial x^3} = \frac{1}{EI(\xi)},\ \xi \ne x_r, \qquad \text{(H.9)}$$
$$r = 0, 1, \ldots, N$$

and

$$\left.\begin{aligned} \frac{\partial^3 G(\xi^+,\xi)}{\partial x^3} &= \frac{\partial^3 G(\xi,\xi^-)}{\partial x^3} \\ \frac{\partial^3 G(\xi^-,\xi)}{\partial x^3} &= \frac{\partial^3 G(\xi,\xi^+)}{\partial x^3} \\ \xi \ne x_r,\quad & r = 0, 1, \ldots, N. \end{aligned}\right\} \qquad \text{(H.10)}$$

The following lemma is useful to verify the existence of this Green's function.

Lemma H.1. The analytical solution of the problem defined by equations (H.2) through (H.5) is identically zero when $f(x) = 0$.

Proof

Suppose $\phi_{1r}(x), \ldots, \phi_{4r}(x)$ represent the four independent solutions of equation (H.6) when $f(x) = 0$ in the interval $x_{r-1} \le x \le x_r$, $r = 1, 2, \ldots, N$. Then the corresponding solution of equation (H.2) can be expressed by

$$y(x) = \sum_{i=0}^{4} a_{ir}\phi_{ir}(x), \quad x_{r-1} \le x \le x_r, \quad r = 1, 2, \ldots, N. \tag{H.11}$$

Substituting (H.11) into the end and interior conditions (H.3) and (H.5) yields

$$\sum_{i=1}^{4} a_{i1} U_{10}[\phi_{i1}] = 0 \tag{H.12}$$

$$\sum_{i=1}^{4} a_{i1} U_{20}[\phi_{i1}] = 0 \tag{H.13}$$

$$\sum_{i=1}^{4} a_{i1} \mathfrak{C}_{11}^{-}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} \mathfrak{C}_{11}^{+}[\phi_{i2}] = 0 \tag{H.14}$$

$$\sum_{i=1}^{4} a_{i1} \mathfrak{C}_{21}^{-}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} \mathfrak{C}_{21}^{+}[\phi_{i2}] = 0 \tag{H.15}$$

$$\sum_{i=1}^{4} a_{i1} U^{-}_{11}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} U^{+}_{11}[\phi_{i2}] = 0 \tag{H.16}$$

$$\sum_{i=1}^{4} a_{i1} U^{-}_{21}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} U^{+}_{21}[\phi_{i2}] = 0 \tag{H.17}$$

$$\vdots$$

$$\sum_{i=1}^{4} a_{i(N-1)} U^{-}_{1(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} U^{+}_{1(N-1)}[\phi_{iN}] = 0 \tag{H.18}$$

$$\sum_{i=1}^{4} a_{i(N-1)} U^{-}_{2(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} U^{+}_{2(N-1)}[\phi_{iN}] = 0 \tag{H.19}$$

$$\sum_{i=1}^{4} a_{i(N-1)} U^{-}_{1(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} U^{+}_{1(N-1)}[\phi_{iN}] = 0 \tag{H.20}$$

$$\sum_{i=1}^{4} a_{i(N-1)} U^{-}_{2(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} U^{+}_{2(N-1}[\phi_{iN}] = 0 \tag{H.21}$$

$$\sum_{i=1}^{4} a_{iN} U_{1N}[\phi_{iN}] = 0 \tag{H.22}$$

and

$$\sum_{i=1}^{4} a_{iN} U_{2N}[\phi_{iN}] = 0. \tag{H.23}$$

Equations (H.12) through (H.23) can be expressed compactly in the matrix form

$$[\Delta]\boldsymbol{a} = 0 \tag{H.24}$$

with

$$\boldsymbol{a} = (a_{11}, \ldots, a_{41}, \ldots, a_{1N}, \ldots, a_{4N})^T. \tag{H.25}$$

Here $[\Delta]$ is a $4N \times 4N$ matrix which consists of the coefficients a_{ir} that are used in equations (H.12) through (H.23). Suppose that the determinant of matrix $[\Delta]$ is zero. Then it is well-known [49] that there exists a set of non-zero coefficients, a_{ir}, such that $y(x)$ is non-zero and satisfies equations (H.2) through (H.5) for $f(x) = 0$. On the other hand, the corresponding variational form of equation (H.2) through (H.5) can be written as

$$B(y, u) = 0 \tag{H.26}$$

for all $u \in B$. In particular, $B(y, y) = 0$ so that $y(x) = 0$ in $0 \le x \le L$ because $B(u, v)$ is an inner product of space B. This conclusion is contrary to the assumption of a non-zero $y(x)$. Hence the determinant of $[\Delta]$ must be non-zero. Consequently, the corresponding analytical solution equals zero.

This completes the proof of Lemma H.1.

By employing Lemma H.1, the existence of the previously described Green's function can be verified straightforwardly. Let I_r denote the open sub-interval: $x_r < x < x_{r+1}$.

Suppose r_0, r_1 and r_2 are three, given positive integers satisfying $1 \leq r_0, r_1, r_2 \leq N$. Then, by following the procedure used in [49], $G(x,\xi)$ can be expressed as

$$G(x,\xi) = \begin{cases} \sum_{i=1}^{4} a_{ir}(\xi)\phi_{ir}(x), & x \in I_{r_1}, \xi \in I_{r_2}, r_1 \neq r_2, \\ \sum_{i=1}^{4} (a_{ir_0}(\xi) + b_{ir_0}(\xi))\phi_{ir_0}(x), & x_{r_0} < x < \xi < x_{r_0+1}, \\ \sum_{i=1}^{4} (a_{ir_0}(\xi) - b_{ir_0}(\xi))\phi_{ir_0}(x), & x_{r_0} < \xi < x < x_{r_0+1}. \end{cases} \tag{H.27}$$

The functions b_{ir_0} are determined by the continuities of $G(x,\xi)$ for $x_{r_0-1} \leq x, \xi \leq x_{r_0}$ as well as those of the first and second partial derivatives of $G(x,\xi)$ with respect to x, i.e. from (I.27),

$$\sum_{i=1}^{4} b_{ir_0} \frac{d^{\nu}\phi_{ir_0}(\xi)}{dx^{\nu}} = 0 \quad for \;\; \nu = 0, 1, 2, \tag{H.28}$$

in addition to the jump condition (H.9), viz

$$\sum_{i=1}^{4} b_{ir_0} \frac{d^3\phi_{ir_0}(\xi)}{dx^3} = -\frac{1}{2EI(x)}. \tag{H.29}$$

$\phi_{1r_0}(x)$, ..., $\phi_{4r_0}(x)$ represent the four independent solutions of equation (H.6) so that the $b_{ir_0}(\xi)$, $i = 1, ..., 4$ can be determined uniquely from equations (H.28) and (H.29). Furthermore, the $b_{ir_0}(\xi)$, $i = 1, ..., 4$ are independent of the end and interior conditions. To determine the $a_{ir}(\xi)$, substitute $G(x,\xi)$ into the end and interior conditions (H.3) and

(H.5) to yield

$$\sum_{i=1}^{4} a_{i1} U_{10}[\phi_{i1}] = 0 \tag{H.30}$$

$$\sum_{i=1}^{4} a_{i1} U_{20}[\phi_{i1}] = 0 \tag{H.31}$$

$$\sum_{i=1}^{4} a_{i1} \mathfrak{C}^{-}_{11}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} \mathfrak{C}^{+}_{11}[\phi_{i2}] = 0 \tag{H.32}$$

$$\sum_{i=1}^{4} a_{i1} \mathfrak{C}^{-}_{21}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} \mathfrak{C}^{+}_{21}[\phi_{i2}] = 0 \tag{H.33}$$

$$\sum_{i=1}^{4} a_{i1} U^{-}_{11}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} U^{+}_{11}[\phi_{i2}] = 0 \tag{H.34}$$

$$\sum_{i=1}^{4} a_{i1} U^{-}_{12}[\phi_{i1}] - \sum_{i=1}^{4} a_{i2} U^{+}_{12}[\phi_{i2}] = 0 \tag{H.35}$$

$$\vdots$$

$$\sum_{i=1}^{4} a_{i(r_0-1)} \mathfrak{C}^{-}_{1(r_0-1)}[\phi_{i(r_0-1)}] - \sum_{i=1}^{4} a_{ir_0} \mathfrak{C}^{+}_{1(r_0-1)}[\phi_{ir_0}] = \sum_{i=1}^{4} b_{ir_0} \mathfrak{C}^{+}_{1(r_0-1)}[\phi_{ir_0}] \tag{H.36}$$

$$\sum_{i=1}^{4} a_{i(r_0-1)} \mathfrak{C}^{-}_{2(r_0-1)}[\phi_{i(r_0-1)}] - \sum_{i=1}^{4} a_{ir_0} \mathfrak{C}^{+}_{2(r_0-1)}[\phi_{ir_0}] = \sum_{i=1}^{4} b_{ir_0} \mathfrak{C}^{+}_{2(r_0-1)}[\phi_{ir_0}] \tag{H.37}$$

$$\sum_{i=1}^{4} a_{i(r_0-1)} U^{-}_{1(r_0-1)}[\phi_{i(r_0-1)}] - \sum_{i=1}^{4} a_{ir_0} U^{+}_{1(r_0-1)}[\phi_{ir_0}] = \sum_{i=1}^{4} b_{ir_0} U^{+}_{1(r_0-1)}[\phi_{ir_0}] \tag{H.38}$$

$$\sum_{i=1}^{4} a_{i(r_0-1)} U^{-}_{2(r_0-1)}[\phi_{i(r_0-1)}] - \sum_{i=1}^{4} a_{ir_0} U^{+}_{2(r_0-1)}[\phi_{ir_0}] = \sum_{i=1}^{4} b_{ir_0} U^{+}_{2(r_0-1)}[\phi_{ir_0}] \tag{H.39}$$

$$\sum_{i=1}^{4} a_{ir_0} \mathfrak{C}^{-}_{1r_0}[\phi_{ir_0}] - \sum_{i=1}^{4} a_{i(r_0+1)} \mathfrak{C}^{+}_{1r_0}[\phi_{i(r_0+1)}] = \sum_{i=1}^{4} b_{ir_0} \mathfrak{C}^{-}_{1r_0}[\phi_{ir_0}] \tag{H.40}$$

$$\sum_{i=1}^{4} a_{ir_0} \mathfrak{C}^{-}_{2r_0}[\phi_{ir_0}] - \sum_{i=1}^{4} a_{i(r_0+1)} \mathfrak{C}^{+}_{2r_0}[\phi_{i(r_0+1)}] = \sum_{i=1}^{4} b_{ir_0} \mathfrak{C}^{-}_{2r_0}[\phi_{ir_0}] \tag{H.41}$$

$$\sum_{i=1}^{4} a_{ir_0} U^{-}_{1r_0}[\phi_{ir_0}] - \sum_{i=1}^{4} a_{i(r_0+1)} U^{+}_{1r_0}[\phi_{i(r_0+1)}] = \sum_{i=1}^{4} b_{ir_0} U^{-}_{1r_0}[\phi_{ir_0}] \tag{H.42}$$

$$\sum_{i=1}^{4} a_{ir_0} U^{-}_{2r_0}[\phi_{ir_0}] - \sum_{i=1}^{4} a_{i(r_0+1)} U^{+}_{2r_0}[\phi_{i(r_0+1)}] = \sum_{i=1}^{4} b_{ir_0} U^{-}_{2r_0}[\phi_{ir_0}] \tag{H.43}$$

$\vdots$

$$\sum_{i=1}^{4} a_{i(N-1)} \mathfrak{C}^{-}_{1(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} \mathfrak{C}^{+}_{1(N-1)}[\phi_{iN}] = 0 \tag{H.44}$$

$$\sum_{i=1}^{4} a_{i(N-1)} \mathfrak{C}^{-}_{2(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} \mathfrak{C}^{+}_{2(N-1)3}[\phi_{iN}] = 0 \tag{H.45}$$

$$\sum_{i=1}^{4} a_{i(N-1)} U^{-}_{1(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} U^{+}_{1(N-1)}[\phi_{iN}] = 0 \tag{H.46}$$

$$\sum_{i=1}^{4} a_{i(N-1)} U^{-}_{2(N-1)}[\phi_{i(N-1)}] - \sum_{i=1}^{4} a_{iN} U^{+}_{2(N-1)}[\phi_{iN}] = 0 \tag{H.47}$$

$$\sum_{i=1}^{4} a_{iN} U_{1N}[\phi_{iN}] = 0 \tag{H.48}$$

and

$$\sum_{i=1}^{4} a_{iN} U_{2N}[\phi_{iN}] = 0 . \tag{H.49}$$

It can be observed that the coefficients a_{ir} in equations (H.30) through (H.49) are the same as those given by equations (H.12) through (H.23). Lemma H.1 demonstrated that the determinant of the matrix of the coefficients is non-zero. Therefore, $a_{ir}(\xi)$, $i = 1, \ldots, 4$ and $r = 1, \ldots, N$ are determined uniquely. This completes the proof of the existence of $G(x, \xi)$. The existence of $G_{1r}(x)$ and $G_{2r}(x)$ can be shown analogously.

It is needed to prove next that the function

$$y(x) = \sum_{r=0}^{N-1} \int_{x_r}^{x_{r+1}} G(x,\xi) f(\xi)\, d\xi + \sum_{r=0}^{N} \left(G_{1r}(x) M_r f(x_r) + G_{2r}(x) J_r f'(x_r)\right) \tag{H.50}$$

is the solution of the following, multiple-point boundary value problem :

$$\mathcal{L}[y] \equiv f(x)\,, \quad x_{r-1} < x < x_r\,, \quad r = 1, \ldots, N \tag{H.51}$$

$$U_{10}[y] \equiv M_0 f(0)\,,\ U_{20}[y] \equiv J_0 f'(0)\,,\ U_{1N}[y] \equiv M_N f(L)\,,\ U_{2N}[y] \equiv J_N f'(L) \tag{H.52}$$

and

$$\left.\begin{aligned} \mathcal{C}^-_{1r}[y] = \mathcal{C}^+_{1r}[y]\,,\quad \mathcal{C}^-_{2r}[y] = \mathcal{C}^+_{2r}[y]\,,\quad U^-_{1r}[y] - U^+_{1r}[y] = M_r f(x_r) \\ U_{2r}[y] - U^+_{2r}[y] = J_r f'(x_r)\,,\quad r = 1, 2, \ldots, N-1\,. \end{aligned}\right\} \tag{H.53}$$

Suppose $x \in I_{r_0}$ for an arbitrary, given positive integer, r_0, satisfying $1 \le r_0 \le N$. Then (H.50) can be rewritten as

$$\begin{aligned} y(x) = {} & \sum_{r=0}^{r_0-1} \int_{x_r}^{x_{r+1}} G(x,\xi) f(\xi)\, d\xi + \int_{x_{r_0}}^{x} G(x,\xi) f(\xi)\, d\xi + \int_{x}^{x_{r_0+1}} G(x,\xi) f(\xi)\, d\xi \\ & + \sum_{r=r_0+1}^{N-1} \int_{x_r}^{x_{r+1}} G(x,\xi) f(\xi)\, d\xi + \sum_{r=0}^{N} \left(G_{1r}(x) M_r f(x_r) + G_{2r}(x) J_r f'(x_r)\right). \end{aligned} \tag{H.54}$$

Using Leibnitz's rule for differentiation, the above equation becomes

$$\frac{dy(x)}{dx} = \sum_{r=0}^{r_0-1} \int_{x_r}^{x_{r+1}} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi + \int_{x_{r_0}}^{x} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi + G(x, x^-) f(x^-)$$
$$+ \int_{x}^{x_{r_0+1}} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi - G(x, x^+) f(x^+) + \sum_{r=r_0+1}^{N-1} \int_{x_r}^{x_{r+1}} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi$$
$$+ \sum_{r=0}^{N} \left(\frac{dG_{1r}(x)}{dx} M_r f(x_r) + \frac{dG_{2r}(x)}{dx} J_r f'(x_r)\right)$$

or

$$\frac{dy(x)}{dx} = \sum_{r=0}^{r_0-1} \int_{x_r}^{x_{r+1}} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi + \int_{x_{r_0}}^{x} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi$$
$$+ \int_{x}^{x_{r_0+1}} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi + \sum_{r=r_0+1}^{N-1} \int_{x_r}^{x_{r+1}} \frac{\partial G(x,\xi)}{\partial x} f(\xi) d\xi \tag{H.55}$$
$$+ \sum_{r=0}^{N} \left(\frac{dG_{1r}(x)}{dx} M_r f(x_r) + \frac{dG_{2r}(x)}{dx} J_r f'(x_r)\right)$$

whilst

$$\frac{d^2y(x)}{dx^2} = \sum_{r=0}^{r_0-1} \int_{x_r}^{x_{r+1}} \frac{\partial^2 G(x,\xi)}{\partial x^2} f(\xi) d\xi + \int_{x_{r_0}}^{x} \frac{\partial^2 G(x,\xi)}{\partial x^2} f(\xi) d\xi + \frac{\partial G(x, x^-)}{\partial x} f(x^-)$$
$$+ \int_{x}^{x_{r_0+1}} \frac{\partial^2 G(x,\xi)}{\partial x^2} f(\xi) d\xi - \frac{\partial G(x, x^+)}{\partial x} f(x^+) + \sum_{r=r_0+1}^{N-1} \int_{x_r}^{x_{r+1}} \frac{\partial^2 G(x,\xi)}{\partial x^2} f(\xi) d\xi$$
$$+ \sum_{r=0}^{N} \left(\frac{d^2 G_{1r}(x)}{dx^2} M_r f(x_r) + \frac{d^2 G_{2r}(x)}{dx^2} J_r f'(x_r)\right)$$

or

$$\frac{d^2y(x)}{dx^2} = \sum_{r=0}^{r_0-1}\int_{x_r}^{x_{r+1}}\frac{\partial^2 G(x,\xi)}{\partial x^2}f(\xi)d\xi + \int_{x_{r_0}}^{x}\frac{\partial^2 G(x,\xi)}{\partial x^2}f(\xi)d\xi$$
$$+\int_{x}^{x_{r_0+1}}\frac{\partial^2 G(x,\xi)}{\partial x^2}f(\xi)d\xi + \sum_{r=r_0+1}^{N-1}\int_{x_r}^{x_{r+1}}\frac{\partial^2 G(x,\xi)}{\partial x^2}f(\xi)d\xi \tag{H.56}$$
$$+\sum_{r=0}^{N}\left(\frac{d^2G_{1r}(x)}{dx^2}M_r f(x_r) + \frac{d^2G_{2r}(x)}{dx^2}J_r f'(x_r)\right)$$

whereas

$$\frac{d^3y(x)}{dx^3}) = \sum_{r=0}^{r_0-1}\int_{x_r}^{x_{r+1}}\frac{\partial^3 G(x,\xi)}{\partial x^3}f(\xi)d\xi + \int_{x_{r_0}}^{x}\frac{\partial^3 G(x,\xi)}{\partial x^3}f(\xi)d\xi$$
$$+\frac{\partial^2 G(x,\, x^-)}{\partial x^2}f(x^-) + \int_{x}^{x_{r_0+1}}\frac{\partial^3 G(x,\xi)}{\partial x^3}f(\xi)d\xi$$
$$-\frac{\partial^2 G(x,\, x^+)}{\partial x^2}f(x^+) + \sum_{r=r_0+1}^{N-1}\int_{x_r}^{x_{r+1}}\frac{\partial^3 G(x,\xi)}{\partial x^3}f(\xi)d\xi$$
$$+\sum_{r=0}^{N}\left(\frac{d^3G_{1r}(x)}{dx^3}M_r f(x_r) + \frac{d^3G_{2r}(x)}{dx^3}J_r f'(x_r)\right.$$

or

$$\frac{d^3y(x)}{dx^3} = \sum_{r=0}^{r_0-1}\int_{x_r}^{x_{r+1}}\frac{\partial^3G(x,\xi)}{\partial x^3}f(\xi)d\xi + \int_{x_{r_0}}^{x}\frac{\partial^3G(x,\xi)}{\partial x^3}f(\xi)d\xi$$

$$+\int_{x}^{x_{r_0+1}}\frac{\partial^3G(x,\xi)}{\partial x^3}f(\xi)d\xi + \sum_{r=r_0+1}^{N-1}\int_{x_r}^{x_{r+1}}\frac{\partial^3G(x,\xi)}{\partial x^3}f(\xi)d\xi \tag{H.57}$$

$$+\sum_{r=0}^{N}\left(\frac{d^3G_{1r}(x)}{dx^3}M_rf(x_r)+\frac{d^3G_{2r}(x)}{dx^3}J_rf'(x_r)\right)$$

and

$$\frac{d^4y(x)}{dx^4} = \sum_{r=0}^{r_0-1}\int_{x_r}^{x_{r+1}}\frac{\partial^4G(x,\xi)}{\partial x^4})f(\xi)d\xi + \int_{x_{r_0}}^{x}\frac{\partial^4G(x,\xi)}{\partial x^4}f(\xi)d\xi$$

$$+\frac{\partial^3G(x,\,x^-)}{\partial x^3}f(x^-)+\int_{x}^{x_{r_0+1}}\frac{\partial^4G(x,\xi)}{\partial x^4}f(\xi)d\xi \tag{H.58}$$

$$-\frac{\partial^3G(x,\,x^+)}{\partial x^3})f(x^+)+\sum_{r=r_0+1}^{N-1}\int_{x_r}^{x_{r+1}}\frac{\partial^3G(x,\xi)}{\partial x^3})f(\xi)d\xi$$

$$+\sum_{r=0}^{N}\left(\frac{d^4G_{1r}(x)}{dx^4}M_rf(x_r)+\frac{d^4G_{2r}(x)}{dx^4}J_rf'(x_r)\right).$$

By employing equations (H.9) and (H.10), the last equation becomes

$$\frac{d^4y(x)}{dx^4}) = \sum_{r=0}^{r_0-1} \int_{x_r}^{x_{r+1}} \frac{\partial^2 G(x,\xi)}{\partial x^4} f(\xi)d\xi + \int_{x_{r_0}}^{x} \frac{\partial^4 G(x,\xi)}{\partial x^4} f(\xi)d\xi$$

$$+ \int_{x}^{x_{r_0+1}} \frac{\partial^4 G(x,\xi)}{\partial x^4}) f(\xi)d\xi + \sum_{r=r_0+1}^{N-1} \int_{x_r}^{x_{r+1}} \frac{\partial^4 G(x,\xi)}{\partial x^4} f(\xi)d\xi \tag{H.59}$$

$$+\sum_{r=0}^{N} (\frac{d^4G_{1r}(x)}{dx^4}) M_r f(x_r) + \frac{d^4G_{2r}(x)}{dx^4} J_r f'(x_r)) + f(x).$$

Substituting equations (H.55), (H.56) (H.57) and (H.59) into the left side of (H.2) leads to

$$\mathfrak{L}[y(x)] = \sum_{r=0}^{r_0-1} \int_{x_r}^{x_{r+1}} \mathfrak{L}[G(x,\xi)]f(\xi)d\xi + \int_{x_{r_0}}^{x} \mathfrak{L}[G(x,\xi)]f(\xi)d\xi$$

$$+ \int_{x}^{x_{r_0+1}} \mathfrak{L}[G(x,\xi)]f(\xi)d\xi + \sum_{r=r_0+1}^{N-1} \int_{x_r}^{x_{r+1}} \mathfrak{L}[G(x,\xi)]f(\xi)d\xi \tag{H.60}$$

$$+\sum_{r=0}^{N} (\mathfrak{L}[G_{1r}(x)]M_r f(x_r) + \mathfrak{L}[G_{2r}(x)]J_r f'(x_r)) + f(x).$$

It can be seen from the definition of $G(x, \xi)$ and equations (H.6) and (H.7) that all the integrals in the last equation are identically zero. Furthermore, I_{r_0} is an arbitrary sub-interval in $0 \le x \le L$. Therefore $y(x)$, which is given by equation (H.50), satisfies

$$\mathfrak{L}[y] = f(x), \quad for\ x \neq x_r, \quad r = 0, 1, ..., N. \tag{H.61}$$

Substituting $y(x)$ into the left side of the first end condition of (H.52) and combining the result with the end conditions (H.3), (H.6) and (H.7), yields

$$U_{10}[y(x)] = \sum_{r=0}^{N-1} \int_{x_r}^{x_{r+1}} U_{10}[G(x,\xi)]f(\xi)d\xi + \sum_{r=0}^{N} (U_{10}[G_{1r}(x)]M_r f(x_r) + U_{10}[G_{2r}(x)]J_r f'(x_r))$$

or

$$U_{10}[y(x)] = M_0 f(x_0). \tag{H.62}$$

Thus $y(x)$, indeed, satisfies the first end condition of (H.52). It can be shown analogously that $y(x)$ also satisfies the remaining end and interior conditions, viz (H.52) and (H.53). Consequently, $y(x)$ is a classical solution of the multiple-point boundary value problem (H.51) through (H.53). This means that the variational equation (3.2.5) has a unique solution of $y(x)$.

To study the continuity of $y(x)$, rewrite equation (H.2) as

$$\begin{aligned} EI(x)\frac{d^4y(x)}{dx^4} = \rho Af(x) - (2\frac{dEI(x)}{dx}\frac{d^3y(x)}{dx^3} + \frac{d^2EI(x)}{dx^2}\frac{d^2y(x)}{dx^2} \\ - \frac{d}{dx}(p(x)\frac{dy}{dx}) + k_e(x)y(x)) \\ x_{r-1} < x < x_r, \quad r = 1, ..., N. \end{aligned} \tag{H.63}$$

It can be found from (H.27) and (H.54) through (H.57) that $y(x)$, $dy(x)/dx$, $d^2y(x)/dx^2$ and $d^3y(x)/dx^3$ are continuous in each sub-interval $V_r: x_r \le x \le x_{r+1}$. Therefore, $EI(x)d^4y(x)/dx^4$ has the same continuity as the function $f(x)$ given on the right side of (H.63). Thus, if $f''(x)$ is square integrable in each V_r, then $d^6y(x)/dx^6$ is also square integrable in V_r because $\rho A(x)$, $EI(x)$, $p(x)$ and $k(x)$ as well as their arbitrarily high derivatives are also continuous. This proves the second part of Lemma 2.1.

Mathematical induction is needed to investigate the continuity of the eigenvectors in each V_r (that is claimed in Theorem 3.2.1). It is known from equation (H.1) that the first derivative, $dw_j(x)/dx$, is continuous in each interval V_r. Suppose $d^k w_j(x)/dx^k$ is continuous in each V_r for all $k \leq n - 1$. Then the induction procedure requires that $d^n w_j(x)/dx^n$ is also shown to be continuous in each V_r. To accomplish this goal, replace $y(x)$ and $f(x)$ in (H.63) by $w_j(x)$ and $\lambda_j w_j(x)$, respectively. That is,

$$EI(x)\frac{d^4 w_j(x)}{dx^4} = \rho A \lambda_j w_j(x) - (2\frac{dEI(x)}{dx}\frac{d^3 w_j(x)}{dx^3} + \frac{d^2 EI(x)}{dx^2}\frac{d^2 w_j(x)}{dx^2} - \frac{d}{dx}(p(x)\frac{dw_j(x)}{dx}) + k_e(x)w_j(x)) \qquad x_{r-1} < x < x_r, \quad r = 1, ..., N. \tag{H.64}$$

Define a function, $g(x)$, as

$$g(x) = \rho A \lambda_j w_j(x) - (2\frac{dEI(x)}{dx}\frac{d^3 w_j(x)}{dx^3} + \frac{d^2 EI(x)}{dx^2}\frac{d^2 w_j(x)}{dx^2} - \frac{d}{dx}(p(x)\frac{dw_j(x)}{dx}) + k_e(x)w_j(x)) \qquad x_{r-1} < x < x_r, \quad r = 1, ..., N. \tag{H.65}$$

It can be noticed from (H.65) that $g(x)$ has $d^3 w_j(x)/dx^3$ as the highest derivative of $w_j(x)$. It is assumed in the induction that $d^k w_j(x)/dx^k$, $k = 0, 1, ..., (n - 1)$, is continuous in each V_r. Furthermore, $\rho A(x)$, $EI(x)$, $p(x)$ and $k(x)$ as well as their arbitrarily high derivatives are also continuous. Consequently, it can be seen from the following equation

$$\frac{d^{n-4}g(x)}{dx^{n-4}} = \sum_{i=0}^{n-4}\binom{n-4}{i}[\lambda_j\frac{d^i\rho A}{dx^i}\frac{d^{n-4-i}w_j}{dx^{n-4-i}} - (2\frac{d^{i+1}EI}{dx^{i+1}}\frac{d^{n-1-i}w_j}{dx^{n-1-i}}$$

$$+\frac{d^{i+2}EI}{dx^{i+2}}\frac{d^{n-2-i}w_j}{dx^{n-2-i}} - \frac{d^ip}{dx^i}\frac{d^{n-2-i}w_j}{dx^{n-2-i}} \tag{H.66}$$

$$-\frac{d^{i+1}p}{dx^{i+1}}\frac{d^{n-3-i}w_j}{dx^{n-3-i}} + \frac{d^ik}{dx^i}\frac{d^{n-4-i}w_j}{dx^{n-4-i}}]$$

where

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \tag{H.67}$$

that $d^{n-4}g(x)/dx^{n-4}$ exists and it is continuous in each sub-interval V_r. On the other hand, it is known from (H.63) and (H.64) that $g(x) = EI(x)d^4w_j(x)/dx^4$. Hence, $d^{n-4}(EI(x)d^4w_j(x)/dx^4)/dx^{n-4}$ is also continuous in each V_r so that

$$\frac{d^{n-4}}{dx^{n-4}}(EI\frac{d^4w_j}{dx^4}) = \sum_{i=1}^{n-4}\binom{n-4}{i}\frac{d^iEI}{dx^i}\frac{d^{n-i}w_j}{dx^{n-i}} + EI(x)\frac{d^nw_j(x)}{dx^n} = \frac{d^{n-4}g(x)}{dx^{n-4}} \tag{H.68}$$

i.e.

$$\frac{d^nw_j}{dx^n} = \frac{1}{EI(x)}[\frac{d^{n-4}g(x)}{dx^{n-4}} - \sum_{i=1}^{n-4}\binom{n-4}{i}\frac{d^iEI}{dx^i}\frac{d^{n-i}w_j}{dx^{n-i}}]. \tag{H.69}$$

This last equation indicates that $d^nw_j(x)/dx^n$ is, indeed, continuous. This completes the proof of the first part of Lemma 2.1.

APPENDIX I

This appendix presents the proofs of Lemmas 3.3.1, I.1, I.2 and I.3 and outlines the two theorems given in [46, 54] which are used to prove Theorem 3.3.1.

It is known from Theorem 3.2.1 that $d^6 z(x)/dx^6$ is square-integrable in each sub-interval V_r when $z(x)$ is an arbitrary solution of equation (3.2.10). Consequently, if $g(x)$ is defined by equations (3.3.2) and (3.3.3), $d^6 g(x)/dx^6$ cannot be fully or piecewise continuous. However, by reference to Definition 3.3.3, it is known that q must satisfy $q \le 5 < 6$. On the other hand, Theorem 3.2.1 shows that $z''(x)$ is piecewise continuous in $0 \le x \le L$. Define $g(x)$ to be identical to $z(x)$. Then Table 2.3 indicates that $g(x)$ has a series expansion with respect to $\{\psi_m(x)\}$ whose first order derivatives can be taken, term by term, without loss of uniform convergence in $0 \le x \le L$, i.e. $q \ge 2 > 1$.

This completes the proof of Lemma 3.3.1. Lemma I.1, which is needed in the proof of Lemma I.2, is demonstrated next.

Lemma I.1. Let i_0, i_1 and q be three positive integers satisfying $i_0 \le i_1 < q$. Suppose that r_{i0} and r_i are two positive integers for a given positive integer i that satisfy $r_{i0} \le r_i \le N$ whilst i satisfies $i_0 \le i \le i_1$. Imagine $\zeta_{i0}(x) \in \mathscr{F}_i^q$, $\zeta_{iN}(x) \in \mathscr{F}_i^{\,q}$ and $\zeta_{ir}(x) \in \Im_i^{\,q}$ where $r_{i0} \le r \le r_i$. Also, $w(x) \in B$ is an arbitrary function that has continuous derivatives upto order q in each sub-interval V_r: $x_{r-1} \le x \le x_r$, $1 \le r \le N$. Moreover, a coefficient h_{ir} is defined by (3.3.4) as

$$
h_{ir} = \begin{cases}
\Big[\dfrac{d^i w(x_r^+)}{dx^i} - \dfrac{d^i w(x_r^-)}{dx^i} - \displaystyle\sum_{k=i_0}^{i-1} h_{kr}\Big(\dfrac{d^i\zeta_{kr}(x_r^+)}{dx^i} \\
\qquad - \dfrac{d^i\zeta_{kr}(x_r^-)}{dx^i}\Big)\Big] / \Big(\dfrac{d^i\zeta_{ir}(x_r^+)}{dx^i} - \dfrac{d^i\zeta_{ir}(x_r^-)}{dx^i}\Big), \\
\qquad\qquad i_0 < i \le i_1,\ 0 < r < N \\[2ex]
\Big(\dfrac{d^i w(x_r^+)}{dx^i} - \dfrac{d^i w(x_r^-)}{dx^i}\Big) / \Big(\dfrac{d^i\zeta_{ir}(x_r^+)}{dx^i} - \dfrac{d^i\zeta_{ir}(x_r^-)}{dx^i}\Big), \\
\qquad\qquad i = i_0,\ 0 < r < N \\[2ex]
\Big[\dfrac{d^i w(x_r)}{dx^i} - \displaystyle\sum_{k=i_0}^{i-1} \dfrac{d^i\zeta_{kr}(x_r)}{dx^i}\Big] / \dfrac{d^i\zeta_{ir}(x_r)}{dx^i}, \\
\qquad\qquad i_0 < i \le i_1,\ r = 0, N,\ \zeta_{ir}(x) \neq 0 \\[2ex]
\dfrac{d^i w(x_r)}{dx^i} / \dfrac{d^i\zeta_{ir}(x_r)}{dx^i}, \quad i = i_0,\ r = 0, N,\ \zeta_{ir}(x) \neq 0.
\end{cases} \tag{I.1}
$$

Given the above statements, there exists a positive constant, c, such that

$$
|h_{ir}| \le c\lceil w \rceil_q \quad where \quad \lceil w \rceil_q = \sum_{i=0}^{q+1} \sum_{r=1}^{N} \Big(\int_{x_{r-1}}^{x_r} \Big(\frac{d^i w}{dx^i}\Big)^2 dx\Big)^{1/2} \tag{I.2}
$$

for $i_0 \le i \le i_1$ and $0 \le r_{i0} \le r \le r_{i1} \le N$.

Proof

First consider r satisfying $0 < r < N$. It can be shown [45] that there exists a constant c_1 such that

$$\left|\frac{d^i w(x_r)}{dx^i}\right| \le c_1 \lceil w \rceil_q, \quad i = 0, 1, ..., q-1 \quad and \quad r = 1, ..., N-1. \tag{I.3}$$

Let c_2 and c_3 be two positive constants that are defined by

$$c_2 = \max\left(\left|\frac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \frac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right|, \quad i = i_0, i_0+1, ..., i_1, \quad r = 1, 2, ..., N-1\right) \tag{I.4}$$

and

$$c_3 = \min\left(\left|\frac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \frac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right| \ne 0, \quad i = i_0, i_0+1, ..., i_1 \quad and \quad r = 1, 2, ..., N-1\right). \tag{I.5}$$

It can be found immediately from the last equation that the inequality

$$c_3 \le \left|\frac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \frac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right| \quad for \quad \left|\frac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \frac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right| \ne 0, \quad i = i_0, i_0+1, ..., i_1 \quad and \quad r = 1, 2, ..., N-1 \tag{I.6}$$

is true. Therefore

$$\frac{1}{c_3} \ge \left|\frac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \frac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right|^{-1}, \quad for \quad \left|\frac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \frac{d^i \zeta_{ir}(x_r^-)}{dx^i}\right| \ne 0, \quad i = i_0, i_0+1, ..., i_1 \quad and \quad r = 1, 2, ..., N-1. \tag{I.7}$$

Before proceeding to prove (I.2), it is helpful to demonstrate that

$$\sum_{i=i_0}^{i_1} |h_{ir}| \le \frac{2c_1 \lceil w \rceil_q}{c_3} \sum_{k=0}^{i-i_0-1} \left(1 + \frac{c_2}{c_3}\right)^k + \left(1 + \frac{c_2}{c_3}\right)^{i-i_0} |h_{i_0 r}|. \tag{I.8}$$

The proof of (I.8) is based upon the use of mathematical induction. It can be found from

(I.1) that, for $0 < r < N$ and $i = i_0 + 1$,

$$|h_{(i_0+1)r}| = |(\frac{d^{i_0+1}w(x_r^+)}{dx^{i_0+1}} - \frac{d^{i_0+1}w(x_r^-)}{dx^{i_0+1}} - h_{i_0 r}(\frac{d^{i_0}\zeta_{i_0 r}(x_r^+)}{dx^{i_0}} - \frac{d^{i_0}\zeta_{i_0 r}(x_r^-)}{dx^{i_0}}))/(\frac{d^{i_0+1}\zeta_{(i_0+1)r}(x_r^+)}{dx^{i_0+1}} - \frac{d^{i_0+1}\zeta_{(i_0+1)r}(x_r^-)}{dx^{i_0+1}})| . \tag{I.9}$$

By employing (I.3) and (I.7), the following inequality can be obtained from (I.9)

$$\begin{aligned}|h_{(i_0+1)r}| &\le \frac{1}{c_3}(|\frac{d^{i_0+1}w(x_r^+)}{dx^{i_0+1}}| + |\frac{d^{i_0+1}w(x_r^-)}{dx^{i_0+1}}| + c_2|h_{i_0 r}|) \\ &\le \frac{1}{c_3}(2c_1 \lceil w \rceil_q + c_2 |h_{i_0 r}|)\end{aligned}$$

or

$$|h_{(i_0+1)r}| \le (\frac{2c_1}{c_3}\lceil w \rceil_q + \frac{c_2}{c_3}|h_{i_0 r}|), \quad 0 < r < N. \tag{I.10}$$

Consequently, the inequality

$$|h_{i_0 r}| + |h_{(i_0+1)r}| \le |h_{i_0 r}| + (\frac{2c_1}{c_3}\lceil w \rceil_q + \frac{c_2}{c_3}|h_{i_0 r}|)$$

or

$$|h_{i_0 r}| + |h_{(i_0+1)r}| \le \frac{2c_1}{c_3}\lceil w \rceil_q + (1 + \frac{c_2}{c_3})|h_{i_0 r}|, \quad 0 < r < N \tag{I.11}$$

can be found from (I.10). This last inequality shows that (I.8) is valid for $i = i_0 + 1$.

Suppose (I.10) also holds for an arbitrary integer, $i = i_2$, that satisfies $i_0 + 1 < i_2 < i_1$, i.e.

$$\sum_{i=i_0}^{i_2} |h_{ir}| \le \frac{2c_1}{c_3} \sum_{k=0}^{i_2-i_0-1} \lceil w \rceil_q (1+\frac{c_2}{c_3})^k + (1+\frac{c_2}{c_3})^{i_2-r_0} |h_{i_0 r}|. \tag{I.12}$$

Then (I.8) is needed to be shown when $i = i_2 + 1$. First, it is known from (I.1) that

$$|h_{(i_2+1)r}| = |(\frac{d^{i_2+1}w(x_r^+)}{dx^{i_2+1}} - \frac{d^{i_2+1}w(x_r^-)}{dx^{i_2+1}} - \sum_{i=i_0}^{i_2} h_{ir}(\frac{d^{i_2}\zeta_{ir}(x_r^+)}{dx^{i_2}} - \frac{d^{i_2}\zeta_{ir}(x_r^-)}{dx^{i_2}}))/(\frac{d^{i_2+1}\zeta_{(i_2+1)r}(x_r^+)}{dx^{i_2+1}} - \frac{d^{i_2+1}\zeta_{(i_2+1)r}(x_r^-)}{dx^{i_2+1}})| \tag{I.13}$$

$$0 < r < N.$$

Then, by employing (I.3) and (I.7) again, the following inequality can be obtained from (I.13) in a similar manner to (I.10), viz

$$|h_{(i_2+1)r}| \le (\frac{2c_1}{c_3}\lceil w \rceil_q + \frac{c_2}{c_3}\sum_{i=i_0}^{i_2} |h_{ir}|),\quad 0 < r < N. \tag{I.14}$$

Consequently,

$$\begin{aligned}\sum_{i=i_0}^{i_2+1} |h_{(i_2+1)r}| &= \sum_{i=i_0}^{i_2} |h_{ir}| + |h_{(i_2+1)r}| \\ &\le (\frac{2c_1}{c_3}\lceil w \rceil_q + (1+\frac{c_2}{c_3})\sum_{i=i_0}^{i_2} |h_{ir}|),\quad 0 < r < N.\end{aligned} \tag{I.15}$$

Combining (I.15) with (I.12) yields

$$\sum_{i=i_0}^{i_2+1} |h_{(i_2+1)r}| \le (\frac{2c_1}{c_3}\lceil w \rceil_q + (1+\frac{c_2}{c_3})\sum_{i=i_0}^{i_2} |h_{ir}|)$$

$$\le [(1+\frac{c_2}{c_3})((1+\frac{c_2}{c_3})^{i_2-r_0}|h_{i_0r}| + \frac{2c_1\lceil w \rceil_q}{c_3}\sum_{k=0}^{i_2-i_0-1}(1+\frac{c_2}{c_3})^k) \tag{I.16}$$

$$+\frac{2c_1}{c_3}\lceil w \rceil_q].$$

or

$$\sum_{i=i_0}^{i_2+1} |h_{(i_2+1)r}| \le (1+\frac{c_2}{c_3})^{i_2-r_0+1}|h_{i_0r}| + \frac{2c_1\lceil w \rceil_q}{c_3}\sum_{k=1}^{i_2-i_0}(1+\frac{c_2}{c_3})^k + \frac{2c_1}{c_3}\lceil w \rceil_q]$$

i.e.

$$\sum_{i=i_0}^{i_2+1} |h_{(i_2+1)r}| \le (1+\frac{c_2}{c_3})^{i_2-r_0+1}|h_{i_0r}| + \frac{2c_1\lceil w \rceil_q}{c_3}\sum_{k=0}^{i_2-i_0}(1+\frac{c_2}{c_3})^k, \quad r_{i0} < r < r_i. \tag{I.17}$$

This last inequality shows that (I.8) is also valid for $i = i_2 + 1$. Therefore, (I.8) holds for any positive integer i satisfying $i_0 \le i \le i_1$. Next, (I.2) can be shown straightforwardly for r satisfying $r_{i0} \le r \le r_i$. First, by employing (I.1), (I.3) and (I.7), the inequality

$$|h_{i_0r}| = |(\frac{d^{i_0}w(x_r^+)}{dx^{i_0}} - \frac{d^{i_0}w(x_r^-)}{dx^{i_0}})/(\frac{d^{i_0}\zeta_{i_0r}(x_r^+)}{dx^{i_0}} - \frac{d^{i_0}\zeta_{i_0r}(x_r^-)}{dx^{i_0}})|$$

$$\le \frac{1}{c_3}(|\frac{d^{i_0}w(x_r^+)}{dx^{i_0}}| + |\frac{d^{i_0}w_{ir}(x_r^-)}{dx^{i_0}}|), \quad 0 \le i_0 \le q-1, \; 0 < r < N, \tag{I.18}$$

or

$$|h_{i_0 r}| \leq 2\frac{c_1}{c_3}\lceil w \rceil_q, \quad 0 \leq i_0 \leq q-1, \ 0 < r < N \tag{I.19}$$

can be obtained. Then it is known from (I.1) that

$$\begin{aligned}|h_{ir}| &= |(\frac{d^i w(x_r^+)}{dx^i} - \frac{d^i w(x_r^-)}{dx^i} + \sum_{k=i_0}^{i-1} h_{kr}(\frac{d^i \zeta_{kr}(x_r^+)}{dx^i} \\ &\qquad - \frac{d^i \zeta_{kr}(x_r^-)}{dx^i}))/(\frac{d^i \zeta_{ir}(x_r^+)}{dx^i} - \frac{d^i \zeta_{ir}(x_r^-)}{dx^i})| \\ &\leq \frac{1}{c_3}|(|\frac{d^i w(x_r^+)}{dx^i}| + |\frac{d^i w(x_r^-)}{dx^i}| + c_2 \sum_{k=i_0}^{i-1} |h_{kr}|).\end{aligned} \tag{I.20}$$

By employing (I.3) and (I.8), (I.20) becomes

$$\begin{aligned}|h_{ir}| &\leq \frac{1}{c_3}[2c_1\lceil w \rceil_q + \frac{2c_1c_2\lceil w \rceil_q}{c_3} \sum_{k=0}^{i-i_0-1} (1+\frac{c_2}{c_3})^k + c_2(1+\frac{c_2}{c_3})^{i-i_0}|h_{i_0 r}|] \\ &\leq \frac{1}{c_3}[2c_1 + \frac{2c_1c_2}{c_3} \sum_{k=0}^{i_1-i_0-1} (1+\frac{c_2}{c_3})^k + c_2(1+\frac{c_2}{c_3})^{i_1-i_0}|h_{i_0 r}|]\lceil w \rceil_q.\end{aligned} \tag{I.21}$$

Substituting (I.19) into (I.21) leads to

$$|h_{ir}| \leq \frac{c_1}{c_3}[2 + \frac{2c_2}{c_3} \sum_{k=0}^{i_1-i_0-1} (1+\frac{c_2}{c_3})^k + \frac{2c_2}{c_3}(1+\frac{c_2}{c_3})^{i_1-i_0}]\lceil w \rceil_q, \quad 0 < r < N. \tag{I.22}$$

Let c be a positive constant defined by

$$c = \frac{c_1}{c_3}[2 + \frac{2c_2}{c_3} \sum_{k=0}^{i_1-i_0-1} (1+\frac{c_2}{c_3})^k + \frac{2c_2}{c_3}(1+\frac{c_2}{c_3})^{i_1-i_0}]. \tag{I.23}$$

Then (I.23) becomes

$$|h_{ir}| \le c\lceil w \rceil_q. \tag{I.24}$$

This last inequality shows that (I.2) holds for $0 < r < N$ and $i_0 \le i \le i_1$. A similar proof can be given for $r = 0$ and $r = N$.

This completes the proof of Lemma I.1. By employing this lemma, the next lemma needs to be shown before Lemma I.3 is finally proved.

Lemma I.2. Let $g(x)$, which is defined by

$$g(x) = w(x) - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x), \quad x \ne x_k, \quad k = 1, \ldots, N-1 \tag{I.25}$$

and

$$\frac{d^j g(x_k)}{dx^j} = \frac{d^j w(x_k^+)}{dx^j} - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir} \frac{d^j \zeta_{ir}(x_k^+)}{dx^j}, \quad j = 2, \ldots, q-1, \quad k = 1, \ldots, N-1, \tag{I.26}$$

have the generalized Fourier series expansion

$$g(x) = \sum_{m=1}^{\infty} d_m \psi_m(x) \tag{I.27}$$

where h_{ir} is given by (I.1) and

$$d_m = \int_0^L g(x)\psi_m(x)\,dx \quad and \quad \left(\int_0^L (\psi_m(x))^2 dx\right)^{1/2} = 1. \tag{I.28}$$

Suppose that the (spatial) derivatives of series (I.25) can be taken, term by term, up to order (q - 1) without loss of uniform convergence in $0 \le x \le L$. Furthermore, imagine that the q th derivative of $g(x)$ is fully or piecewise continuous. Then d_m can be rewritten as

$$d_m = \frac{(-1)^q}{(\Omega_m/L)^{q+1+\nu}} \sum_{r=r_{i0}}^{r_i-1} \left(\frac{d^q g}{dx^q} \frac{d^\nu \psi_m}{dx^\nu} \Big|_{x_{r+1}^-} - \frac{d^q g}{dx^q} \frac{d^\nu \psi_m}{dx^\nu} \Big|_{x_r^+} - \int_{x_r}^{x_{r+1}} \frac{d^{q+1} g}{dx^{q+1}} \frac{d^\nu \psi_m}{dx^\nu} dx \right) \tag{I.29}$$

where $\nu = 3 - q \; \boldsymbol{mod} \; 4$. (The ***mod*** term is the integer remainder from dividing q by 4 [51].) Furthermore, a positive constant, c_4, exists such that

$$|d_m| \le c_4 m^{-(q+1)} \|w\|_q \tag{I.30}$$

for a sufficiently large m.

Proof

It has been shown in [55] that q can be expressed in the form

$$q = 4k + q \; \boldsymbol{mod} \; 4 \tag{I.31}$$

where k is the positive integer quotient obtained when q is divided by 4. On the other hand, the following relation can be found from [50], viz

$$\frac{1}{(\Omega_m/L)^{4k+4}} \frac{d^{4k+4} \psi_m(x)}{dx^{4k+4}} = \psi_m(x). \tag{I.32}$$

Substituting (I.32) into the first equation labelled (I.28) leads to

$$d_m = \frac{1}{(\Omega_m/L)^{4k+4}} \int_0^L g(x) \frac{d^{4k+4} \psi_m(x)}{dx^{4k+4}} dx. \tag{I.33}$$

Integrating (I.33) by parts and using (I.1) yields

$$d_m = \frac{(-1)^q}{(\Omega_m/L)^{4k+4}} \sum_{r=r_{i0}}^{r_i-1} \Big(\frac{d^q g}{dx^q} \frac{d^{4k+4-(q+1)}\psi_m}{dx^{4k+4-(q+1)}} \Big|_{x_{r+1}^-} - \frac{d^q g}{dx^q} \frac{d^{4k+4-(q+1)}\psi_m}{dx^{4k+4-(q+1)}} \Big|_{x_{r+1}^+}$$

$$- \int_{x_r}^{x_{r+1}} \frac{d^{q+1} g}{dx^{q+1}} \frac{d^{4k+4-(q+1)}\psi_m}{dx^{4k+4-(q+1)}} dx \Big). \tag{I.34}$$

Let $\nu = 3 - q \; mod \; 4$. Then, by employing (I.31), the following relations can be shown

$$\nu = 3 - q \; mod \; 4 = 3 + (4k - q) = (4k+4) - (q+1). \tag{I.35}$$

This last equation yields

$$4k + 4 = q + 1 + \nu. \tag{I.36}$$

Substituting (I.35) and (I.36) into (I.34) produces (I.29).

Now consider inequality (I.30). First, it can be shown [45] that there exists a positive constant c_5 such that

$$\Big| \frac{d^i g(x_r)}{dx^i} \Big| \le c_5 \lceil g \rceil_q, \quad i = 0, 1, ..., q-1 \quad and \quad r = 0, 1, ..., N. \tag{I.37}$$

Furthermore, it is known [42] that the generic inequality

$$\Big(\sum_{k=k_0}^{k_1} b_k \Big)^2 \le (k_1 - k_0 + 1) \sum_{k=k_0}^{k_1} (b_k)^2 \tag{I.38}$$

holds for b_k, $k = k_0, k_0 + 1, ..., k_1$ where b_k is an arbitrary real value. The k_0 and k_1 are two positive integers. By employing (I.38), the inequality

$$(g(x))^2 = (w(x) - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x))^2 \le 2[(w)^2 + (\sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x))^2]$$

$$\le 2[(w(x))^2 + (i_1 - i_0 + 1)\sum_{i=i_0}^{i_1} (\sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x))^2] \tag{I.39}$$

$$\le 2[(w(x))^2 + (i_1 - i_0 + 1)(r_i - r_{i0} + 1)\sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} (h_{ir})^2(\zeta_{ir}(x))^2].$$

can be obtained from (I.25). Moreover, the inequality

$$(\frac{d^i g(x)}{dx^i})^2 \le 2[(\frac{d^i w(x)}{dx^i})^2 + (i_1 - i_0 + 1)(r_i - r_{i0} + 1)\sum_{k=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} (h_{kr})^2(\frac{d^i\zeta_{kr}(x)}{dx^i})^2] \tag{I.40}$$

can be found similarly for $x_{r-1} \le x \le x_r$, $r = 1, 2, \ldots, N$. Thus,

$$(\lceil g \rceil_q)^2 = \sum_{i=0}^{q+1} \sum_{r=1}^{N} (\int_{x_{r-1}}^{x_r} (\frac{d^i g}{dx^i})^2 dx)^{1/2}$$

$$\le 2[\sum_{i=0}^{q+1} \sum_{r=1}^{N} (\int_{x_{r-1}}^{x_r} (\frac{d^i w}{dx^i})^2 dx)^{1/2}$$

$$+ (i_1 - i_0 + 1)(r_i - r_{i0} + 1)\sum_{k=i_0}^{i_1} \sum_{t=r_{i0}}^{r_i} (h_{kr})^2(\sum_{i=0}^{q+1} \sum_{r=1}^{N} (\int_{x_{r-1}}^{x_r} (\frac{d^i\zeta_{kt}}{dx^i})^2 dx)^{1/2})]$$

or

$$(\lceil g \rceil_q)^2 \le 2[(\lceil w \rceil_q)^2 + (i_1 - i_0 + 1)(r_i - r_{i0} + 1)\sum_{k=i_0}^{i_1} \sum_{t=r_{i0}}^{r_i} (h_{kr})^2(\lceil \zeta_{kt} \rceil_q)^2] \tag{I.41}$$

can be obtained straightforwardly. By using (I.2), (I.41) becomes

$$\lceil g \rceil_q \le c_6 \lceil w \rceil_q \tag{I.42}$$

where c_6 is a positive constant which is given by

$$c_6 = 2^{1/2}[1 + c^2(i_1 - i_0 + 1)(r_i - r_{i0} + 1)\sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} \lceil \zeta_{ir} \rceil_q)^2]^{1/2}. \tag{I.43}$$

Additionally, it is known [46] that the ith order derivative of $\psi_m(x)$ satisfies

$$(\frac{L}{\Omega_m})^i \frac{d^i \psi_m(x)}{dx^i} = Q_{1m}(\frac{L}{\Omega_m})^i \frac{d^i \cos(\Omega_m x/L + \vartheta_m)}{dx^i} + (-1)^i Q_{2m} \exp(-\Omega_m x/L) + Q_{3m} \exp(-\Omega_m (L-x)/L). \tag{I.44}$$

where

$$Q_{1m} = (\frac{2}{L})^{1/2}, \quad 0 \le \lim_{m\to\infty} |Q_{2m}| \le \frac{2}{L^{1/2}}, \quad 0 \le \lim_{m\to\infty} |Q_{3m}| \le \frac{2}{L^{1/2}} \tag{I.45}$$

whilst

$$\lim_{m\to\infty} \vartheta_m = \begin{cases} 0 & (\textit{sliding-sliding and sliding-pinned ends}) \\ -\dfrac{\pi}{2} & (\textit{pinned-pinned ends}) \\ \dfrac{\pi}{4} & (\textit{all other standard end conditions}). \end{cases} \tag{I.46}$$

Consequently, it can be shown in a similar manner to (F.1) that there exists a positive constant, c_7, such that

$$(\int_{x_{r-1}}^{x_r} ((\frac{L}{\Omega_m})^i \frac{d^i \psi_m(x)}{dx^i})^2 dx)^{1/2} \le (\int_0^L ((\frac{L}{\Omega_m})^i \frac{d^i \psi_m(x)}{dx^i})^2 dx)^{1/2} \le c_7, \; i_0 \le i \le i_1. \tag{I.47}$$

After the above preliminaries, (I.30) can be shown straightforwardly. First, it is seen from (I.29) that

$$|d_m| \le \frac{1}{(\Omega_m/L)^{q+1}} \sum_{r=r_{i0}}^{r_i-1} [\,|\frac{d^q g(x_{r+1}^-)}{dx^q}|\,|\frac{d^\nu \psi_m(x_{r+1}^-)}{(\Omega_m/L)^\nu dx^\nu}| + |\frac{d^q g(x_r^+)}{dx^q}|\,|\frac{d^\nu \psi_m(x_r^+)}{(\Omega_m/L)^\nu dx^\nu}| + |\int_{x_r}^{x_{r+1}} \frac{d^{q+1} g}{dx^{q+1}} \frac{d^\nu \psi_m}{(\Omega_m/L)^\nu dx^\nu} dx)\,|\,]. \tag{I.48}$$

By employing Schwarz's inequality [45] i.e.

$$|\int_{x_{r-1}}^{x_r} f(x) g(x) dx| \le (\int_{x_{r-1}}^{x_r} f(x)^2 dx)^{1/2} (\int_{x_{r-1}}^{x_r} g(x)^2 dx)^{1/2} \tag{I.49}$$

as well as the inequalities [50]

$$|\frac{d^i \psi_m(x_r)}{(\Omega_m/L)^i dx^i}| \le 6/L^{1/2} \quad and \quad 2\pi m > \Omega_m > m(1-\frac{3}{m})\pi \tag{I.50}$$

for an arbitrary integer i and a sufficiently large m, (I.48) becomes

$$|d_m| \le \frac{1}{(\Omega_m/L)^{q+1}} \sum_{r=r_{i0}}^{r_i-1} [\,|\frac{d^q g(x_{r+1}^-)}{dx^q}|\,|\frac{d^\nu \psi_m(x_{r+1}^-)}{(\Omega_m/L)^\nu dx^\nu}| + |\frac{d^q g(x_r^+)}{dx^q}|\,|\frac{d^\nu \psi_m(x_r^+)}{(\Omega_m/L)^\nu dx^\nu}| + (\int_{x_r}^{x_{r+1}} (\frac{d^{q+1} g}{dx^{q+1}})^2 dx)^{1/2} (\int_{x_r}^{x_{r+1}} (\frac{d^\nu \psi_m}{(\Omega_m/L)^\nu dx^\nu})^2 dx)^{1/2}]. \tag{I.51}$$

By employing (I.47) and (I.50), (I.51) can be simplified to

$$|d_m| \le \frac{1}{(\Omega_m/L)^{q+1}} \sum_{r=r_{i0}}^{r_i-1} [\,|\frac{d^q g(x_{r+1}^-)}{dx^q}|\,6L^{-1/2} + |\frac{d^q g(x_{r+1}^+)}{dx^q}|\,6L^{-1/2} \tag{I.52}$$

$$+ (\int_{x_r}^{x_{r+1}} (\frac{d^{q+1}g}{dx^{q+1}})^2 dx)^{1/2} c_7]$$

i.e.

$$|d_m| \le \frac{1}{(\Omega_m/L)^{q+1}} 6L^{-1/2} [\sum_{r=r_{i0}}^{r_i-1} |\frac{d^q g(x_{r+1}^-)}{dx^q}| + \sum_{r=r_0}^{r_i-1} |\frac{d^q g(x_{r+1}^+)}{dx^q}| \tag{I.53}$$

$$+ L^{1/2}(c_7/6) \sum_{r=r_{i0}}^{r_i-1} (\int_{x_r}^{x_{r+1}} (\frac{d^{q+1}g}{dx^{q+1}})^2 dx)^{1/2}].$$

By using (I.37), (I.53) can be simplified further to

$$|d_m| \le \frac{1}{(\Omega_m/L)^{q+1}} 6L^{-1/2} [(c_5(r_i - r_{i0}) \lceil g \rceil_q + (r_i - r_{i0}) c_5 \lceil g \rceil_q) \tag{I.54}$$

$$+ L^{1/2}(c_7/6) \lceil g \rceil_q].$$

Finally, because $N > r_i - r_{i0}$, the application of (I.42) to (I.54) leads to

$$|d_m| \le \frac{1}{(\Omega_m/L)^{q+1}} 6L^{-1/2} [2c_5 c_6 (r_i - r_{i0}) \lceil w \rceil_q + L^{1/2}(c_6 c_7)/6 \lceil w \rceil_q]$$

or

$$|d_m| \le \frac{c_6}{(\Omega_m/L)^{q+1}} 6L^{-1/2} [2c_5 N + L^{1/2}(c_7/6)] \lceil w \rceil_q. \tag{I.55}$$

By defining a positive constant, c_4, as

$$c_4 = 12L^{q+1/2}(2c_5N+L^{1/2}(c_7/6))c_6, \tag{I.56}$$

(I.55) can be rewritten as

$$|d_m| \le \frac{c_4}{m^{q+1}}\lceil w \rceil_q. \tag{I.57}$$

The last two inequalities show that, indeed, (I.30) holds.

This completes the proof of Lemma I.2. The next result can be obtained by using this lemma.

Lemma I.3. Let $S_n \subset B$ be an n-dimensional subspace spanned by m_1 linearly independent functions $\{\zeta_{ir}(x)\}$ of set (3.3.1) as well as $\{\psi_m(x): m = 1, ..., n - m_1\}$. Suppose $\{\zeta_{ir}(x), 2 \le i_0 \le i \le i_1 \le (q_1 - 1), 0 \le r_{i0} \le r \le r_i\}$ form a set of q_1-GFM functions with respect to $\{\psi_m(x)\}$ and an eigenvector $w_j(x) \in M(\lambda_j)$. If

$$\varepsilon = \inf_{u \in S_n} \| w_j(x) - u \|_B, \qquad w_j(x) \in M(\lambda_j), \tag{I.58}$$

then a positive constant, c_8, exists which is independent of n and $w_j(x)$ and such that

$$\varepsilon \le c_8 n^{-q_2} \lceil w_j \rceil_{q_1} \tag{I.59}$$

for a sufficiently large n. The q_2 and $\lceil w_j \rceil_{q_1}$ are given by equation (3.3.6). Furthermore, suppose P is an orthogonal projection of space B on S_n and, for an arbitrary $\varphi \in B$ and all $u \in S_n$,

$$B(\varphi - P\varphi, u) = 0. \tag{I.60}$$

If $\{\zeta_{ir}(x)\}$ form a set of q_3-GFM functions with respect to $\{\psi_m(x)\}$ and equation (3.2.10), then

$$\|\varphi - P\varphi\|_D \leq \eta \|\varphi - P\varphi\|_B \quad and \quad \eta = c_9 n^{-q_4} \tag{I.61}$$

where q_4 is given by equation (3.3.10) and c_9 is a positive constant that is independent of φ.

Proof

To prove inequality (I.59), let $w(x) \equiv w_j(x)$ and $q = q_1$. Then the $g(x)$ defined by equations (3.3.2) and (3.3.3) has the generalized Fourier series expansion (I.27) and the coefficient d_m is given by (I.29). It has been shown in Lemma I.2 that a positive constant, c_4, exists such that

$$|d_m| \leq c_4 m^{-(q_1+1)} \lceil w_j \rceil_{q_1} \tag{I.62}$$

for a sufficiently large m. By using the inequality [40]

$$\sum_{m=n-m_1+1}^{\infty} m^{-s} \leq \frac{(n-m_1+1)^{-s+1}}{s-1} \leq (1/2)^{-s+1} \frac{n^{-s+1}}{s-1}, \quad s > 1, \tag{I.63}$$

for a sufficiently large n, the following inequalities can be shown

$$\begin{aligned} |g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x)| &= |\sum_{m=n-m_1+1}^{\infty} d_m \psi_m(x)| \leq \sum_{m=n-m_1+1}^{\infty} |\psi_m| \, |d_m| \\ &\leq \frac{6c_4 \lceil w_j \rceil_{q_1}}{L^{1/2}} \sum_{m=n-m_1+1}^{\infty} m^{-(q+1)} \leq \frac{6c_4 \lceil w_j \rceil_{q_1}}{L^{1/2}} (1/2)^{-q} n^{-q}/q \end{aligned}$$

i.e.

$$|g(x)-\sum_{m=1}^{n-m_1} d_m\psi_m(x)| \le \frac{6(2)^q c_4}{qL^{1/2}}\lceil w_j \rceil_{q_1} n^{-q}. \tag{I.64}$$

Similarly

$$\begin{aligned}
|g'(x)-\sum_{m=1}^{n-m_1} d_m\psi'_m(x)| &= |\sum_{m=n-m_1+1}^{\infty} d_m\psi'_m(x)| \\
&\le \sum_{m=n-m_1+1}^{\infty} |\frac{L}{\Omega_m}\psi'_m(x)|\,|d_m\frac{\Omega_m}{L}| \\
&\le \frac{6c_4\lceil w_j \rceil_{q_1}}{L^{3/2}} \sum_{m=n-m_1+1}^{\infty} 2\pi m m^{-(q+1)} \\
&\le \frac{12\pi c_4}{L^{3/2}}\lceil w_j \rceil_{q_1} \sum_{m=n-m_1+1}^{\infty} m^{-q} \\
&\le \frac{12\pi c_4\lceil w_j \rceil_{q_1}}{L^{3/2}(q-1)}(1/2)^{-(q-1)}n^{-(q-1)}
\end{aligned}$$

i.e.

$$|g'(x)-\sum_{m=1}^{n-m_1} d_m\psi'_m(x)| \le \frac{12(2)^{(q-1)}\pi c_4}{L^{3/2}(q-1)}\lceil w_j \rceil_{q_1} n^{-(q-1)} \tag{I.65}$$

for any point x satisfying $0 \le x \le L$. On the other hand, Parseval's identity applied to the right side of (I.27) for $g''(x)$ produces [40]

$$\int_0^L (g''(x) - \sum_{m=1}^{n-m_1} d_m \psi''_m(x))^2 dx = \int_0^L (g''(x) - \sum_{m=1}^{n-m_1} d_m \|\psi''_m\|_H \frac{\psi''_m(x)}{\|\psi''_m\|_H})^2 dx$$

$$= \sum_{m=n-m_1+1}^{\infty} (|d_m| \, \|\psi''_m\|_H)^2 \tag{I.66}$$

where $\|\psi_m''\|_H$ is the norm of $\psi_m(x)$ in a Hilbert space, H, [45]. It is given by [50]

$$\|\psi''_m\|_H = (\int_0^L (\psi''_m(x))^2 dx)^{1/2} = (\Omega_m/L)^2. \tag{I.67}$$

By employing (I.50), (I.62) and (I.67), the inequality

$$\int_0^L (g''(x) - \sum_{m=1}^{n-m_1} d_m \psi''_m(x))^2 dx = \sum_{m=n-m_1+1}^{\infty} (|d_m| \, \|\psi''_m\|_H)^2$$

$$\leq \frac{(2\pi)^4}{L^4} \sum_{m=n-m_1+1}^{\infty} |d_m|^2 m^4 \tag{I.68}$$

can be demonstrated. By using (I.30), the last inequality becomes

$$\int_0^L (g''(x) - \sum_{m=1}^{n-m_1} d_m \psi''_m(x))^2 dx \leq \frac{(2\pi)^4}{L^4} c_4^2 (\|w_j\|_{q_1})^2 \sum_{m=n-m_1+1}^{\infty} m^{-(q-1)} m^4$$

$$\leq \frac{(2\pi)^4}{L^4} c_4^2 (\|w_j\|_{q_1})^2 \sum_{m=n-m_1+1}^{\infty} m^{-(2q-2)}$$

i.e.

$$\int_0^L (g''(x) - \sum_{m=1}^{n-1} d_m \psi''_m(x))^2 dx \leq \frac{(2\pi)^4}{L^4} \frac{c_4^2 2^{(2q-3)}}{2q-3} (\|w_j\|_{q_1})^2 n^{-(2q-3)}. \tag{I.69}$$

On the other hand, it is known from (3.2.7) that

$$\| g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x) \|_B^2 = \int_0^L [EI(g''(x) - \sum_{m=1}^{n-m_1} d_m \psi''_m(x))^2$$
$$+ p(g'(x) - \sum_{m=1}^{n-m_1} d_m \psi'_m(x))^2$$
$$+ k_e(g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x))^2] dx \tag{I.70}$$
$$+ \sum_{r=0}^{N} [K_r(g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x))^2$$
$$+ \beta_r(g'(x) - \sum_{m=1}^{n-m_1} d_m \psi'_m(x))^2].$$

Substituting (I.64), (I.65) and (I.69) into (I.70) yields

$$\| g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x) \|_B^2 \le \max(EI(x)) \int_0^L (g''(x) - \sum_{m=1}^{n-m_1} d_m \psi''_m(x))^2 dx$$
$$+ \max(p(x)) \int_0^L (g'(x) - \sum_{m=1}^{n-m_1} d_m \psi'_m(x))^2 dx$$
$$+ \max(k_e(x)) \int_0^L (g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x))^2 dx$$
$$+ \sum_{r=0}^{N} [K_r \frac{(6(2^q) c_4)^2}{(qL^{1/2})^2} (\| w_j \|_{q_1})^2 n^{-2q}$$
$$+ \beta_r \frac{[12(2^{(q-1)}) \pi c_4]^2}{(L^{3/2}(q-1))^2} (\| w_j \|)_{q_1})^2 n^{-2(q-1)}]$$

or

$$
\begin{aligned}
\| g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x) \|_B^2 \le\ & \max(EI(x)) \frac{(2\pi)^4}{L^4} \frac{c^4 2^{(2q-3)}}{2q-3} (\|w_j\|_{q_1})^2 n^{-(2q-3)} \\
& + \max(p(x)) L \frac{[12(2^{(q-1)})\pi c_4]^2}{(L^{3/2}(q-1))^2} (\|w_j\|_{q_1})^2 n^{-2(q-1)} \\
& + \max(k_e(x)) L \left(\frac{6(2^q)c_4}{qL^{1/2}}\right)^2 (\|w_j\|_{q_1})^2 n^{-2q} \\
& + \sum_{r=0}^{N} \Big[K_r \left(\frac{6(2^q)c_4}{qL^{1/2}}\right)^2 (\|w_j\|_{q_1})^2 n^{-2q} \\
& + \beta_r \left(\frac{12(2^{(q-1)})\pi c_4}{(L^{3/2}(q-1))}\right)^2 (\|w_j\|_{q_1})^2 n^{-2(q-1)}\Big].
\end{aligned}
\tag{I.71}
$$

It can be shown straightforwardly that

$$
n^{-(2q-3)} > n^{-2q} \quad \textit{and} \quad n^{-(2q-3)} > n^{-2(q-1)} \tag{I.72}
$$

for $n > 1$ so that (I.71) can be simplified to

$$
\begin{aligned}
\| g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x) \|_B^2 \le\ & \Big(\max(EI(x)) \frac{(2\pi)^4}{L^4} \frac{c_4 2^{(2q-3)}}{2q-3} \\
& + \max(p(x)) \frac{[12(2^{(q-1)})\pi c_4]^2}{L^2(q-1)^2} \\
& + \max(k_e(x)) \frac{6(2^q)c_4}{q^2} + \sum_{r=0}^{N} \Big[K_r \left(\frac{6(2^q)c_4}{qL^{1/2}}\right)^2 \\
& + \beta_r \left(\frac{12(2^{(q-1)})\pi c_4}{L^{3/2}(q-1)}\right)^2\Big] \Big) (\|w_j\|_{q_1})^2 n^{-2(q-3)}.
\end{aligned}
\tag{I.73}
$$

By defining a positive constant, c_8, as

$$(c_8)^2 = \max(EI(x))\frac{(2\pi)^4}{L^4}\frac{c_4 2^{(2q-3)}}{2q-3} + \max(p(x))\frac{[12(2^{(q-1)})\pi c_4]^2}{L^2(q-1)^2}$$

$$+\max(k_e(x))\frac{6(2^q)c_4}{q^2} + \sum_{r=0}^{N}\ [K_r(\frac{6(2^q)c_4}{qL^{1/2}})^2 + \beta_r(\frac{12(2^{(q-1)})\pi c_4}{L^{3/2}(q-1)})^2], \tag{I.74}$$

(I.73) becomes

$$\| g(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x) \|_B \le c_8 n^{-(2q_1-3)/2} \| w_j \|_{q_1}. \tag{I.75}$$

Combining this last relation with equations (3.3.2) and (3.3.3) produces

$$\| w_j(x) - \sum_{i=i_0}^{i_1} \sum_{r=r_{i0}}^{r_i} h_{ir}\zeta_{ir}(x) - \sum_{m=1}^{n-m_1} d_m \psi_m(x) \|_B \le c_8\, n^{-(2q_1-3)/2} \| w_j \|_{q_1}. \tag{I.76}$$

Hence, it can be seen from (I.76) that there exists a positive constant, c_8 such that required inequality (I.59), viz

$$\epsilon \le c_8 n^{-(2q_1-3)/2} \| w_j \|_{q_1} = c_8 n^{-q_2} \| w_j \|_{q_1}, \tag{I.77}$$

holds.

To prove inequality (I.61), let $f(x) = \varphi - P\varphi$ for an arbitrary $\varphi \in B$ and suppose that $z(x)$ is a solution of equation (3.2.10). By choosing $u = f(x)$, equation (3.2.10) gives [56]

$$B(z,f) = \| f \|_D^2. \tag{I.78}$$

It is seen easily from equations (3.2.6) and (I.60) that

$$B(z-\hbar, f) = \|f\|_D^2 \tag{I.79}$$

for all $\hbar(x) \in S_n$. Applying Schwarz's inequality to the right side of the last equation yields

$$\|f\|_D^2 \leq (B(z-\hbar, z-\hbar))^{1/2}(B(f,f))^{1/2}. \tag{I.80}$$

Now $\{\zeta_{ir}(x)\}$ constitutes a set of q_3-GFM functions with respect to $\{\psi_m(x)\}$ and equation (3.2.10). Let $w(x) \equiv z(x)$ so that the $g(x)$ defined by equations (3.3.2) and (3.3.3) has the generalized Fourier series expansion (I.27) whose coefficients are given by equation (I.29) in which q is replaced by q_3. For $q_3 = 5$, it can be found from (I.30) that a positive constant, c_4, exists such that

$$|d_m| \leq c_4 m^{-6} \|\varphi - P\varphi\|_B. \tag{I.81}$$

On the other hand, by choosing $\hbar(x)$ as

$$\hbar(x) = z(x) - \sum_{m = n-m_1+1}^{\infty} d_m \psi_m(x), \tag{I.82}$$

it can be seen from (I.63) and (I.81), in a manner similar to the derivation of (I.75), that a positive constant, c_9, exists such that

$$\|z-\hbar\|_B \leq c_9 n^{-7/2} \|\varphi - P\varphi\|_B \tag{I.83}$$

for a sufficiently large n. Combining inequalities (I.80) and (I.83) yields

$$\|\varphi - P\varphi\|_D \leq c_9 n^{-7/4} \|\varphi - P\varphi\|_B. \tag{I.84}$$

A similar proof can also be given for $q_3 = 2$, 3 and 4.

In addition to Lemma I.3, the following theorem is needed in Theorem 3.2.1 to estimate the eigenvalue and eigenvector errors in spaces B and D.

Theorem I.1 [46]. There exists a positive constant, c, such that, for the eigenvalues and eigenvectors of variational problem (3.2.6),

$$\lambda_j^n - \lambda_j \leq c\,\varepsilon^2 \tag{I.85}$$

and

$$\|w_j^n - w_j\|_B \leq c\varepsilon \quad \textit{whilst} \quad \|w_j^n - w_j\|_D \leq c\eta\varepsilon \tag{I.86}$$

The ε and η in the last inequalities are given by relations (I.59) and (I.61), respectively.

The next theorem is used to derive pointwise error estimates in Theorem 3.4.1 for an eigenvector and its higher spatial derivatives.

Theorem I.2 [54]. Imagine u and a sequence of functions $\{u_n : n = 1, 2, \ldots\}$ belong to $W^{(k)}(0, L)$, a Sobolev space in which every element and its derivatives have absolutely continuous derivatives upto order (k - 1) whilst the kth derivative is square integrable in $0 \leq x \leq L$. Suppose

$$\left.\begin{aligned} &\Big(\int_0^L |u - u_n|^2 dx\Big)^{1/2} \leq A_n\,, \quad \Big(\int_0^L \Big|\frac{d^k u}{dx^k} - \frac{d^k u_n}{dx^k}\Big|^2 dx\Big)^{1/2} \leq B_n \\ \textit{and} \quad & \\ &A_n^2 + B_n^2 = E_n^2. \end{aligned}\right\} \tag{I.87}$$

If

$$A_n \to 0,\ A_n^{k-i-\frac{1}{2}} B_n^{i+\frac{1}{2}} \to 0 \ \textit{and} \ (A_n/E_n)^{1/k} \to 0 \quad as\ n \to \infty \qquad i = 0, 1, \ldots, k-1, \tag{I.88}$$

then the sequence $\{d^i u_n/dx^i\}$ converges uniformly to $d^i u/dx^i$ in $0 \le x \le L$ such that

$$\left| \frac{d^i(u_n - u)}{dx^i} \right| \le c(i)(A_n^{k-i-\frac{1}{2}} E_n^{i+\frac{1}{2}})^{1/k}. \tag{I.89}$$

The $c(i)$, $i = 0, 1, \ldots, (k - 1)$, are positive constants that depend upon i.

APPENDIX J

This appendix presents Lemma J.1, which is needed to prove Theorem 3.4.1, and sketches the proof of Corollary 3.4.1.

(1) Lemma J.1

Reference [57] estimates the positive integer, h, used in the inequality

$$\|f - f_{n-1}\|_H \geq c n^{-h}. \tag{J.1}$$

Here $f \equiv f(x)$ is a continuous function and $f_{n-1}(x)$ is an orthogonal projection onto an $(n - 1)$ dimensional subspace, $\mathcal{H}_{n-1}$, spanned by an arbitrary, orthogonal series in a Hilbert space, H, whose norm is given by

$$\|f\|_H \equiv (\langle f, f \rangle_H)^{1/2} = (\int_0^L f^2 dx)^{1/2}. \tag{J.2}$$

Moreover, c is a positive value that is independent of n. Inequality (3.4.9) indicates that an estimate of h is needed for a continuous or a discontinuous $\zeta''_{st}(x)$ in an $(n - 1)$ dimensional subspace, B_{n-1}, spanned by a non-orthogonal series that contains the orthogonal eigenvectors, $\{\psi''_m(x)\}$, and the GFM functions $\{\zeta''_{ir}(x)\}$ which exclude $\zeta''_{st}(x)$. The s and t are two given positive integers.

It is known [40] that the value and sign of the coefficients in an orthogonal series expansion dictate the expansion's convergence rate. These coefficients are determined by the orthogonal eigenvectors themselves as well as their derivatives at the discontinuous points of $\zeta_{ir}(x)$. On the other hand, the ratio of the m th characteristic value, Ω_m, to m tends to π in (3.2.12) as $m \to \infty$. Moreover, (3.2.12) indicates that, for a sufficiently large

m, a uniform beam's eigenvectors and their derivatives are dominated by either $cos(m\pi x/L + \theta_0(x))$ or $sin(m\pi x/L + \theta_0(x))$ where $\theta_0(x)$ is presented in Table J.1 for different standard end conditions. To find how these trigonometric functions change for different m at a given $x = x_r$, a ray OA_r is introduced in the χ_0-χ_1 plane shown in Figure J.1. This ray has a (constant) unit length and runs from the origin, O, to an arbitrary point, A_r. It rotates counterclockwise about O and, after m equal stepped increments of $\pi x_r/L$ from the initial angle $\theta_0(x_r)$, the ray forms the angle

$$\theta_m(x_r) = \frac{m\pi x_r}{L} + \theta_0(x_r) \tag{J.3}$$

relative to the χ_0-axis. By using the particular locations of OA_r that correspond to $r = 0, 1, ..., N$, as well as the following Lemma, inequality (J.1) can be demonstrated to apply to $\zeta''_n(x)$ in the $(n - 1)$ dimensional sub-space B_{n-1}.

Lemma J.1. Imagine $\zeta_{i0}(x) \in \mathscr{S}_i^q$, $\zeta_{iN}(x) \in \mathscr{F}_i^q$, $\zeta_{ir}(x) \in \Im_i^q$ and

$$\left.\begin{aligned} &\frac{d^{i+1}\zeta_{i0}(x)}{dx^{i+1}} = 0, \quad \frac{d^{i+1}\zeta_{iN}(x)}{dx^{i+1}} = 0, \quad 0 \le x \le L \\ &\frac{d^{i+1}\zeta_{ir}(x)}{dx^{i+1}} = 0, \quad 0 \le x < x_r \quad and \quad x_r < x \le L, \end{aligned}\right\} \tag{J.4}$$

where $r = 1, 2, ..., N - 1$ and $i = 2, 3$. The sets $\mathscr{S}_i^q$, $\mathscr{F}_i^q$ and $\Im_i^q$ are defined in Definitions 3.3.1 and 3.3.2. Let S_n be spanned by n functions consisting of the m_1 linearly independent functions $\{\zeta_{ir}(x)\}$ in addition to $\{\psi_m(x), m = 1, ..., n - m_1\}$. Suppose that $\zeta_{st}(x)$ ($\neq 0$) $\in S_n$ where s and t are two (known) positive integers that satisfy $2 \le s \le 3$

and $0 \le t \le N$. Then, for a non-zero $\zeta_{st}(x)$ and a set of arbitrary real constants $\{b_{tr}\}$ in which $b_{st} = 1$ and $b_{tr} = 0$ if $\zeta_{tr}(x) \notin S_n$, there exists a positive constant, c_1, which is independent of n and such that, for a sufficiently large n,

$$\| \sum_{t=2}^{3} \sum_{r=0}^{N} b_{tr}\zeta''_{tr} - \sum_{m=1}^{n-m_1} d_m \frac{\psi''_m}{\|\psi''_m\|_H} \|_H > c_1 n^{-h}, \quad h = (s-1) \quad and \quad b_{st} = 1, \tag{J.5}$$

whilst

$$\left. \begin{aligned} d_m &= \|\psi''_m\|_H^{-1} \int_0^L (\sum_{t=2}^{3} \sum_{r=0}^{N} b_{tr}\zeta''_{tr})\psi''_m dx \\ \textit{and} \qquad \|\psi''_m\|_H &= (\int_0^L (\psi''_m(x))^2 dx)^{1/2} = (\frac{\Omega_m}{L})^2 \end{aligned} \right\} \tag{J.6}$$

if (1) for a set of arbitrary constants, $\{\varrho_{kr}, k = 0, 1 \text{ and } r = 0, 1, ..., N\}$, satisfying $\varrho_{vt} \neq 0$, $\varrho_{0r} = 0$ for $\zeta_{3r}(x) \notin S_n$ and $\varrho_{1r} = 0$ for $\zeta_{2r}(x) \notin S_n$, there is a positive integer, $m_0 < n$, such that the rays OA_r, with $\zeta_{2r}(x) \in S_n$ or $\zeta_{3r}(x) \in S_n$, $1 < r < N$, can be rotated synchronously into the plain regions defined in Figure J.1 with the exception that just one ray OA_{r_0}, corresponding to a rational number x_{r_0}/L, may coincide with the χ_v-axis of this figure, where v is a non-negtive integer satisfying $v = (3 - s)$ ***mod*** 3; and if

(2) the relations

$$\varrho_{vt}\alpha^{(v)}_{tm_0} \neq 0 \quad and \quad (\varrho_{vt}\alpha^{(v)}_{tm_0})(\varrho_{vr}\alpha^{(v)}_{rm_0}) \ge 0 \tag{J.7}$$

hold for all r with $r \neq t$ and $\zeta_{rt}(x)$ $(\neq 0) \in S_n$. In (J.7),

$$
\alpha^{(i)}_{tm_0} = \begin{cases} \lim\limits_{k\to\infty}\left(\dfrac{L}{\Omega_{m_0+2k}}\right)^i \dfrac{d^i\psi_{m_0+2k}(x_t)}{dx^i}, & i = 0,\ 1 \textit{ and } t = 0,\ N \\[2ex] \left(\dfrac{2}{L}\right)^{1/2}\left(\dfrac{L}{m_0\pi}\right)^i \dfrac{d^i\cos\left(\dfrac{m_0\pi x_t}{L}+\theta_0(x_t)\right)}{dx^i}, & i = 0,\ 1 \textit{ and } 0 < t < N \end{cases} \tag{J.8}
$$

where k is a positive integer and d^0/dx^0 implies the function itself. However,

$$
\left.\begin{aligned} &\mathfrak{Q}_{v_1r_2}\alpha^{(v_1)}_{r_2m_0}\left(\varrho_0\,\mathfrak{Q}_{v_10}\alpha^{(v_1)}_{0m_0}+\mathfrak{B}_{r_1}\,\mathfrak{Q}_{v_1r_1}\alpha^{(v_1)}_{r_1m_0}+\varrho_N\,\mathfrak{Q}_{v_1N}\alpha^{(v_1)}_{Nm_0}\right) \geq 0 \\ &\textit{if } \left(\varrho_0\,\mathfrak{Q}_{v_10}\alpha^{(v_1)}_{0m_0}+\mathfrak{B}_{r_1}\,\mathfrak{Q}_{v_1r_1}\alpha^{(v_1)}_{r_1m_0}+\varrho_N\,\mathfrak{Q}_{v_1N}\alpha^{(v_1)}_{Nm_0}\right) \neq 0,\ \zeta_{v_2r_2}(x)\ (\neq 0) \in S_n. \end{aligned}\right\} \tag{J.9}
$$

also,

$$
\left.\begin{aligned} &(\mathfrak{Q}_{v_1r_3}\alpha^{(v_1)}_{r_3m_0})(\mathfrak{Q}_{v_1r_4}\alpha^{(v_1)}_{r_4m_0}) \geq 0,\ \zeta_{v_2r_3}(x)\ (\neq 0) \in S_n,\ \zeta_{v_2r_4}(x)\ (\neq 0) \in S_n \\ &\textit{if } \left(\varrho_0\,\mathfrak{Q}_{v_10}\alpha^{(v_1)}_{0m_0}+\mathfrak{B}_{r_1}\,\mathfrak{Q}_{v_1r_1}\alpha^{(v_1)}_{r_1m_0}+\varrho_N\,\mathfrak{Q}_{v_1N}\alpha^{(v_1)}_{Nm_0}\right) = 0, \end{aligned}\right\} \tag{J.10}
$$

and

$$
\left.\begin{aligned} &\left(\varrho_0\,\mathfrak{Q}_{v_10}\alpha^{(v_1)}_{0m_0}+\mathfrak{B}_{r_1}\,\mathfrak{Q}_{v_1r_1}\alpha^{(v_1)}_{r_1m_0}+\varrho_N\,\mathfrak{Q}_{v_1N}\alpha^{(v_1)}_{Nm_0}\right) \gtrless 0 \\ &\textit{if } \zeta_{v_2r_1}(x)\ (\neq 0) \in S_n,\ \zeta_{v_2r}(x) \notin S_n \quad \textit{providing } r \neq r_1,\ 0 < r < N \end{aligned}\right\} \tag{J.11}
$$

for all positive integers $r_l\,(l = 1, 2, 3, 4)$ satisfying $0 < r_l < N$. Furthermore, $r_l \neq r_1$ for $l = 2, 3, 4$. Moreover, the corresponding $x_{r_1}/L = j_{r_1}/j$ is rational when j_{r_1} and j are two positive integers. Also, $v_1 = (s + 1)$ ***mod*** 3, $v_2 = 1 + (7s \bmod 4)$. Moreover, $r_0 = r_1$ for a uniform free-sliding beam or a free-pinned beam; and

$$\rho_r = \begin{cases} 1 \text{ if } \zeta_{v_2 r}(x)\ (\neq 0) \in S_n,\ \ r = 0, N \\ 0 \text{ if } \zeta_{v_2 r}(x) \equiv 0 \text{ or } \zeta_{v_2 r}(x)\ (\neq 0) \notin S_n,\ \ r = 0, N \end{cases} \tag{J.12}$$

whilst

$$\mathfrak{B}_{r_1} = \begin{cases} 1 \text{ if } \zeta_{v_2 r_1}(x) \in S_n \text{ and } r_1 \neq 0, N \\ 0 \text{ if } r_1 = 0, N. \end{cases} \tag{J.13}$$

Proof

Parseval's identity applied to the left side of (J.5) produces

$$\| \sum_{i=2}^{3} \sum_{r=0}^{N} b_{ir} \zeta''_{ir} - \sum_{m=1}^{n-m_1} d_m \frac{\psi''_m}{\|\psi''_m\|_H} \|_H^2 = \sum_{m=n-m_1+1}^{\infty} |d_m|^2. \tag{J.14}$$

Thus, if there exist a positive constant c and a positive integer m_2 ($\geq n$) such that

$$d_{m_2} > cn^{-(s-1)}, \tag{J.15}$$

then

$$\left\| \sum_{i=2}^{3} \sum_{r=0}^{N} b_{ir}\zeta''_{ir} - \sum_{m=1}^{n-m_1} d_m \frac{\psi''_m}{\|\psi''_m\|_H} \right\|_H = \Big(\sum_{m=n-m_1+1}^{\infty} |d_m|^2 \Big)^{1/2}$$

$$= \Big(\sum_{m=n-m_1+1}^{m_2-1} |d_m|^2 + |d_{m_2}|^2 + \sum_{m=m_2+1}^{\infty} |d_m|^2 \Big)^{1/2} \tag{J.16}$$

$$\geq (|d_{m_2}|^2)^{1/2} > c^{1/2} n^{-(s-1)}.$$

By taking $c_1 = c^{1/2}$, the last inequality indicates that Lemma J.1 holds. All the following development is needed to show the existence of c and m_2 so that, indeed, (J.15) holds.

Suppose that all x_r/L are rational. They are denoted by [59]

$$x_r/L = j_r/j, \quad r = 1, \ldots, N-1 \tag{J.17}$$

where j_r and j are two positive integers. Then, by employing (J.17) for a given integer m_0 such that $1 \leq m_0 < n$, it is known [55] that

$$\begin{aligned}(2j+m_0)\pi x_r/L \bmod 2\pi &= (2j_r\pi + m_0\pi x_r/L) \bmod 2\pi \\ &= m_0\pi x_r/L \bmod 2\pi, \\ r &= 1, 2, \ldots, N-1.\end{aligned} \tag{J.18}$$

This last equation indicates that, regardless of r, the ray OA_r returns to its initial position after $2j$ stepped increments (i.e. the periodicity is $2j$).

Define, next, a function, $f_{st}(x)$, as

$$f_{st}(x) = \sum_{r=0}^{N} (b_{2r}\zeta''_{2r}(x) + b_{3r}\zeta''_{3r}(x)), \quad b_{st} = 1, \tag{J.19}$$

for a set of given real constants $\{b_{ir}\}$, $i = 2, 3$. Suppose that $f_{st}(x)$ has the generalized

Fourier series expansion

$$f_{u}(x) = \sum_{m=1}^{\infty} d_m \frac{\psi''_m(x)}{\|\psi''_m\|_H} \tag{J.20}$$

where d_m is defined by the first equation denoted (J.6). Integrating the right side of this equation by parts leads to

$$d_m = L^2\left(\frac{T_{1,m}}{\Omega_m} + \frac{T_{2,m}}{\Omega_m^2}\right) \tag{J.21}$$

where

$$T_{1,m} = \sum_{r=0}^{N} a_{2r} \frac{1}{\Omega_m} \psi'_m(x_r) \quad and \quad T_{2,m} = \sum_{r=0}^{N} a_{3r} \psi_m(x_r) \tag{J.22}$$

whilst

$$\left.\begin{array}{l}
a_{20} = -b_{20}\dfrac{d^2\zeta_{20}(0)}{dx^2}, \quad a_{2N} = b_{2N}\dfrac{d^2\zeta_{2N}(L)}{dx^2} \\
\textit{but} \\
a_{2r} = -b_{2r}\left(\dfrac{d^2\zeta_{2r}(x_r^-)}{dx^2} - \dfrac{d^2\zeta_{2r}(x_r^+)}{dx^2}\right), \quad r = 1, 2, \ldots, N-1 \\
\textit{whilst} \\
a_{30} = b_{30}\dfrac{d^3\zeta_{30}(0)}{dx^3}, \quad a_{3N} = -b_{3N}\dfrac{d^3\zeta_{3N}(L)}{dx^3} \\
\textit{and} \\
a_{3r} = b_{3r}\left(\dfrac{d^3\zeta_{3r}(x_r^+)}{dx^3} - \dfrac{d^3\zeta_{3r}(x_r^-)}{dx^3}\right), \quad r = 1, 2, \ldots, N-1.
\end{array}\right\} \tag{J.23}$$

It can be seen from (J.21) and (J.22) that the validity of (J.15) depends upon the analytical properties of $T_{1,m}$ and $T_{2,m}$ which are functions of $\psi_m(x_r)$ and $\psi'_m(x_r)$. Thus, to prove Lemma J.1, some analytical properties of $T_{1,m}$ and $T_{2,m}$, as well as $\psi_m(x_r)$ and $\psi'_m(x_r)$, are needed. Consider, for example, a free-sliding uniform beam. Suppose $s = 2$. Let ϱ_{0r} and ϱ_{1r} in requirement (1) of Lemma J.1 be given by $\varrho_{0r} = a_{3r}$ and $\varrho_{1r} = a_{2r}$, respectively. If requirement (1) is to be satisfied, there must exist an integer, m_0, such that the rays OA_r, with $\zeta_{2r}(x) \in S_n$ or $\zeta_{3r}(x) \in S_n$, $1 < r < N$, can be rotated synchronously into one of the four plain regions of Figure J.1 with the possible exception that just one OA_{r_0} coincides with the χ_1-axis. It is known, on the other hand, from Lemma K.5 of Appendix K that

$$|\frac{L}{\Omega_m}\psi'_{m_0+2jk}(x_r)| > \frac{1}{10}(\frac{2}{L})^{1/2} \tag{J.24}$$

for all r satisfying $1 \le r \le N - 1$, a sufficiently large, positive integer k and a fixed, positive integer j. Furthermore, if OA_{r_0} coincides with the χ_1-axis, then

$$|\psi_{m_0+2jk}(x_r)| > \frac{1}{10}(\frac{2}{L})^{1/2}, \quad r \neq r_0, \tag{J.25}$$

where $\psi_{m_0+2jk}(x)$ is the $(m_0 + 2kj)$ th eigenvector of the free-sliding beam.

It is known [50] that $\psi''_m(L) \neq 0$. Hence, Definition 3.3.1 indicates that $\zeta_{2N}(x) \equiv 0$. This leads to $t \neq N$ in Lemma J.1 for a free-sliding uniform beam. Thus, if (J.7) holds in addition to (J.24) and (J.25) then, when $s = 2$, $\nu = (3 - s)$ ***mod*** $3 = 1$ so that

$$a_{2t}\alpha^{(1)}_{tm_0} \neq 0, \; t \neq N, \; (a_{2t}\alpha^{(1)}_{tm_0})(a_{2r}\alpha^{(1)}_{rm_0}) \ge 0, \; \zeta_{2r}(x) \neq 0. \tag{J.26}$$

Furthermore, it is seen from (K.3), (K.61) of Lemma K.5 and Remark K.1 that there exists a positive constant, m_3, such that

$$\psi'_m(x_N) = \psi'_m(L) \equiv 0 \quad and \quad \alpha^{(1)}_{rm_0}\psi'_{m_0+2kj}(x_r) > 0,\ r \neq N \tag{J.27}$$

for all m and $m_0 + 2kj > m_3$. Inequality (J.27), when combined with (J.26), leads then to

$$(a_{2t}\alpha^{(1)}_{tm_0})(a_{2r}\psi'_{m_0+2kj}(x_r)) \geq 0 \quad for\ all\ r. \tag{J.28}$$

This last inequality as well as (J.22) and (J.24), together with the periodicity, $2j$, of rotation of ray OA_r, lead immediately to the inequalities

$$\left.\begin{aligned} & |T_{1,(m_0+2kj)}| \geq |\frac{1}{L}a_{2t}\frac{L}{\Omega_{m_0+2kj}}\psi'_{m_0+2kj}(x_t)| > \frac{1}{10L}(\frac{2}{L})^{1/2}|a_{2t}| \\ and \quad & \\ & T_{1,(m_0+2kj)}T_{1,(m_0+2k_1j)} > 0 \end{aligned}\right\} \tag{J.29}$$

where k and k_1 are two arbitrary positive integers that satisfy $m_0 + 2kj > m_3$ and $m_0 + 2k_1 j > m_3$. Also, Ω_{m_0+2kj} is the $(m_0 + 2kj)$ th characteristic value of the free-sliding beam. Furthermore, when $s = 2$, $\nu_1 = (s + 1)$ ***mod*** $3 = 0$ and $\nu_2 = 1 + (7s \bmod 4) = 3$. Suppose that (J.9) is true and $\zeta_{30}(x) \in S_n$, $\zeta_{3r}(x) \in S_n$, $\zeta_{3r_1}(x) \in S_n$ and $\zeta_{3N}(x) \in S_n$. Then, by employing (J.8) as well as (J.12) and (J.13), the inequality

$$\left.\begin{aligned} & a_{3r}\alpha^{(0)}_{rm_0}(a_{30}\alpha^{(0)}_{0m_0} + a_{3r_1}\alpha^{(0)}_{r_1m_0} + a_{3N}\alpha^{(0)}_{Nm_0}) \geq 0 \\ & if\ (a_{30}\alpha^{(0)}_{0m_0} + a_{3r_1}\alpha^{(0)}_{r_1m_0} + a_{3N}\alpha^{(0)}_{Nm_0}) \neq 0 \end{aligned}\right\} \tag{J.30}$$

can be shown for $\varrho_{00} = a_{30}$, $\varrho_{0r} = a_{3r}$, $\varrho_{0r_1} = a_{3r_1}$ and $\varrho_{0N} = a_{3N}$. Here the notation $d^0\psi_{m_0}(x_r)/dx^0 \equiv \psi_{m_0}(x_r)$ is used again whilst $r_2 \equiv r$ and $r_1 \equiv r_0$ are two arbitrary integers satisfying $1 \leq r < N$, $1 \leq r_1 < N$ and $r_1 \neq r$. It is known from (K.22), (K.24) and (K.25) that

$$\alpha_{0m}^{(0)} = \frac{2}{L^{1/2}}, \quad \alpha_{r_1 m}^{(0)} = \left(\frac{2}{L}\right)^{1/2} \cos\left(\frac{m\pi x_{r_1}}{L} + \theta_0(x_{r_1})\right) \tag{J.31}$$

and

$$\alpha_{Nm_0}^{(0)} = (-1)^{m_0+1} \left(\frac{2}{L}\right)^{1/2} \tag{J.32}$$

where, from (K.23),

$$\theta_0(x_r) = \left(1 - \frac{5j_r}{j}\right)\frac{\pi}{4}. \tag{J.33}$$

On the other hand, it is known from (K.28) that $\psi_m(x_r)$ can be written, for an arbitrary and sufficiently large positive integer m, in the form

$$\psi_m(x_r) = \alpha_{rm}^{(0)} + \mathbf{A}_{rm} \tag{J.34}$$

where, from (K.26),

$$\mathbf{A}_{0m} \equiv 0, \quad \mathbf{A}_{rm} = \frac{1}{L^{1/2}} \exp\left(-\Omega_m \frac{x_r}{L}\right)\left[1 + O(\exp(-\Omega_m))\right], \quad r \neq 0. \tag{J.35}$$

Thus, by employing (J.31) and (J.34) as well as (J.35), the $T_{2,m}$ defined in (J.22) can be expanded as

$$\begin{aligned} T_{2,m} = \sum_{r=0}^{N} a_{3r}\psi_m(x_r) &= (a_{30}\alpha_{0m}^{(0)} + a_{30}\mathbf{A}_{0m}) + \sum_{r=1}^{r_1-1} a_{3r}\psi_m(x_r) + (a_{3r_1}\alpha_{r_1 m}^{(0)} + a_{3r_1}\mathbf{A}_{r_1 m}) \\ &\quad + \sum_{r=r_1+1}^{N-1} a_{3r}\psi_m(x_r) + (a_{3N}\alpha_{Nm}^{(0)} + a_{3N}\mathbf{A}_{Nm}) \end{aligned}$$

or

$$T_{2,m} = \frac{2}{L^{1/2}}a_{30} + \sum_{r=1}^{r_1-1} a_{3r}\Psi_m(x_r) + (a_{3r_1}\alpha_{r_1 m}^{(0)} + a_{3r_1}\mathbf{A}_{r_1 m})$$
$$+ \sum_{r=r_1+1}^{N-1} a_{3r}\Psi_m(x_r) + (a_{3N}\alpha_{Nm}^{(0)} + a_{3N}\mathbf{A}_{Nm}). \tag{J.36}$$

This last equation leads to

$$T_{2,m} - (a_{3r_1}\mathbf{A}_{r_1 m} + a_{3N}\mathbf{A}_{Nm}) = \frac{2}{L^{1/2}}a_{30} + \sum_{r=1}^{r_1-1} a_{3r}\Psi_m(x_r) + a_{3r_1}\alpha_{r_1 m}^{(0)}$$
$$+ \sum_{r=r_1+1}^{N-1} a_{3r}\Psi_m(x_r) + a_{3N}\alpha_{Nm}^{(0)}. \tag{J.37}$$

Let k_2 be another arbitrary positive integer satisfying $m_0 + 2k_2 j > m_3$. Then, the inequalities

$$\Psi_{(m_0+2k_1 j)}(x_r)\,\alpha_{rm_0}^{(0)} > 0 \quad and \quad \Psi_{(m_0+2k_2 j)}(x_r)\,\alpha_{rm_0}^{(0)} > 0 \tag{J.38}$$

can be found from (K.60) of Lemma K.5. Furthermore, by employing (K.29), it is known that

$$\frac{2}{L^{1/2}}a_{30} + a_{3r_1}\alpha_{r_1(m_0+2kj)}^{(0)} + a_{3N}\alpha_{N(m_0+2kj)}^{(0)} = \frac{2}{L^{1/2}}a_{30} + a_{3r_1}\alpha_{r_1 m_0}^{(0)} + a_{3N}\alpha_{Nm_0}^{(0)} \tag{J.39}$$

for any positive integer k. Therefore, by employing (J.30), (J.38) and (J.39), it can be shown that

$$a_{3r}\Psi_{m_0+2k_i j}(x_r)\left(\frac{2}{L^{1/2}}a_{30} + a_{3r_1}\alpha_{r_1(m_0+2k_i j)}^{(0)} + a_{3N}\alpha_{N(m_0+2k_i j)}^{(0)}\right) \geq 0 \tag{J.40}$$

so that

$$
\begin{aligned}
&\Big(\frac{2}{L^{1/2}}a_{30}+\sum_{r=1}^{r_1-1} a_{3r}\Psi_{m_0+2k_i j}(x_r)+a_{3r_1}\alpha^{(0)}_{r_1(m+2k_i j)} \\
&\quad+\sum_{r=r_1+1}^{N-1} a_{3r}\Psi_{m_0+2k_i j}(x_r)+a_{3N}\alpha^{(0)}_{N(m+2k_i j)}\Big)\times \\
&\Big(\frac{2}{L^{1/2}}a_{30}+a_{3r_1}\alpha^{(0)}_{r_1(m+2k_i j)}+a_{3N}\alpha^{(0)}_{N(m+2k_i j)}\Big) \ge 0
\end{aligned}
\tag{J.41}
$$

with $i = 1$ and 2. Consequently, the inequality

$$
\begin{aligned}
&\Big(\frac{2}{L^{1/2}}a_{30}+\sum_{r=1}^{r_1-1} a_{3r}\Psi_{m_0+2k_1 j}(x_r)+a_{3r_1}\alpha^{(0)}_{r_1(m+2k_1 j)} \\
&\quad+\sum_{r=r_1+1}^{N-1} a_{3r}\Psi_{m_0+2k_1 j}(x_r)+a_{3N}\alpha^{(0)}_{N(m+2k_1 j)}\Big)\times \\
&\Big(\frac{2}{L^{1/2}}a_{30}+\sum_{r=1}^{r_1-1} a_{3r}\Psi_{m_0+2k_2 j}(x_r)+a_{3r_1}\alpha^{(0)}_{r_1(m+2k_2 j)} \\
&\quad+\sum_{r=r_1+1}^{N-1} a_{3r}\Psi_{m_0+2k_2 j}(x_r)+a_{3N}\alpha^{(0)}_{N(m+2k_2 j)}\Big) \ge 0
\end{aligned}
$$

or, from (J.37),

$$
\begin{aligned}
&[T_{2.(m_0+2k_1 j)}-(a_{3r_1}\mathbf{A}_{r_1(m_0+2k_1 j)}+a_{3N}\mathbf{A}_{N(m_0+2k_1 j)})][T_{2.(m_0+2k_2 j)} \\
&\qquad -(a_{3r_1}\mathbf{A}_{r_1(m_0+2k_2 j)}+a_{3N}\mathbf{A}_{N(m_0+2k_2 j)})] \ge 0
\end{aligned}
\tag{J.42}
$$

can be shown by employing (J.39) and (J.41). In addition to (J.29) and (J.42), a positive integer ℓ, defined by

$$
\ell = 2k_0 j,
\tag{J.43}
$$

is also needed in the proof of (J.15). Here k_0 is the positive quotient obtained when n is divided by $2j$. It is known, however, from the proof of Lemma K.1 given in Appendix K that ℓ satisfies the inequality

$$3n > m_0 + \ell > n \tag{J.44}$$

for a sufficiently large n.

(J.15) can be proved now by employing (J.24), (J.25), (J.27), (J.42) and (J.44). First, it is known from the analogous principle of real numbers [55] that, for two arbitrary but given finite real values a and b, either $a \geq b$ or $b < a$. Now $|d_{m_0+\ell}|$ and $(L^2/2\Omega_{m_0+\ell})|T_{1,(m_0+\ell)}|$ are two given, finite non-negative values. Thus, either

$$\frac{L^2}{2}\frac{|T_{1,(m_0+\ell)}|}{\Omega_{m_0+\ell}} \leq |d_{m_0+\ell}| = L^2\left|\frac{T_{1,(m_0+\ell)}}{\Omega_{m_0+\ell}} + \frac{T_{2,(m_0+\ell)}}{\Omega^2_{m_0+\ell}}\right| \tag{J.45}$$

by using (J.21) or

$$0 \leq |d_{m_0+\ell}| = L^2\left|\frac{T_{1,(m_0+\ell)}}{\Omega_{m_0+\ell}} + \frac{T_{2,(m_0+\ell)}}{\Omega^2_{m_0+\ell}}\right| < \frac{L^2}{2}\frac{|T_{1,(m_0+\ell)}|}{\Omega_{m_0+\ell}}. \tag{J.46}$$

If (J.45) holds, then it is seen from (J.29) that

$$|d_{m_0+\ell}| \geq \frac{L^2}{2}\frac{|T_{1,(m_0+\ell)}|}{\Omega_{m_0+\ell}} > \frac{L|a_{2\ell}|}{20}\left(\frac{2}{L}\right)^{1/2}\frac{1}{\Omega_{m_0+\ell}}. \tag{J.47}$$

On the other hand, it can be seen from (K.38) that

$$\Omega_{m_0+\ell} < 3n\pi \tag{J.48}$$

for a sufficiently large $(m_0 + \ell)$. Consequently, it can be found from (J.47) and (J.48) that

$$|d_{m_0+\ell}| > \frac{L|a_{2\ell}|}{20}\left(\frac{2}{L}\right)^{1/2}\frac{1}{\Omega_{m_0+\ell}} > \frac{L|a_{2\ell}|}{20}\left(\frac{2}{L}\right)^{1/2}\frac{1}{3n\pi} = \frac{L|a_{2\ell}|}{60\pi}\left(\frac{2}{L}\right)^{1/2}\frac{1}{n}. \tag{J.49}$$

It is known from (J.5) and (J.26) that $b_{2t} = 1$ when (*i*) $s = 2$ and (ii) a given positive integer t satisfies $0 \le t < N$. Substituting $b_{2t} = 1$, $0 \le t < N$, into (J.23) leads to

$$a_{2t} = \begin{cases} -\dfrac{d^2\zeta_{20}(0)}{dx^2}, & t = 0 \\ -\left(\dfrac{d^2\zeta_{2t}(x_t^-)}{dx^2} - \dfrac{d^2\zeta_{2r}(x_t^+)}{dx^2}\right), & t \ne 0,\ N. \end{cases} \tag{J.50}$$

Let

$$c = \frac{L\min(|a_{2t}|,\ t = 0,\ \dots,\ N-1)}{60\pi}\left(\frac{2}{L}\right)^{1/2} \quad and \quad m_2 = m_0 + \ell \tag{J.51}$$

so that (J.49) can be simplified to

$$|d_{m_2}| > cn^{-1}. \tag{J.52}$$

This last inequality means that, because $s = 2$, (J.15) is true. Consequently (J.5), indeed, holds.

A further study is needed when (J.46) rather than (J.45) holds. First, consider $T_{1,(m_0+\ell)} > 0$. Suppose

$$\left.\begin{aligned} & T_{1,(m_0+\ell)}\left(\frac{2}{L^{1/2}}a_{30} + a_{3r_1}\alpha^{(0)}_{r_1(m_0+\ell)} + a_{3N}\alpha^{(0)}_{N(m_0+\ell)}\right) \le 0 \\ and \quad & T_{1,(m_0+\ell)}\left(a_{3r_1}\mathbf{A}_{r_1(m_0+\ell)} + a_{3N}\mathbf{A}_{N(m_0+\ell)}\right) \le 0, \\ & T_{1,(m_0+\ell)}a_{3r_1}\mathbf{A}_{r_1(m_0+\ell)} \le 0. \end{aligned}\right\} \tag{J.53}$$

Then, by replacing m with $(m_0 + 3\ell)$ in (J.21), the equality

$$|d_{m_0+3\ell}| = L^2 \left| \frac{T_{1.(m_0+3\ell)}}{\Omega_{m_0+3\ell}} + \frac{T_{2.(m_0+3\ell)}}{\Omega^2_{m_0+3\ell}} \right|$$

$$= L^2 \left| \frac{T_{1.(m_0+\ell)} + T_{1.(m_0+3\ell)} - T_{1.(m_0+\ell)}}{\Omega_{m_0+3\ell}} \right. \tag{J.54}$$

$$+ \frac{T_{2.(m_0+\ell)} + T_{2.(m_0+3\ell)} - \Lambda_{m_0+3\ell} - (T_{2.(m_0+\ell)} - \Lambda_{m_0+\ell})}{\Omega^2_{m_0+3\ell}}$$

$$\left. + \frac{\Lambda_{m_0+3\ell} - \Lambda_{m_0+\ell}}{\Omega^2_{m_0+3\ell}} \right|$$

can be obtained because

$$T_{1.(m_0+\ell)} - T_{1.(m_0+\ell)} \equiv T_{2.(m_0+\ell)} - T_{2.(m_0+\ell)} \equiv \Lambda_{m_0+\ell} - \Lambda_{m_0+\ell} \equiv 0. \tag{J.55}$$

Moreover,

$$\left.\begin{aligned} \Lambda_{m_0+\ell} &= a_{3r_1} \mathbf{A}_{r_1(m_0+\ell)} + a_{3N} \mathbf{A}_{N(m_0+\ell)} \\ \text{and} \qquad \Lambda_{m_0+3\ell} &= a_{3r_1} \mathbf{A}_{r_1(m_0+3\ell)} + a_{3N} \mathbf{A}_{N(m_0+3\ell)}. \end{aligned}\right\} \tag{J.56}$$

Let

$$\delta_{1.(m_0+\ell)} = T_{1.(m_0+3\ell)} - T_{1.(m_0+\ell)} \tag{J.57}$$

and

$$\delta_{2.(m_0+\ell)} = (T_{2.(m_0+3\ell)} - \Lambda_{m_0+3\ell}) - (T_{2.(m_0+\ell)} - \Lambda_{m_0+\ell}). \tag{J.58}$$

Then, by employing the identities

$$(\frac{\Omega_{m_0+\ell}}{\Omega_{m_0+3\ell}})^2\frac{\Omega_{m_0+3\ell}}{\Omega_{m_0+\ell}}\frac{1}{\Omega_{m_0+\ell}} \equiv \frac{1}{\Omega_{m_0+3\ell}} \quad \textit{and} \quad (\frac{\Omega_{m_0+\ell}}{\Omega_{m_0+3\ell}})^2\frac{1}{\Omega^2_{m_0+\ell}} \equiv \frac{1}{\Omega^2_{m_0+3\ell}}, \tag{J.59}$$

(J.54) can be simplified to

$$\begin{aligned} |d_{m_0+3\ell}| &= L^2(\frac{\Omega_{m_0+\ell}}{\Omega_{m_0+3\ell}})^2 \left| \frac{\Omega_{m_0+3\ell}}{\Omega_{m_0+\ell}} \frac{T_{1.(m_0+\ell)}+\delta_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}} \right. \\ &\left. + \frac{T_{2.(m_0+\ell)}+\delta_{2.(m_0+\ell)}}{\Omega^2_{m_0+\ell}} + \frac{\Lambda_{m_0+3\ell}-\Lambda_{m_0+\ell}}{\Omega^2_{m_0+\ell}} \right|, \end{aligned} \tag{J.60}$$

To apply (J.60) in proving (J.15), the following inequalities are needed

$$|\delta_{1.(m_0+\ell)}| < \frac{1}{500}|T_{1.(m_0+\ell)}| \tag{J.61}$$

and

$$|\delta_{2.(m_0+\ell)}| < \frac{1}{500}|T_{2.(m_0+\ell)}-\Lambda_{m_0+\ell}|. \tag{J.62}$$

To derive (J.61), the following equality, which is obtained by employing (K.25) and (K.26),

$$\begin{aligned} \psi_{m_0+3\ell}(x_r)-\psi_{m_0+\ell}(x_r) &= \alpha^{(0)}_{r(m_0+3\ell)}+\mathbf{\Lambda}_{r(m_0+3\ell)}-(\alpha^{(0)}_{r(m_0+\ell)}+\mathbf{\Lambda}_{r(m_0+\ell)}) \\ &= \mathbf{\Lambda}_{r(m_0+3\ell)}-\mathbf{\Lambda}_{r(m_0+\ell)} \end{aligned}$$

or, from (K.26),

$$\psi_{m_0+3\ell}(x_r)-\psi_{m_0+\ell}(x_r) = \frac{1}{L^{1/2}}(\exp(-\Omega_{m_0+3\ell}\frac{x_r}{L})[1+O(\exp(-\Omega_{m_0+3\ell}))]-$$
$$\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})[1+O(\exp(-\Omega_{m_0+\ell}))]) \tag{J.63}$$

is useful. In fact, by employing (K.66), it can be found that

$$|\psi_{m_0+3\ell}(x_r)-\psi_{m_0+\ell}(x_r)| \le \frac{2}{L^{1/2}}(|\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})|+|\exp(-\Omega_{m_0+3\ell}\frac{x_r}{L})|).$$

Hence, from (K.44),

$$|\psi_{m_0+3\ell}(x_r)-\psi_{m_0+\ell}(x_r)| < 10^{-4}(\frac{1}{L})^{1/2} < 10^{-4}(\frac{2}{L})^{1/2} \tag{J.64}$$

for a sufficiently large $m_0+\ell$. Similarly, from (K.21) rather than (K.20),

$$|\frac{L}{\Omega_{m_0+3\ell}}\psi'_{m_0+3\ell}(x_r)-\frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r)| \le (\frac{2}{L})^{1/2}(|\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})|$$
$$+|\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})|) \tag{J.65}$$
$$< 10^{-4}(\frac{2}{L})^{1/2}.$$

By employing (J.64) and (J.65) with (J.25) and (J.24), respectively, and remembering that $\ell = 2jk_0$, it can be seen that

$$|\psi_{m_0+3\ell}(x_r)-\psi_{m_0+\ell}(x_r)| < 10^{-3}\frac{1}{10}(\frac{2}{L})^{1/2}$$
$$< 10^{-3}|\psi_{m_0+\ell}(x_r)|$$

or

$$|\psi_{m_0+3\ell}(x_r) - \psi_{m_0+\ell}(x_r)| < \frac{1}{500}|\psi_{m_0+\ell}(x_r)| \tag{J.66}$$

and, similarly,

$$\begin{aligned}|\frac{L}{\Omega_{m_0+3\ell}}\psi'_{m_0+3\ell}(x_r) - \frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r)| &< 10^{-3}\frac{1}{10}(\frac{2}{L})^{1/2} \\ &< 10^{-3}|\frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r)\ |\end{aligned}$$

or

$$|\frac{L}{\Omega_{m_0+3\ell}}\psi'_{m_0+3\ell}(x_r) - \frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r)| < \frac{1}{500}|\frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r)|. \tag{J.67}$$

Consequently, it can be shown from the definition of $\delta_{1,(m_0+\ell)}$, i.e. (J.57), and (J.22) as well as (J.67) that

$$\begin{aligned}500|\delta_{1,(m_0+\ell)}| &= 500|T_{1,(m_0+3\ell)} - T_{1,(m_0+\ell)}| \\ &= \frac{500}{L}|\sum_{r=0}^{N} a_{2r}(\frac{L}{\Omega_{m_0+3\ell}}\psi'_{m_0+3\ell}(x_r) - \frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r))| \\ &\le \frac{500}{L}\sum_{r=0}^{N}|a_{2r}|\,|(\frac{L}{\Omega_{m_0+3\ell}}\psi'_{m_0+3\ell}(x_r) - \frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r))| \\ &< \frac{1}{L}\sum_{r=0}^{N}|a_{2r}|\,|\frac{L}{\Omega_{m_0+\ell}}\psi'_{m_0+\ell}(x_r)| = \sum_{r=0}^{N}|a_{2r}(\frac{1}{\Omega_{m_0+\ell}})\psi'_{m_0+\ell}(x_r)|\end{aligned}$$

or, from (J.22) again,

$$|\delta_{1,(m_0+\ell)}| < \frac{1}{500}|T_{1,(m_0+\ell)}|. \tag{J.68}$$

This last inequality is just (J.61). A similar proof can be given for inequality (J.62). On the other hand, it is known from (K.74) that

$$\mathring{\Lambda}_{r_1(m_0+\ell)} - \mathring{\Lambda}_{r_1(m_0+3\ell)} > 0 \tag{J.69}$$

and

$$\mathring{\Lambda}_{N(m_0+\ell)} - \mathring{\Lambda}_{N(m_0+3\ell)} > 0 \tag{J.70}$$

for a sufficiently large $(m_0 + \ell)$. On the other hand, it can be found from (J.35) that $\mathring{\Lambda}_{r_1 m}$ and $\mathring{\Lambda}_{Nm}$ are both positive for a sufficiently large m. Thus, for $T_{1,(m_0+\ell)} > 0$, inequality (J.53) indicates that $a_{3r_1} \leq 0$. Then, by employing (J.56), (J.69) and (J.70), it can be shown that, if $a_{3N} \leq 0$, the $(\Lambda_{m_0+3\ell} - \Lambda_{m_0+\ell})$ used in (J.60) satisfies

$$\begin{aligned} \Lambda_{m_0+3\ell} - \Lambda_{m_0+\ell} &= (a_{3r_1}\mathring{\Lambda}_{r_1(m_0+3\ell)} + a_{3N}\mathring{\Lambda}_{N(m_0+3\ell)}) - (a_{3r_1}\mathring{\Lambda}_{r_1(m_0+\ell)} + a_{3N}\mathring{\Lambda}_{N(m_0+\ell)}) \\ &= -a_{3r_1}(\mathring{\Lambda}_{r_1(m_0+\ell)} - \mathring{\Lambda}_{r_1(m_0+3\ell)}) - a_{3N}(\mathring{\Lambda}_{N(m_0+\ell)} - \mathring{\Lambda}_{N(m_0+3\ell)}) \geq 0 \end{aligned}$$

i.e.

$$\Lambda_{m_0+3\ell} - \Lambda_{m_0+\ell} \geq 0. \tag{J.71}$$

The equality in (J.71) holds, of course, when $a_{3r_1} = a_{3N} = 0$. Furthermore, it is known from the elementary algebraic theory [60] that $|a| \leq b$ is equivalent to $-b \leq a \leq b$ whilst $-c \leq a + b \leq c$ is equivalent to $-(c + b) \leq a \leq (c - b)$. Here the generic b is an arbitrary finite <u>positive</u> value whilst a and c are arbitrary finite real values. Thus, the inequality

$$-\frac{1}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}} \le \frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}} + \frac{T_{2.(m_0+\ell)}}{\Omega^2_{m_0+\ell}} \le \frac{1}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}$$

or

$$-\frac{3}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}} \le \frac{T_{2.(m_0+\ell)}}{\Omega^2_{m_0+\ell}} \le -\frac{1}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}} \tag{J.72}$$

can be obtained straightforwardly from (J.46) for $T_{1.(m_0+\ell)} > 0$. Moreover

$$T_{2.(m_0+\ell)} \equiv T_{2.(m_0+\ell)} - \Lambda_{m_0+\ell} + \Lambda_{m_0+\ell} \tag{J.73}$$

so that

$$-\frac{3}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}} \le \frac{T_{2.(m_0+\ell)} - \Lambda_{m_0+\ell}}{\Omega^2_{m_0+\ell}} + \frac{\Lambda_{m_0+\ell}}{\Omega^2_{m_0+\ell}} \le -\frac{1}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}. \tag{J.74}$$

On the other hand, it is known from (J.37) that,

$$\begin{aligned} T_{2.(m_0+\ell)} - (a_{3r_1}\mathbf{A}_{r_1(m_0+\ell)} + a_{3N}\mathbf{A}_{N(m_0+\ell)}) &= \frac{2}{L^{1/2}}a_{30} + \sum_{r=1}^{r_1-1} a_{3r}\Psi_{m_0+\ell}(x_r) + a_{3r_1}\alpha^{(0)}_{r_1(m_0+\ell)} \\ &\quad + \sum_{r=r_1+1}^{N-1} a_{3r}\Psi_{m_0+\ell}(x_r) + a_{3N}\alpha^{(0)}_{N(m_0+\ell)}. \end{aligned}$$

Hence, from the definition (J.56) of Λ_{rm},

$$T_{2,(m_0+\ell)} - \Lambda_{m_0+\ell} = \frac{2}{L^{1/2}} a_{30} + \sum_{r=1}^{r_1-1} a_{3r} \Psi_{m_0+\ell}(x_r) + a_{3r_1} \alpha^{(0)}_{r_1(m_0+\ell)} + \sum_{r=r_1+1}^{N-1} a_{3r} \Psi_{m_0+\ell}(x_r) + a_{3N} \alpha^{(0)}_{N(m_0+\ell)} . \tag{J.75}$$

Furthermore, it is known from (J.43) that $\ell = 2k_0 j$. Consequently, when $m_0 + \ell > m_3$, it is seen from (J.41) that

$$(T_{2,(m_0+\ell)} - \Lambda_{m_0+\ell})(\frac{2}{L^{1/2}} a_{30} + a_{3r_1} \alpha^{(0)}_{r_1 m_0} + a_{3N} \alpha^{(0)}_{N m_0}) \geq 0. \tag{J.76}$$

However, (J.53) indicates that

$$T_{1,(m_0+\ell)}(\frac{2}{L^{1/2}} a_{30} + a_{3r_1} \alpha^{(0)}_{r_1(m_0+\ell)} + a_{3N} \alpha^{(0)}_{N(m_0+\ell)}) \leq 0 \tag{J.77}$$

and

$$T_{1,(m_0+\ell)}(a_{3r_1} \mathbf{A}_{r_1(m_0+\ell)} + a_{3N} \mathbf{A}_{N(m_0+\ell)}) \leq 0 \tag{J.78}$$

or, from (J.56),

$$T_{1,(m_0+\ell)} \Lambda_{m_0+\ell} \leq 0. \tag{J.79}$$

Thus, when $T_{1,(m_0+\ell)} > 0$, (J.74) and (J.76) through (J.79) lead to

$$\frac{(T_{2,(m_0+\ell)} - \Lambda_{m_0+\ell})}{\Omega^2_{m_0+\ell}} \leq 0 \quad and \quad \frac{\Lambda_{m_0+\ell}}{\Omega^2_{m_0+\ell}} \leq 0. \tag{J.80}$$

Consequently, by employing (J.80), the inequality

$$|d_{m_0+3\ell}| \geq \frac{L^2}{15}[\frac{9}{10}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}+\frac{1}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}-|\frac{\delta_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}|-|\frac{\delta_{2.(m_0+\ell)}}{\Omega^2_{m_0+\ell}}|] \quad (J.84)$$

for $a_{3r_1} \leq 0$ and $a_{3N} \leq 0$. By using (J.61) and (J.62), (J.84) becomes

$$|d_{m_0+3\ell}| > \frac{L^2}{15}[\frac{9}{10}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}+\frac{1}{2}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}-\frac{1}{500}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}} - \frac{1}{500}|\frac{T_{2.(m_0+\ell)}-\Lambda_{m_0+\ell}}{\Omega^2_{m_0+\ell}}|]$$

or

$$|d_{m_0+3\ell}| > \frac{L^2}{15}[\frac{9}{10}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}+\frac{1}{4}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}-\frac{1}{500}|\frac{T_{2.(m_0+\ell)}-\Lambda_{m_0+\ell}}{\Omega^2_{m_0+\ell}}|] \quad (J.85)$$

because (1/2 - 1/500) > 1/4. Moreover, the last inequality, when combined with (J.78), yields

$$|d_{m_0+3\ell}| > \frac{L^2}{15}(\frac{9}{10}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}+\frac{1}{4}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}}-\frac{3}{1000}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}})$$

$$> \frac{L^2}{15}(\frac{9}{10}\frac{T_{1.(m_0+\ell)}}{\Omega_{m_0+\ell}})$$

because (1/4 - 3/1000) > 0. In summary,

$$|d_{m_0+3\ell}| > \frac{3}{500} L^2 \frac{T_{1,(m_0+\ell)}}{\Omega_{m_0+\ell}}. \tag{J.86}$$

Then it can be seen from (J.29) that, as $\ell = 2k_0 j$,

$$|d_{m_0+3\ell}| \geq \frac{3}{500} L^2 \frac{T_{1,(m_0+\ell)}}{\Omega_{m_0+\ell}} \geq \frac{3}{500} L^2 \frac{|a_{2t}|}{10L} (\frac{2}{L})^{1/2} \frac{1}{\Omega_{m_0+\ell}}. \tag{J.87}$$

By using (K.40), (J.87) can be simplified to

$$|d_{m_0+3\ell}| \geq \frac{L|a_{2t}|}{5000\pi} (\frac{2}{L})^{1/2} \frac{1}{n}. \tag{J.88}$$

Let

$$m_2 = m_0 + 3\ell \quad and \quad c = \frac{L \min(|a_{2t}|,\ t = 0,\ ...,\ N-1)}{5000\pi} (\frac{2}{L})^{1/2} \tag{J.89}$$

so that (J.88) becomes

$$|d_{m_2}| \geq \frac{c}{n}. \tag{J.90}$$

This last inequality means that (J.15), indeed, holds when $s = 2$ so that (J.16) is valid. That is, Lemma J.1 is true when (J.53) holds and $a_{3N} \leq 0$. A similar proof can also be given for $a_{3N} > 0$ as well as for the following cases:

(i)

$$\textit{while}\qquad \left.\begin{array}{l} T_{1,(m_0+\ell)}(2a_{30}+a_{3r_1}\alpha^{(0)}_{r_1 m_0}+a_{3N}\alpha^{(0)}_{N(m_0+\ell)}) \lessgtr 0 \\ \\ T_{1,(m_0+\ell)}(a_{3r_1}\mathbf{A}_{r_1(m_0+\ell)}+a_{3N}\mathbf{A}_{N(m_0+\ell)}) \leq 0, \\ \\ T_{1,(m_0+\ell)}a_{3r_1}\mathbf{A}_{r_1(m_0+\ell)} \geq 0 \end{array}\right\} \tag{J.91}$$

(ii)

$$\textit{while}\qquad \left.\begin{array}{l} T_{1,(m_0+\ell)}(2a_{30}+a_{3r_1}\alpha^{(0)}_{r_1 m_0}+a_{3N}\alpha^{(0)}_{N(m_0+\ell)}) \gtrless 0 \\ \\ T_{1,(m_0+\ell)}(a_{3r_1}\mathbf{A}_{r_1(m_0+\ell)}+a_{3N}\mathbf{A}_{N(m_0+\ell)}) \leq 0, \\ \\ T_{1,(m_0+\ell)}a_{3r_1}\mathbf{A}_{r_1(m_0+\ell)} \leq 0. \end{array}\right\} \tag{J.92}$$

(iii)

$$2a_{30}+a_{3r_1}\alpha^{(0)}_{r_1 m_0}+a_{3N}\alpha^{(0)}_{N(m_0+\ell)} = 0. \tag{J.93}$$

Moreover, when $T_{1,(m_0+\ell)} < 0$, let

$${}^1T_{1,(m_0+\ell)} = -\,{}^1T_{1,(m_0+\ell)} \quad \textit{and} \quad {}^1T_{2,(m_0+\ell)} = -\,{}^1T_{2,(m_0+\ell)}. \tag{J.94}$$

Then it can be shown straightforwardly from (J.94) that

$$\left|\frac{{}^1T_{1,(m_0+\ell)}}{\Omega_{m_0+\ell}}+\frac{{}^1T_{2,(m_0+\ell)}}{\Omega^2_{m_0+\ell}}\right| = \left|\frac{T_{1,(m_0+\ell)}}{\Omega_{m_0+\ell}}+\frac{T_{2,(m_0+\ell)}}{\Omega^2_{m_0+\ell}}\right|. \tag{J.95}$$

Thus, it can be found from (J.21) and (J.95) that

$$|d_{m_0+\ell}| = L^2\left|\frac{{}^1T_{1,(m_0+\ell)}}{\Omega_{m_0+\ell}}+\frac{{}^1T_{2,(m_0+\ell)}}{\Omega^2_{m_0+\ell}}\right|, \quad {}^1T_{1,(m_0+\ell)} > 0. \tag{J.96}$$

Hence Lemma J.1 can be shown to hold in a similar manner to $T_{1,(m_0+\ell)} > 0$ for $s = 2$.

The previous analysis is based upon the assumption that the $x_r/L = j_r/j$, $r = 1, \ldots, N - 1$, are all rational. Suppose, conversely, that the x_r/L, $1 \le r < N$, are irrational. Then it is known [59] that, for a sufficiently large n, there exists a rational number, denoted by

$$x_{r+N}/L = l_r/n, \tag{J.97}$$

such that

$$|x_r/L - l_r/n| \le n^{-2}. \tag{J.98}$$

Here l_r is a positive integer. Let $\theta_0(x_r)$ and $\theta_0(x_{r+N})$ be the initial angles of rays OA_r and OA_{r+N}, respectively. Then it can be seen from (J.3) that $\theta_m(x_r)$ and $\theta_m(x_{r+N})$ are given, for $m = 2jn$, by

$$\theta_{2jn}(x_r) = (2jn\pi x_r/L)+\theta_0(x_r) \quad \textit{and} \quad \theta_{2jn}(x_{r+N}) = (2jn\pi l_r/n)+\theta_0(x_{r+N}) \tag{J.99}$$

where, from Table J.1 for a free-sliding beam,

$$\theta_0(x_r) = (1 - \frac{5x_r}{L})\frac{\pi}{4} \quad \textit{and} \quad \theta_0(x_{r+N}) = (1 - \frac{5l_r}{n})\frac{\pi}{4} \tag{J.100}$$

after $2jn$ simultaneously stepped increments. Thus, it can be shown from (J.97) through (J.100) that

$$\begin{aligned} |\theta_{2jn}(x_r) - \theta_{2jn}(x_{r+N})| &= |2jn\pi(x_r/L - l_r/n) + (\theta_0(x_r) - \theta_0(l_r))| \\ &\le 2jn\pi|\frac{x_r}{L} - \frac{l_r}{n}| + |(1 - \frac{5x_r}{L})\frac{\pi}{4} - (1 - \frac{5l_r}{n})\frac{\pi}{4}| \\ &\le 2j\pi n^{-1} + \frac{5\pi}{4}|\frac{x_r}{L} - \frac{l_r}{n}| \le 2j\pi n^{-1} + \frac{5\pi}{4}n^{-2}. \end{aligned} \tag{J.101}$$

Furthermore, it is known from equation (J.17) and (J.97) that all rational x_r/L can be expressed by

$$x_r/L = j_r/j = (j_r n)/(jn) \quad \textit{and} \quad x_{r+N}/L = l_r/n = (l_r j)/(jn), \qquad 0 < r < N. \tag{J.102}$$

Consequently, it can be shown from (J.18) that all the OA_r which correspond to rational x_r/L return to their initial positions after $2jn$ increments. (Note that j is a known positive integer for a given set of x_r, $r = 1, 2, ..., N - 1$.) Define $\ell = 2jn$. Then, inequality (J.5) can be shown to hold for $s = 2$ by employing inequality (J.102) and the same procedure as before - even if some of the x_r/L are irrational as well as rational.

Lemma J.1 can be proved analogously for $s = 3$ as well as for the eigenvectors, $\{\psi_m(x)\}$, of a uniform beam having the other end conditions given in Table 3.1. Then the analytical properties of the eigenvectors listed in Tables J.2 and J.3 are useful.

(2) **Proof of Corollary 3.4.1**

The following result is helpful in proving Corollary 3.4.1. It is written in the form of a lemma.

Lemma J.2. Suppose that m_0 and m_1 are an even and odd integer, respectively. Then a ray, OA_1, that has an end point A_1 which does not coincide with $x_1/L = 1/4$, 1/3, 1/2, 2/3 or 3/4 can be rotated into one of the four plain regions defined in Figure J.1 by taking either m_0 or m_1 stepped increments from any initial position. This statement is also true for a beam having other than pinned-pinned or sliding-sliding ends when $x_1/L = 1/4$ and 3/4.

Proof

Consider the ray OA_1. It has the angle $\theta_m(x_1)$, relative to the positive χ_0-axis of Figure J.1, which is given by

$$\theta_m(x_1) = (m\pi x_1/L) + \theta_0(x_1) \tag{J.103}$$

after m stepped increments from its initial angle $\theta_0(x_1)$. Suppose a point x_1 satisfies $0 < x_1/L \le 1/5$. Then it can be found from (J.103) that, for any positive integer m,

$$\begin{aligned}\theta_{m+1}(x_1) - \theta_{m-1}(x_1) &= [(m+1)\pi x_1/L + \theta_0(x_1)] - [(m-1)\pi x_1/L + \theta_0(x_1)] \\ &= 2\pi x_1/L \le \frac{2\pi}{5} < 23\pi/54,\end{aligned} \tag{J.104}$$

a value corresponding to the angular width of each plain region of Figure J.1. Imagine that ray OA_1 is rotated into a plain region after m_0 stepped increments from $\theta_0(x_1)$. The corresponding angle $\theta_{m_0}(x_1)$ can be found from (J.103) to be

$$\theta_{m_0}(x_1) = (m_0 \pi x_1/L) + \theta_0(x_1). \qquad (J.105)$$

On the other hand, (J.103) also shows that

$$\theta_{m_0-1}(x_1) = (m_0 - 1)\pi x_1/L + \theta_0(x_1) \qquad (J.106)$$

and

$$\theta_{m_0+1}(x_1) = (m_0 + 1)\pi x_1/L + \theta_0(x_1) \qquad (J.107)$$

are the angles after $(m_0 - 1)$ and $(m_0 + 1)$ stepped increments, respectively. If neither $\theta_{m_0-1}(x_1)$ nor $\theta_{m_0+1}(x_1)$ is in the same plain region as $\theta_{m_0}(x_1)$ then

$$\theta_{m_0+1}(x_1) - \theta_{m_0-1}(x_1) > 23\pi/54. \qquad (J.108)$$

This conclusion contradicts (J.104) for any positive integer m. Hence, the ray OA_1, which lies at either the angle $\theta_{m_0-1}(x_1)$ or $\theta_{m_0+1}(x_1)$, must stay in the same plain region as $\theta_{m_0}(x_1)$. Moreover, it is known [55] that m_0 is an even (odd) integer if $(m_0 - 1)$ and $(m_0 + 1)$ are odd (even) integers. Thus, Lemma J.2 must hold for $0 < x_1/L \le 1/5$.

Consider next a point x_1 satisfying $1/5 < x_1/L \le 43/180$. The coordinate transformation $x_2/L = x_1/L - 1/5$ is useful in proving Lemma J.2. It can be found from this transformation that x_2/L satisfies $0 < x_2/L < 30/180 = 1/6 < 1/5$. Furthermore, it can be determined from (J.103) and the transformation that $\theta_m(x_1)|_{m=5k} = 5k\pi + k\pi(5x_2/L) + \theta_0(x_1)$ where k is a positive integer. Introduce a ray OA_2 having the end point A_2 that corresponds to x_2/L. This ray's angle, relative to the positive χ_0-axis, is

$$\theta_k^*(x_2) = k\pi(5x_2/L) + \theta_0(x_1) \tag{J.109}$$

after k stepped increments from the initial angle $\theta_0(x_1)$. As x_2 satisfies $0 < x_2/L < 1/5$, it can be shown in a similar manner to that for OA_1 that there must exist an even integer, k_0, and an odd integer, k_1, such that ray OA_2 can be rotated into any one of the plain regions of Figure J.1. On the other hand, the periodicity of rotation is 2π so that, when k equals k_0, it can be found from (J.103) and (J.109) that

$$(\theta_m(x_1)\,|_{m=5k_0} - \theta_m^*(x_2)\,|_{k=k_0}) \; \boldsymbol{mod} \; 2\pi = 0. \tag{J.110}$$

That is, OA_2 coincides with OA_1. However, when k equals k_1, rather than k_0, the direction of OA_2 is 180^0 out phase with the direction of ray OA_1 because

$$(\theta_m(x_1)\,|_{m=5k_1} - \theta_m^*(x_2)\,|_{k=k_1}) \; \boldsymbol{mod} \; 2\pi = \pi. \tag{J.111}$$

This last equality means that there exists an even integer, $m_0 = 5k_0$, and an odd integer, $m_1 = 5k_1$, such that the ray OA_1 can be rotated into any of the plain regions of Figure J.1 after m_0 and m_1 stepped increments from its initial angle. A similar procedure can also be applied for any point satisfying $x_1/L < 1/2$ as well as $x_1/L \neq 1/4$ and $x_1/L \neq 1/3$. In particular, the coordinate transformations $x_2/L = 1/4 - x_1/L$ and $x_2/L = x_1/L - 1/4$ are needed for $43/180 < x_1/L < 1/4$ and $1/4 < x_1/L \leq 3/10$, respectively. Similarly, the coordinate transformations $x_2/L = 1/3 - x_1/L$ and $x_2/L = x_1/L - 1/3$ are required for $3/10 < x_1/L < 1/3$ and $1/3 < x_1/L \leq 5/12$ whilst $x_2/L = 1/2 - x_1/L$ is useful for

$5/12 < x_1/L < 1/2$. Moreover, the coordinate transformation $x_2/L = 1 - x_1/L$ as well as the result of Lemma J.2 for $0 < x_1/L < 1/2$ can be used to prove Lemma J.2 for $x_1/L > 1/2$ and $x_1/L \neq 2/3$ and $x_1/L \neq 3/4$.

Table J.1 indicates that, for $x_1/L = 1/4$, the initial angle, $\theta_0(x_1)$, of ray OA_1 is $-\pi/2$ and $-\pi/4$ for a pinned-pinned beam and a sliding-sliding beam, respectively. Thus, the first eight locations of OA_1 can be obtained analytically from [50]. They are tabulated for convenience in Tables J.4 and J.5. It can be observed from these tables that after 1, 3, 5 and 7 stepped increments from its initial position, ray OA_1 lies in the plain regions IV, I, II and III for a pinned-pinned beam and the plain regions I, II, III and IV for a sliding-sliding beam. Furthermore, it can be found from (J.18) that the periodicity of rotation is eight because $j = 4$. Therefore, there is no even integer, m_0, that permits OA_1 to be rotated into the plain regions from its initial angle. This confirms Lemma J.1 for a pinned-pinned beam or a sliding-sliding beam providing $x_1/L = 1/4$. Lemma J.2 can be obtained similarly from Table J.4 and Table J.5 for a pinned-pinned beam or a sliding-sliding beam providing $x = 1/3$, $1/2$, $2/3$ and $3/4$.

The first eight locations of OA_1 can be obtained similarly for a beam having other end conditions. They are presented in Tables J.6 through J.13. By employing the periodicity of rotation for OA_1, it can be concluded that, for any one of the plain regions, there must exist an even integer, m_0, and an odd integer, m_1, such that a ray OA_1, which corresponds to $x_1/L = 1/4$ and $3/4$, can be rotated into the plain region after m_0 and m_1 stepped increments. However, Tables J.6 through J.13 indicate that the same conclusion does not hold for OA_1 if $x_1/L = 1/3$, $1/2$ and $2/3$.

This completes the proof of Lemma J.2.

By employing this lemma and Theorem 3.4.1, Corollary 3.4.1 can be shown straightforwardly. In fact, once the set of constants $\varrho_{i\,r}$, $i = 0, 1$, used in (J.7) through (J.10) are given, the signs of

$$\varrho_{00}\alpha_{0m}^{(0)},\ \varrho_{10}\alpha_{0m}^{(1)},\ \varrho_{02}\alpha_{2m}^{(0)}\ and\ \varrho_{12}\alpha_{2m}^{(1)}$$

are determined. It can be found from Table J.2 that the signs of these values depend only upon whether m is an even or an odd integer because $N = 2$ here. In order for (J.7), (J.9) through (J.11) to be satisfied, for example, the required signs of

$$\alpha_{1m}^{(0)}\ and\ \alpha_{1m}^{(1)}$$

can be determined for given ϱ_{01} and ϱ_{11}. They must involve one of the following four combinations

$$\left.\begin{matrix}\alpha_{1m}^{(0)} \geq 0\\ and\\ \alpha_{1m}^{(1)} \geq 0\end{matrix}\right\},\quad \left.\begin{matrix}\alpha_{1m}^{(0)} \geq 0\\ and\\ \alpha_{1m}^{(1)} \leq 0\end{matrix}\right\},\quad \left.\begin{matrix}\alpha_{1m}^{(0)} \leq 0\\ and\\ \alpha_{1m}^{(1)} \geq 0\end{matrix}\right\}\quad or\quad \left.\begin{matrix}\alpha_{1m}^{(0)} \leq 0\\ and\\ \alpha_{1m}^{(1)} \leq 0\end{matrix}\right\} \tag{J.112}$$

It is well known [42] that, for a ray OA_1 located in plain region I after m stepped increments from its initial angle, i.e. in part of the first quadrant,

$$\alpha_{1m}^{(0)} = (\frac{2}{L})^{1/2}\cos(\theta_m(x_1)) > 0\quad and\quad \alpha_{1m}^{(1)} = -(\frac{2}{L})^{1/2}\sin(\theta_m(x_1)) < 0 \tag{J.113}$$

and, for OA_1 in the plain region II or part of the second quadrant,

$$\alpha_{1m}^{(0)} = (\frac{2}{L})^{1/2}\cos(\theta_m(x_1)) < 0 \quad and \quad \alpha_{1m}^{(1)} = -(\frac{2}{L})^{1/2}\sin(\theta_m(x_1)) < 0 \tag{J.114}$$

Furthermore, for a ray OA_1 in plain region III,

$$\alpha_{1m}^{(0)} = (\frac{2}{L})^{1/2}\cos(\theta_m(x_1)) < 0 \quad and \quad \alpha_{1m}^{(1)} = -(\frac{2}{L})^{1/2}\sin(\theta_m(x_1)) > 0 \tag{J.115}$$

and, for OA_1 in plain region IV,

$$\alpha_{1m}^{(0)} = (\frac{2}{L})^{1/2}\cos(\theta_m(x_1)) > 0 \quad and \quad \alpha_{1m}^{(1)} = -(\frac{2}{L})^{1/2}\sin(\theta_m(x_1)) > 0 \tag{J.116}$$

Also, Lemma J.2 indicates that, for each plain region of Figure J.1, there exist an even integer, m_0, and an odd integer, m_1, such that OA_1 can be rotated into that plain region with the possible exception of (i) the eigenvectors of a pinned-pinned and sliding-sliding beam at $x_1/L = 1/4$, 1/3, 1/2, 2/3 and 3/4 or (ii) a beam having the end conditions given in Tables J.6 through J.13 when $x_1/L = 1/3$, 1/2 and 2/3. Thus, it is seen from (K.22) that any one of the four combinations given in (J.109) holds sometimes except possibly at the stated, isolated points. This means that assumption (1) of Lemma J.1 as well as (J.7) and (J.9) through (J.11) are, indeed, valid. It follows from Theorem 3.4.1 that Corollary 3.4.1 holds except maybe at $x_1/L = 1/4$, 1/3, 1/2, 2/3 and 3/4. The remaining part of this section demonstrates that Lemma J.1 holds even at these points.

Consider $x_1/L = 1/4$, for instance, when $\psi_m(x)$ is the m th eigenvector of a sliding-sliding beam. Suppose that the ϱ_{ir}, $i = 0$, 1 and $r = 0$, 1, 2, are arbitrary but given constants. Now, it can be found from Table J.2 that

$$\alpha^{(0)}_{0(2m+1)} = \alpha^{(0)}_{2(2m+1)} = (\frac{2}{L})^{1/2} > 0 \tag{J.117}$$

so that, when $m = 5$ in (J.117),

$$\varrho_{00}\alpha^{(0)}_{05} \geq 0 \quad and \quad \varrho_{02}\alpha^{(0)}_{25} \geq 0 \tag{J.118}$$

for $\varrho_{00} \geq 0$ and $\varrho_{02} \geq 0$. Furthermore, Table J.4 indicates that the corresponding ray, OA_1, can be rotated to coincide with the χ_0-axis after five stepped increments so that

$$\alpha^{(0)}_{05} > 0 \quad and \quad \varrho_{01}\alpha^{(0)}_{05} > 0 \quad for\ \varrho_{01} > 0. \tag{J.119}$$

On the other hand, it is known from Table J.2 that

$$\alpha^{(1)}_{0m} = \alpha^{(1)}_{2m} = 0 \tag{J.120}$$

for any positive integer m. Consequently,

$$\varrho_{10}\alpha^{(1)}_{05} + \varrho_{12}\alpha^{(1)}_{25} = 0. \tag{J.121}$$

Let $r_1 = 0$ and $r_3 = r_4 = 1$ in (J.9). Then it can be found straightforwardly that (J.9) holds no matter whether $\varrho_{11}\,\psi_5'(x_1)$ is positive, zero or negative. Thus, there exists a positive integer $m_0 \equiv 5$ such that assumption (1) of Lemma J.1 and (J.10) are true. Hence Lemma J.1 is valid for $s = 3$ and $t = 1$ when $\varrho_{01} > 0$, $\varrho_{00} \geq 0$ and $\varrho_{02} \geq 0$. When ϱ_{01}, ϱ_{00} and ϱ_{02} have different signs, a similar analysis can be followed and the final results are summarized next:

$$\left.\begin{array}{l} \mathcal{Q}_{11}\alpha_{13}^{(1)} \geq 0 \quad \textit{or} \quad \mathcal{Q}_{11}\alpha_{13}^{(1)} \leq 0 \\ \mathcal{Q}_{01}\alpha_{13}^{(0)} > 0,\ \mathcal{Q}_{00}\alpha_{03}^{(0)} \geq 0,\ \mathcal{Q}_{02}\alpha_{23}^{(0)} \geq 0, \quad \textit{if}\ \mathcal{Q}_{00} \geq 0,\ \mathcal{Q}_{02} \geq 0,\ \mathcal{Q}_{01} < 0 \end{array}\right\} \tag{J.122}$$

$$\left.\begin{array}{l} \mathcal{Q}_{11}\alpha_{15}^{(1)} \geq 0 \quad \textit{or} \quad \mathcal{Q}_{11}\alpha_{15}^{(1)} \leq 0 \\ \mathcal{Q}_{01}\alpha_{15}^{(0)} < 0,\ \mathcal{Q}_{00}\alpha_{05}^{(0)} \leq 0,\ \mathcal{Q}_{02}\alpha_{25}^{(0)} \leq 0, \quad \textit{if}\ \mathcal{Q}_{00} \leq 0,\ \mathcal{Q}_{02} \leq 0,\ \mathcal{Q}_{01} < 0 \end{array}\right\} \tag{J.123}$$

$$\left.\begin{array}{l} \mathcal{Q}_{11}\alpha_{13}^{(1)} \geq 0 \quad \textit{or} \quad \mathcal{Q}_{11}\alpha_{13}^{(1)} \leq 0 \\ \mathcal{Q}_{01}\alpha_{13}^{(0)} < 0,\ \mathcal{Q}_{00}\alpha_{03}^{(0)} \leq 0,\ \mathcal{Q}_{02}\alpha_{23}^{(0)} \leq 0, \quad \textit{if}\ \mathcal{Q}_{00} \leq 0,\ \mathcal{Q}_{02} \leq 0,\ \mathcal{Q}_{01} > 0 \end{array}\right\} \tag{J.124}$$

$$\left.\begin{array}{l} \mathcal{Q}_{11}\alpha_{12}^{(1)} \geq 0 \quad \textit{or} \quad \mathcal{Q}_{11}\alpha_{12}^{(1)} \leq 0 \\ \mathcal{Q}_{01}\alpha_{12}^{(0)} > 0,\ \mathcal{Q}_{00}\alpha_{02}^{(0)} \geq 0,\ \mathcal{Q}_{02}\alpha_{22}^{(0)} \geq 0, \quad \textit{if}\ \mathcal{Q}_{00} \geq 0,\ \mathcal{Q}_{02} \leq 0,\ \mathcal{Q}_{01} > 0 \end{array}\right\} \tag{J.125}$$

$$\left.\begin{array}{l} \mathcal{Q}_{11}\alpha_{16}^{(1)} \geq 0 \quad \textit{or} \quad \mathcal{Q}_{11}\alpha_{16}^{(1)} \leq 0 \\ \mathcal{Q}_{01}\alpha_{16}^{(0)} > 0,\ \mathcal{Q}_{00}\alpha_{06}^{(0)} \geq 0,\ \mathcal{Q}_{02}\alpha_{26}^{(0)} \geq 0, \quad \textit{if}\ \mathcal{Q}_{00} \geq 0,\ \mathcal{Q}_{02} \leq 0,\ \mathcal{Q}_{01} < 0 \end{array}\right\} \tag{J.126}$$

$$\left.\begin{array}{l} \mathcal{Q}_{11}\alpha_{16}^{(1)} \geq 0 \quad \textit{or} \quad \mathcal{Q}_{11}\alpha_{16}^{(1)} \leq 0 \\ \mathcal{Q}_{01}\alpha_{16}^{(0)} < 0,\ \mathcal{Q}_{00}\alpha_{06}^{(0)} \leq 0,\ \mathcal{Q}_{02}\alpha_{26}^{(0)} \leq 0, \quad \textit{if}\ \mathcal{Q}_{00} \leq 0,\ \mathcal{Q}_{02} \geq 0,\ \mathcal{Q}_{01} > 0 \end{array}\right\} \tag{J.127}$$

and

$$\left.\begin{array}{l} \mathcal{Q}_{11}\alpha_{12}^{(1)} \geq 0 \quad \textit{or} \quad \mathcal{Q}_{11}\alpha_{12}^{(1)} \leq 0 \\ \mathcal{Q}_{01}\alpha_{12}^{(0)} < 0,\ \mathcal{Q}_{00}\alpha_{02}^{(0)} \leq 0,\ \mathcal{Q}_{02}\alpha_{22}^{(0)} \leq 0, \quad \textit{if}\ \mathcal{Q}_{00} \leq 0,\ \mathcal{Q}_{02} \geq 0,\ \mathcal{Q}_{01} < 0 \end{array}\right\} \tag{J.128}$$

for $s = 3$, $t = 1$ and $c_{01} \neq 0$. Again, (J.122) through (J.128) indicate that there exists an integer $m_0 = 2$ or 3 or 5 or 6 such that assumption (1) of Lemma J.1 as well as (J.7) and (J.9) through (J.11) are satisfied. Therefore, Lemma J.1 holds for $s = 3$ and $t = 1$. The validity of Lemma J.1 can be shown similarly for $s = 3$ and $t = 0, 2$ as well as for $s = 2$ and $t = 1$. Thus, Theorem 3.4.1 confirms that Corollary 3.4.1 holds even when $x_1/L = 1/4$ for the sliding-sliding beam. A similar proof can be derived for a beam having the end conditions given in Table 3.1 or stated in Tables J.4 through J.13 when $x_1/L = 1/4, 1/3, 1/2, 2/3$ and $3/4$.

This completes the proof of Corollary 3.4.1.

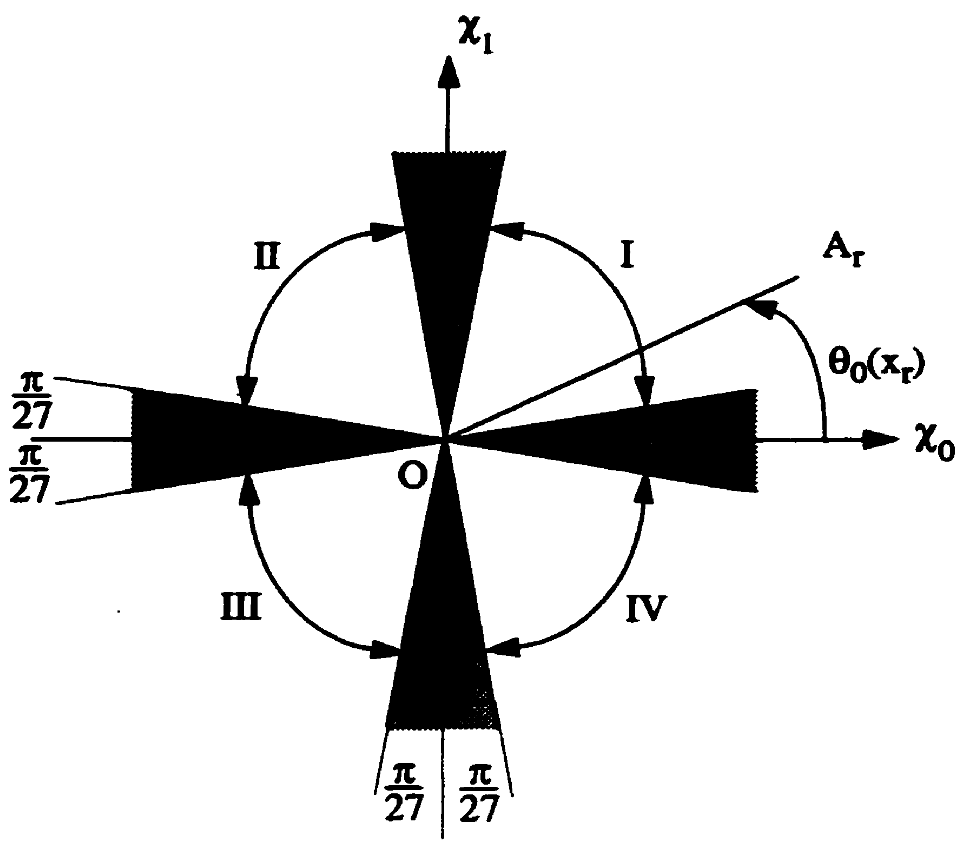

Figure J.1. Defining plain regions I through IV $\square$.

Table J.1. Values of $\theta_0(x)$, $0 < x < L$.

End Supports	$\theta_0(x)$
pinned-pinned	$-\pi/2$
sliding-sliding	$-\pi x/L$
clamped-clamped	$(x/L + 1/2)\pi/2$
free-free	$(-3x/L + 1/2)\pi/2$
clamped-free	$(-x/L + 1/2)\pi/2$
clamped-pinned	$(x/L + 1)\pi/4$
free-pinned	$(-3x/L + 1)\pi/4$
sliding-pinned	$-x\pi/2L$
clamped-sliding	$(-x/L + 1)\pi/4$
free-sliding	$(-5x/L + 1)\pi/4$

Table J.2. $\alpha_{rm}^{(0)}$ and $\alpha_{rm}^{(1)}$, $r = 0, N$, for a uniform Euler-Bernoulli beam.

End Supports	$\alpha_{rm}^{(0)}$ and $\alpha_{rm}^{(1)}$, $r = 0, N$
pinned-pinned	$\alpha_{0m}^{(0)} = \alpha_{Nm}^{(0)} = 0,\ \alpha_{0m}^{(1)} = \alpha_{N(2m)}^{(1)} = -\alpha_{N(2m+1)}^{(1)} = L^{-1/2}$
sliding-sliding	$\alpha_{0m}^{(0)} = \alpha_{N(2m+1)}^{(0)} = -\alpha_{N(2m)}^{(0)} = 1,\ \alpha_{0m}^{(1)} = \alpha_{Nm}^{(1)} = 0$
clamped-clamped	$\alpha_{0m}^{(0)} = \alpha_{Nm}^{(0)} = \alpha_{0m}^{(1)} = \alpha_{Nm}^{(1)} = 0$
free-free	$\alpha_{0(2m)}^{(0)} = -\alpha_{N(2m)}^{(0)} = \alpha_{0(2m+1)}^{(0)} = \alpha_{N(2m+1)}^{(0)} = (2/L)^{1/2}$ $\alpha_{0(2m)}^{(1)} = \alpha_{N(2m)}^{(1)} = \alpha_{0(2m+1)}^{(1)} = -\alpha_{N(2m+1)}^{(1)} = (2/L)^{1/2}$
clamped-free	$\alpha_{0m}^{(0)} = \alpha_{Nm}^{(0)} = 0,\ \alpha_{N(2m)}^{(0)} < 0,\ \alpha_{N(2m+1)}^{(0)} > 0,$ $\alpha_{N(2m)}^{(1)} < 0,\ \alpha_{N(2m+1)}^{(1)} > 0$
clamped-pinned	$\alpha_{0m}^{(0)} = \alpha_{Nm}^{(0)} = \alpha_{0m}^{(1)} = 0,\ \alpha_{N(2m)}^{(1)} > 0,\ \alpha_{N(2m+1)}^{(1)} < 0$
free-pinned	$\alpha_{0m}^{(0)} > 0,\ \alpha_{0m}^{(1)} < 0,\ \alpha_{Nm}^{(0)} = 0,\ \alpha_{N(2m)}^{(1)} > 0,\ \alpha_{N(2m-1)}^{(1)} < 0$
sliding-pinned	$\alpha_{0m}^{(0)} = L^{-1/2},\ -\alpha_{N(2m)}^{(1)} = \alpha_{N(2m+1)}^{(1)} = L^{-1/2},\ \alpha_{0m}^{(1)} = \alpha_{Nm}^{(0)} = 0$
clamped-sliding	$\alpha_{0m}^{(0)} = \alpha_{0m}^{(1)} = \alpha_{Nm}^{(1)} = 0,\ \alpha_{N(2m)}^{(0)} < \alpha_{N(2m-1)}^{(0)} > 0$
free-sliding	$\alpha_{0m}^{(0)} > 0,\ \alpha_{0m}^{(1)} < 0,\ \alpha_{N(2m)}^{(0)} < 0,\ \alpha_{N(2m+1)}^{(0)} > 0$

Table J.3. $\mathring{A}_{rm}$ for a uniform Euler-Bernoulli beam. x_r satisfies $0 < x_r < L$.

End supports	$\mathring{A}_{rm}$
pinned-pinned sliding-pinned sliding-sliding	0
clamped-clamped	$L^{-1/2}\exp(-\Omega_m x_r/L)[1 + O(\exp(-\Omega_m(L - 2x_r)/L)]$, $0 < x_r/L \le 1/2$ $L^{-1/2}\exp(-\Omega_m(1 - x_r/L))[sin(2m + 1)\pi/2$ $+ O(\exp(-\Omega_m(2x_r - L)/L))]$, $1/2 < x_r/L < 1$
free-free	$L^{-1/2}\exp(-\Omega_m x_r/L)[1 + O(\exp(-\Omega_m(L - 2x_r)/L)]$, $0 < x_r/L \le 1/2$ $L^{-1/2}\exp(-\Omega_m(1 - x_r/L))[sin(2m - 3)\pi/2$ $+ O(\exp(-\Omega_m(2x_r - L)/L))]$, $1/2 < x_r/L < 1$
clamped-free	$L^{-1/2}\exp(-\Omega_m x_r/L)[1 + O(\exp(-\Omega_m(L - 2x_r)/L)]$, $0 < x_r/L \le 1/2$ $L^{-1/2}\exp(-\Omega_m(1 - x_r/L))[-sin(2m - 1)\pi/2$ $+ O(\exp(-\Omega_m(2x_r - L)/L))]$, $1/2 < x_r/L < 1$
clamped-pinned clamped-sliding free-pinned free-sliding	$L^{-1/2}\exp(-\Omega_m x_r/L)[1 + O(\exp(-\Omega_m)]$

Table J.4. Locations for a pinned-pinned beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. ($1 \leq r \leq N-1$)

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	"Coincidence" with χ_1-axis
1/4	3	5	7	1	2, 6*	4, 8*
1/3	2	4	5	1		3, 6*
1/2					1, 3*	2, 4*
2/3	1, 4	2, 5				3*, 6*
3/4	1	7	5	3	2*, 6	4, 8*

* indicates that the direction of ray OA_r is 180^0 out phase with the positive χ_0 or χ_1-axis. Otherwise it is in phase. (This symbol has the same implication in Table J.5 through J.13.)

Table J.5. Locations for a sliding-sliding beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. ($1 \leq r \leq N-1$)

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	"Coincidence" with χ_1-axis
1/4	2	4	6	8	5*, 9	3, 7*
1/3	2	3	5	6	4*, 7	
1/2					3*, 5	2, 4*
2/3		2, 5	3, 6		4, 7	
3/4	4	2	8	6	5*, 9	3*, 7

Table J.6. Locations for a clamped-clamped beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. ($1 \le r \le N-1$)

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	χ_1-axis
1/4	1, 8	2, 3	4, 5	6, 7		
1/3	5, 6	1	2, 3	4		
1/2					1*, 3	2*, 4
2/3		3, 6	1, 4	2, 5		
3/4	2, 5	3, 8	1, 6	4, 7		

Table J.7. Locations for a free-free beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. ($1 \le r \le N-1$)

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	χ_1-axis
1/4	3, 10	4, 5	6, 7	8, 9		
1/3	7, 8	3	4, 5	6		
1/2					3*, 5	4*, 6
2/3		5, 9	3, 6	4, 7		
3/4	4, 7	5, 10	3, 8	6, 9		

Table J.8. Locations for a clamped-free beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. $(1 \le r \le N - 1)$

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	χ_1-axis
1/4	1, 2	3, 4	5, 6	7, 8		
1/3	6	2	3	5	1, 4*	
1/2					2*, 4	1, 3*
2/3		2, 5	3, 6	1, 4		
3/4	3, 6	1, 4	2, 7	5, 8		

Table J.9. Locations for a clamped-pinned beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. $(1 \le r \le N - 1)$

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	χ_1-axis
1/4	7, 8	1, 2	3, 4	5, 6		
1/3	6	1	3	4		2*, 5
1/2	4	1	2	3		
2/3	3, 6		1, 4	2, 5		
3/4	5, 8	3, 6	1, 4	2, 7		

Table J.10. Locations for a free-pinned beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. $(1 \le r \le N-1)$

	Value of m					
					"Coincidence" with	
x_r/L	Region I	Region II	Region III	Region IV	χ_0-axis	χ_1-axis
1/4	8, 9	2, 3	4, 5	6, 7		
1/3	7	2	4	5	3*, 6	
1/2	5	2	3	4		
2/3	4, 7		2, 5	3, 6		
3/4	6, 9	4, 7	2, 5	3, 8		

Table J.11. Locations for a sliding-pinned beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. $(1 \le r \le N-1)$

	Value of m					
					"Coincidence" with	
x_r/L	Region I	Region II	Region III	Region IV	χ_0-axis	χ_1-axis
1/4	1, 2	3, 4	5, 6	7, 8		
1/3	1	3	4	6		2, 5*
1/2	1	2	3	4		
2/3	3, 6	1, 4	2, 5			
3/4	3, 8	1, 6	4, 7	2, 5		

Table J.12. Locations for a clamped-sliding beam of OA_r after m stepped increments from the initial angle $\theta_0(x_r)$. ($1 \le r \le N-1$)

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	"Coincidence" with χ_1-axis
1/4	1, 2	3, 4	5, 6	7, 8		
1/3	6	2	3	5	1, 4*	
1/2	1	2	3	4		
2/3	3, 6	1, 4	2, 5			
3/4	3, 8	1, 6	4, 7	2, 5		

Table J.13. Locations for a free-sliding beam of OA_r after m stepped increments from the initial angle $\theta_0(x)$. ($1 \le r \le N-1$)

x_r/L	Value of m					
	Region I	Region II	Region III	Region IV	"Coincidence" with χ_0-axis	"Coincidence" with χ_1-axis
1/4	2, 3	4, 5	6, 7	8, 9		
1/3	7	3	4	6		2, 5*
1/2	2	3	4	5		
2/3	4, 7	2, 5	3, 6			
3/4	4, 9	2, 7	5, 8	3, 6		

APPENDIX K

This appendix gives details of the results that are used without proof in Appendix J. First, following a similar procedure to that employed in [33], the asymptotic expressions of the m th eigenvector, $\psi_m(x)$, and corresponding characteristic value, Ω_m, of a free-sliding Euler-Bernoulli beam are presented. It is known [50] that the analytical expressions of $\psi_m(x)$ and its first spatial derivative for a free-sliding uniform beam are given by [50]

$$\psi_m(x) = \frac{1}{L^{1/2}}[\cosh\Omega_m\frac{x}{L}+\cos\Omega_m\frac{x}{L}-\tanh\Omega_m(\sinh\Omega_m\frac{x}{L}+\sin\Omega_m\frac{x}{L})] \tag{K.1}$$

and

$$\frac{L}{\Omega_m}\psi'_m(x) = \frac{1}{L^{1/2}}[\sinh\Omega_m\frac{x}{L}-\sin\Omega_m\frac{x}{L}-\tanh\Omega_m(\cosh\Omega_m\frac{x}{L}+\cos\Omega_m\frac{x}{L})] \tag{K.2}$$

where $m = 2, 3, \ldots$. Free-sliding end conditions at $x = 0$ and $x = L$, respectively, correspond to [50]

$$\psi''_m(0) = \psi'''_m(0) = \psi'_m(L) = \psi'''_m(L) = 0. \tag{K.3}$$

On other hand, Ω_m satisfies [50]

$$\tan\Omega_m = \tanh\Omega_m, \quad m \geq 2. \tag{K.4}$$

Furthermore, it is known [50] that

$$\Omega_m \to m\pi - \frac{5\pi}{4} \tag{K.5}$$

for a sufficiently large m. Let

$$\nabla_m = \Omega_m - (m\pi - \frac{5\pi}{4}). \tag{K.6}$$

Then (K.4) can be rewritten as

$$\tan(m\pi - \frac{5\pi}{4} + \nabla_m) = \tanh\Omega_m. \tag{K.7}$$

Substituting the expansion [42]

$$\tanh\Omega_m = \frac{1-\exp(-2\Omega_m)}{1+\exp(-2\Omega_m)} \tag{K.8}$$

and

$$\tan(m\pi - \frac{5\pi}{4} + \nabla_m) = \frac{\tan(m\pi - \frac{5\pi}{4}) + \tan\nabla_m}{1 - \tan(m\pi - \frac{5\pi}{4})\tan\nabla_m} = \frac{1+\tan\nabla_m}{1-\tan\nabla_m} \tag{K.9}$$

into (K.7) leads to

$$\frac{1+\tan\nabla_m}{1-\tan\nabla_m} = \frac{1-\exp(-2\Omega_m)}{1+\exp(-2\Omega_m)}. \tag{K.10}$$

This last equation leads, after algebraic manipulation, to

$$\tan\nabla_m = -\exp(-2\Omega_m)$$

or

$$\nabla_m = \tan^{-1}(-\exp(-2\Omega_m)). \quad \text{(K.11)}$$

On the other hand, it is known [42] that

$$\tan^{-1}a = a - \frac{a^3}{3} + \frac{a^5}{5} - \frac{a^7}{7} + \ldots \quad \text{(K.12)}$$

for an arbitrary real value a satisfying $|a| < 1$. Furthermore, (K.5) indicates that $-\exp(-2\Omega_m) \rightarrow 0$ as $m \rightarrow \infty$. Let $a = -\exp(-2\Omega_m)$. Consequently, the following equality can be found from (K.11) and (K.12)

$$\nabla_m = \tan^{-1}(-\exp(-\Omega_m)) = -\exp(-2\Omega_m) + \frac{\exp(-6\Omega_m)}{3} - \frac{\exp(-10\Omega_m)}{5} + \frac{\exp(-14\Omega_m)}{7} - \ldots$$

or

$$\nabla_m = -\exp(-2\Omega_m)\left(1 - \frac{\exp(-4\Omega_m)}{3} + \frac{\exp(-8\Omega_m)}{5} - \frac{\exp(-12\Omega_m)}{7} + \ldots\right). \quad \text{(K.13)}$$

It can be observed that the terms

$$-\frac{\exp(-4\Omega_m)}{3}+\frac{\exp(-8\Omega_m)}{5}-\frac{\exp(-12\Omega_m)}{7}+\ldots \tag{K.14}$$

form an alternating series because they are alternately negative and positive. Consequently, it is known [42] that series (K.14) satisfies the inequality

$$\begin{aligned} \left|-\frac{\exp(-4\Omega_m)}{3}+\frac{\exp(-8\Omega_m)}{5}-\frac{\exp(-12\Omega_m)}{7}+\ldots\right| &\leq \frac{\exp(-4\Omega_m)}{3} \\ &< \frac{\exp(-\Omega_m)}{3}. \end{aligned} \tag{K.15}$$

That is, series (K.14) tends to zero like the order of the term $\exp(-\Omega_m)$. Thus, by employing Landau's notation [58], series (K.14) can be denoted as

$$-\frac{\exp(-4\Omega_m)}{3}+\frac{\exp(-8\Omega_m)}{5}-\frac{\exp(-12\Omega_m)}{7}+\ldots = O(\exp(-\Omega_m)) \tag{K.16}$$

so that (K.13) can be rewritten more succinctly as

$$\nabla_m = -\exp(-2\Omega_m)(1+O(\exp(-\Omega_m))). \tag{K.17}$$

By substituting (K.17) into (K.6) and rearranging terms, the asymptotic form of Ω_m can be obtained as

$$\Omega_m = m\pi-\frac{5\pi}{4}-\exp(-2\Omega_m)[1+O(\exp(-\Omega_m))]. \tag{K.18}$$

By substituting (K.18) and the expressions [42]

$$\left.\begin{aligned} \sinh\Omega_m &= \frac{1}{2}(\exp(\Omega_m)-\exp(-\Omega_m)) \\ \textit{and} \qquad & \\ \cosh\Omega_m &= \frac{1}{2}(\exp(\Omega_m)+\exp(-\Omega_m)) \end{aligned}\right\} \tag{K.19}$$

into (K.1) and (K.2), the asymptotic forms of $\psi_m(x)$ and $\psi'_m(x)$ can be obtained, for a sufficiently large m and $0 \le x \le L$, as

$$\psi_m(x) = \begin{cases} \dfrac{2}{L^{1/2}}, & \textit{for } x = 0 \\ (\dfrac{2}{L})^{1/2}\cos[(m-\dfrac{5}{4})\pi\dfrac{x}{L}+\dfrac{\pi}{4}] + \dfrac{1}{L^{1/2}}\exp(-\Omega_m\dfrac{x}{L})[1+ & \\ \qquad +O(\exp(-\Omega_m))], & \textit{for } 0 < x \le L \end{cases} \tag{K.20}$$

and

$$\frac{L}{\Omega_m}\psi'_m(x) = \begin{cases} -\dfrac{2}{L^{1/2}}+\dfrac{4}{L^{1/2}}\exp(-6\Omega_m)(1+O(\exp(-\Omega_m))), & \textit{for } x = 0 \\ -(\dfrac{2}{L})^{1/2}\sin[(m-\dfrac{5}{4})\pi\dfrac{x}{L}+\dfrac{\pi}{4}] - \dfrac{1}{L^{1/2}}\exp(-\Omega_m\dfrac{x}{L})[1+ & \\ \qquad +O(\exp(-\Omega_m))], & \textit{for } 0 < x < L \\ 0, & \textit{for } x = L. \end{cases} \tag{K.21}$$

Now

$$\alpha_{rm}^{(i)} = (\frac{2}{L})^{1/2}(\frac{m\pi}{L})^{-i}\frac{d^i\cos(\dfrac{m\pi x_r}{L}+\theta_0(x_r))}{dx^i}, \quad i = 0,\ 1 \textit{ and } 0 < x_r < L \tag{K.22}$$

where

$$\theta_0(x_r) = (1 - \frac{5x_r}{L})\frac{\pi}{4} \tag{K.23}$$

whilst

$$\left.\begin{aligned} \alpha_{0m}^{(0)} = \lim_{k\to\infty}\psi_{m+2k}(0) = \frac{2}{L^{1/2}},\ \alpha_{0m}^{(1)} = \lim_{k\to\infty}\frac{L}{\Omega_{m+2k}}\psi'_{m+2k}(0) = -\frac{2}{L^{1/2}} \\ \alpha_{Nm}^{(1)} = \lim_{k\to\infty}\frac{L}{\Omega_{m+2k}}\psi'_{m+2k}(L) = 0 \end{aligned}\right\} \tag{K.24}$$

and

$$\begin{aligned} \alpha_{Nm}^{(0)} &= \lim_{k\to\infty}\psi_{m+2k}(L) = \lim_{k\to\infty}(\frac{2}{L})^{1/2}\cos[(m+2k-\frac{5}{4})\pi\frac{L}{L}+\frac{\pi}{4}] \\ &= \lim_{k\to\infty}(\frac{2}{L})^{1/2}\cos[2k\pi+(m-1)\pi] \end{aligned}$$

or

$$\alpha_{Nm}^{(0)} = (-1)^{m+1}(\frac{2}{L})^{1/2} \tag{K.25}$$

Let

$$\Lambda_{rm} = \begin{cases} 0, & \textit{for } r = 0 \\ \frac{1}{L^{1/2}}\exp(-\Omega_m\frac{x_r}{L})[1+O(\exp(-\Omega_m))], & \textit{for } 0 < r \le N \end{cases} \tag{K.26}$$

and

$$\bar{\Lambda}_{rm} = \begin{cases} \frac{4}{L^{1/2}}\exp(-6\Omega_m))(1+O(\exp(-\Omega_m))), & for\ r = 0 \\ -\frac{1}{L^{1/2}}\exp(-\Omega_m\frac{x_r}{L})[1+O(\exp(-\Omega_m))], & for\ 0 < r < N \\ 0, & for\ r = N. \end{cases} \quad (K.27)$$

Then the asymptotic forms (K.20) and (K.21) may be rewritten, at $x = x_r$, as

$$\psi_m(x_r) = \alpha^{(0)}_{rm} + \Lambda_{rm} \quad and \quad \frac{L}{\Omega_m}\psi'_m(x_r) = \alpha^{(1)}_{rm} + \bar{\Lambda}_{rm},\ 0 \le r \le N. \quad (K.28)$$

When $x_r/L = j_r/j$ ($0 < r < N$) is rational, i.e. j_r and j are two positive integers, it can be found from (J.18) and (K.22) that

$$\alpha^{(i)}_{r(m+2kj)} = (\frac{2}{L})^{1/2}(\frac{m\pi}{L})^{-i}\frac{d^i\cos(2kj_r\pi + \frac{m\pi x_r}{L} + \theta_0(x_r))}{dx^i}$$

$$= (\frac{2}{L})^{1/2}(\frac{m\pi}{L})^{-i}\frac{d^i\cos(\frac{m\pi x_r}{L} + \theta_0(x_r))}{dx^i}$$

or

$$\alpha^{(i)}_{r(m+2kj)} = \alpha^{(i)}_{rm} \quad (K.29)$$

for any positive integer k. Moreover, it can be seen from (K.24) and (K.25) that (K.29) is also valid when $r = 0$ and $r = N$. After obtaining the asymptotic expressions (K.18), (K.20) and (K.21), the next result is needed to demonstrate (J.44).

Lemma K.1. Suppose m_0 and j are two given finite positive integers that satisfy $0 < m_0 < n$ and $0 < j < n$. Here n is also a positive integer but it can increase to infinity. Then there exists a positive integer, k_0, such that the inequality

$$3n > m_0 + \ell > n \tag{K.30}$$

where

$$\ell = 2k_0 j \tag{K.31}$$

holds for a sufficiently large n.

Proof

It is known that j is a given finite positive integer whilst n tends to infinity. Thus, n can be increased such that

$$n > 10j \tag{K.32}$$

is satisfied. Let k_0 be the positive quotient obtained when $2n$ is divided by $2j$. Then it is known [50] that

$$2n = k_0(2j) + 2n \bmod 2j \tag{K.33}$$

and

$$0 \le 2n \bmod 2j \le (2j-1). \tag{K.34}$$

Thus, by using (K.32) and (K.34), it can be found from (K.33) that

$$2n = k_0(2j) + 2n \bmod 2j \ge k_0(2j) \tag{K.35}$$

and

$$k_0(2j) = 2n - 2n \ mod \ 2j \geq 2n-(2j-1) > 2n-n = n$$

or

$$k_0(2j) > n. \tag{K.36}$$

Let ℓ be defined by (K.31), i.e. $\ell = 2k_0 j$. It can be found from (K.35) and (K.36) in conjunction with $0 < m_0 < n$ that $m_0 + \ell$ must satisfy

$$3n > m_0 + \ell > \ell > n \tag{K.37}$$

for a sufficiently large n.

This completes the proof of Lemma K.1. By using Lemma K.1, the following result can be obtained.

Lemma K.2. Suppose that the conditions of Lemma K.1 hold. Then the inequality

$$\frac{1}{2}n\pi < \Omega_{m+\ell} < 3n\pi \tag{K.38}$$

is valid for a sufficiently large n.

Proof

It is known from [50] that, regardless of the standard end conditions,

$$\Omega_m > 2 \quad for \ m \geq 3. \tag{K.39}$$

Consequently,

$$\exp(-2\Omega_m) < \exp(-\Omega_m) < \exp(-2) < \frac{1}{6}. \tag{K.40}$$

By employing (K.40) and (K.15) through (K.17), it can be shown that

$$|\nabla_m| < \exp(-2\Omega_m)(1+\exp(-\Omega_m)) < \frac{1}{6}+\frac{1}{36} = \frac{7}{36}$$

or

$$|\nabla_m| < \frac{9}{36} = \frac{1}{4}. \tag{K.41}$$

Hence, it can be found from (K.6) that

$$\Omega_{m_0+\ell} = (m_0+\ell)\pi - \frac{5\pi}{4} + \nabla_{m_0+\ell} < (m_0+\ell)\pi - (\frac{5\pi}{4}-\frac{1}{4}) < (m_0+\ell)\pi. \tag{K.42}$$

Combining the last inequality with (K.30) leads immediately to the rightmost inequality (K.38). On the other hand, the leftmost inequality (K.38) can be obtained straightforwardly from the inequality

$$\begin{aligned} 0\Omega_{m_0+\ell} &= (m_0+\ell)\pi - \frac{5\pi}{4} + \nabla_{m_0+\ell} > (m_0+\ell)\pi - (\frac{5\pi}{4}+\frac{1}{4}) \\ &> n\pi - 2\pi > \frac{n\pi}{2} \qquad \textit{for } n > 4. \end{aligned} \tag{K.43}$$

This completes the proof of Lemma K.2. By using Lemma K.1 and K.2, the following result, which is also needed in Appendix J, can be obtained.

Lemma K.3. Suppose the requirements for Lemma K.1 hold. Then the inequality

$$\exp(-\Omega_{m_0+\ell}\frac{x_r}{L}) < \frac{10^{-4}}{4}, \quad r = 1, 2, \ldots, N-1 \tag{K.44}$$

is valid when n satisfies

$$n > \max\left(\frac{2}{\pi}\frac{L}{x_r}\ln\left(\frac{10^{-4}}{4}\right)^{-1},\quad r = 1, \ldots, N-1\right). \tag{K.45}$$

Proof

It can be seen that the value of

$$\frac{2}{\pi}\frac{L}{x_r}\ln\left(\frac{10^{-4}}{4}\right)^{-1} \tag{K.46}$$

is a positive constant for a given point x_r, $0 < r < N$. On the other hand, positive integers have no upper bound. Therefore, it is reasonable to let n_r be a positive integer which satisfies

$$n_r \geq \frac{2}{\pi}\frac{L}{x_r}\ln\left(\frac{10^{-4}}{4}\right)^{-1}. \tag{K.47}$$

This last inequality leads, after algebraic manipulation, to

$$\frac{n_r}{2}\pi\frac{x_r}{L} \geq \ln\left(\frac{10^{-4}}{4}\right)^{-1} \tag{K.48}$$

so that

$$\exp\left(\frac{n_r}{2}\pi\frac{x_r}{L}\right) \geq \exp\left(\ln\left(\frac{10^{-4}}{4}\right)^{-1}\right) = \left(\frac{10^{-4}}{4}\right)^{-1}$$

or

$$\left(\exp\left(-\frac{n_r}{2}\pi\frac{x_r}{L}\right)\right)^{-1} \leq \frac{10^{-4}}{4}$$

i.e.

$$\exp(-\frac{n_r}{2}\pi\frac{x_r}{L}) \le \frac{10^{-4}}{4}. \tag{K.49}$$

Let $n > max(n_r, r = 1, 2, ..., N - 1)$, then it was shown in [42] that

$$\exp(-\frac{n}{2}\pi\frac{x_r}{L}) \le \exp(-\frac{n_r}{2}\pi\frac{x_r}{L}) \le \frac{10^{-4}}{4}. \tag{K.50}$$

By employing the leftmost inequality of (K.38) and (K.50)

$$\exp(-\Omega_{m_0+\ell}\frac{x_r}{L}) \le \exp(-\frac{n}{2}\pi\frac{x_r}{L}) \le \frac{10^{-4}}{4} \tag{K.51}$$

which is the required relationship (K.44).

The following result is also needed in Appendix J.

Lemma J.4. Suppose Lemma K.1 holds. Then the inequalities

$$\frac{\Omega_{m_0+\ell}}{\Omega_{m_0+3\ell}} > \frac{3}{10}, \quad (\frac{\Omega_{m_0+\ell}}{\Omega_{m_0+3\ell}})^2 > \frac{9}{100} \quad \textit{and} \quad \frac{\Omega_{m_0+3\ell}}{\Omega_{m_0+\ell}} > \frac{29}{10} \tag{K.52}$$

are true for a sufficiently large $(m_0 + \ell)$.

Proof

When $n > 15$, the inequality

$$\frac{3}{2n} < \frac{1}{10}. \tag{K.53}$$

must hold. This last inequality, together with (K.6), (K.30) and (K.41), leads to

$$\Omega_{m_0+\ell} = (m_0+\ell)\pi[1-\frac{1}{(m_0+\ell)}(\frac{5}{4}-\frac{\nabla_{m_0+\ell}}{\pi})] > (m_0+\ell)\pi[1-\frac{1}{n}(\frac{5}{4}+\frac{1}{4})] \tag{K.54}$$

or

$$\Omega_{m_0+\ell} > (m_0+\ell)\pi(1-\frac{3}{2n}) > (m_0+\ell)\pi(1-\frac{1}{10}) = (m_0+\ell)\pi\frac{9}{10} \qquad \text{(K.55)}$$

i.e.

$$\Omega_{m_0+\ell} > (m_0+\ell)\pi\frac{9}{10}. \qquad \text{(K.56)}$$

On the other hand, a similar proof can be given for the inequality

$$\Omega_{m_0+3\ell} < 3(m_0+\ell)\pi. \qquad \text{(K.57)}$$

Consequently, by employing (K.56) and (K.57), the inequality

$$\frac{\Omega_{m_0+\ell}}{\Omega_{m_0+3\ell}} > \frac{9(m_0+\ell)}{30(m_0+\ell)} = \frac{3}{10} \qquad \text{(K.58)}$$

can be obtained immediately. It can be seen from the last inequality that

$$(\frac{\Omega_{m_0+\ell}}{\Omega_{m_0+3\ell}})^2 > \frac{9}{100}. \qquad \text{(K.59)}$$

The last inequality given in (K.52) can be shown analogously.

This completes the proof of Lemma K.4. In addition to Lemma K.1 through K.4, the following result is also needed in Appendix J.

Lemma K.5. Let $x_r/L = j_r/j$ be rational. (The j_r and j are two given positive integers.) Suppose that there exists a positive integer, m_0, such that the ray OA_r, which has an initial angle $\theta_0(x_r)$ given by (K.23), can be rotated into one of the plain regions defined in Figure J.1. Alternatively, it may coincide with the χ_0-axis. Then, regardless,

$$|\psi_{m_0+2jk}(x_r)| > \frac{1}{10}(\frac{2}{L})^{1/2} \quad and \quad \alpha^{(0)}_{rm_0}\psi_{m_0+2kj}(x_r) > 0 \tag{K.60}$$

can be demonstrated for a sufficiently large k and a fixed positive integer j. On the other hand, Suppose OA_r is rotated either into one of the plain regions or it coincides with the χ_1-axis after m_0 stepped increments from $\theta_0(x_r)$. Then

$$|\frac{L}{\Omega_m}\psi'_{m_0+2jk}(x_r)| > \frac{1}{10}(\frac{2}{L})^{1/2} \quad and \quad \alpha^{(1)}_{rm_0}\psi'_{m_0+2kj}(x_r) > 0. \tag{K.61}$$

Proof

Suppose that the ray OA_r either lies in one of the plain regions defined in Figure J.1 or it coincides with the χ_0-axis after m_0 stepped increments from $\theta_0(x_r)$. According to (J.3) or (K.23) and the periodicity $2j$ of OA_r, which can be derived from (J.18), the inequality

$$|\cos\theta_{m_0+2kj}(x_r)| > \frac{11}{100} \tag{K.62}$$

holds for any positive integer k and a given j. This last inequality, when combined with (K.22), leads to

$$|\alpha^{(0)}_{rm}| = (\frac{2}{L})^{1/2}|\cos\theta_{m_0+2kj}(x_r)| > (\frac{2}{L})^{1/2}\frac{11}{100}. \tag{K.63}$$

On the other hand, the notation $O(\exp(-\Omega_m))$ in (K.20) means that there exist a positive constant, c, and a positive integer, n_0, such that, when $m > n_0$,

$$|O(\exp(-\Omega_m))| \le c\exp(-\Omega_m). \tag{K.64}$$

Combining (K.64) with (K.38) leads to

$$|O(\exp(-\Omega_{m_0+2kj}))| \le c\exp(-\Omega_{m_0+2kj}) < c\exp(-\frac{n\pi}{2}) \tag{K.65}$$

for $m_0 + 2kj \ge n_0$. Let $n > max(2\ln(2c)/\pi,\ 10j,\ n_0)$. Then, the inequality

$$|O(\exp(-\Omega_{m_0+2kj})| < c\exp(-\frac{n\pi}{2}) < c\exp(-\ln(2c)) = \frac{c}{2c}$$

or,

$$|O(\exp(-\Omega_{m_0+2kj}))| < \frac{1}{2} \tag{K.66}$$

can be found for $m_0 + 2kj \ge n$. Thus, it is known from (K.26) and (K.66) that, when $r \ne 0$,

$$\begin{aligned}\hat{A}_{r(m_0+2kj)} &= \frac{1}{L^{1/2}}\exp(-\Omega_{m_0+2kj}\frac{x_r}{L})[1+O(\exp(-\Omega_{m_0+2kj}))] \\ &< \frac{2}{L^{1/2}}\exp(-\Omega_{m_0+2kj}\frac{x_r}{L}).\end{aligned}$$

Otherwise (K.26) indicates that if $r = 0$

$$\hat{A}_{r(m_0+2kj)} = 0 < \frac{2}{L^{1/2}}\exp(-\Omega_{m_0+2kj}\frac{x_r}{L}).$$

Consequently, from (K.38),

$$\hat{A}_{r(m_0+2kj)} < \frac{2}{L^{1/2}}\exp(-\frac{n}{2}\frac{x_r}{L}\pi) \tag{K.67}$$

holds for $m_0 + 2kj \ge n$. Furthermore, if

$$n > \max\left(\left(\frac{2}{\pi}\right)\left(\frac{x_r}{L}\right)^{-1}\ln(100(2)^{1/2}),\ \frac{2}{\pi}\ln(2c),\ 10j,\ n_0\right)$$

then, by employing (K.67),

$$\mathbf{\Lambda}_{r(m_0+2kj)} < \frac{2}{L^{1/2}}\exp\left(-\frac{n}{2}\frac{x_r}{L}\pi\right) < \frac{2}{L^{1/2}}\exp(-\ln(100(2)^{1/2}))$$

or

$$\mathbf{\Lambda}_{r(m_0+2kj)} < \left(\frac{2}{L}\right)^{1/2}\frac{1}{100}. \tag{K.68}$$

Consequently, by using (K.63) and (K.68), the following inequality can be found from (K.28)

$$|\psi_{m_0+2kj}(x_r)| \geq |\alpha^{(0)}_{r(m_0+2kj)}| - \mathbf{\Lambda}_{r(m_0+2kj)} > \left(\frac{2}{L}\right)^{1/2}\left(\frac{11}{100}-\frac{1}{100}\right) = \left(\frac{2}{L}\right)^{1/2}\frac{1}{10}. \tag{K.69}$$

This last inequality proves the first inequality of (K.60). On the other hand, it is known from elementary algebraic theory that $|a| - b > 0$ is equivalent to $a < -b$ or $a > b$. Here the generic b is an arbitrary finite positive value and a is an arbitrary finite real value. Thus, the inequalities

$$\alpha^{(0)}_{r(m_0+2kj)} + \mathbf{\Lambda}_{r(m_0+2kj)} < 0, \quad \textit{if}\ \alpha^{(0)}_{r(m_0+2kj)} < 0 \tag{K.70}$$

and

$$\alpha^{(0)}_{r(m_0+2kj)} + \mathbf{\Lambda}_{r(m_0+2kj)} > 0, \quad \textit{if}\ \alpha^{(0)}_{r(m_0+2kj)} > 0 \tag{K.71}$$

can be demonstrated. The last two inequalities lead immediately to

$$\alpha^{(0)}_{r(m_0+2kj)}(\alpha^{(0)}_{r(m_0+2kj)}+\hat{\Lambda}_{r(m_0+2kj)}) > 0. \tag{K.72}$$

Consequently, the second inequality of (K.60), i.e.

$$\alpha^{(0)}_{rm_0}\psi_{m_0+2kj}(x_r) > 0 \tag{K.73}$$

can be obtained by employing (K.29) and (K.72). A similar proof can be given for (K.61). This completes the proof of Lemma K.5.

(*Remark* K.1. It can be shown straightforwardly from (K.24) and (K.25) that (K.60) is also true when $r = 0$ and $r = N$ whilst (K.61) holds just for $r = 0$.)

Finally, the following result is needed in Appendix J.

Lemma K.6. Suppose that Lemma K.1 holds and $x_r/L = j_r/j$ is rational where j_r and j are two given positive integers. Then the inequalities

$$\hat{\Lambda}_{r(m_0+\ell)} - \hat{\Lambda}_{r(m_0+3\ell)} > 0 \tag{K.74}$$

hold for a sufficiently large $(m_0 + \ell)$ and $0 < r \le N$ where m_0 is positive integer.

Proof

It is known from the proof of Lemma K.5 that there exists a n_0 such that, when $n > n_0$, (K.66) holds so that

$$|O(\exp(-\Omega_{m_0+\ell}))| \le \frac{1}{2}. \tag{K.75}$$

This last inequality leads, additionally, to the inequality

$$|O(\exp(-\Omega_{m_0+3\ell}))| \le |O(\exp(-\Omega_{m_0+\ell}))| \le \frac{1}{2}. \tag{K.76}$$

By employing (K.75) and (K.76), the inequalities

$$\begin{aligned} \mathbf{A}_{r(m_0+3\ell)} &= \frac{1}{L^{1/2}}\exp(-\Omega_{m_0+3\ell}\frac{x_r}{L})[1+O(\exp(-\Omega_{m_0+3\ell}))] \\ &\le \frac{3}{2L^{1/2}}\exp(-\Omega_{m_0+3\ell}\frac{x_r}{L}) \end{aligned} \tag{K.77}$$

and

$$\begin{aligned} \mathbf{A}_{r(m_0+\ell)} &= \frac{1}{L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})[1+O(\exp(-\Omega_{m_0+\ell}))] \\ &\ge \frac{1}{2L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L}) \end{aligned} \tag{K.78}$$

can be obtained from (K.26) for $0 < r \le N$. Consequently,

$$\mathbf{A}_{r(m_0+\ell)} - \mathbf{A}_{r(m_0+3\ell)} \ge \frac{1}{2L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L}) - \frac{3}{2L^{1/2}}\exp(-\Omega_{m_0+3\ell}\frac{x_r}{L})$$

or

$$\mathbf{A}_{r(m_0+\ell)} - \mathbf{A}_{r(m_0+3\ell)} \ge \frac{1}{2L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})(1-3\exp(-(\Omega_{m_0+3\ell}-\Omega_{m_0+\ell})\frac{x_r}{L}) \tag{K.79}$$

can be shown from (K.77) and (K.78). On the other hand, it can be shown from (K.41) that, because $\pi/4 > 1/4$,

$$\frac{5\pi}{4} + \nabla_m < \frac{5\pi}{4} + \frac{\pi}{4} = \frac{6\pi}{4} < 3\pi. \tag{K.80}$$

This last inequality, when combined with (K.6), leads to

$$\Omega_{m_0+3\ell} > (m_0+3\ell)\pi - 3\pi. \tag{K.81}$$

Furthermore, it is known from (K.42) that

$$\Omega_{m_0+\ell} < (m_0+\ell)\pi < 2(m_0+\ell)\pi. \tag{K.82}$$

Thus, by employing (K.81) and (K.82), the inequality

$$\Omega_{m_0+3\ell} - \Omega_{m_0+\ell} > (m_0+3\ell)(1-\frac{3}{m_0+3\ell})\pi - 2\pi(m_0+\ell)$$

or

$$\Omega_{m_0+3\ell} - \Omega_{m_0+\ell} > \pi[\ell-(m_0+3)]. \tag{K.83}$$

can be demonstrated. Furthermore, it is known from (K.31) and (K.36) that $\ell > n$ and, hence, $\ell - (m_0 + 3) > n - (m_0 + 3)$. When $n > max((x_r/L)^{-1}(\ln 6)/\pi,\ n_0)$, the following inequality can be found from (J.79) and (J.83), viz

$$\begin{aligned}
\mathbf{A}_{r(m_0+\ell)} - \mathbf{A}_{r(m_0+3\ell)} &\geq \frac{1}{2L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})(1-3\exp(-(\Omega_{m_0+3\ell}-\Omega_{m_0+\ell})\frac{x_r}{L}) \\
&> \frac{1}{2L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})(1-3\exp(-\pi(\ell-(m_0+3))\frac{x_r}{L}) \\
&> \frac{1}{2L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})(1-3\exp(-\pi(n-(m_0+3))\frac{x_r}{L}) \\
&> \frac{1}{2L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L})(1-3\exp(-\ln 6))
\end{aligned}$$

or

$$\mathbf{A}_{r(m_0+\ell)} - \mathbf{A}_{r(m_0+3\ell)} > \frac{1}{4L^{1/2}}\exp(-\Omega_{m_0+\ell}\frac{x_r}{L}) > 0. \tag{K.84}$$

This completes the proof of Lemma K.6.

APPENDIX L

It is shown here that the conditions required by Theorem 3.4.1 are satisfied by the numerical example. In the example, $N = 3$, $x_1/L = 0.25$ and $x_2/L = 0.5$ are the two points where the second derivatives of the GFM functions, $\zeta_{21}(x)$ and $\zeta_{22}(x)$, are discontinuous. On the other hand, the third derivative of $\zeta_{32}(x)$ is discontinuous at $x_2 = 0.5L$. The n-dimensional subspace, S_n, in Theorems 3.3.1 and 3.4.1 is formed by $\zeta_{21}(x)$, $\zeta_{22}(x)$ and $\zeta_{32}(x)$ in addition to the $(n - 3)$ eigenvectors $\{\psi_m(x),\ m = 1, \ldots, (n - 3)\}$ of a uniform cantilevered beam. Suppose that OA_1 and OA_2 are two rays that have the initial angles $\theta_0(x_1) = \pi/8$ and $\theta_0(x_2) = 0$, respectively, to the η-axis. These rays rotate counterclockwise about the origin, O, shown in Figure J.1 of Appendix J in increments of $x_1\pi/L$ (OA_1) and $x_2\pi/L$ (for OA_2). Then it can be found from Table J.8 that the following inequalities

$$\left.\begin{aligned} & \varrho_{11}\alpha_{11}^{(1)} \neq 0, \ \ (\varrho_{11}\alpha_{11}^{(1)})(\varrho_{12}\alpha_{21}^{(1)}) \geq 0, \ \ \mathit{if} \ \ \varrho_{11}\varrho_{12} \geq 0 \\ & \varrho_{02}\alpha_{21}^{(0)} \geq 0 \ \ \mathit{or} \ \ \varrho_{02}\alpha_{21}^{(0)} \leq 0 \end{aligned}\right\} \qquad \text{(L.1)}$$

and

$$\left.\begin{aligned} & \varrho_{11}\alpha_{13}^{(1)} \neq 0, \ \ (\varrho_{11}\alpha_{13}^{(1)})(\varrho_{12}\alpha_{23}^{(1)}) \geq 0, \ \ \mathit{if} \ \ \varrho_{11}\varrho_{12} \leq 0 \\ & \varrho_{02}\alpha_{23}^{(0)} \geq 0 \ \ \mathit{or} \ \ \varrho_{02}\alpha_{23}^{(0)} \leq 0 \end{aligned}\right\} \qquad \text{(L.2)}$$

hold for $s = 2$ and $t = 1$. Also,

$$\left.\begin{array}{l} \ell_{12}\alpha_{21}^{(1)} \neq 0, \quad (\ell_{12}\alpha_{21}^{(1)})(\ell_{11}\alpha_{11}^{(1)}) \geq 0, \quad \textit{if} \quad \ell_{11}\ell_{12} \geq 0 \\ \\ \ell_{02}\alpha_{21}^{(0)} \geq 0 \quad \textit{or} \quad \ell_{02}\alpha_{21}^{(0)} \leq 0 \end{array}\right\} \tag{L.3}$$

and

$$\left.\begin{array}{l} \ell_{12}\alpha_{23}^{(1)} \neq 0, \quad (\ell_{12}\alpha_{23}^{(1)})(\ell_{11}\alpha_{13}^{(1)}) \geq 0, \quad \textit{if} \quad \ell_{11}\ell_{12} \leq 0 \\ \\ \ell_{02}\alpha_{23}^{(0)} \geq 0 \quad \textit{or} \quad \ell_{02}\alpha_{23}^{(0)} \leq 0 \end{array}\right\} \tag{L.4}$$

when $s = 2$ and $t = 2$. Furthermore,

$$\left.\begin{array}{l} \ell_{02}\alpha_{22}^{(0)} \neq 0 \\ \ell_{11}\alpha_{12}^{(1)} \leq 0, \quad \textit{if}\ \ell_{11} \geq 0 \quad \textit{and} \quad \ell_{11}\alpha_{12}^{(1)} \geq 0, \quad \textit{if}\ \ell_{11} \leq 0, \ \ell_{12}\alpha_{22}^{(1)} \equiv 0 \end{array}\right\} \tag{L.5}$$

holds for $s = 3$ and $t = 2$. Thus, (J.7) and (J.9) through (J.11) of Lemma J.1 are satisfied. Moreover, the functions form a set of 4-GFM functions so that Theorem 3.4.1 applies.

APPENDIX M

This appendix derives equations (2.2.1) through (2.2.3) of the freely vibrating beam shown in Figure 2.1 by employing the Euler-Bernoulli beam theory. The notation of chapter 2 is used in this appendix. First, consider the equation of motion of an element of the beam that is given in Figure 2.1 and which is also shown in Figure M.1. Let $w(x, \hat{t})$ and $\theta(x, \hat{t})$ be the transverse deflection and rotation of the beam whilst $V_f(x, \hat{t})$ and $M_f(x, \hat{t})$ denote the shear force and bending moment acting at a point, x, and at an instant of time, $\hat{t}$. Furthermore, $p(x)$ and $A_f(x)$ represent the time independent axial force and external, distributed load along the x axis. Then transverse equilibrium yields

$$-(V_f+\frac{\partial V_f}{\partial x}dx)+V_f+(p+\frac{\partial p}{\partial x}dx)\sin(\theta+\frac{\partial \theta}{\partial x}dx)-p\sin\theta = \rho A dx\frac{\partial^2 w}{\partial \hat{t}^2}. \tag{M.1}$$

Rotational equilibrium about the centroid, O_x, of the cross-sectional area is shown in Figure M.1. It produces

$$\begin{aligned}(M_f+\frac{\partial M_f}{\partial x}dx)-M_f-(V_f+\frac{\partial V_f}{\partial x}dx)dx+A_f dx\frac{dx}{2}\sin\theta \\ +(p+\frac{dp}{dx}dx)dx\sin(\frac{\partial\theta}{\partial x}dx) = 0\end{aligned} \tag{M.2}$$

whilst the longitudinal equilibrium creates

$$(p+\frac{dp}{dx}dx)\cos(\theta+\frac{\partial\theta}{\partial x}dx)-p\cos\theta-A_f dx = 0. \tag{M.3}$$

By employing the following approximations in the Euler-Bernoulli beam theory [50]

$$\left.\begin{aligned}&\sin(\theta+\frac{\partial\theta}{\partial x}dx) = \theta+\frac{\partial\theta}{\partial x}dx = \frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}dx\\&\sin(\frac{\partial\theta}{\partial x}dx) = \frac{\partial\theta}{\partial x}dx,\ \cos(\theta+\frac{\partial\theta}{\partial x}dx) = 1,\ \ \cos\theta = 1\end{aligned}\right\} \tag{M.4}$$

and neglecting higher-order terms involving $(dx)^2$, equations (M.1) can be simplified to

$$-\frac{\partial V_f}{\partial x}dx+p\frac{\partial\theta}{\partial x}dx+\theta\frac{\partial p}{\partial x}dx = \rho A dx\frac{\partial^2 w}{\partial \hat{t}^2}$$

or

$$-\frac{\partial V_f}{\partial x}+\frac{\partial(p\theta)}{\partial x} = \rho A\frac{\partial^2 w}{\partial \hat{t}^2}$$

or

$$-\frac{\partial V_f}{\partial x}+\frac{\partial}{\partial x}(p\frac{\partial w}{\partial x}) = \rho A\frac{\partial^2 w}{\partial \hat{t}^2}. \tag{M.5}$$

Moreover, (M.2) and (M.3) can be simplified to

$$V_f = \frac{\partial M_f}{\partial x} \tag{M.6}$$

and

$$\frac{dp}{dx} = A_f. \tag{M.7}$$

On the other hand, it is known from the Euler-Bernoulli beam theory [50] that

$$M_f(x,t) = EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}. \tag{M.8}$$

Substituting the last equation into (M.5) and (M.6) produces

$$-\frac{\partial^2}{\partial^2}(EI\frac{\partial^2 w}{\partial x^2}) + \frac{\partial}{\partial x}(p\frac{\partial w}{\partial x}) = \rho A\frac{\partial^2 w}{\partial \hat{t}^2}. \tag{M.9}$$

Now consider the equation of motion of the lumped mass, M_1, and the rotary inertia, J_1, in Figure 2.1. The corresponding free body diagram is shown in Figure M.2. $M_f(L, \hat{t})$, $V_f(L, \hat{t})$ and $p(L)$ are the bending moment, shear force and axial force of the beam whilst $K_1 (w(L, \hat{t}) + \eta_1 \theta(L, \hat{t}))$ and $\beta_1 \theta(L, \hat{t})$ represent the transverse force and torsional moment due to the deflection of the linear spring K_1 and torsional spring β_1 shown in Figure 2.1. The P_1 in Figure M.2 is a time-independent, concentrated external load acting along the x axis at $x = L + e_1$. Transverse equilibrium leads to

$$\begin{aligned} -K_1(w(L,\hat{t}) + \eta_1\theta(L,\hat{t})) + V_f(L,\hat{t}) - p(L)\sin\theta = M_1\frac{\partial^2}{\partial \hat{t}^2}(w(L,\hat{t}) \\ + e_1\frac{\partial w(L,\hat{t})}{\partial x}). \end{aligned} \tag{M.10}$$

On the other hand, rotational equilibrium about O_1, the center of gravity of the lumped mass M_1, yields

$$(e_1 - \eta_1)K_1(w(L,\hat{t}) + \eta_1\theta(L,\hat{t})) - e_1 V_f(L,\hat{t}) + e_1 p(L)\sin\theta \\ -\beta_1\theta(L,\hat{t}) - M_f(L,\hat{t}) = J_1\frac{\partial^2}{\partial\hat{t}^2}(\frac{\partial w(L,\hat{t})}{\partial x}). \tag{M.11}$$

Furthermore, longitudinal equilibrium gives

$$p(L)\cos\theta = P_1. \tag{M.12}$$

By employing (M.4), (M.10) through (M.12) can be simplified to

$$-K_1(w(L,\hat{t}) + \eta_1\frac{\partial w(L,\hat{t})}{\partial x}) + p(L)\frac{\partial w(L,\hat{t})}{\partial x} + \frac{\partial^2}{\partial x^2}(EI(L)\frac{\partial^2 w(l,\hat{t})}{\partial x^2}) \\ = M_1\frac{\partial^2}{\partial\hat{t}^2}(w(L,\hat{t}) + e_1\frac{\partial w(L,\hat{t})}{\partial x}) \tag{M.13}$$

$$-e_1\frac{\partial}{\partial x}(EI(L)\frac{\partial^2 w(L,\hat{t})}{\partial x^2}) + (e_1 - \eta_1)K_1(w(L,\hat{t}) + \eta_1\frac{\partial w(L,\hat{t})}{\partial x}) - EI(L)\frac{\partial^2 w(L,\hat{t})}{\partial x^2} \\ + e_1 p(L)\frac{\partial w(L,\hat{t})}{\partial x} - \beta_1\frac{\partial w(L,\hat{t})}{\partial x} = J_1\frac{\partial^2}{\partial\hat{t}^2}(\frac{\partial w(L,\hat{t})}{\partial x}) \tag{M.14}$$

and

$$p(L) = P_1. \tag{M.15}$$

By employing a similar procedure, the following equation can be derived from Figure M.3 for the lumped mass, M_0, and the rotary inertia, J_0, illustrated in Figure 2.1

$$\begin{aligned}
&-K_0(w(0,\hat{t})-\eta_0\frac{\partial w(0,\hat{t})}{\partial x})+p(0)\frac{\partial w(0,\hat{t})}{\partial x}-\frac{\partial^2}{\partial x^2}(EI(0)\frac{\partial^2 w(0,\hat{t})}{\partial x^2}) \\
&\qquad = M_0\frac{\partial^2}{\partial \hat{t}^2}(w(0,\hat{t})-e_0\frac{\partial w(0,\hat{t})}{\partial x})
\end{aligned} \tag{M.16}$$

$$\begin{aligned}
&-e_0\frac{\partial}{\partial x}(EI(0)\frac{\partial^2 w(0,\hat{t})}{\partial x^2})+(e_0-\eta_0)K_0(w(0,\hat{t})-\eta_0\frac{\partial w(0,\hat{t})}{\partial x})+EI(0)\frac{\partial^2 w(0,\hat{t})}{\partial x^2} \\
&\qquad +e_0 p(0)\frac{\partial w(0,\hat{t})}{\partial x}-\beta_0\frac{\partial w(0,\hat{t})}{\partial x} = J_0\frac{\partial^2}{\partial \hat{t}^2}(\frac{\partial w(0,\hat{t})}{\partial x})
\end{aligned} \tag{M.17}$$

and

$$p(0) = -P_0. \tag{M.18}$$

P_0 is a time-independent, concentrated external load acting along the x axis at $x = -e_0$. Suppose the beam of Figure 2.1 is in free vibrations corresponding to the j th natural frequency, ω_j. Then $w(x, \hat{t})$ can be expressed by [50]

$$w(x,\hat{t}) = w_{1j}(x)(A\cos\omega_j\hat{t}+B\sin\omega_j\hat{t}) \tag{M.19}$$

where $w_{1j}(x)$ is the corresponding eigenvector. Substituting (M.19) into (M.9), (M.13), (M.14), (M.16) and (M.17) and letting $\lambda_j = \omega_j^2$ leads to the equation of free vibration of the beam shown in Figure 2.1 as

$$\frac{\partial^2}{\partial^2}(EI\frac{\partial^2 w_{1j}(x)}{\partial x^2}) - \frac{\partial}{\partial x}(p\frac{\partial w_{1j}(x)}{\partial x}) = \rho A \lambda_j w_{1j}(x) \tag{M.20}$$

$$K_0(w_{1j}(0) - \eta_0\frac{\partial w_{1j}(0)}{\partial x}) - p(0)\frac{\partial w_{1j}(0)}{\partial x} + \frac{\partial^2}{\partial x^2}(EI(0)\frac{\partial^2 w_{1j}(0)}{\partial x^2}) = M_0\lambda_j(w_{1j}(0) - e_0\frac{\partial w_{1j}(0)}{\partial x}) \tag{M.21}$$

$$e_0\frac{\partial}{\partial x}(EI(0)\frac{\partial^2 w_{1j}(0)}{\partial x^2}) + (e_0 - \eta_0)K_0(w_{1j}(0) - \eta_0\frac{\partial w_{1j}(0)}{\partial x}) - EI(0)\frac{\partial^2 w_{1j}(0)}{\partial x^2} - e_0 p(0)\frac{\partial w_{1j}(0)}{\partial x} + \beta_0\frac{\partial w_{1j}(0)}{\partial x} = J_0\lambda_j\frac{\partial w_{1j}(0)}{\partial x}. \tag{M.22}$$

$$K_1(w_{1j}(L) + \eta_1\frac{\partial w_{1j}(L)}{\partial x}) + p(L)\frac{\partial w_{1j}(L)}{\partial x} - \frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 w_{1j}(L)}{\partial x^2}) = M_1\lambda_j(w_{1j}(L) + e_1\frac{\partial w_{1j}(L)}{\partial x}) \tag{M.23}$$

and

$$e_1\frac{\partial}{\partial x}(EI(L)\frac{\partial^2 w_{1j}(L)}{\partial x^2}) - (e_1 - \eta_1)K_1(w_{1j}(L) + \eta_1\frac{\partial w_{1j}(L)}{\partial x}) + EI(L)\frac{\partial^2 w_{1j}(L)}{\partial x^2} - e_1 p(L)\frac{\partial w_{1j}(L)}{\partial x} + \beta_1\frac{\partial w_{1j}(L)}{\partial x} = J_1\lambda_j\frac{\partial w_{1j}(L)}{\partial x} \tag{M.24}$$

Let τ_n, $n = 1, 2, \ldots, 5$ be linear maps that are defined by

$$\tau_1 w_{1j} \equiv (\rho A)^{-1} \frac{\partial^2}{\partial^2}\left(EI \frac{\partial^2 w_{1j}}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(p \frac{\partial w_{1j}}{\partial x}\right) \tag{M.25}$$

$$\tau_2 w_{2j} \equiv M_0^{-1}\left[K_0\left(w_{1j}(0) - \eta_0 \frac{\partial w_{1j}(0)}{\partial x}\right) - p(0)\frac{\partial w_{1j}(0)}{\partial x} + \frac{\partial^2}{\partial x^2}\left(EI(0)\frac{\partial^2 w_{1j}(0)}{\partial x^2}\right)\right] \tag{M.26}$$

$$\begin{aligned}\tau_3 w_{3j} \equiv J_0^{-1}\Big[&e_0 \frac{\partial}{\partial x}\left(EI(0)\frac{\partial^2 w_{1j}(0)}{\partial x^2}\right) + (e_0 - \eta_0)K_0\left(w_{1j}(0) - \eta_0 \frac{\partial w_{1j}(0)}{\partial x}\right)\\ &- EI(0)\frac{\partial^2 w_{1j}(0)}{\partial x^2} - e_0 p(0)\frac{\partial w_{1j}(0)}{\partial x} + \beta_0 \frac{\partial w_{1j}(0)}{\partial x}\Big]\end{aligned} \tag{M.27}$$

$$\tau_4 w_{4j} = M_1^{-1}\left[K_1\left(w_{1j}(L) + \eta_1 \frac{\partial w_{1j}(L)}{\partial x}\right) + p(L)\frac{\partial w_{1j}(L)}{\partial x} - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w_{1j}(L)}{\partial x^2}\right)\right] \tag{M.28}$$

and

$$\begin{aligned}\tau_5 w_{5j} = J_1^{-1}\Big[&e_1 \frac{\partial}{\partial x}\left(EI(L)\frac{\partial^2 w_{1j}(L)}{\partial x^2}\right) - (e_1 - \eta_1)K_1\left(w_{1j}(L) + \eta_1 \frac{\partial w_{1j}(L)}{\partial x}\right)\\ &+ EI(L)\frac{\partial^2 w_{1j}(L)}{\partial x^2} - e_1 p(L)\frac{\partial w_{1j}(L)}{\partial x} + \beta_1 \frac{\partial w_{1j}(L)}{\partial x}\Big]\end{aligned} \tag{M.29}$$

where

$$\left.\begin{array}{ll} w_{2j} \equiv w_{1j}(0) - e_0 w'_{1j}(0), & w_{3j} \equiv w'_{1j}(0), \\ \textit{and} & \\ w_{4j} \equiv w_{1j}(L) + e_1 w'_{1j}(L), & w_{5j} \equiv w'_{1j}(L). \end{array}\right\} \tag{M.30}$$

Consequently, equations (M.20) through (M.21) can be rewritten as

$$\tau_1 w_{1j} = \lambda_j w_{1j}, \quad 0 < x < L \tag{M.31}$$

$$\tau_2 w_{2j} = \lambda_j w_{2j} \tag{M.32}$$

$$\tau_3 w_{3j} = \lambda_j w_{3j} \tag{M.33}$$

$$\tau_4 w_{4j} = \lambda_j w_{4j} \tag{M.34}$$

and

$$\tau_5 w_{5j} = \lambda_j w_{5j}. \tag{M.35}$$

It can be found that equations (2.2.1) through (2.2.3) are just concise forms of equations (M.25) through (M.35).

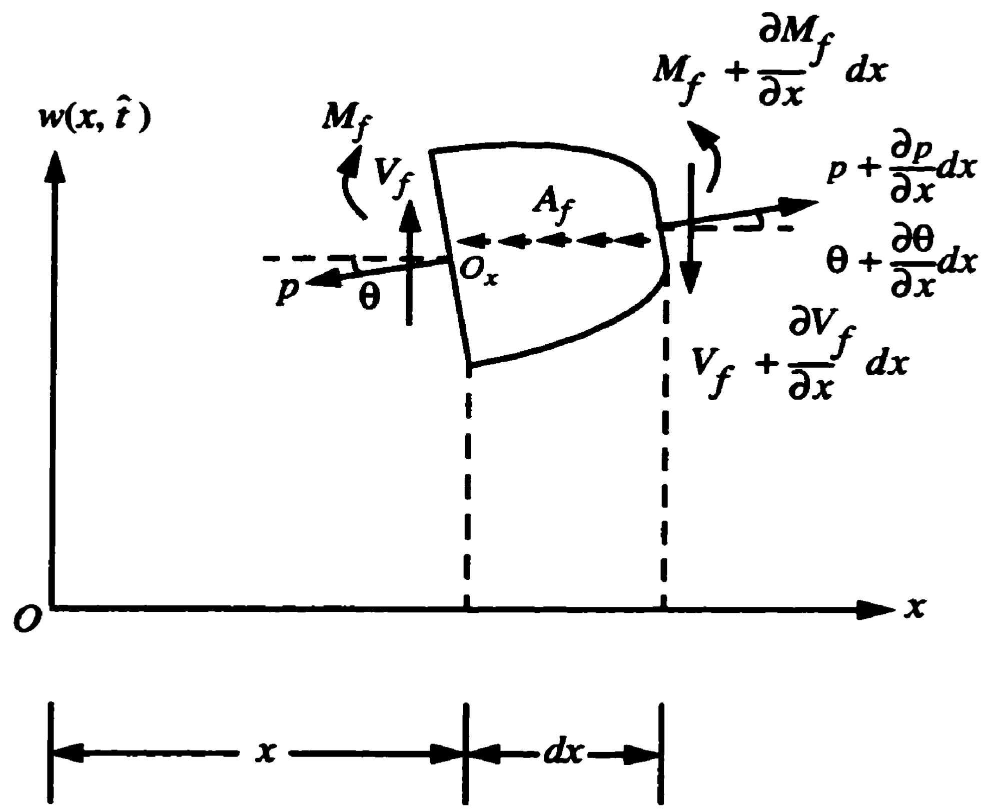

Figure M.1. Free-body diagram of an element of the beam shown in Figure 2.1.

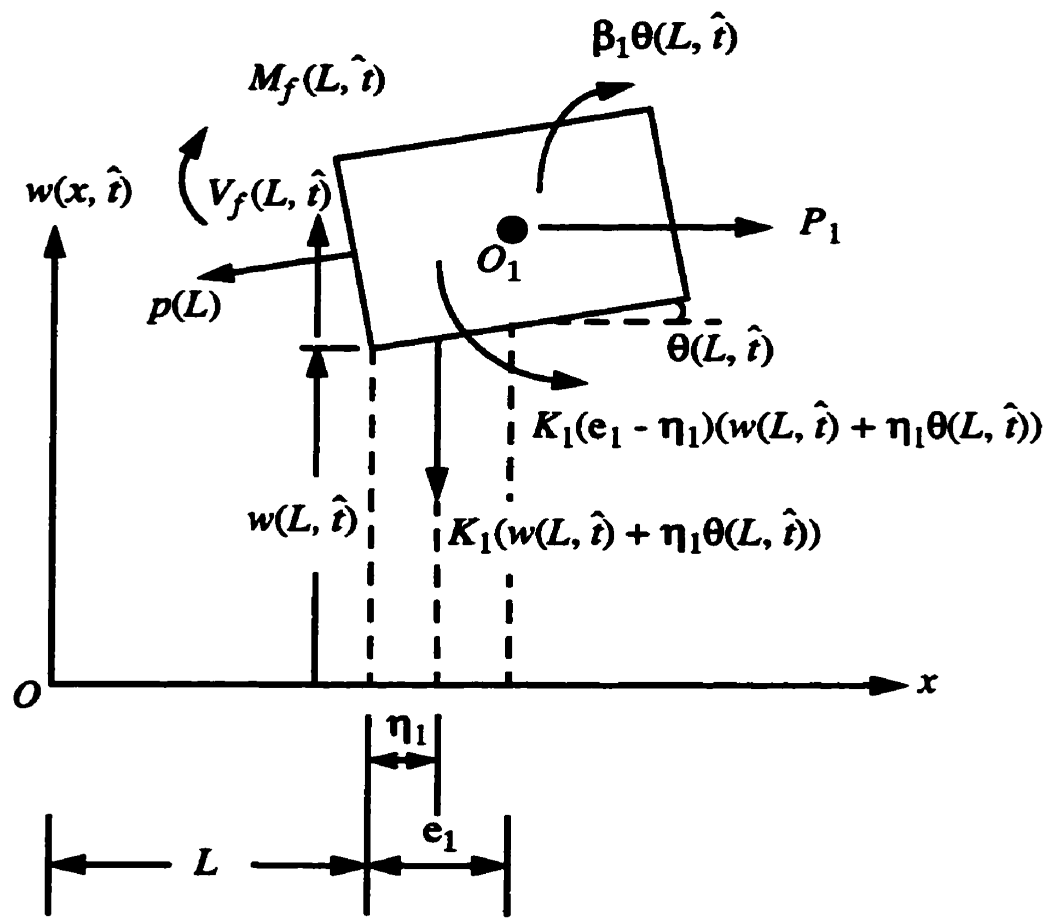

Figure M.2. Free-body diagram of the lumped mass, M_1, and the rotary inertia, J_1, shown in Figure 2.1.

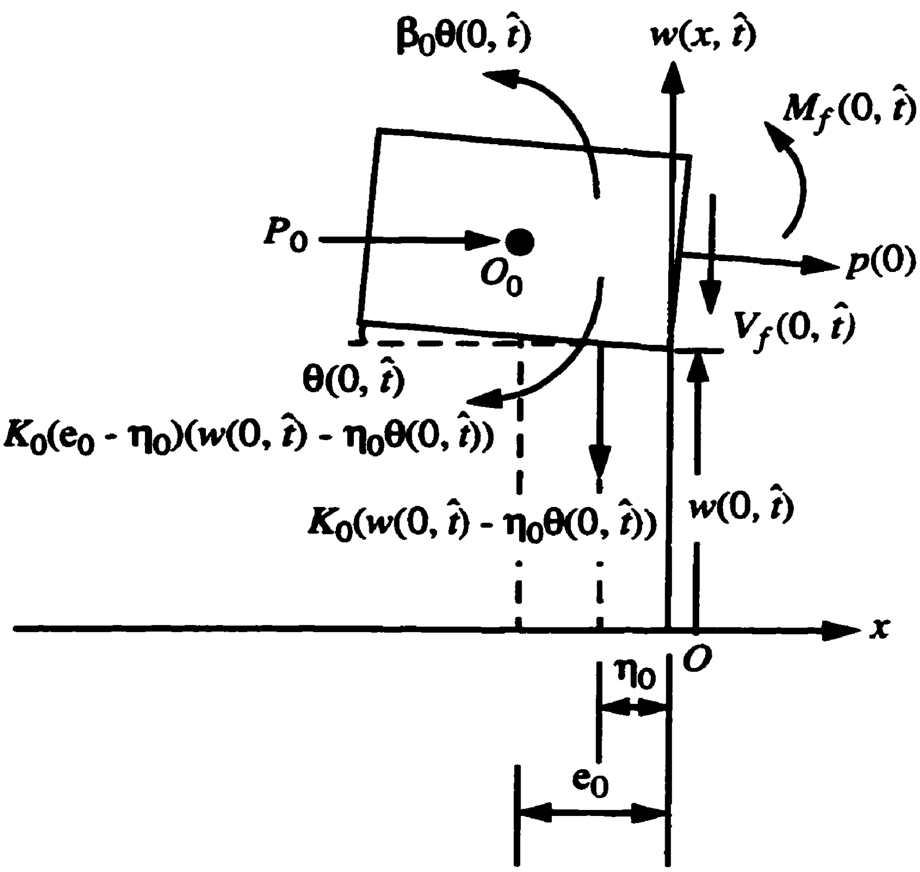

Figure M.3. Free-body diagram of the lumped mass, M_0, and the rotary inertia, J_0, shown in Figure 2.1.