SOME MATHEMATICAL ASPECTS OF VALUATION

by

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SOME MATHEMATICAL ASPECTS OF VALUATION

Chapter I

Reserve Standards for Life Insurance in the United States and Canada

The early history of Valuation in the United States centres around those standards set in New York and Massachusetts.

<u>Massachusetts</u>

In 1854 life insurance companies doing business in Massachusetts were required to report annually on their business, but no basis was specified and apparently no effort was made to check the returns submitted. As a result of the tireless efforts of Elizur Wright, one of the Insurance Commissioners in 1858, a law was passed providing that life insurance companies should furnish the Commissioners with the data necessary for calculating the value of all outstanding policies and charging the Commissioners to compute annually the policy liabilities of each company doing business in the State. The basis of valuation was not specified but the Commissioners adopted net level premium valuation on the basis of the Combined Experience or Actuaries Table, with interest at 4%.

The Act of 1861 specified this basis for calculating surrender values, and in 1863 it was specified for testing the solvency of a company and as the standard for the reserve to be held before any surplus could be distributed.

In 1887 the Massachusetts Insurance Law required valuations on the Combined Experience 4% basis, and in 1900 amended this Law to provide for the valuation of policies issued subsequent to December 31, 1900 on the American Experience table, with 3½% or 3% interest. In 1923 Massachusetts, by ruling, permitted Illinois Standard valuation.

New York

The New York Insurance Department began operation in 1860 under William Barnes, as Superintendent of Insurance. In 1863 he offered to reinstate the American Mutual valuing policies on the Carlisle 5% basis, using gross premiums less 25%, thus wavering towards a Gross Premium Valuation. In 1866 New York, by law, established the English Life Table No. 3, for males, with 5% interest, as the reserve basis, and required quinquennial valuations. Net premium valuation or the use of gross premiums less one-sixth was allowed until 1879 when net premium valuation was unconditionally required.

In 1869 New York adopted the American Experience 4½% basis but Wright of Massachusetts refused to lower his standard to make it the same.

Between this date and 1884, out of thirty-six companies which were insolvent under the Massachusetts standard but solvent on the New York basis, all but four went out of business. In 1884 New York changed to the net premium basis on the Combined Experience 4% basis. At this time a double standard for solvency was set up. (a) If the company's assets were less than its liabilities on the American Experience 4½% basis a receiver could be appointed. (b) If the assets were as large as the above but less than the liability on the American Experience 4% basis, the company could issue no new business until its position had sufficiently improved. By this Act "Capital Stock" was not considered a liability. In New York it was not so considered until 1901.

In 1901 the valuation section was amended to provide a minimum reserve standard. For issues prior to January 1, 1901, this minimum was the Combined Experience 4%, while for issues of January1, 1901, and later, the minimum was the American Experience 3½%. From 1901 to 1906 New York permitted preliminary term valuation but Massachusetts did not.

Following the Armstrong Investigation, in 1906, preliminary term valuation was prohibited and the minimum standard fixed by the select and ultimate method based on the American Experience table, with 3½% interest, for policies issued subsequent to December 31, 1906. In 1923 the New York law was again amended to permit as a minimum standard, reserves on either the select and ultimate method or according to the Illinois modified preliminary term basis.

<u>Other States</u>

Other States followed fairly closely what was done in New York and Massachusetts. Ohio and New Jersey used modified preliminary term bases only slightly different from the Illinois Standard.

<u>Canada</u>

A discussion of reserve standards in Canada is included in Chapter II.

Chapter II

Methods of Valuation

Gross Premium Method

Policies were first valued by what is known as the Gross Premium Method. Under this method no provision was made for future expenses and the whole gross premium was assumed to be available to pay future claims. Hence the reserve was much less than was necessary and if a large number of claims were presented there was danger of the company becoming insolvent.

Net Premium Method

To remedy this evil there was introduced the Net Premium Method of Valuation. Under such, a level expense charge is assumed throughout the whole duration of the policy and this is taken care of by the loading on the policy. Two further assumptions are also made. Firstly that none of the loading is ever available to meet claims and secondly, that the whole of the net premium is always so available.

But none of these assumptions are quite in accordance with fact. The first year's expenses are large relative to those of subsequent years. This is particularly true since the practice of paying large initial commissions has become popular. Therefore, the loading is not sufficient to pay the first year's expenses and is too great to pay the expenses of later years. If, then, a company were required to put up the full net premium reserve at the end of the first year, it would have to get some of it from accumulated surplus and therefore younger companies would be forced to restrict the amount of new business they could issue.

Thus the Net Premium Method of calculating reserves made the establishment of young companies very difficult. This was made still more difficult about 1860, for in that year a stricter interpretation was made as to what might constitute the assets of a company. Many items of assets previously allowed were disallowed and many young companies were forced out of business.

Full Preliminary Term Method

It was largely to enable young companies to become established that there was introduced, in 1863, what was called the Full Preliminary Term plan of computing reserves. This plan provides for term insurance for one year with the right of renewal on the Life or Endowment plan. Thus no reserve is required at the end of the first policy year. The whole of the first year's premium so becomes available for claims (which are small) and for expenses. The companies actually wrote their policies as Term Insurances for one year and Life or Endowment thereafter.

Under the Full Preliminary Term plan the valuation premiums are considered as:

Net premium first year $P^1_{x\overline{1}|}$ on all policies.

Net premium second and future years:

P_{x+1} on whole-life policies

${}_{k-1}P_{x+1}$ on limited payment whole-life policies

$P_{x+1:\overline{n-1}|}$ on n-year endowment policies.

The reserve at the end of t years is:

${}_{t}V'_{x} = {}_{t-1}V_{x+1}$ for a whole-life policy

${}^{k}_{t}V'_{x} = {}^{k-1}_{t-1}V_{x+1}$ for a k-payment whole-life policy

${}_{t}V'_{x:\overline{n}|} = {}_{t-1}V_{x+1:\overline{n-1}|}$ for an n-year endowment policy

${}_{t}V'^{1}_{x:\overline{n}|} = {}_{t-1}V^{1}_{x+1:\overline{n-1}|}$ for an n-year term policy.

The two main objections to this method of valuation are firstly that the extra allowance for first year expenses is unnecessarily large in the case of short premium payment policies and secondly, policy-holders, under the higher premium plans, pay much more for their first year's insurance than policy-holders under lower premium plans. For example, a policy-holder having a ten year endowment policy pays about three times as much for his first year's insurance as one having an Ordinary Life policy.

It has also been criticized on the ground that the entire premium paying period is taken to make up the deficiency in the reserve. Also, if the premium paying period is short there is a considerable strain on the office to make up the deficiency. The main objection is, however, that under high premium policies the large allowance for first year expenses may ~~leads~~ lead to extravagance in the acquisition of new business.

Another objection is that the principle is capable of extension and may be applied to a preliminary term of two or even three years. Thus a very large amount would be available for initial expenses. At one time advantage was taken of this by companies in certain states.

Modified Preliminary Term Plans

Several modifications of the Full Preliminary Term Method have been introduced in the United States and Canada so that some reserve is held at the end of the first year on the higher premium policies and a smaller amount is thus available for initial expense.

Among these plans have been

1. the Ohio Method
2. the Illinois Method
3. the New Jersey Method
4. the Select and Ultimate Method of New York
5. the Canadian Method.

1. Ohio Method

Under the first of these modifications known as the Ohio Method the limit of the first year's expenditure is based upon the Whole Life Policy Premium.

(a) In the case of Ordinary Whole Life policies and n-payment Life policies ($n \geqq 20$) the Full Preliminary Term Method is allowed. (See page 6).

(b) Under any policy whose premium is greater than that of the corresponding Twenty Payment Life premium the valuation premium is the same as that of an Ordinary Whole

Life policy under the Full Preliminary Term plan plus an amount whose accumulation with the benefit of survivorship will, at the end of the premium paying period, produce a Pure Endowment equal to the difference between the Net Premium reserve on the policy at that time and the reserve according to the Full Preliminary Term Method.

Thus in the case of an n-payment Whole Life policy ($n<20$) the valuation premium for the first year is equal to $P^1_{x\,\overline{1|}}+\pi_1$ and thereafter $P_{x+1}+\pi_1$ where

$$\pi_1 = \frac{D_{x+n}\left(A_{x+n} - {}_{n-1}V_{x+1}\right)}{N_x - N_{x+n}}$$

The reserve at the end of t years is equal to

$${}_{t-1}V_{x+1} + \pi'\left(\frac{N_x - N_{x+t}}{D_{x+t}}\right) \qquad (t<n)$$

$$A_{x+t} \qquad (t \geqq n)$$

(c) In the case of an m-year Endowment Insurance:

Net valuation premium for the first year $= P^1_{x\,\overline{1|}} + \pi_2$

Net valuation premium thereafter $= P_{x+1} + \pi_2$

where $\pi_2 = \dfrac{D_{x+m}\left(1 - {}_{m-1}V_{x+1}\right)}{N_x - N_{x+n}}$ since $\pi_2 \dfrac{N_x - N_{x+m}}{D_{x+m}} + {}_{m-1}V_{x+1} = 1$

Reserve at the end of t years ($t<m$)

$$= {}_{t-1}V_{x+1} + \pi_2 \frac{N_x - N_{x+t}}{D_{x+t}}.$$

(d) In the case of an n-payment m-year Endowment Insurance:

Net valuation premium for the first year $= P^1_{x\,\overline{1|}} + \pi_3$

Net valuation premium thereafter $= P_{x+1} + \pi_3$

where $\pi_3 = \dfrac{D_{x+n}\left(A_{x+n:\,\overline{m-n|}} - {}_{n-1}V_{x+1}\right)}{N_x - N_{x+n}}$

Reserve at the end of t years ($t<n$)

$$= {}_{t-1}V_{x+1} + \pi_3 \frac{N_x - N_{x+t}}{D_{x+t}}$$

the Full Net Premium reserve being held from the nth year onward.

This method of valuation is used in Ohio and a few other states but it has been objected to on the ground that it is not sufficiently liberal to enable newly-formed companies to operate successfully in competition with companies that have been longer established and have available larger surplus funds.

2. Illinois Method

The second method, provided by the State of Illinois, makes the Twenty-Payment Life Policy the basis of its calculations.

Here the maximum allowance for initial expense, in addition to the loading in the office premium, is the difference between the full net premium for a Nineteen Payment Whole Life Policy at age $x+1$ and the cost of the first year's risk. Whole Life policies by twenty or more payments are valued by the Full Preliminary Term Method.

The net valuation premium and reserves under this method are as follows:

(a) For Ordinary Whole Life policies, n-payment Life policies ($n \geqq 20$) and all other policies for which the annual premiums are less than those of the Twenty Payment Life policy the net valuation premiums and reserves are the same as under the Full Preliminary Term plan.

(b) For n-payment Whole Life policies ($n < 20$)

net valuation premium for first year $= P_{\overset{1}{x}\,\overline{1}|} + \pi_4$

net valuation premium thereafter $= {}_{19}P_{x+1} + \pi_4$

where $\pi_4 = \dfrac{D_{x+n}(A_{x+n} - {}^{19}_{n-1}V_{x+1})}{N_x - N_{x+n}}$ since ${}^{19}_{n-1}V_{x+1} + \pi_4 \dfrac{N_x - N_{x+n}}{D_{x+n}}$

Reserve at the end of t years $(t \geqq n) = A_{x+t}$

Reserve at the end of t years $(t < n) = {}^{19}_{t-1}V_{x+1} + \pi_4 \dfrac{N_x - N_{x+t}}{D_{x+t}}$

(c) For m-year Endowment policies:

If $P_{x+1:\overline{m-1|}} < {}_{19}P_{x+1}$ the Full Preliminary Term plan is allowed.

Otherwise: If $m \leqq 20$

net valuation premium first year $= P^1_{x\overline{1|}} + \pi_5$

net valuation premium thereafter $= {}_{19}P_{x+1} + \pi_5$

where $\pi_5 = \dfrac{D_{x+m}(1 - {}^{19}_{m-1}V_{x+1})}{N_x - N_{x+m}}$

Reserve at the end of t years $(t < m)$

$$= {}^{19}_{t-1}V_{x+1} + \pi_5 \frac{N_x - N_{x+t}}{D_{x+t}}$$

If $m > 20$ and $P_{x+1:\overline{m-1|}} > {}_{19}P_{x+1}$

net valuation premium first year $= P^1_{x\overline{1|}} + \pi_6$

net valuation premium second to twentieth years

$$= {}_{19}P_{x+1} + \pi_6$$

net valuation premium thereafter $= P_{x\overline{m|}}$

where $\pi_6 = \dfrac{D_{x+20}({}_{20}V_{x\overline{m|}} - A_{x+20})}{N_x - N_{x+20}}$

Reserve at the end of t years $(t < 20)$

$$= {}^{19}_{t-1}V_{x+1} + \pi_6 \frac{N_x - N_{x+t}}{D_{x+t}}$$

Reserve at the end of t years where $t \geqq 20 = {}_tV_{x:\overline{m|}}$

ie. the Full Net Premium reserve.

Under this plan the full net premium reserve is attained not later than the twentieth year except under contracts where the full preliminary term net premium is less than the nineteen payment life premium at age $x+1$, ie. except where the full preliminary term valuation is permitted, in which cases the net premium reserve is not made up until the end of the premium paying period.

The net premiums and reserves by the Illinois Method are the same as under the Full Preliminary Term plan on the Ordinary Life and Twenty Payment Life plans as well as on other contracts having a net premium equal to or less than the Nineteen Payment Life premium at an age one year older. On other contracts, the reserves lie between the Full Preliminary Term reserves and the Modified reserves as provided by the Ohio Method. Some reserve is held during the first year in addition to the cost of the current year's risk, but the initial expense allowances are larger than under the Ohio Method.

The Illinois Method is the legal standard in Illinois, Idaho, Indiana, Maryland, Michigan, North Dakota, South Dakota and Tennessee; and is voluntarily adopted in many other states. It was made permissive in New York in 1923 as an alternative to the Select and Ultimate Method and it was made permissive in Massachusetts in 1923.

3. <u>New Jersey Method</u>

The original "New Jersey" method provided for a modification of the full preliminary term plan upon the full net premium basis, as follows:-

If the net premium under the particular contract be equal to or greater than the corresponding Whole Life net premium the following deductions may be made from the full reserve.

First year -- excess of the full reserve for a Whole Life policy over the reserve on a One Year Term policy.

Second year - five-sixths of such excess.

Third year -- four-sixths of such excess.

Fourth year - three-sixths of such excess.

Fifth year -- two-sixths of such excess.

Sixth year -- one-sixth of such excess, the full net premium reserve being held thereafter.

Under contracts where the net premium is less than the corresponding whole life net premium, the reserve refunded during the first year is the reserve for a One Year Term policy, and the full net premium reserve for the particular contract has to be reached at the end of five years.

Due to two conditions attached by the State of New Jersey to the use of such reserves, namely that only companies having less than a certain amount of reserves might avail themselves of this provision, and that if any company did take advantage of such reduced reserves, then the sum of the first year's expenses and modified reserves must not exceed the amount of the first year's premiums, the larger companies were under no restriction as to the amount of their expenses and smaller companies who took advantage of the valuation law had to keep within a certain limit. The larger companies were thus able to pay much more in the way of commissions and it was therefore easy to see that the benefit of the relief provided by the reduction in the reserves was largely offset by the difficulty of obtaining business in competition.

This plan has been considerably modified, however, to give the following basis for valuation.

Present New Jersey Method

The net valuation premiums and reserves are the same as under the Illinois Method for the following classes of policies:

(a) All limited payment life policies whose premium payment periods do not exceed twenty years.

(b) All endowment policies for which the full preliminary term net renewal premiums are greater than the full preliminary term net renewal premium for a twenty payment life policy issued at the same age and for the same amount.

(c) All policies whose full preliminary term net renewal premiums are less than the full preliminary term net renewal premium for a twenty payment life policy issued at the same age and for the same amount and whose full preliminary term net renewal premiums do not exceed 150 per cent of the net one year term premium for the age at issue ($P'_{x\overline{1}|}$).

For all other classes of policies all reserves must be brought up to the level net premium reserves at the end of twenty years by means of a pure endowment accumulation extending from the second to the twentieth policy years.

Thus, for policies such as the ordinary life, twenty-five payment life and thirty payment life whose full preliminary term net renewal premiums are less than the full preliminary term net renewal premium for a twenty payment life policy, the level net premium reserve must be attained by the end of the twentieth year.

Generally:

Let P'_{x+1} be the net full preliminary term renewal premium for any form of policy.

Then, if ${}_{19}P_{x+1} \geqq P'_{x+1} > 1.50\,P_{\dot{x}\,\overline{1}|}$

net premium for first year $= P_{\dot{x}\,\overline{1}|}$

net premium for second to twentieth years $= P'_{x+1} + \pi_7$

where $\pi_7 = \dfrac{({}_{20}V'_x - {}_{19}V'_{x+1})\,D_{x+20}}{N_{x+1} - N_{x+20}}$

and ${}_{20}V'_x$ is the level net premium terminal reserve of the twentieth year and ${}_{19}V'_{x+1}$ is the full preliminary term terminal reserve of the twentieth year for the policy under consideration.

net premium after twenty years = level net premium for the policy under consideration.

4. <u>Select and Ultimate Method of New York</u>

The office premiums were based on the American Experience Table. The purpose of this Method was to offset the expense of new business, not covered by the loading in the office premium, by the mortality profit derived from new entrants, consequent upon the office premiums being based upon an "ultimate" mortality table. It was assumed that the mortality experience during the first five years after issue would be 50, 65, 75, 85 and 95 per cent respectively of the rates of mortality shown in the American Experience Table. The profit thus expected to be derived from the lighter mortality during the first five years after entry was discounted and applied toward the initial expense. The full net premium reserves on the Ultimate basis had to be attained by the end of the fifth year.

The reserve at the end of t years according to this method would be

$$_{t}V_{[x]} = A_{[x]+t} - P_{x} \cdot a_{[x]+t}$$

where the net annual premium, P_x, is based on the ultimate table.

The main objection to this method was that it was faulty in theory. If the assumed rates of mortality in the first few years were proper modifications of the standard mortality table, the net level premiums and reserves should be based on such a modified mortality table. Then the resulting reserves would be greater than the full net level premium reserves based on the original mortality table.

5. Canadian Method

The Canadian Insurance Act of 1910 provided another modification of the Full Net Premium method of valuation, a deduction being allowed during the first four years in the case of all policies having a net premium at least equal to the whole life net premium at the same age. This allowance was the same under all forms of policies, and was based upon the difference between the whole life net premium and the net premium for a one-year term policy calculated by the $O^{M(5)}$ Table with $3\frac{1}{2}\%$ interest. The whole of this difference was allowed as a deduction from the first year's reserve, and three-quarters, one half, and one-quarter respectively of the first year's allowance from the second, third, and fourth years' reserves. The full net premium reserve was to be carried from the fifth year onwards.

The Canadian Insurance Act of 1927 made several changes in the method of valuation of the policy liabilities of life insurance companies as provided by the former Act. These changes were as follows:

1. The maximum rate of interest allowable is $3\frac{1}{2}\%$. No minimum is set.

2. Any one of the following tables of mortality may be employed:

(a) Canadian Men Table, $C^{M(5)}$

(b) British Offices Life Tables, 1893, $O^{M(5)}$

(c) British Offices Life Tables, 1893, O^{M}

(d) British Offices Life Tables, 1893, $O^{[M]}$

(e) Institute of Actuaries of Great Britain, H^{M}

(f) American Men Table, $AM^{(5)}$

(g) American Experience Table, Am. Exp.

Note the freedom allowed in the choice of table.

3. No deficiency reserve is required as under the

former Act.

4. A company may use any method of valuation provided that the reserve calculated thereby shall not be less at any duration than the reserve calculated on a modified preliminary term valuation basis described as follows:

(a) Under policies with premiums equal to or less than the Whole Life premium the Full Preliminary Term Method of computing reserves is allowed.

eg. Whole Life Policy

$$_{n}V'_{x} = A_{x+n} - P_{x+1} \cdot \ddot{a}_{x+n}$$

Ten-Year Term Policy $(n<10)$

$$_{n}V'_{x:\overline{10}|} = A'_{x+n:\overline{10-n}|} - P'_{x+1\,\overline{9}|}\, \ddot{a}_{x+n:\overline{10-n}|}$$

(b) Under policies with premiums equal to or greater than the Whole Life premium the value of the policy is to be equal to the value of the sum assured less the present value of a valuation premium which shall be obtained by adding to each net level annual premium, excluding the first, such an amount assumed to be payable at the beginning of the second and subsequent years for which premiums are payable, as is equal in value as of the date of issue of the policy to the difference between a corresponding Whole Life premium and a One-Year Term premium.

eg. Twenty-Payment Life Policy $(n<20)$

$$^{20}_{n}V'_{x} = A_{x+n} - ({}_{20}P_{x} + \pi)\, \ddot{a}_{x+n:\overline{20-n}|}$$

where $\pi a_{x\,\overline{19}|} = P_{x} - P_{x\,\overline{1}|}$

Twenty Year Endowment Policy

$$ {}_nV'_{x:\overline{20|}} = A_{x+n:\overline{20-n|}} - (P_{x\overline{20|}} + \pi_1)\ddot{a}_{x+n:\overline{20-n|}} $$

$$ \text{where} \quad \pi_1\, a_{x\overline{19|}} = P_x - P_{x\overline{1|}} $$

Ten-Payment Twenty Year Endowment $(n < 10)$

$$ {}^{10}_{n}V'_{x:\overline{20|}} = A_{x+n:\overline{20-n|}} - ({}_{10}P_{x\overline{20|}} + \pi_2)\ddot{a}_{x+n:\overline{10-n|}} $$

$$ \text{where} \quad \pi_2\, a_{x\overline{9|}} = P_x - P'_{x\overline{1|}} $$

This method is similar to the "Ohio" method of modified preliminary term valuation. There are, however, the following differences:

An allowance is made under the Canadian Act in the case of policies with premiums less than a Whole Life premium but greater than a One-Year Term premium. eg. a Ten-Year Term policy.

Under policies with premiums equal to or greater than the Ordinary Life Premium both methods allow a deduction in the initial reserve for the first year of the difference between the Ordinary Life net premium and the One-Year Term premium at the age of issue. Both methods arrange for the accumulation of the full reserve at the end of the premium paying period.

Under the Canadian Method this difficiency in the reserve is accumulated from the beginning of the second policy year, the terminal reserve at the end of the first policy year having no provision included in it for such accumulation.

Under the Ohio method the accumulation begins in the first policy year, and while the initial reserve at the beginning of the first policy year is the same as under the Canadian method, the terminal reserve of the first policy year provides for the accumulation of a Pure Endowment.

Chapter III

Year End Reserves

Office Methods of Valuation

There are four main methods of Office Valuation, namely, the Seriatim Method, the Group Method, the Retrospective Method and the Attained-Age Method.

I. Seriatim Method

Each policy is valued separately. This method is convenient if the number of policies is small. It is therefore generally used as the principal method by small companies. It is also used by large companies for valuing special types of policies the number of which is usually small.

II. Group Method

Under this method one may value at once all policies of the same plan, year of issue and age at issue.

If there are very many plans of insurance issued the Group Method loses some of its superiority over the Seriatim Method. Where the groups are small and scattered, valuation is almost as lengthy as on the Seriatim basis.

III. Retrospective Method

Where the groups are small and scattered the retrospective method of valuation may prove more efficient than the Group Method.

This method groups together all policies of the same age and year of issue, irrespective of plan. By summing for the group the amount of insurance (S) and the net premium (^sP) the corresponding terminal reserve for all plans is

$$\Sigma\, {}_nV_x = (\Sigma\, {}^sP_x)\frac{N_x - N_{x+n}}{D_{x+n}} - (\Sigma S)\frac{M_x - M_{x+n}}{D_{x+n}}$$

where ^sP_x is the net premium for the amount of insurance under the individual policy issued at age x.

The above formula holds only for level-premium, level-insurance policies during the premium paying period. Full-paid and paid-up policies can be valued by the Attained-Age or Group Method.

IV. <u>Attained-Age Methods</u>

There are two general types.

(1) The Accumulation type.

(2) The Karup Method.

(1) The reserve of the year is found by carrying forward by some accumulation formula the preceding year's reserve grouped under attained ages, with appropriate adjustment for the reserve on incoming and outgoing policies. Each year's valuation is dependent on that for the previous year, with the consequent increased likelihood of error. (For the development of an Accumulation formula see page 23.)

(2) The Karup Method produces the exact reserve with the minimum of labor. All policies of the same attained age are grouped together irrespective of plan, age at issue, or duration, and totals made of the amounts of insurance, net premiums and valuation constants for the group. Each

of these totals is multiplied by a simple multiplier which depends only on the attained age and is independent of the plan of insurance, age at issue or duration. The general formula for obtaining the terminal reserve for the group with attained age (y) is

$$\sum_{n=1}^{n=y} {}_nV_{y-n} = (\Sigma S)A_y + (\Sigma\, {}^s\Theta)\frac{1}{D_y} - (\Sigma\, {}^sP)a_y$$

The valuation constant Θ is based on the plan of insurance and the age at issue and when multiplied by the amount of insurance (S) on the policy becomes ${}^s\Theta$ where $\Theta_x \equiv (\pi_x - P_x)N_x$ on premium paying policies where π_x is the net premium for the particular plan of insurance and P_x the corresponding ordinary life net premium. In order to reduce Θ to a workable size it is the custom in practice to divide it by an arbitrary number such as 100,000 or D_{40} and to multiply the $(\frac{1}{D_y})$ in the valuation formula by the same number so that the resulting reserve will not be affected.

(See Chapter IV for the development of the Attained-Age formulae.)

__Fackler's Accumulation Formula__

If ${}_tV_x$ be the terminal reserve for the tth year on an insurance of 1, and π_x be the net annual premium, then ${}_tV_x + \pi_x$ is the initial reserve for the (t+1)th year.

Therefore the aggregate reserve at the beginning of the (t+1)th year for the l_{x+t} persons insured is $l_{x+t}({}_tV_x + \pi_x)$.

At the end of the year this will have amounted to $l_{x+t}({}_tV_x + \pi_x)(1+i)$ at a rate of interest i per annum.

During the $(t+1)$th year d_{x+t} persons will die and d_{x+t} will have to be paid out in death claims. Hence the amount left after these claims have been paid will be

$$l_{x+t}({}_tV_x+\pi_x)(1+i) - d_{x+t}.$$

which represents the aggregate policy reserves belonging to the l_{x+t+1} survivors.

Therefore

$$l_{x+t+1} \cdot {}_{t+1}V_x = l_{x+t}({}_tV_x+\pi_x)(1+i) - d_{x+t}$$

Hence,

$$\begin{aligned}
{}_{t+1}V_x &= \frac{l_{x+t}({}_tV_x+\pi_x)(1+i)}{l_{x+t+1}} - \frac{d_{x+t}}{l_{x+t+1}} \\
&= \frac{v^{x+t}\,l_{x+t}}{v^{x+t+1}\,l_{x+t+1}}({}_tV_x+\pi_x) - \frac{v^{x+t+1}\,d_{x+t}}{v^{x+t+1}\,l_{x+t+1}} \\
&= \frac{D_{x+t}}{D_{x+t+1}}({}_tV_x+\pi_x) - \frac{C_{x+t}}{D_{x+t+1}} \\
&= u_{x+t}({}_tV_x+\pi_x) - k_{x+t}
\end{aligned}$$

where $u_x = \dfrac{D_x}{D_{x+1}}$, $k_x = \dfrac{C_x}{D_{x+1}}$

The u's and k's are tabulated.

This formula is known as Fackler's Accumulation Formula. The u_{x+t} and k_{x+t} are independent of the form of the policy. For a varying amount of insurance from year to year, the factor multiplying k_x would, of course, change.

In finding the reserve for the first policy year $t=0$ and ${}_0V_x = 0$. Therefore,

$${}_1V_x = u_x\,\pi_x - k_x.$$

The Fackler formula is perhaps the most widely used of any accumulation formula. It is used probably more than any other formula in preparing complete tables of policy reserves. Its chief advantage lies in the fact that the reserve calculation at the end of any policy year is made to depend on that of the previous year; and, thus, by checking the calculations of the reserve for a given policy year we have an automatic check for the earlier policy years.

Since Life Office Valuations in Canada and the United States are made as at December 31, for valuation purposes all policies are assumed to be issued as at July 1 at integral ages, so that the Office Valuations require the use of half ages (attained ages). Accordingly the average policy value needed for this purpose is a mean between the initial value $(V_0 + \pi)$ and the terminal value (V_1), or in symbols the general reserve value--$MV_{\frac{1}{2}}$-- is given by the formula, $MV_{\frac{1}{2}} = \frac{1}{2}(V_0 + \pi + V_1)$

Individual mean and terminal policy values on a prospective basis are tabulated by plan, age at entry and duration of insurance. If a valuation method requires the calculation of reserves, the mean values of the functions required for the valuation factors are prepared beforehand from the initial and terminal values thus:

Reversionary Value, $A_{\frac{1}{2}} = \frac{1}{2}(A_0 + A_1)$;

Annuity Value, $\ddot{a}_{\frac{1}{2}} = \frac{1}{2}(\ddot{a}_0 + a_1) = \frac{1}{2}(1 + a_0 + a_1)$;

$D_{\frac{1}{2}} = \frac{1}{2}(D_0 + D_1)$;

$M_{\frac{1}{2}} = \frac{1}{2}(M_0 + M_1)$;

$C_{\frac{1}{2}} = \frac{1}{2}(C_0 + C_1)$; etc.

These remarks regarding Life Office Valuations will be considered to apply equally as well to the subject matter of Chapter IV.

Chapter IV

Attained-Age Method of Valuation

Derivation of General Formula

This is an exact valuation method and depends on the principle that the reserve on a policy can be expressed in terms of functions some of which depend only on the age attained, the others being constant while the status of the policy remains unchanged.

We may arrive at the general formula in either of two ways, prospectively or retrospectively.

Prospectively:

The tth reserve on any special policy under which unity is payable on the death of the insured or on the maturity of the policy, during, say, n years, must be the same as the tth reserve on the corresponding Whole Life Policy together with an annuity foreborne for t years for the difference between the annual premium on the special policy (π_x) and the annual premium on the corresponding Whole Life Policy (P_x); that is, the tth reserve on the special policy

$$= {}_tV_x + (\pi_x - P_x)\frac{N_x - N_{x+t}}{D_{x+t}}$$

$$= A_{x+t} - P_x\ddot{a}_{x+t} + (\pi_x - P_x)\frac{N_x - N_{x+t}}{D_{x+t}}$$

$$= A_{x+t} - P_x\ddot{a}_{x+t} - (\pi_x - P_x)\ddot{a}_{x+t} + (\pi_x - P_x)\frac{N_x}{D_{x+t}}$$

$$= A_{x+t} - \pi_x\ddot{a}_{x+t} + \frac{(\pi_x - P_x)N_x}{D_{x+t}}.$$

Retrospectively:

The value of any policy of unit amount at the end of t years, the $(t+1)$th premium being due and unpaid, is equal to the accumulation of the premiums paid, less the accumulated claims, divided among the survivors l_{x+t}.

$$\text{ie. the value} = \frac{1}{l_{x+t}}\Big[\pi_x\left\{(1+i)^t l_x + (1+i)^{t-1} l_{x+1} + \cdots\cdots + (1+i)\,l_{x+t-1}\right\} - \left\{(1+i)^{t-1} d_{x+t} + (1+i)^{t-2} d_{x+t} + \cdots\cdots + d_{x+t-1}\right\}\Big]$$

$$= \frac{\pi_x(N_x - N_{x+t}) - (M_x - M_{x+t})}{D_{x+t}}$$

$$= A_{x+t} - \pi_x \ddot{a}_{x+t} + \frac{\pi_x N_x - M_x}{D_{x+t}}.$$

But $P_x = \dfrac{M_x}{N_x}$ and therefore $M_x = P_x N_x$

Therefore the value $= A_{x+t} - \pi_x \ddot{a}_{x+t} + \dfrac{(\pi_x - P_x) N_x}{D_{x+t}}$

Hence the formula

$$V = A_{x+t} - \pi_x \ddot{a}_{x+t} + (\pi_x - P_x) N_x \cdot \frac{1}{D_{x+t}}$$

gives the reserve on any policy after t years duration whether it be a Whole Life, Limited Payment Life, Term or Endowment Assurance. The amount of insurance, the net premium for the plan and the difference between the net premium for the plan and the Ordinary Whole Life premium are independent of the duration t and are therefore constant throughout the duration of the policy. The value of the whole life single premium A_{x+t}; the whole life annuity $\ddot{a}_{x+t}$ and D_{x+t} depend only on the attained age $x+t$. Except in so far as it enters into π_x the plan or number of premiums yet to be paid on the policy does not enter into the calculation of the policy value.

Therefore all policies with a level premium and level benefit, where only one life is at risk, may be grouped together by their attained age at valuation irrespective of plan or duration.

For term insurances $(\pi_x - P_x)$ is negative.

If the attained age $x+t$ be denoted by y the formula becomes $V = A_y - \pi_x \ddot{a}_y + \Theta \cdot \frac{1}{D_y}$ where $\Theta = (\pi_x - P_x) N_x$.

__Attained-Age Method as Applied to the Illinois Method of Valuation__

According to the Illinois Method of Valuation the reserves at the end of t years would be:

(a) for an Ordinary Whole Life policy

$$V = {}_{t-1}V_{x+1} = A_{x+t} - P_{x+1} \cdot \ddot{a}_{x+t}.$$

(b) for an n-payment Whole Life policy

$$V = {}^{n-1}_{t-1}V_{x+1} = A_{x+t} - {}_{n-1}P_{x+1} \cdot \ddot{a}_{x+t} + \frac{({}_{n-1}P_{x+1} - P_{x+1}) N_{x+1}}{D_{x+t}} \quad (n \geqq 20)$$

$$= {}^{19}_{t-1}V_{x+1} + \pi_4 \frac{N_x - N_{x+t}}{D_{x+t}} \qquad (n < 20)$$

$$= A_{x+t} - ({}_{19}P_{x+1} + \pi_4) \ddot{a}_{x+t} + \frac{({}_{19}P_{x+1} - P_{x+1}) N_{x+1} + \pi_4 N_x}{D_{x+t}}$$

(c) for an m-year Endowment

If $P_{x+1:\overline{m-1}|} < {}_{19}P_{x+1}$ the Full Preliminary Term Plan is allowed.

ie. $${}_tV'_{x:\overline{m}|} = {}_{t-1}V_{x+1:\overline{m-1}|}$$

$$= A_{x+t} - P_{x+1:\overline{m-1}|} \ddot{a}_{x+t} + \frac{(P_{x+1:\overline{m-1}|} - P_{x+1}) N_{x+1}}{D_{x+t}}$$

Otherwise:

$m \leqq 20$, $(t < m)$ $$V = {}^{19}_{t-1}V_{x+1} + \pi_5 \frac{N_x - N_{x+t}}{D_{x+t}}$$

$$= A_{x+t} - ({}_{19}P_{x+1} + \pi_5) \ddot{a}_{x+t} + \frac{({}_{19}P_{x+1} - P_{x+1}) N_{x+1} + \pi_5 N_x}{D_{x+t}}$$

$m > 20$, $P_{x+1:\overline{m-1}|} > {}_{19}P_{x+1}$

$$V = {}^{19}_{t-1}V_{x+1} + \pi_6 \frac{N_x - N_{x+t}}{D_{x+t}} \qquad (t<20)$$

$$= A_{x+t} - ({}_{19}P_{x+1} + \pi_6)\ddot{a}_{x+t} + \frac{({}_{19}P_{x+1} - P_{x+1})N_{x+1} + \pi_6 N_x}{D_{x+t}}$$

or

$$V = {}_tV_{x\,\overline{m}|} \qquad (t \geqq 20)$$

$$= A_{x+t} - P_{x\,\overline{m}|}\,\ddot{a}_{x+t} + \frac{(P_{x\,\overline{m}|} - P_x)N_x}{D_{x+t}}$$

Attained-Age Method as Applied to the Canadian Method of Valuation

According to the Canadian Method of Preliminary Term Valuation the reserves at the end of t years would be:

(a) for policies with premiums equal to or less than the Whole Life premium the full preliminary term plan of valuation is allowed.

eg. for a Whole Life Policy

$$ {}_tV'_x = {}_{t-1}V_{x+1}$$

$$= A_{x+t} - P_{x+1} \cdot \ddot{a}_{x+t}$$

for a ten year Term Policy

$$ {}_tV'^{1}_{x\,\overline{10}|} = {}_{t-1}V^{1}_{x+1\,\overline{9}|}$$

$$= A_{x+t} - P^{1}_{x+1\,\overline{9}|}\,\ddot{a}_{x+t} + \frac{(P^{1}_{x+1\,\overline{9}|} - P_{x+1})N_{x+1}}{D_{x+t}}$$

(b) for policies with premiums equal to or greater than the Whole Life premium

eg. Twenty Payment Life Policy $(t<20)$

$$ {}^{20}_{t}V'_x = A_{x+t} - ({}_{20}P_x + \pi)\ddot{a}_{x+t:\overline{20-t}|}$$

$$= A_{x+t} - ({}_{20}P_x + \pi)\ddot{a}_{x+t} + \frac{({}_{20}P_x + \pi)N_{x+20}}{D_{x+t}}$$

Twenty Year Endowment Policy

$$ {}_tV'_{x:\overline{20}|} = A_{x+t:\overline{20-t}|} - (P_{x:\overline{20}|} + \pi_1)\ddot{a}_{x+t:\overline{20-t}|}$$

$$= A_{x+t} - P_{x\,\overline{20}|}\,\ddot{a}_{x+t} + \frac{(P_{x\,\overline{20}|} - P_x)N_x}{D_{x+t}}$$

$$- \pi_1\,\ddot{a}_{x+t:\overline{20-t}|}$$

$$= A_{x+t} - (P_{x:20} + \pi_1)\ddot{a}_{x+t} + \frac{(P_{x:20} - P_x)N_x + \pi_1 N_{x+20}}{D_{x+t}}$$

Ten-Payment Twenty Year Endowment ($t < 10$)

$${}^{10}_{t}V'_{x:\overline{20|}} = A_{x+t:\overline{20-t|}} - ({}_{10}P_{x:\overline{20|}} + \pi_2)\ddot{a}_{x+t:\overline{10-t|}}$$

$$= A_{x+t} - {}_{10}P_{x:\overline{20|}}\ddot{a}_{x+t} + \frac{({}_{10}P_{x:\overline{20|}} - P_x)N_x}{D_{x+t}} - \pi_2 \ddot{a}_{x+t:\overline{10-t|}}$$

$$= A_{x+t} - ({}_{10}P_{x:\overline{20|}} + \pi_2)\ddot{a}_{x+t} + \frac{({}_{10}P_{x:\overline{20|}} - P_x)N_x + \pi_2 N_{x+10}}{D_{x+t}}$$

__Henderson's Extension of Karup's Method__

Robert Henderson extended Karup's Attained-Age Method to include the case of an m-payment, n-year Endowment, net premium π, the amount payable in case of death being __a__ and in case of survival __b__.

If x be the age at issue, then the reserve at the end of t years ($t < m$) would be (prospectively)

$$V_{x+t} = a\frac{M_{x+t} - M_{x+n}}{D_{x+t}} + b\frac{D_{x+n}}{D_{x+t}} - \pi\frac{N_{x+t} - N_{x+m}}{D_{x+t}}$$

$$= aA_{x+t} - \pi\ddot{a}_{x+t} + \frac{\pi N_{x+m} + bD_{x+n} - aM_{x+n}}{D_{x+t}}$$

Now $\pi = \dfrac{a(M_x - M_{x+n}) + bD_{x+n}}{N_x - N_{x+m}}$

or $\pi N_x - \pi N_{x+m} = aM_x - aM_{x+n} + bD_{x+n}$

or $\pi N_x - aM_x = \pi N_{x+m} - aM_{x+n} + bD_{x+n}$

Then $\pi N_{x+m} + bD_{x+n} - aM_{x+n} = (\pi - aP_x)N_x$

$$\therefore V_{x+t} = aA_{x+t} - \pi\ddot{a}_{x+t} + \frac{(\pi - aP_x)N_x}{100{,}000} \times \frac{100{,}000}{D_{x+t}}$$

If $t > m$ the reserve would be

$$V_{x+t} = a\frac{M_{x+t} - M_{x+n}}{D_{x+t}} + b\frac{D_{x+n}}{D_{x+t}}$$

$$= aA_{x+t} + \frac{bD_{x+n} - aM_{x+n}}{100{,}000} \times \frac{100{,}000}{D_{x+t}}.$$

Retrospectively:

$$V_{x+t} = \pi \frac{N_x - N_{x+t}}{D_{x+t}} - a \frac{M_x - M_{x+t}}{D_{x+t}}$$

$$= a A_{x+t} - \pi \ddot{a}_{x+t} + \frac{\pi N_x - a M_x}{D_{x+t}}$$

$$= a A_{x+t} - \pi \ddot{a}_{x+t} + \frac{(\pi - a P_x) N_x}{100,000} \times \frac{100,000}{D_{x+t}}$$

This general case may be made to cover Ordinary and Limited Payment Whole Life policies, Term, Endowment and Pure Endowment policies, by the proper selection of the values of a, b, m and n. In fact it may be applied to any case where the amount insured is constant over a specified period by taking for b the reserve and for x+n the attained age at the end of that period.

Reserves on Policies with Increasing and Decreasing Premiums

If the premium for the first year be π and for renewal years $\pi + r$ the reserve at the end of any renewal year t may be written

$$V = \frac{\pi (N_x - N_{x+t}) + r (N_{x+1} - N_{x+t}) - (M_x - M_{x+t})}{D_{x+t}}$$

$$= A_{x+t} - (\pi + r) \ddot{a}_{x+t} + \frac{(\pi - P_x) N_x + r N_{x+1}}{D_{x+t}}$$

Generally, if the net annual premium be:

π for the first m years,

$\pi + r$ from the (m+1)th to the nth year,

$\pi + r + s$ from the (n+1)th to the qth year,

the reserve at the end of any year t will be

$$(t \leq m) \quad {}_tV_x = \frac{1}{D_{x+t}} \left[\pi (N_x - N_{x+t}) - (M_x - M_{x+t}) \right]$$

$$= A_{x+t} - \pi\, a_{x+t} + \frac{(\pi - P_x) N_x}{D_{x+t}}$$

$$(m < t \leqq n) \quad {}_tV_x = \frac{1}{D_{x+t}}\left[\pi(N_x - N_{x+t}) + r(N_{x+m} - N_{x+t}) - (M_x - M_{x+t})\right]$$

$$= A_{x+t} - (\pi + r) a_{x+t} + \left[(\pi - P_x) N_x + r\, N_{x+m}\right] \cdot \frac{1}{D_{x+t}}$$

$$(n < t \leqq q) \quad {}_tV_x = \frac{1}{D_{x+t}}\left[\pi(N_x - N_{x+t}) + r(N_{x+m} - N_{x+t}) + s(N_{x+n} - N_{x+t}) - (M_x - M_{x+t})\right] \cdot \frac{1}{D_{x+t}}$$

$$= A_{x+t} - (\pi + r + s) a_{x+t} + \left[(\pi - P_x) N_x + r\, N_{x+m} + s\, N_{x+n}\right] \cdot \frac{1}{D_{x+t}}$$

<u>Attained-Age Method Applied to finding the Reserve on an Increasing Coupon or Decreasing Premium Policy</u>

If the coupon is for level amounts no additional reserve is required.

If, however, the coupon increases by a constant from year to year the reserve **may** be found as follows:

Assume an n-payment life policy with n - 1 coupons, each, after the first, increasing by a constant, and to be valued by the Illinois Standard.

The first coupon is payable when and if the second annual premium is paid; the (n+1)th is payable when the nth annual premium is paid.

Let r be the annual increase in the coupon and π_i be the net annual premium to purchase the increasing portion of the benefit.

Let ${}_tV_2|(IA)_x$ = the reserve at the end of the tth policy year ($1 < t \leqq n$).

Then ${}_tV_2|(IA)_x = \dfrac{\pi_i(N_{x+1} - N_{x+t}) - r(S_{x+2} - S_{x+t} - \overline{t-2}\, N_{x+t})}{D_{x+t}}$

$$= \frac{1}{D_{x+t}}\left[\pi_i\, N_{x+1} - \pi_i\, N_{x+t} - r\, S_{x+2} + r\, S_{x+t} + r(t-2)\, N_{x+t}\right]$$

Add and subtract $2r\,N_{x+t} + r\,x\,N_{x+t}$

Therefore $_tV_2|(IA)_x = -[\pi_i + r(x+2)]\,\ddot{a}_{x+t} + \dfrac{\pi_i N_{x+1} - r\,S_{x+2}}{D_{x+t}}$

$$+ \frac{r[S_{x+t} + (x+t)N_{x+t}]}{D_{x+t}}$$

which is in a form suitable for an attained-age valuation.

<u>Attained-Age Method Applied to Finding the Value of a Policy with a Return-Premium Benefit</u>

Assume a return-premium benefit for n years.

Let $_tV(IA)_x$ be the <u>extra</u> reserve at the end of the tth policy year ($t \leqq n$) for this benefit.

Let π' be the gross annual premium to be returned and π_ℓ be the extra net level annual premium for the benefit.

Then $_tV(IA)_x = \dfrac{\pi_\ell(N_x - N_{x+t}) - \pi'(R_x - R_{x+t} - t\,M_{x+t})}{D_{x+t}}$

$$= \frac{\pi_\ell N_x - \pi_\ell N_{x+t} - \pi' R_x + \pi' R_{x+t} + \pi' t\,M_{x+t}}{D_{x+t}}$$

Add and subtract $\dfrac{\pi' x\,M_{x+t}}{D_{x+t}}$

Therefore

$$_tV(IA)_x = [\pi_\ell N_x - \pi_\ell N_{x+t} - \pi' R_x + \pi' R_{x+t} + \pi' t\,M_{x+t} + \pi' x\,M_{x+t} - \pi' x\,M_{x+t}] \cdot \frac{1}{D_{x+t}}$$

$$= -\pi' x\,A_{x+t} - \pi_\ell\,\ddot{a}_{x+t} + \frac{\pi_\ell N_x - \pi' R_x}{D_{x+t}} + \frac{\pi'[R_{x+t} + (x+t)M_{x+t}]}{D_{x+t}}.$$

This is now in a form suitable for Attained-Age valuation. It expresses the extra reserve in terms of constants and functions varying only with the attained-age of the insured.

If it is desired to value this benefit on the full preliminary term basis (which complies with the Illinois Standard) the reserve at the end of t years (putting π_r

for the net annual renewal extra premium for the benefit) will be:

$$_{t-1}V(IA)_{x+1} = \frac{\pi_r(N_{x+1} - N_{x+t}) - \pi'(R_{x+1} - R_{x+t} - \overline{t-1}\,M_{x+t})}{D_{x+t}}$$

Add and subtract $\pi' x M_{x+t}$ to the numerator of the right-hand side.

Therefore

$$_{t-1}V(IA)_{x+1} = -(x+1)\,\pi' A_{x+t} - \pi_r\,\ddot{a}_{x+t} + \frac{\pi_r N_{x+1} - \pi' R_{x+1}}{D_{x+t}}$$

$$+ \frac{\pi'[R_{x+t} + (x+t)M_{x+t}]}{D_{x+t}}$$

This is now in a form suitable for an attained-age valuation for the renewal years on the full preliminary term basis.

Pure Endowments

Bonus additions to sums assured are sometimes made contingent upon the survival of the life assured to an agreed age. These are therefore pure endowments. In this case the value of the bonus is $\frac{D_{x+n}}{D_{x+t}}$ where x+n is the endowment age.

Pure endowments secured by annual premiums may be included with the other classes of policies for attained-age valuation by means of the following formula:

$$_tV_{x:\overset{1}{\overline{n|}}} = -P_{x:\overset{1}{\overline{n|}}}\cdot\ddot{a}_{x+t} + \frac{P_{x:\overset{1}{\overline{n|}}}\cdot N_x}{D_{x+t}}$$

This is identical with the formula of the Prospective Method for

$$_tV_{x:\overset{1}{\overline{n|}}} = P_{x:\overset{1}{\overline{n|}}}\left(\frac{N_x}{D_{x+t}} - \ddot{a}_{x+t}\right)$$

$$= \frac{D_{x+n}}{N_x - N_{x+n}}\left(\frac{N_x - N_{x+t}}{D_{x+t}}\right)$$

$$= \frac{D_{x+n}(N_x - N_{x+n} - N_{x+t} + N_{x+n})}{D_{x+t}(N_x - N_{x+n})}$$

$$= A_{x+t:\overset{1}{\overline{n-t}|}} - P_{x:\overset{1}{\overline{n}|}}\, a_{x+t:\overline{n-t}|}$$

Valuation of Bonuses or Paid-up Policies

Bonus additions to a policy are really small paid-up additions to the amount of the policy. In this respect they may therefore be valued as paid-up policies.

Let A'_x be the single premium paid under the policy.

Then the reserve for the policy at the end of t years will be equal to the accumulated value of the premium less the accumulated claims divided among the survivors l_{x+t}.

ie. $$V = \frac{A'_x (1+i)^t l_x - [d_x (1+i)^{t-1} + d_{x+1} (1+i)^{t-2} + \dots + d_{x+t-1}]}{l_{x+t}}$$

$$= \frac{A'_x D_x - (M_x - M_{x+t})}{D_{x+t}}$$

$$= A_{x+t} - (A'_x - A_x) D_x \cdot \frac{1}{D_{x+t}}$$

since $A_x = \frac{M_x}{D_x}$

The only expressions made use of are the whole life single premium at the advanced age and a correction $\frac{(A'_x - A_x)D_x}{D_{x+t}}$ whose numerator is constant throughout the duration of the policy and whose denominator depends on the attained age $x+t$.

Extended Term Insurances

These are granted as a non-forfeiture option and the cash value at the date to which premiums have been paid is used as a single net premium to purchase term insurance for face amount of policy plus any dividend additions less any indebtedness for as long a period as it will.

These are ordinary paid-up term insurances and for year end valuation are grouped according to attained age and at each attained age according to the unexpired term.

Chapter V

Approximate Methods of Valuation

Approximate methods of valuation are used extensively in Great Britain to check the main valuation but are not much used on this continent.

If $\omega_1, \omega_2, \cdots\cdots, \omega_n$ represent the quantities to be valued and $f(1), f(2), \cdots\cdots, f(n)$ the valuation factors, a short process of approximating to $\sum \omega_x \cdot f(x)$ is required.

<u>A. E. King's Method</u>

Existing methods of valuation group policies with a common Office year of birth or attained age, year of maturity, or year of issue according to the class of policy. King shows that it is possible to group the particulars in respect of a number of consecutive attained ages, years of maturity, or years of issue. This method may be used as a check for or even in place of existing methods.

Figures for valuation are arranged according to attained age. The sums assured, etc. are divided into groups of a fixed number, say 10, consecutive ages and the sum of the figures for each group recorded. A single average factor for each group is then sought by which the total in the group may be multiplied to give the valuation for the group. The weighted age $\bar{x}$ is found and the factor at that age used. That is, find $\bar{x}$ where

$$\bar{x} = \frac{\sum x \cdot \omega_x}{\sum \omega_x}$$

and then evaluate $f(\bar{x}) \cdot \sum \omega_x \cdot$

Assume for the purpose of obtaining a correction to $f(\bar{x})$ that the quantities to be valued are equal at each age of the group. Then the corrected factor would be $F(\bar{x})$ where $F(\bar{x}) = f(\bar{x}) + \frac{n^2-1}{24}\delta^2 f(\bar{x})$.

Hence King's valuation rule is:

(1) Find $\bar{x} = \frac{\sum x.\omega_x}{\sum \omega_x}$

(2) then find $F(\bar{x}).\sum\omega_x$ where $F(\bar{x}) = f(\bar{x}) + \frac{n^2-1}{24}.\delta^2 f(\bar{x})$

If a suitable range of ages for grouping is definitely fixed upon, tables of $F(\bar{x})$ can be prepared and tabulated to correspond to each value of $\bar{x}$.

Henry's Method

Henry assumes that $f(x)$ can be expressed as $a+bx+cx^2$.

Then
$$\begin{aligned}\sum\omega_x.f(x) &= \sum\omega_x(a+bx+cx^2)\\ &= a\sum\omega_x + b\sum x.\omega_x + c\sum x^2.\omega_x\\ &= a\sum\omega_x + (b-c)\sum x.\omega_x + 2c\sum\frac{x(x+1)}{2}\omega_x\\ &= a\sum\omega_x + (b-c)\Sigma^2\omega_x + 2c\,\Sigma^3\omega_x.\end{aligned}$$

In this form the valuation can be expressed in terms of the continuous summation of the ω_x column from the bottom upwards. If satisfactory values of a, b, and c can be found the valuation may be performed without difficulty.

Henry's second paper gives a modification of the above method whereby the third summation is not necessary. The data is divided into two (or, in exceptional cases, three) sections. He assumes that for each of these sections $f(x)$ can be expressed in the form $a+bx$. In this case for each section

$$\begin{aligned}\sum\omega_x.f(x) &= a\sum\omega_x + b\sum x.\omega_x\\ &= a\sum\omega_x + b\,\Sigma^2\omega_x\end{aligned}$$

The amount of arithmetical work is further reduced by treating the data in quinary groups.

To find a and b, only two equations are necessary. Instead of making an arbitrary selection, Henry used all the equations connecting them in conjunction with a set of weights $w_1, w_2, \ldots\ldots, w_n$, chosen so as to conform roughly with the expected trend of the quantities $\omega_1, \omega_2, \cdots, \omega_n$. These weighted equations in respect of each quinary group were then combined in two groups of ages so as to form the final equations from which the values of a and b were calculated.

<u>Trachtenberg's Method</u>

The underlying idea of this method is the valuation of the sum assured (S) in groups of n years by multiplying the total sum by a tabulated function α and $\sum_{x=-\frac{n-1}{2}}^{x=\frac{n-1}{2}} S_x$ by a tabulated β. A similar arrangement could be adopted for premiums, etc.

Thus Trachtenberg's formula is

$$\alpha \sum_{-\frac{n-1}{2}}^{\frac{n-1}{2}} u_x + \beta \sum_{-\frac{n-1}{2}}^{\frac{n-1}{2}} x \cdot u_x$$

where α and β are constants depending on the valuation factors and u_x are the quantities to be valued.

The values of α and β may be obtained from the equations

$$n\alpha = \sum_{-\frac{n-1}{2}}^{\frac{n-1}{2}} f(x)$$

$$\frac{n^2}{4}\beta = \sum_{\frac{1}{2}}^{\frac{n-1}{2}} \left\{ f(x) - f\left(x - \frac{n}{2}\right) \right\}$$

when n is even. When n is odd the values of α and β may be obtained from the equations

$$n\alpha = \Sigma_{-\frac{n-1}{2}}^{\frac{n-1}{2}} f(x)$$

$$\frac{n^2-1}{4}\beta = \Sigma_{1}^{\frac{n-1}{2}}\left\{f(x) - f\left(x - \frac{n+1}{2}\right)\right\}$$

With these values the approximation reproduces the true value as far as second differences in cases where u_x is linear.

Kenchington's Method

This method is an adaption of the method of approximation of Henry.

If w_t be the valuation data for which $f(t)$ is the corresponding valuation factor, then the sum of the products for a range of n terms is represented by

$$w_1 f(1) + w_2 f(2) + \cdots + w_n f(n) = \sum_{t=1}^{t=n} w_t f(t)$$

Assuming that for a limited range of terms $f(t)$ may be written as approximately equal to $a+bt$, we have

$$\sum_{t=1}^{t=n} w_t f(t) = \sum_{t=1}^{t=n} w_t (a+bt)$$

$$= a\sum_{t=1}^{t=n} w_t + b\sum_{t=1}^{t=n} t\,w_t = a\sum_{t=1}^{t=n} w_t + b\sum_{t=1}^{t=n}{}^2 w_t .$$

There are n equations connecting a and b:

$$f(1) = a + b$$

$$f(2) = a + 2b$$

$$- - - - - - - -$$

$$f(n) = a + nb$$

To find the values of a and b Kenchington made two summations from the bottom upwards and used the resulting two equations:

$$na + \frac{n(n+1)}{2}b = \Sigma f$$

$$\frac{n(n+1)}{2}a + \frac{n(n+1)(2n+1)}{6}b = \Sigma^2 f$$

Solving, $a = \dfrac{2\{(2n+1)\Sigma f - 3\Sigma^2 f\}}{n(n+1)}$

$$b = \frac{6\{2\Sigma^2 f - (n+1)\Sigma f\}}{n(n^2-1)}$$

Kenchington restricted the range of summations to 10 and when the values of a and b had once been found for a group they could be used year after year. The results were very accurate.

Elderton's Approximate Method of Valuing Endowment Assurances

${}_tV_{x\overline{n}|}$ is approximately constant for any particular values of t and n regardless of the values of x. Therefore, if we assume that all policies mature at a certain age, say 55, and value them for their correct terms, the error in a total valuation will not be very large.

Lidstone's Method of Valuing Endowment Assurances

All policies having the same unexpired term are grouped together and for each group valuation factors based upon a mean age are used. Various methods of finding this mean age have been advanced but we will consider only what is known as Lidstone's Z Method.

Since $A_{x\overline{n}|} = 1 - d(1 + a_{x\,\overline{n-1}|})$ it is evident that a mean age can be found by reference to the annuity values at the various ages appearing in a group of policies. If we weight these values according to the sums assured under the respective policies the resulting mean age will be found from the following equation

$$a_{y:\overline{n}|}\left[S_x + S_{x+1} + \cdots\right] = S_x a_{x\overline{n}|} + S_{x+1} a_{x+1\,\overline{n}|} + \cdots\cdots \quad (1)$$

where S_x, S_{x+1}, ... are the sums assured at ages x, x+1, ... in a group of policies with an unexpired term of n years, and y is the mean valuation age required for that group.

But to find y from this equation would be laborious as all the products $S_x a_{x\overline{n}|}$, $S_{x+1} a_{x+1\,\overline{n}|}$, etc. would have to be found for each valuation.

Let us examine the form of the differences of $a_{x\overline{n}|}$ to find a more suitable method.

Assuming Makeham's Law and dealing with $\frac{d}{dx}\bar{a}_{x\overline{n}|}$ instead of $\Delta_x a_{x\overline{n}|}$, we have

$$\begin{aligned}\frac{d}{dx}\,\bar{a}_{x\overline{n}|} &= \frac{d}{dx}\int_0^n v^t\,{}_tp_x\,dt \\ &= \int_0^n v^t\,{}_tp_x\,(\mu_x-\mu_{x+t})\,dt \\ &= -Bc^x\int_0^n v^t(c^t-1)\,{}_tp_x\,dt.\end{aligned}$$

The general tendency of the value of this definite integral is to diminish with increasing rapidity as the age increases. If we assume that it decreases in geometrical progression, the differential coefficients and therefore the first differences of $a_{x\overline{n}|}$ are increasing in geometrical progression with a common ratio r, say, which is less than c. Hence the annuity values will be of the form $\alpha-\beta r^x$.

Substituting in (1) we get

$$(\alpha-\beta r^y)\sum S_x = S_x(\alpha-\beta r^x)+S_{x+1}(\alpha-\beta r^{x+1})+\dots,$$

whence $$r^y\sum S_x = \sum S_x r^x$$

As the expression now stands it is necessary to obtain the product $S_x r^x$ at each valuation for each policy. If, however, this product can be made to depend upon the age at maturity it will be constant throughout the duration of the policy. Multiplying both sides of the expression by r^n where n is the number of years from the valuation date to the date of maturity, we have

$$r^{y+n} \Sigma S_x = \Sigma S_x r^{x+n}$$

or

$$r^{M'} \Sigma S_x = \Sigma S_x r^M$$

where $x + n$ or M is the age at the end of the year of maturity and is therefore constant, and $y + n$ or M' is the mean "Maturity Age".

The value of $S \times r^M$ will be constant for each policy. Multiply both sides of the equation by r^{-55}. Then

$$r^{M'-55} \Sigma S_x = \Sigma S_x r^{M-55} = \Sigma Z_M \quad \text{where } S \times r^{M-55} = Z_M$$

and the Mean value of Z per unit assured is equal to $\frac{\Sigma Z_M}{\Sigma S_x}$

If a table of Z_M be constructed showing the value of Z for each maturity age, the mean value of M may be obtained by entering the table inversely with the mean value of Z for the group of policies; whence by deducting n, we have

y = Mean valuation age = (Mean value of M) - n.

With a Makeham Table r is slightly less than c, but for tables following Makeham's Law, the value of c is sufficiently accurate to be used, and since it slightly increases the mean age and therefore the reserves to be held by the Office this value has been adopted.

For other tables a rough graduation by Makeham's formula for the ages most important to endowment assurances will generally give a suitable indication of the value of c that should be employed.

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