FRITZ JOHN OPTIMALITY CONDITIONS AND DUALITY FOR SOME GENERALIZED CONVEX PROGRAMMING PROBLEMS

by

© Meena Kumari Bector

A Thesis presented to the University of Manitoba in partial fulfillment of the requirements for the degree of Master of Science in the Department of Actuarial and Management Science, Faculty of Management.

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BY

MEENA KUMARI BECTOR

A thesis submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

MASTER OF SCIENCE

DEDICATED TO

MY PARENTS

MR. A. N. MONGA AND MRS. S. MONGA

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CONTENTS

ABSTRACT

Optimization problems appear in a variety of fields. The problem may involve construction of an optimal or most economical design, planning optimal inventory, allocation of scarce resources or finding an optimal trajectory of a rocket. During the past five decades there has been very rapid growth of optimization models and efforts to solve them. To understand the optimization system as a whole it is important to understand the components and their interactions. Study of nonlinear programmong (NLP) problems stems from the fact that they form an important component of the optimization system.

In (NLP) optimality coditions and duality have played very important role both theoretically as well as computationally and have been studied by several researchers under different types of generalized convexity assumptions. Based on optimality condition several computational procedures for solving nonlinear programs have been devised or are being devised. A dual problem provides alternative ways of solving a nonlinear program and serves the purpose of generating the bounds for the optimal solution of the primal problem.

In the present thesis we prove, under generalized convexity assumptions, optimalty conditions and introduce duality models for a variety of NLP problems. In Chapter 2 we prove the Fritz John type sufficient optimality theorem and various duality results for the Mond-Weir dual problem assuming quasiconvexity assumption of the

objective and strict pseudoconcavity of a linear combination of components of the constraints. An advantage of Fritz John type optimality conditions is that we do not require any form of constraint qualifications imposed on the constraints to prove their necessity. This in turn implies that we do not require the assumption of constraint qualifications in the proofs of Direct and Strictly Converse duality theorems. In Chapter 3, we further extend the results of Chapter 2 under slightly more general and different conditions of generalized convexity of the constraints.

Chapter 4 deals with generalized minmax programming problem. In this chapter, under weaker convexity assumptions, we (i) prove the Fritz John type sufficient optimality conditions, (ii) introduce a Mond and Weir type dual program and using the Fritz John type conditions prove duality theorems, (iii) apply the results proved to obtain the duality of a generalized fractional programming problem.

In Chapter 5 we introduce second order duality for a quasibonvex programming problem and under generalised bonvexity assumptions, prove Weak, Direct and Strict Converse duality theorems using Fritz John conditions. Chapter 6 deals with Mond-Weir type duality for multiobjective programming problems in which the constraints are pseudolinear with respect to the same kernel function and the objective functions are pseudolinear with different kernel functions.

CHAPTER 1

INTRODUCTION, NOTATIONS AND DEFINITIONS, NONLINEAR PROGRAMMING PROBLEMS AND SUMMARY OF THE THESIS

1. INTRODUCTION

Nonlinear Programming (NLP). The (NLP) problem arises in a number of different forms and may be found in the natural sciences, physical sciences, business and government, engineering, economics, and mathematics. In its most abstract form, we minimize (or maximize) something subject to certain type of limitations. Suppose, for example, we minimize the cost of both producing an item and holding it in stock by selecting an appropriate production schedule. However, the schedule cannot be chosen freely. Perhaps we must meet certain demand requirements for the item or there may also be a limitation on the amount that can be produced at any one time. Thus we must select a production schedule that minimizes the total cost and also satisfies certain constraints.

Generally speaking, for the NLP problem something must be minimized (or maximized). However, limitations in the form of constraints restrict the actions we may take to achieve this minimum (or maximum).

Mathematically, we are given a function $f : X^0 \rightarrow R$, where, R is the set of reals and X^0 is an open set in R^n, the n-dimensional space such that if $x \in X^0$ then f(x) is a point in R. Our goal is to select an x to minimize f(x). For this reason we call f(x) the *objective function* . However, the point x is not chosen arbitrarily because we may be given m *constraint functions*, where $g_i : X^0 \rightarrow R$ or $g : X^0 \rightarrow R^m$, and x must satisfy

$$g_i(x) \geq \mathbf{0} , \ i = 1, 2, \ldots, m.$$

Consequently, we have the following problem (called primal problem, denoted by (P)) in which we attempt to

(P) $\displaystyle \underset{x \in S}{\text{minimize}} \ f(x)$

so that

$$S = \{x \mid x \in X^0 , \ g_i(x) \geq \mathbf{0} , \ i = 1, 2, \ldots, m\}$$

$$= \{x \mid x \in X^0 , \ g(x) \geq \mathbf{0} \}.$$

A point that achieves the *minimum* is called an *optimal point* . Any x that satisfies all the constraints is termed *feasible* .

2. APPLICATIONS OF NONLINEAR PROGRAMMING

Bazaraa and Shetty [4], Schaible [66], Crouzeix et al [33] and Zangwill [82] etc. have given plenty of applications of the NLP in a variety of fields. In the present section we mention some of those applications to get a feeling for the usefulness of the NLP problem.

(i) Economics. The NLP is closely related to the science of allocating scarce resources in a manner either to maximize efficiency or, if one is dealing with consumers, to maximize the utility. The objective function can indicate efficiency, which we attempt to maximize, while the constraint can specify the limitations imposed by the scarce resources. Similarly, the objective function can be consumer utility and the constraints can be specified on consumer's limited income.

In a general setting, it may be impossible to determine the precise form of the fuctions; nevertheless, in specific applications precise formulation is often straight forward. For example, consider a particular industrial plant, say a leather shoe manufacturer. Here, efficiency may become profit, and the constraints may be interpreted as manpower, space available, machine capacities etc. In such a specific situation the quantitative data are often available, and the NLP problem can then be precisely formulated and solved.

(ii) Cost-Benefit Analysis. The important concept of cost-benefit analysis falls within the context of NLP problem . In cost-benefit analysis there is no profit function but instead there is a general welfare-benefit function. Two closely related NLP problems arise here, either maximize benefit subject to a cost limitation or minimize cost with the benefit above a minimum level. With the vast amount of data available to various government agencies, particular cost-benefit analyses can often be well modeled by NLP.

(iii) Scientific Applications. NLP problems often arise in the physical sciences as well. In physics, for example, the objective function could be potential energy and the constraints the various equations of motion. Minimizing the objective function would determine a stable configuration of the system. Correspondingly, by changing the objective function, we can determine the configuration with the largest thermal energy, kinetic energy, etc. Similarly a problem in chemistry is to determine the molecular structure that minimizes the Gibbs free energy.

3. DUALITY FOR NLP PROBLEM AND ITS INTERPRETATION

In duality we have another problem (called the Mond-Weir [58] dual problem, denoted by (D)) which is related to (P). With $y \in R^m$, this problem is stated as:

(P) Maximize $f(u)$

subject to

$$\nabla f(u) - y^t \nabla g(u) = 0$$

$$y^t g(u) \leqq 0$$

$$y \geqq 0$$

Let the NLP problem (P)4 represent an industry's objective of minimizing its cost of producing a particular product. Thus, the objective function $f(x)$ represents the net cost of production when the industry is operating at a level x. In addition, the industry requires m raw materials but starts production with an amount b_i , $i = 1, 2, \ldots, m$ of raw material i available. Define

$$r_i(x)$$

as the quantity of raw material i used when operating the industry at level x. Also,

$$g_i(x) = b_i - r_i(x)$$

If

$$g_i(x) > 0$$

then the excess raw material is left over with the industry after production, while should $g_i(x) < 0$, then insufficient raw material was initially available to the industry to achieve production level x. In NLP terms the industry's problem of minimizing the final net production cost becomes,

$$\text{Minimize} \quad f(x)$$

subject to

$$g_i(x) \geq 0 \quad i = 1, 2, \ldots, m.$$

Now let $y_i \geq 0$ be the unit price at which raw material i is purchased or sold at the market. Should $g_i(x) > 0$, by selling raw material to the industry the market can realize a revenue of

$$-\, y_i g_i(x)$$

from the producer.

Interpretation [82]. We now interpret the dual problem as follows. When the industry operates at an optimum level $x = \bar{u}$,

(i) $f(\bar{u})$ represents the net cost to the industry,

(ii) $\nabla f(\bar{u})$ represents the marginal cost of production of the product,

(iii) $-\nabla(\bar{y}^t g(\bar{u}))$ represents the total marginal revenue to the market,

then, economically, at optimum level (or at the equilibrium) ,the marginal cost of production must be equal to the marginal revenue. This gives

$$\nabla f(\bar{u}) - \nabla(\bar{y}^t g(\bar{u})) = 0$$

which is the first dual constraint. Second interest of the market is to keep

$$y_i g_i(x) \leq o \qquad \text{for} \quad i = 1, 2, \ldots\ldots m$$

so that there is never any excess material left with the industry to sell it back to the market to reduce its (market's) revenue. This constraint leads to

$$y^t g(\bar{u}) \leq 0$$

which is the second constraint of the dual.

Again as given in Zangwill [82], at the optimal solution

$$\frac{\partial f(\bar{u}(b))}{\partial b_k} = \bar{y}_k$$

This shows that $\overline{y}_k$ is the marginal change in the optimal value of the objective function $f(x)$ at $x = \overline{u}$. On an intuitive basis $\overline{y}_k$, for which $\overline{y}_k \geqq 0$ for all $k = 1, 2, \ldots, m$, indicates the approximate increase in the objective function per unit increase in the availability of resource $k = 1, 2, \ldots, m$. Thus the $\overline{y}_k$ be interpreted [82] as the imputed price of resource k.

Thus we see that by using the dual problem at the optimal solution, the market can control the movements of the industry's net cost and its own sales revenue from the sales of the raw marerials to the industry, with the help of the knowledge about $\overline{y}_k$'s.

Therefore, n the present thesis we study the duality of different NLP problems. For this purpose, first of all we introduce some mathematical notation and definitions used in the sequel, then the different problems dealt with in the thesis and the results proved for them by other researchers. Lastly, in this chapter, we give a summary of the work done in this thesis.

4. NOTATION AND DEFINITIONS

For $x, y \in R^n$, by $x \leqq y$ we mean $x_i \leqq y_i$ for all i; $x \leq y$ means $x_i \leqq y_i$ for all i and $x_s < y_s$ for at least one s, $1 \leq s \leq n$. By $x < y$ we mean $x_i < y_i$ for all i. Consider numerical function $h : D \rightarrow R$ defined on some open set D in R^n containing the convex set $S \subseteq D$. Let C denote the class of single valued continuous functions and let C^p, $p = 1, 2, \ldots,$

(p being finite) denote the subclass of all those $h \in C$, of which every p-th order partial derivative exists and is continuous. For $h \in C^1$ let $\nabla_x h$ denote the gradient of h and for $h \in C^2$ let $\nabla^2{}_x h$ denote the Hessian of h.

We define a kernel function $K: S \times S \times [0,1] \rightarrow R_+$ such that $K = K[x, y, \lambda] > 0$ and is continuous in λ, where $\lambda \in [0,1]$. Define $k : S \times S \rightarrow R_+$ by $k = \lim_{\lambda \rightarrow 0} K$, so that $k = k(x, y) > 0$. We assume that the kernel function is uniformly bounded. Let $(x)^t$ denote transpose of vector x.

Then at $x \in S$ the function h is called

1. **Beevex** (*Bector [5], Bector and Singh [20]*) with respect to

(i) K if for all y in S and $\lambda \in [0,1]$

$$h(\lambda y + (1 - \lambda)x) \leq K \lambda h(y) + (1 - K \lambda) h(x)$$

(ii) k if for all y in S, and h differentiable,

$$k [h(y) - h(x)] \geq (y - x)^t \nabla_x h(x)$$

Note. It is easy to see that every convex function at $x \in S$ is a Beevex function for K=1(or k=1).

2. **Bonvex** (*Mond and Weir [59], Bector and Chandra [13]*) if for all y in S and for all $\eta \in R^n$,

$$h(y) - h(x) \geq (y - x)^t [\nabla_x h(x) + \nabla^2_x h(x) \eta] - 1/2\, \eta^t \nabla^2_x h(x) \eta$$

3. **Pseudobonvex** (*Mond and Weir [59], Bector and Chandra [13]*) if for all y in S and for all $\eta \in R^n$,

$$(y - x)^t [\nabla_x h(x) + \nabla^2_x h(x) \eta] \geq 0 \Rightarrow h(y) \geq h(x) - 1/2\, \eta^t \nabla^2_x h(x) \eta$$

4. **Strictly Pseudobonvex** (*Bector and Bector [14]*) if for all y in S and for all $\eta \in R^n$,

$$(y - x)^t [\nabla_x h(x) + \nabla^2_x h(x) \eta] \geq 0 \Rightarrow h(y) > h(x) - 1/2\, \eta^t \nabla^2_x h(x) \eta$$

5. **Quasibonvex** if for all y in S and for all $\eta \in R^n$,

   (i) (*Mond and Weir [59], Bector and Chandra [13]*)

$$h(y) \leq h(x) - 1/2\, \eta^t \nabla^2_x h(x) \eta \Rightarrow (y - x)^t [\nabla_x h(x) + \nabla^2_x h(x) \eta] \leq 0,$$

   or

   (ii) (*Bector and Bector [14]*)

$$h(y) < h(x) - 1/2\, \eta^T \nabla^2_x h(x) \eta \Rightarrow (y - x)^t [\nabla_x h(x) + \nabla^2_x h(x) \eta] \leq 0 \;,$$

h is **strictly beevex (strictly bonvex) at** $x \in S$ if in the definition of a **beevex (bonvex)** function strict inequality holds. h is called beecave (strictly beecave) at $x \in S$ with respect to K (or k respectively, when h is differentiable) iff $-h$ is beevex (strictly beevex) at $x \in S$ with respect to K (or k respectively, when h is differentiable). A function h is pseudolinear at $x \in S$ when it is both beevex and beecave at $x \in S$. We similarly define boncave and generalized pseudoboncave functions and assume that the reader is familiar with other definitions of (generalized) convex functions [Mangasarian [54] and Craven [31].

A function h is said to be a convex(bonvex)-like (convex and generalized convex, bonvex and generalized bonvex) function on the set S if it is convex(bonvex)-like at every point of S.

5. SOME NLP PROBLEMS

(i) Nonlinear Programming Problem. Along the line of Mond and Weir [58] consider the following nonlinear programming problems,

$$\text{(P)} \qquad \underset{x \in S}{\text{Minimize}} \quad f(x) \tag{1}$$

and

$$\text{(PE)} \qquad \underset{x \in X}{\text{Minimize}} \quad f(x) \tag{2}$$

where,

(i) X^o is an open set of R^n ;

(ii) $f : X^0 \to R$, $g : X^0 \to R^m$ and $h : X^0 \to R^k$ are differentiable functions.

(iii) $S = \{x;\ x \in X^0,\ \ g(x) \geqq 0\}$; (3)

(iv) $X = \{x;\ x \in X^0,\ \ g(x) \geqq 0,\ h(x) = 0\}$; (4)

Evidently, if equality constraints in X are absent, (PE) becomes (P).

Optimization problems appear in a variety of fields. The problem may involve construction of an optimal or most economical design, planning optimal inventory, allocation of scarce resources or finding an optimal trajectory of a rocket [4]. During the past five decades there has been very rapid growth of optimization models and efforts to solve them. Engineers, mathematicians, scientists, managers, social scientists and behavioral scientists have been called upon to solve many complex problems. To understand the optimization system as a whole it is important to understand the components and their interactions. Study of nonlinear programming problems stems from the fact that they form an important component of the optimization system.

Several computational procedures for solving nonlinear programs have been devised or are being devised [4]. Duality plays an imprtant role in nonlinear programming. A dual problem provides alternative ways of solving a nonlinear program and serves the purpose of generating the bounds for the optimal solution of the primal problem.

Sufficient optimality conditions for such problems are important both theoretically as well as computationally and have been studied by Kuhn-Tucker [50], Mangasarian [54], Bector and Grover [9], Bector and Gulati [11], Singh [69] and Skarpness and Sposito [73]. Mangasarian [54] assuming f to be pseudoconvex, g_I $(I = \{i;\ g_i(\bar{x}) = 0,\ \bar{x} \in X\})$ to be quasiconcave and h to be both quasiconvex and quasiconcave a $\bar{x} \in X$, showed that, if $(\bar{x}, \bar{y}, \bar{z})$ is a solution to the following Kuhn-Tucker [50] type conditions :

$$\nabla[f(x) - y^t g(x) - z^t h(x)] = 0 \tag{5}$$

$$y^t g(x) = 0 \tag{6}$$

$$g(x) \geqq 0 \tag{7}$$

$$h(x) = 0 \tag{8}$$

$$y \in R^m, \quad z \in R^k, \qquad y \geqq 0\ , \tag{9}$$

then $\bar{x}$ is (PE)-optimal.

Bhatt and Misra [22], assuming f, g and h to be convex at $\bar{x} \in X$, showed that the above conditions (5)-(9) with the additional restriction

$z \geqq 0$, are sufficient for $\bar{x}$ to be (PE)-optimal.

Assuming f to be convex and g to be strictly concave at $\bar{x} \in S$, Mangasarian [54] showed that, if $(\bar{x}, \bar{y}_0, \bar{y})$ is a solution to the following Fritz John type [54] conditions

$$\nabla[y_0 f(x) - y^t g(x)] = 0 \tag{10}$$

$$y^t g(x) = 0 \tag{11}$$

$$g(x) \geqq 0 \tag{12}$$

$$y_0 \in R, \quad y \in R^m, \qquad (y_0, y) \geq 0, \tag{13}$$

then $\bar{x}$ is (P)-optimal.

Assuming f to be pseudoconvex at $\bar{x} \in X$ and g_I and h to be strictly pseudoconcave at $\bar{x} \in X$, Bector and Gulati [11] and Skarpness and Sposito [73] proved that, if $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z})$ is a solution to the Fritz John type conditions

$$\nabla[y_0 f(x) - y^t g(x) - z^t h(x)] = 0 \tag{14}$$

$$y^t g(x) = 0 \tag{15}$$

$$g(x) \geqq 0 \tag{16}$$

$$h(x) = 0 \tag{17}$$

$$y_0 \in R, \quad y \in R^m, z \in R^k, \qquad (y_0, y) \geq 0, \tag{18}$$

$$z \in R^k, \qquad z \geqq 0, \tag{19}$$

then $\bar{x}$ is (PE)-optimal.

Recently, Bector and Bector [15] showed that if $(\bar{x}, \bar{y}, \bar{z})$ is a solution to (5)-(9), f is quasi convex at $\bar{x} \epsilon X$ and $\bar{y}^t g + \bar{z}^t h$ is strictly pseudoconcave at $\bar{x} \epsilon X$, then $\bar{x}$ is (PE)-optimal. Furthermore, they proved Weak, Direct and Converse duality theorems for the Mond-Weir dual problem.

(ii) Minmax Programming Problem. We now consider the following nonlinear minmax program as the primal problem (P) whose Fritz John type optimality conditions and duality we want to discuss.

(P) $$\underset{x \epsilon S}{\text{Minimize}} \quad \underset{1 \leq i \leq p}{\text{Maximum}} \quad \{f_i(x)\} \tag{20}$$

where,

(i) $S = \{x \epsilon R^n;\ h_k(x) \leq 0,\ k = 1, 2, \ldots, m\}$ is nonempty and compact,

(ii) f_i $(1 \leq i \leq p)$, h_k $(1 \leq k \leq m)$ are real valued and differentiable functions.

Minmax programs are encountered in discrete approximation where the Chebychev norm is used [3]. Furthermore, minmax programs arise in goal programming where a decision maker wishes to bring several ratios as close as possible to certain predetermined values [48]. The individual goal functions are usually ratios of economic or technical terms such as revenue, cost, profit, time, amounts, etc. More details on the applications of minmax fractional programming are given in [65,66]. A goal program involving ratios gives rise to a minmax fractional program if the Chebychev norm is used, as discussed in [26]. Charnes and Cooper [26] stress that this norm has a natural appeal for problems of equity or equality. This is demonstrated, for example, by Vogt's development of an 'Equal Employment Opportunity Index' [77] and by the problem of allocating state funds to educational institutions as discussed by Charnes, Cox and Lane [27].

An application of minmax fractional programming is discussed by Ashton and Atkins in [1]. The authors consider ratios that are used in financial planning such as liquidity, return on capital, earnings cover, dividend cover and earnings per share.

The first duality results for minmax programs ($p>1$) were given by J. von Neumann [76] in his paper on an expanding economy. A treatment of this model using recent quasiconvex duality results can be found in Crouzeix [32] and Crouzeix et al [33]. Among other authors who have studied minmax fractional programs we mention Rubinshtein [62] who examined a special linear fractional program using a geometric concept of duality, Gol'stein [39] who examined the case that f_i are ratios of nonnegative convex functions to positive affine functions using saddle-point results and Passy and Keslassy [60] who investigated certain fractional programs using duality results based on a generalization of Legendre's transformation.

Recently, some duality results have been obtained for minmax fractional programming problems involving several ratios. Of particular interest are those by Jagannathan and Schaible [45], Chandra, Craven and Mond [25], Bector, Chandra and Bector [17] and Bector and Suneja [19]. Jagannathan and Schaible [45] obtained the duality results via Farkas' Lemma while Chandra, Craven and Mond [25] have studied the duality of such problems through a ratio game approach. Bector, Chandra and Bector [17], using a result proved by Crouzeix, Ferland and Schaible [34,35] and using Kuhn-Tucker conditions [50], established the duality results and related different duals obtained for a minmax fractional programming problem. Bector and Suneja [19] used a Lagrangian approach [4] to obtain the duality results for nondifferentiable generalized minmax fractional programs.

(iii) Bonvex Programming. Lately, there has been a trend to generalize and refine convexity of functions and sets and apply them in mathematical programming problems. Hanson [40,41], Craven [31], Mangasarian [54] and Martos [56] respectively introduced the concepts of functionally convex [40], invex [41,31], pseudoconvex [54] and explicit quasiconvex [56] functions. (Other interesting references are Ponstein [61], Bector [5,8], Mond [57], Mond and Weir [59], Bector and Chandra [13], Bector and Singh [20], Vial [75], Avriel, Diewert, Schaible and Ziemba [2], Zang [81], etc.).

Mangasarian [54] first formulated a second order dual to a nonlinear program and established appropriate duality theorems for both second and higher order duality. Mond [57] introduced a class of second order functions (named bonvex and boncave functions by Bector and Chandra [13]) and proved appropriate duality and symmetric duality results for the class of second order duality. Later, Mond and Weir [59] introduced the class of second order pseudo and quasi convex functions (independently named pseudo and quasi bonvex functions by Bector and Chandra [13]) and proved duality results under more general conditions of second order generalized convexity (i.e. generalized bonvexity).

Recently, Bector and Bector [15], assuming quasiconvexity ofthe objective function and strict pseudoconvexity of the constraints, proved a Kuhn-Tucker sufficient optimality theorem and established duality results for the Mond and Weir dual [58] using Kuhn-Tucker conditions. Furthermore, Bector and Bector [14] extended, on the lines of [15] the duality results contained in [15,58] for second order duality under weaker convexity conditions.

(iv) Multiobjective Pseudolinear Programming. We now consider the following pseudolinear multiobjective program (P) and call it the primal problem .

(P) V-minimize $(f_1(x),\ f_2(x), \ldots, f_p(x))$ (21)

subject to

$$g_i(x) \leq 0, \qquad (i = 1, 2, \ldots, m). \tag{22}$$

where,

(i) the symbol "V-minimize" stands for vector minimization, minimality being taken in terms of "efficient points" or "Pareto optimal solutions" as defined below,

(ii) $f : R^n \to R^p$, $g : R^n \to R^m$, with $f = (f_1, f_2, \ldots, f_p)$ and $g = (g_1, g_2, \ldots, g_m)$ respectively,

(iii) the function f_i, $i = 1,2,\ldots, p$, is pseudolinear with respect to a kernel function k_i (i.e. $k_i(x, y) > 0$) and the function g_j, $j=1, 2,\ldots, m$, is pseudolinear w.r.t. a kernel function G (i.e.$G(x, y) > 0$).

In the recent past, duality in multiobjective programming, involving nonlinear functions, has been of much interest and various contributions have been made in this field by different researchers (e.g. Bitran [23], Brumelle [24], Craven [30], Ivanov and Neshe [42], Kawasaki [49], Lai and Ho [51], Singh [71,72], Tanino and Sawaragi [74] and Weir [78]). These studies differ in their approaches aor in the sense in which "optimality" is defined for a multiobjective programming problem.

Recently, Bector, Chandra and Durgaprasad [18] discussed the duality of a pseudolinear multiobjective programming problem in which the constraints are linear and the objective functions are pseudolinear [Bector [5,8], Bector and Singh [20], Chew and Choo [22]). The purpose of the present chapter is to study Mond-Weir [58] type duality for multiobjective programming problems in which the constraints are pseudolinear with respect to the same proportional (kernel) function and the objective functions are pseudolinear with different proportional (kernel) functions. The main difference between the problem (21), (22) and the problem considered by Bector et al [18] is that in Bector et al [18] the kernel functions k_i of the corresponding functions f_i in the primal objective are assumed to be the same (i.e. $k_i = k$) whereas in the present work we assume those kernels k_i's to be different .

6. SUMMARY OF THE THESIS.

The results proved in the thesis are contained in Chapters 2 - 6. We summarize them as follows.

(i) Chapter 2. Mond-Weir Duality. The purpose of the present chapter is to extend the results proved by Bector and Bector [15] further by proving for (PE) the Fritz John type sufficient optimality theorem and various duality results for the Mond-Weir dual problem assuming quasiconvexity assumption of f and strict pseudoconcavity of a linear combination of components of g and h. An advantage of Fritz John type optimality conditions is that we do not require any form of constraint qualifications imposed on the constraints (e.g. see [15]) to prove their necessity. This in turn implies that we do not require the assumption of constraint qualifications in the proofs of Direct and Strictly Converse duality theorems as is done in Bector and Bector [15].

(ii) Chapter 3. Sufficient Optimality Conditions and Duality for a Quasiconvex Programming Problem. In this chapter we extend the results proved by Bector and Bector [15] further by proving for (PE) the Fritz John type sufficient optimality theorem and various duality results for the Mond-Weir dual problem assuming the quasiconvexity of f, quasiconcavity of g and h and strict pseudoconcavity of one component of g. The results proved in this chapter are different from those of Chapter 2 because when the components of g and h are individualy quasiconcave their nonnegative linear combination may not be strictly pseudoconcave.

(iii) Chapter 4. Generalized Minmax Programming Problem. In this chapter we consider a minmax problem and, under weaker convexity assumptions,

(i) prove the Fritz John type sufficient optimality conditions,

(ii) introduce a Mond and Weir [58] type dual program and using the Fritz John type conditions prove duality theorems.

(iii) apply the results proved to obtain the duality of a generalized fractional programming problem.

(iv) Chapter 5. Second Order Duality for a Quasibonvex Programming Problem. The purpose of this chapter is to introduce a second order dual problem for (P) given by (1) , on the lines of Mond and Weir [58] and Bector and Bector [14], and under generalised bonvexity assumptions, to prove Weak, Direct and Strict Converse duality theorems using Fritz John [54] conditions.

(v) Chapter 6. Duality for Multiobjective Pseudolinear Programming. The purpose of this chapter is to study Mond-Weir [58] type duality for multiobjective programming problems in which the constraints are pseudolinear with respect to the same kernel function and the objective functions are pseudolinear with different kernel functions. The main difference between the problem (21), (22) and the problem considered by Bector et al [18] is that in Bector et al [18] the *kernel functions* k_i's of the corresponding functions f_i's in the primal objective are assumed to be the same (i.e. $k_i = k$) where as in the present work we assume those kernels k_i's to be different .

Remark. A formula in Chapter A will be numbered as (X) and a formula of Chapter B used in Chapter A will be referred to as (B. Y).

CHAPTER - 2

FRITZ JOHN DUALITY IN NONLINEAR PROGRAMMING

1. INTRODUCTION

On the lines of Mond and Weir [58] consider the following nonlinear programming problems,

$$\text{(P)} \qquad \underset{x \in S}{\text{Minimize}} \quad f(x) \tag{1}$$

and

$$\text{(PE)} \qquad \underset{x \in X}{\text{Minimize}} \quad f(x) \tag{2}$$

where,

(i) X^0 is an open set of R^n ;

(ii) $f : X^0 \to R$, $g : X^0 \to R^m$ and $h : X^0 \to R^k$ are differentiable functions.

(iii) $S = \{x;\ x \in X^0,\ g(x) \geqq 0\}$; (3)

(iv) $X = \{x;\ x \in X^0,\ g(x) \geqq 0,\ h(x) = 0\}$; (4)

Evidently, if equality constraints in X are absent, (PE) becomes (P).

In the present chapter we prove for (PE), under differentiability assumptions, Fritz John sufficient optimality conditions for a nonlinear program in which the objective function is assumed to be quasiconvex, and the linear combination of the components of the constraint functions is assumed to be strictly pseudoconcave. Furthermore, we establish duality theorems for Mond-Weir type duality under the above generalized convexity assumptions. An advantage of Fritz John type optimality conditions is that we do not require any form of constraint qualifications imposed on the constraints to prove their necessity (e.g. see [54]). This in turn implies that we do not require the assumption of constrtaint qualifications in the proofs of Direct and Strictly Converse duality theorems as in Bector and Bector [4].

2. OPTIMALITY

We now prove the following Fritz John sufficient optimality theorems for (PE) assuming the quasiconvexity of f and the strictly pseudoconcavity assumption of a linear combination of the constraint functions g and h.

Theorem 1 (Sufficient Optimality Theorem). Let $\bar{y}_0 \in R$, $\bar{y} \in R^m$, $\bar{z} \in R^k$ and let $\bar{x} \in X$ along with $\bar{y}_0$, $\bar{y}$, $\bar{z}$ satisfy the Fritz John type conditions (1. 14) - (1. 18). If, at $\bar{x}$ with respect to X, (i) f is quasiconvex ([61], 3f, p.116) and (ii) $\bar{y}^t g + \bar{z}^t h$ is strictly pseudoconcave, then $\bar{x}$ is a global optimal solution to (PE).

Proof. If $\bar{x}$ is not a global minimum for (PE), let $x^0 \in X$ be such that

$$f(x^0) < f(\bar{x}) \tag{5}$$

Since f is QX ([61, 3f, p. 116) at $\bar{x}$ therefore, (5) yields

$$(x^0 - \bar{x})^t \nabla f(\bar{x}) \leqq 0$$

Using $\bar{y}_0 \geq 0$, this implies,

$$(x^0 - \bar{x})^t \nabla \bar{y}_0 f(\bar{x}) \leqq 0 \tag{6}$$

Using (1. 18), (4), (1. 15) and (1. 17) and $x^0 \in X$ we get,

$$\bar{y}^t g(x^0) + \bar{z}^t h(x^0) \geqq \bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x}) \tag{7}$$

Since $\bar{y}^t g + \bar{z}^t h$ is SPCV at $\bar{x} \in X$, (7) gives

$$(x^0 - \bar{x})^t \nabla [\, \bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x}) \,] > 0 \tag{8}$$

From (1. 14) and (8) we obtain

$$(x^0 - \bar{x})^t \nabla \bar{y}_0 f(\bar{x}) > 0 \tag{9}$$

(9) contradicts (6). Hence the result follows.

Remarks 1.

(i) It is very important to observe here that the proof of Theorem 1 requires the strict pseudo concavity of $\bar{y}^t g + \bar{z}^t h$ without which the strict inequality (8) and hence (9) will not hold .

(ii) If we assume $\overline{y_0} > 0$ in (1. 14) and (1. 18) then we recover the theorem of Bector and Bector ([15], Theorem 3.1)

(iii) It is well known [54] that, when a continuous quasiconvex function is minimized on an appropriately restricted convex set, a local minimum may not be global. However, the assumptions in Theorem 1 always yield a global minimum.

3. DUALITY

We consider the following dual (DE) suggested by Mond and Weir [58] for (PE).

(DE) Maximize $f(u)$

subject to

$$\nabla[y_0 f(u) - y^t g(u) - z^t h(u)] = 0 \tag{10}$$

$$y^t g(u) + z^t h(u) \leqq 0 \tag{11}$$

$$y_0 \in R, \quad y \in R^m, \ z \in R^k \qquad y_0, \ y \geqq 0 \tag{12}$$

We now prove the following theorems relating (DE) to (PE).

Theorem 2 (Weak Duality). Let x be (PE)-feasible and (u, y_0, y, z) be (DE)-feasible. If, for all feasible solutions (x, u, y_0, y, z), f is quasi-convex ([61], 3f, p.116), and $y^t g + z^t h$ is strictly pseudoconcave then,

Infimum (PE) $\geqq$ Supremum (DE)

Proof. If possible let $f(x) < f(u)$. Since for all feasible solutions (x, u, y_0, y, z) f is QX , we have ([61], 3f, p.116)

$$(x - u)^t \nabla f(u) \leqq 0$$

or using $y_0 \geq 0$

$$(x - u)^t \nabla y_0 f(u) \leqq 0 \tag{13}$$

Since x is (PE)-feasible, (4), (12) and (11) yield

$$y^t g(x) + z^t h(x) \geq y^t g(u) + z^t h(u) . \tag{14}$$

Strong pseudo concavity of $y^t g + z^t h$ for all feasible solutions (x, u, y_0, y, z) and (14) give

$$(x - u)^t \nabla [y^t g(u) + z^t h(u)] > 0 \tag{15}$$

From (10) and (15) we have $(x - u)^t \nabla y_0 f(u) > 0$. This contradicts (13). Hence the result follows.

Corollary 1. Let $\overline{x}$ be (PE)-feasible and let $(\overline{u}, \overline{y}_0\ \overline{y}, \overline{z})$ be (DE)-feasible such that $f(\overline{x}) = f(\overline{u})$ and let the hypotheses of Theorem 1 hold. Then $\overline{x}$ is a global optimum for (PE) and $(\overline{u}, \overline{y}_0, \overline{y}, \overline{z})$ is global optimum for (DE) with the corresponding optimal objective value $f(\overline{x})$ and $f(\overline{u})$ respectively.

Theorem 3 (Direct Duality). Let $\overline{x} \in X$ be a local or global optimum of (PE). Then there exists $(\overline{y}_0, \overline{y}, \overline{z})$ such that $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z})$ is feasible for (DE) and the corresponding values of the objective functions of (PE) and (DE) are equal. If, also, the hypotheses of Theorem 1 are satisfied, then $\overline{x}$ and $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z})$ are, respectively, global optima for (PE) and (DE).

Proof. Since $\overline{x} \in X$ is a local or global optimum of (PE), there exist $\overline{y}_0 \in R$, $\overline{y} \in R^m$, $\overline{z} \in R^k$ such that the conditions (1. 14)-(1. 18) are satisfied [54].

Now from (1. 15), (1. 17) and (1. 18) we have

$$\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x}) = 0 \tag{16}$$

Therefore, (1. 14), (16) and (1. 18) yield that $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z})$ is (DE)-feasible. Equality of the objectives follows from the fact that each of them is equal to $f(\overline{x})$. If the hypotheses of Theorem 2 are satisfied then, using the equality of the two objectives and Corollary 1, we see that $\overline{x}$ and $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z})$ are, respectively, global optima for (PE) and (DE).

Theorem 4 (Strict Converse Duality). Let (PE) have an optimal solution $\overline{x}$ and let the hypotheses of Theorem 2 hold. Let $(\overline{u}, \overline{y}_0, \overline{y}, \overline{z})$ be an optimal solution of (DE). If, for all feasible solutions (x, u, y_0, y, z), f is quasiconvex, and $\overline{y}^t g + \overline{z}^t h$ is strictly pseudoconcave, then $\overline{u} = \overline{x}$, that is, $\overline{u}$ is an optimal solution of (PE).

Proof. We assume $\overline{u} \neq \overline{x}$ and exhibit a contradiction. Since $\overline{x}$ is an optimal solution of (PE), Theorem 3 yields that there exist $y_0 \in R$, $y \in R^m$, $z \in R^k$ such that $(\overline{x}, y_0, y, z)$ is an optimal solution for the dual problem (DE). Since $(\overline{u}, \overline{y}_0, \overline{y}, \overline{z})$ is also optimal for (DE), it follows that

$$f(\overline{x}) = f(\overline{u}) \tag{17}$$

From (4), (12) and (11) we have, for feasible $\overline{x}$ and $(\overline{u}, \overline{y}_0, \overline{y}, \overline{z})$,

$$\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x}) \geqq 0 \geqq \overline{y}^t g(\overline{u}) + \overline{z}^t h(\overline{u}) \tag{18}$$

Since for all feasible solutions (x, u, y_0, y, z), $\overline{y}^t g(\overline{x}) + \overline{z}^t h$ is strictly pseudoconcave, (18) gives

$$(\overline{x} - \overline{u})^t \nabla[\overline{y}^t g(\overline{u}) + \overline{z}^t h(\overline{u})] > 0 \tag{19}$$

(19), along with (10), gives

$$(\overline{x} - \overline{u})^t \nabla \overline{y}_0 f(\overline{u}) > 0 \tag{20}$$

In (17), using the quasiconvexity of f for all feasible solutions (x, u, y_0, y, z), we have

$$(\overline{x} - \overline{u})^t \nabla f(\overline{u}) \leqq 0$$

Using $\overline{y}_0 \geq 0$

$$(\overline{x} - \overline{u})^t \nabla \overline{y}_0 f(\overline{u}) \leqq 0 \tag{21}$$

(20) and (21) contradict each other. Hence the result follows.

Remark 2. It is important to point out here that we did not make any constraint qualification assumption in Theorems 3 and 4 . This is because we made use of Fritz John necessary optimality conditions [54] in the proofs of those theorems. However, in Theorems 3.7, 3.8 of ([15], a constraint qualification was assumed to hold.

Theorem 5 (Converse Duality). Let $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z})$ be a local or global optimum of (DE). Let $f, g, h \in C^2$, f be QX, $\overline{y}^t g + \overline{z}^t h$ be SPCV for all feasible solutions of (PE) and (DE) and let $\nabla[\, \overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})\,] \neq 0$. If, in addition, the $n \times n$ matrix $\nabla^2[\, \overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,]$ is positive or negative definite, then $\overline{y}_0 > 0$, and $\overline{x}$ is an optimal solution of (PE).

Proof. As in Mond and Weir ([58], Theorem 6, p. 272) we have, by the generalized Fritz John Theorem given by Mangasarian and Fromovitz [53], that there exist $\tau \in R$, $v \in R^n$, $w \in R$, $s_0 \in R$ and $s \in R^m$ such that

$$\tau \nabla f(\bar{x}) - \nabla^2[\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})]v - w\nabla[\bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x})] = 0 \tag{22}$$

$$v^t \nabla f(\bar{x}) + s_0 = 0 \tag{23}$$

$$\forall\ i \in M = \{1,2, \ldots, m\}, \qquad v^t \nabla g_i(\bar{x}) - w g_i(\bar{x}) + s_i = 0 \tag{24}$$

$$\forall\ k \in K = \{1, 2, \ldots, k\}, \qquad v^t \nabla h_k(\bar{x}) - w h_k(\bar{x}) = 0 \tag{25}$$

$$v^t \nabla[\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})] = 0 \tag{26}$$

$$w[\bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x})] = 0 \tag{27}$$

$$\nabla[\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})] = 0 \tag{28}$$

$$\bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x}) \leq 0 \tag{29}$$

$$(\bar{y}_0, \bar{y}, \bar{z}) \neq 0 \tag{30}$$

$$(\bar{y}_0, \bar{y}) \geq 0 \tag{31}$$

$$s_0 \bar{y}_0 = 0 \tag{32}$$

$$(\forall\ i \in M) \qquad s_i \bar{y}_i = 0 \tag{33}$$

$$(\tau, v, w, s_0, s) \neq 0 \tag{34}$$

and

$$(\tau, w, s_0, s) \geq 0 \tag{35}$$

Multiplying both sides of (23) by $\overline{y}_0$ and using (32) we obtain

$$v^t\overline{y}_0\nabla f(\overline{x}) = 0 \tag{36}$$

(26) and (36) yield,

$$v^t\nabla[\,\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})\,] = 0 \tag{37}$$

Premultiplying (22) by v^t and using (37) we get

$$\tau v^t\nabla f(\overline{x}) - v^t\nabla^2[\,\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,]v = 0 \tag{38}$$

We now claim that

$$\tau > 0\,. \tag{39}$$

Otherwise from (38) we would have $v^t\nabla^2[\,\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,]v = 0$, which, since the matrix $\nabla^2[\,\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,]$ is positive or negative definite by hypothesis, yields $v = 0$ and this in turn from (22) gives $w\nabla[\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})] = 0$. With the hypothesis $\nabla[\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})] \neq 0$, this gives $w = 0$ and, puuting $v = 0$, $w = 0$ in (23) and (24) we get, $s_0 = 0$, $s_i = 0$ $\forall\ i \in M$. Therefore, $\tau = 0 \Rightarrow v = 0,\ w = 0,\ s_0 = 0,\ s = 0$ which contradicts (34). Hence (39) holds.

Again, multiplying (38) by $\overline{y}_0$ and using (36), we obtain

$$\overline{y}_0 v^t\nabla^2[\,\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,]v = 0$$

or
$$(\overline{y}_0 v^t)\nabla^2[\,\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,](\overline{y}_0 v) = 0 \tag{40}$$

Using the hypothesis that $\nabla^2[\,\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,]$ is positive or negative definite we get from (40)

$$\overline{y}_0 v = 0 \quad \text{with} \quad \overline{y}_0 \geq 0 \tag{41}$$

We claim $\overline{y}_0 > 0$, otherwise (10) gives $\nabla[\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})] = 0$, which contradicts the hypothesis that $\nabla[\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})] \neq 0$. Hence $\overline{y}_0 > 0$ and, therefore, from (41) we have $v = 0$. This, in conjunction with (22), yields,

$$\tau \nabla f(\overline{x}) - w \nabla[\,\overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,] = 0. \tag{42}$$

From (28) we have

$$\overline{y}_0 \nabla f(\overline{x}) - \nabla[\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})] = 0 \tag{43}$$

(42) and (43) give

$$(\tau - \overline{y}_0 w) \nabla[\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})] = 0 \tag{44}$$

Since $\nabla[\overline{y}^t g(\overline{x}) + \overline{z}^t h(\overline{x})] \neq 0$, (44) yields

$$w = \tau / \overline{y}_0 > 0 \qquad \text{(because } \tau > 0,\ \overline{y}_0 > 0\text{)} \tag{45}$$

Since $v = 0$, therefore, (45) and (24), (25) gives

$$g_i(\overline{x}) \geq 0 \qquad \forall\ i \in M$$

and

$$h_k(\overline{x}) = 0 \qquad \forall\ k \in K$$

i.e. $\overline{x}$ is (PE)-feasible and hence using the hypothesis of the theorem and Corollary 1, we see that $\overline{x}$ is optimal for (PE) as the values of the two objectives at $\overline{x}$ and $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z})$ are equal.

4. CONCLUDING REMARKS.

(i) Fritz John sufficient optimality criteria have been established for differentiable functions under generalized convexity assumptions. The objective function f is assumed to be quasi convex ([61], 3f, p.116) and $\bar{y}^t g + \bar{z}^t h$ is assumed to be strictly pseudoconcave . Under these generalized assumptions various duality theorems are proved for Mond-Weir duality. These results are generalizations of results of Bector and Bector [15], Mangasarian [54], Bector and Grover [9], Bector and Gulati [11], Singh [69], Skarpness and Sposito [73].

(ii) Similar results can also be established under the assumptions of quasiinvexity [30] on the objective function and quasiinvexity/ strict pseudoinvexity on the constraints.

(iii) If we replace (1. 14) by the inequality $(x - \bar{x})^t[\bar{y}_0 \nabla f(\bar{x}) - \nabla(\bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x}))] \geq 0$ for $x \in \bar{X}^0$ where, as in [54], $\bar{X}^0$ is the closure of X^0 and we make slight modifications in the statement of Theorem 1, we can still prove it following the same lines of proof.

CHAPTER 3

SUFFICIENT OPTIMALITY CONDITIONS AND DUALITY FOR A QUASICONVEX PROGRAMMING PROBLEM

1. INTRODUCTION.

On the lines of Mond and Weir [58], consider the following nonlinear programming problems:

$$\text{(P)} \qquad \underset{x \in S}{\text{minimize}} \quad f(x)\,, \tag{1}$$

$$\text{(PE)} \qquad \underset{x \in X}{\text{Minimize}} \quad f(x)\,, \tag{2}$$

where

(i) $M = \{1,2, \ldots, m\}$, $K = \{1, 2, \ldots, k\}$,

(ii) X^0 is an open set of R^n,

(iii) $f : X^0 \to R$, $g : X^0 \to R^m$ and $h : X^0 \to R^k$ are differentiable functions.

(iv) $S = \{x;\ x \in X^0,\ g(x) \geqq 0\}$

$$= \{x;\ x \in X^0,\ g_i(x) \geqq 0,\ i \in M\}\,, \tag{3}$$

(v) $X = \{x;\ x \in X^0,\ g(x) \geqq 0,\ h(x) = 0\}$ (4)

$$= \{x;\ x \in X^0,\ g_i(x) \geqq 0,\ i \in M,\ h_k(x) = 0,\ k \in K\}\,, \tag{5}$$

Evidently, if $K = \phi$ (null set), (PE) becomes (P).

Mangasarian [54], assuming f to be pseudoconvex, g_I (with $I = \{i; g_i(\overline{x}) = 0, i \in M\}$) to be quasiconcave and h to be both quasiconvex and quasiconcave at $\overline{x} \in X$ showed that, if $(\overline{x}, \overline{y}, \overline{z})$ satisfies the following Kuhn-Tucker type conditions:

$$\nabla[f(x) - y^t g(x) - z^t h(x)] = 0 \, , \tag{6}$$

$$y^t g(x) = 0 \, , \tag{7}$$

$$g(x) \geqq 0 \, , \tag{8}$$

$$h(x) = 0 \, , \tag{9}$$

$$y \in R^m, \;\; z \in R^k, \qquad y \geq 0 \, , \tag{10}$$

then $\overline{x}$ is (PE)-optimal.

Bhatt and Misra [22] assuming all of f, g, h to be convex at $\overline{x} \in X$, showed that the above conditions (6)-(10), with the additional restriction $z \geqq 0$, are sufficient for $\overline{x}$ to be (PE)-optimal.

Assuming f to be convex and g to be strictly concave at $\overline{x} \in S$, Mangasarian [54] showed that, if $(\overline{x}, \overline{y}_0, \overline{y})$ satisfies the following Fritz John type conditions:

$$\nabla[y_0 f(x) - y^t g(x)] = 0 \ , \tag{11}$$

$$y^t g(x) = 0 \ , \tag{12}$$

$$g(x) \geq 0 \ , \tag{13}$$

$$y_0 \in R, \ \ y \in R^m, \qquad (y_0, y) \geq 0, \tag{14}$$

then $\overline{x}$ is (P)-optimal.

Assuming f to be pseudoconvex at $\overline{x} \in X$ and g_I (where $I = \{i;\ g_i(\overline{x})=0\}$) and h to be strictly pseudo concave at $\overline{x} \in X$, Bector and Gulati [11] and Skarpness and Sposito [73] proved that, if $(\overline{x}, \overline{y_0}, \overline{y}, \overline{z})$ satisfies the following Fritz John type conditions:

$$\nabla[y_0 f(x) - y^t g(x) - z^t h(x)] = 0 \ , \tag{15}$$

$$y^t g(x) = 0 \ , \tag{16}$$

$$g(x) \geq 0 \ , \tag{17}$$

$$h(x) = 0 \ , \tag{18}$$

$$y_0 \in R, \ \ y \in R^m, z \in R^k, (y_0, y) \geq 0, \tag{19}$$

$$z \in R^k, \qquad z \geq 0, \tag{20}$$

then $\overline{x}$ is (PE)-optimal.

Recently, Bector and Bector [15] showed that, if $(\overline{x}, \overline{y}, \overline{z})$ satisfies (15)-(19), f is quasi convex at $\overline{x} \in X$ and $\overline{y}^t g + \overline{z}^t h$ is strictly pseudoconcave at $\bar{x} \in X$, then $\bar{x}$ is (PE)-optimal. Furthermore, they proved weak, direct and converse duality theorems for the Mond-Weir dual problem. The purpose of the present chapter is to extend the results proved by Bector and Bector [15] further by proving the Fritz John type sufficient optimality theorem and various duality theorems for the Mond-Weir dual problem under quasi convexity assumption on f and quasi concavity/strict pseudo concavity on components of g and h. This problem is different from the problem considered in Chapter 2 in that a linear combination of quasiconvex/strictly pseudoconvex functions may not be a strictly pseudoconvex function.

2. OPTIMALITY

We now prove the following Fritz John sufficient optimality theorems for (PE) under the quasiconvexity assumption on f and generalized concavity assumptions in different forms on g and h.

Theorem 1 (Sufficient Optimality Theorem). Let $\bar{y}_0 \in R$, $\bar{y} \in R^m$, $\bar{z} \in R^k$ and let $\bar{x} \in X$ along with $\bar{y}_0, \bar{y}, \bar{z}$ satisfy the Fritz John type conditions (15)-(19). If at $\bar{x}$ with respect to X, (i) f is quasiconvex

(QX) ([61], 3f, p.116), (ii) for $i \in I$ $(I = \{i;\ g_i(\overline{x}) = 0,\ i \in M\})$, $i \neq s$, g_i is QV, but for $i = s$, g_s is strictly pseudoconcave (SPCV) with $\overline{y}_s > 0$ and (iii) $\forall\ k \in \mathbf{K}$, $\overline{z}_k\ h_k$ is quasiconcave (QV), then $\overline{x}$ is a global optimal solution to (PE).

Proof. If $\overline{x}$ is not a global minimum for (PE), let $x^0 \in X$ be such that

$$f(x^0) < f(\overline{x}) \tag{21}$$

Since f is QX ([61], 3f, p. 116) at $\overline{x}$ therefore, (21) yields

$$(x^0 - \overline{x})^t \nabla f(\overline{x}) \leqq 0$$

With $\overline{y}_0 \geq 0$ this implies,

$$(x^0 - \overline{x})^t \nabla \overline{y}_0 f(\overline{x}) \leqq 0 \tag{22}$$

Let

$$I = \{i;\ g_i(\overline{x}) = 0,\ i \in M\},\ J = \{i;\ g_i(\overline{x}) < 0,\ i \in M\} \text{ such that } I \cup J = M$$

This, in view of (16), (17), and (19) yields

$$\overline{y}_J = 0 \tag{23}$$

Again, as in ([54], p. 152), we have

$$g_i\ (x^0) \geqq 0 = g_i(\overline{x}) \quad \text{for} \quad i \in I \tag{24}$$

Since $\forall\, i \in I,\ i \neq s,\ g_i$ is QV and for $i = s,\ g_s$ is SPCV

with $\bar{y}_s > 0$, it follows that from (24)

$$(x^0 - \bar{x})^t \nabla g_i(\bar{x}) \geqq 0 \qquad \forall\, i \in I,\ i \neq s$$

$$(x^0 - \bar{x})^t \nabla g_s(\bar{x}) > 0 \qquad \text{for } i = s,$$

and hence,

$$(x^0 - \bar{x})^t \nabla \bar{y}_I{}^t g_I(\bar{x}) > 0 \tag{25}$$

Since $\bar{y}_J = 0$ from (23), we have that

$$(x^0 - \bar{x})^t \nabla \bar{y}_J{}^t g_J(\bar{x}) = 0 \tag{26}$$

(25) and (26) yield

$$(x^0 - \bar{x})^t \nabla \bar{y}^t g(\bar{x}) > 0 \tag{27}$$

Now $\forall\, k \in K$ $\qquad \bar{z}_k h_k(x^0) = \bar{z}_k h_k(\bar{x})$

It follows from the quasiconcavity of $\bar{z}_k h_k$, $\forall\, k \in K$, that

$$(x^0 - \bar{x})^t \nabla \bar{z}_k h_k(\bar{x}) \geqq 0 \qquad \forall\, k \in K \tag{28}$$

(27) and (28) yield

$$(x^0 - \bar{x})^t \nabla [\bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x})] > 0 \tag{29}$$

(15) and (29) together give

$$(x^0 - \bar{x})^t \nabla \bar{y}_0 f(\bar{x}) > 0 \tag{30}$$

This contradicts (22). Hence the result follows.

Remarks 1.

(i) It is very important to observe here that in the proof of the Theorem 1 it is the strict pseudo concavity of g_s with $y_s > 0$ without which the strict inequality (25) and hence (30) is not possible. Thus, for (PE) global optimality of $\overline{x}$ satisfying conditions (15)-(20) with QX objective function is solely dependent upon the strict pseudoconcavity of one of the constraint functions with positive multiplier. Hence, the importance of the theorem lies in the fact that, once we have solved a nonlinear programming problem with QX objective and QV constraints, all we need to ensure the global optimality of the solution (and this is crucial) is to show that at the optimal solution just one constraint function corresponding to a positive multiplier is SPCV.

(ii) If we assume $\overline{y}_0 > 0$ in (15) and (19) then the Theorem 1 gives a Kuhn-Tucker type sufficient optimality theorem under weaker convexity assumptions on the objective function and constraints that are different from Bector and Bector [15] and Mangasarian [54].

(iii) We can also prove the Theorem 1 by replacing the assumption '$\forall\, k \in K$, $\bar{z}_k h_k$ is QV' by '$\forall\, k \in K$, h_k is both QV and QX '.

(iv) Taking $K = \phi$ in Theorem 3.1 we recover the Fritz John sufficient optimality theorem for (P) that generalizes the results of Bector and Grover [9].

(v) It is well known [54] that when a continuous QV function is minimized on an appropriately restricted convex set, a local minimum, in general, may not be global. However, the assumptions in Theorem 1 always yield a global minimum.

3. DUALITY

We consider the following two of the dual (DE) suggested by Mond and Weir [58] for (PE).

(DE) maximize $f(u)$,

subject to

$$\nabla[y_0 f(u) - \sum_{i \in M} y_i g_i(u) - \sum_{k \in K} z_k h_k(u)] = 0 , \tag{31}$$

$$y_i g_i(u) \leq 0 , \quad \forall\, i \in M \tag{32}$$

$$z_k h_k(u) \leq 0 , \quad \forall\, k \in K \tag{33}$$

$$y_0,\, y_i \geq 0 , \quad \forall\, i \in M, \forall\, k \in K. \tag{34}$$

We now prove the following theorems relating (DE) to (PE).

Theorem 2 (Weak Duality). Let x be (PE)-feasible and (u, y_0, y, z) be (DE)-feasible. If, for all feasible solutions (x, u, y_0, y, z),

(i) f is QX ([61], 3f, p.116), and

(ii) $\forall\, i \in M,\ i \neq s,\ g_i$ is QV, but for $i = s$, g_s is SPCV with $y_s > 0$ and $\forall\, k \in K,\ z_k h_k$ is QV,

then, infimum (PE) $\geq$ supremum (DE)

Proof. If possible let $f(x) < f(u)$. Since for all feasible solutions (x, u, y_0, y, z), f is QX, therefore ([61], 3f, p.116),

$$(x - u)^t \nabla f(u) \leqq 0$$

With $y_0 \geq 0$ this gives

$$(x - u)^t \nabla y_0 f(u) \leqq 0 \tag{35}$$

Using (34), (4) and (32) we have,

$$y_i g_i(x) \geqq 0 \geqq y_i g_i(u)\,, \quad \forall\, i \in M\,. \tag{36}$$

Since for all feasible solutions (x, u, y_0, y, z) and for $i \in M,\ i \neq s$, g_i is QV $\forall\, i \in M$ but for $s \in M$, g_s is SPCV with $y_s > 0$, (36) gives

$$\forall\, i \in M,\ i \neq s \qquad (x - u)^t \nabla y_i g_i(u) \geqq 0 \tag{37}$$

$$s \in M \qquad (x - u)^t \nabla y_s g_s(u) > 0 \tag{38}$$

Similarly, using (4), (33) and the quasiconcavity of $z_k h_k$ $\forall\, k \in K$ we have

$$\forall\, k \in K \qquad (x - u)^t \nabla z_k h_k\,(u) \geqq 0\,. \tag{39}$$

(37), (38) and (39) yield

$$(x - u)^t \left[\sum_{i \in M} y_i g_i(u) + \sum_{k \in K} z_k h_k(u)\right] > 0 \tag{40}$$

(31) and (40) give $(x - u)^t \nabla y_0 f(u) > 0$, which contradicts (35). Hence the result follows.

Corollary 1. Let $\bar{x}$ be (PE)-feasible and $(\bar{u}, \bar{y}_0\ \bar{y}, \bar{z})$ (DE)-feasible such that $f(\bar{x}) = f(\bar{u})$ and let the hypotheses of Theorem 2 hold. Then $\bar{x}$ is a global optimum for (PE) and $(\bar{u}, \bar{y}_0, \bar{y}, \bar{z})$ is a global optimum for (DE) with the corresponding optimal objective values $f(\bar{x})$ and $f(\bar{u})$ respectively.

Theorem 3 (Direct Duality). Let $\bar{x} \in X$ be a local or global optimum of (PE). Then there exists $(\bar{y}_0, \bar{y}, \bar{z})$ such that $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z})$ is feasible for (DE) and the corresponding values of the objective functions of (PE) and (DE) are equal. If, also, the hypotheses of Theorem 2 are satisfied, then $\bar{x}$ and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z})$ are, respectively, global optima for (PE) and (DE).

Proof. Since $\bar{x} \in X$ is a local or global optimum of (PE), there exist $\bar{y}_0 \in R$, $\bar{y} \in R^m$, $\bar{z} \in R^k$ such that the conditions (15)-(19) are satisfied ([54], p.170).

From (16), (17), (18) and (19) we have

$$\forall\ i \in M, \quad \bar{y}_i g_i(\bar{x}) = 0, \quad \forall\ i \in M, \quad \bar{z}_k h_k(\bar{x}) = 0 \tag{41}$$

Therefore, (15), (41) and (19) imply that $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z})$ is (DE)-feasible. Equality of the objectives follows from the fact that each of them is equal to $f(\bar{x})$ at $\bar{x}$ and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z})$ respectively. If the hypotheses of

Theorem 2 are satisfied, then using the equality of the two objectives and Corollary 1 implies that $\bar{x}$ and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z})$ are, respectively, global optima for (PE) and (DE).

Theorem 4 (Strict Converse Duality). Let (PE) have an optimal solution $\bar{x}$ and let the hypotheses of Theorem 2 hold. Let $(\bar{u}, \bar{y}_0, \bar{y}, \bar{z})$ be an optimal solution of (DE). If, for all feasible solutions (x, u, y_0, y, z),

(i) f is QX, and

(ii) $\forall\ i \in M,\ i \neq s,\ g_i$ is QV, but for $i = s \in M$, g_s is SPCV with $\bar{y}_s > 0$

and $\forall\ k \in K$, $\bar{z}_k h_k$ is QV ,

then $\bar{u} = \bar{x}$, that is, $\bar{u}$ is an optimal solution for (PE).

Proof. We assume $\bar{u} \neq \bar{x}$ and exhibit a contradiction. Since $\bar{x}$ is an optimal solution of (PE), therefore, Theorem 3 yields that there exist $y_0 \in R$, $y \in R^m$, $z \in R^k$ such that $(\bar{x}, y_0, y, z)$ is an optimal solution for the dual problem (DE). Since $(\bar{u}, \bar{y}_0, \bar{y}, \bar{z})$ is also optimal for (DE), it follows that

$$f(\bar{x}) = f(\bar{u}) \tag{42}$$

Using the quasiconvexity of f and $y_0 \geq 0$, we obtain from (42)

$$(\bar{x} - \bar{u})^t \nabla y_0 f(\bar{u}) \leqq 0 \tag{43}$$

For feasible $(\bar{x}, \bar{u}, \bar{y}_0, \bar{y}, \bar{z})$, from (4), (32), (33), (34) and using the quasiconcavity of g_i $\forall\ i \in M,\ i \neq s$, strict pseudo concavity of g_s with $\bar{y}_s > 0$ for $i = s,\ i \in M$ and quasiconcavity of $\bar{z}_k h_k$ $\forall\ k \in K$, we have (as in Theorem 2)

$$(\overline{x} - \overline{u})^t \left[\sum_{i \in M} \overline{y}_i g_i(\overline{u}) + \sum_{k \in K} \overline{z}_k h_k(\overline{u})\right] > 0 \tag{44}$$

(31), the feasibility of $(\overline{u}, \overline{y_0}, \overline{y}, \overline{z})$ and (44) give $(\overline{x} - \overline{u})^t \nabla y_0 f(\overline{u}) > 0$, which contradicts (43). Hence the result follows.

Remark 2. It is important to point out here that we did not make any constraint qualification assumption in Theorems 3 and 4. This is because we made use of Fritz John necessary optimality conditions [54] in the proofs of those theorems. However, in Theorems 3.7 and 3.8 of [15] a constraint qualification was assumed to hold.

Theorem 5 (Converse Duality). Let $(\overline{x}, \overline{y_0}, \overline{y}, \overline{z})$ be a local or global optimum of (DE). Let $f, g, h \in C^2$, for all feasible solutions (x, u, y_0, y, z), f be QX, g_i be QV $\forall\, i \in M$ $i \neq s$, but for $i = s$ let g_s be SPCV with $\overline{y}_s > 0$, let $\overline{z}_k h_k$ be QV $\forall\, k \in K$ and let for all $i \in M$, all $k \in K$ the vectors $\nabla \overline{y}_i g_i(\overline{x})$, $\nabla \overline{z}_k h_k(\overline{x})$ be linearly independent.

If, in addition the $n \times n$ matrix $\nabla^2[\,\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})\,]$ is positive or negative definite, then $\overline{y}_0 > 0$ and $\overline{x}$ is optimal solution to (PE).

Proof. As in Mond and Weir ([58], Theorem 6, p. 272), by the generalized Fritz John Theorem given by Mangasarian and Fromovitz

[53], there exist $\tau \in R$, $v \in R^n$, $w \in R^m$, $p \in R^k$, $s_0 \in R$ and $s \in R^m$ such that

$$\tau \nabla f(\bar{x}) - \nabla^2[\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})]v$$

$$- \nabla\Big[\sum_{i \in M} w_i \bar{y}_i g_i(\bar{x}) + \sum_{k \in K} p_k \bar{z}_k h_k(\bar{x})\Big] = 0 \qquad (45)$$

$$v^t \nabla f(\bar{x}) + s_0 = 0 \qquad (46)$$

$$v^t \nabla g_i(\bar{x}) - w_i g_i(\bar{x}) + s_i = 0 \text{ , } (\forall\ i \in M) \qquad (47)$$

$$v^t \nabla h_k(\bar{x}) - p_k h_k(\bar{x}) = 0 \text{ , } (\forall\ k \in K) \qquad (48)$$

$$v^t \nabla[\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})] = 0 \qquad (49)$$

$$w_i \bar{y}_i g_i(\bar{x}) = 0 \text{ , } (\forall\ i \in M) \qquad (50)$$

$$p_k \bar{z}_k h_k(\bar{x}) = 0 \text{ , } (\forall\ k \in K) \qquad (51)$$

$$\nabla[\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})] = 0 \qquad (52)$$

$$\bar{y}_i g_i(\bar{x}) \leqq 0 \text{ , } \qquad (\forall\ i \in M), \qquad (53)$$

$$\bar{z}_k h_k(\bar{x}) \leqq 0 \text{ , } \qquad (\forall\ k \in K), \qquad (54)$$

$$(\bar{y}_0, \bar{y}_i) \geq 0, \ (\bar{y}_0, \bar{y}_i, \bar{z}_k) \neq 0 \text{ , } \qquad (\forall\ i \in M, \ \forall\ k \in K), \qquad (55)$$

$$s_0 \bar{y}_0 = 0 \qquad (56)$$

$$s_i \bar{y}_i = 0 \text{ , } \qquad (\forall\ i \in M), \qquad (57)$$

$$(\tau, v, w, p, s_0, s) \neq 0 \qquad (58)$$

$$(\tau, w, p, s_0, s) \geqq 0 \qquad (59)$$

Multiplying (46) by $\bar{y}_0$ and using (56), (47) by $\bar{y}_i$ and using (57) and (48) by $\bar{z}_k$ and using (51) we obtain

$$v^t \bar{y}_0 \nabla f(\bar{x}) = 0 \tag{60}$$

$$v^t \nabla \bar{y}_i g_i(\bar{x}) = 0\,, \qquad (\forall\ i \in M), \tag{61}$$

$$v^t \nabla \bar{z}_k h_k(\bar{x}) = 0\,, \qquad (\forall\ k \in K). \tag{62}$$

From (61) and (62) we have

$$v^t \nabla [\sum_{i \in M} w_i \bar{y}_i g_i(\bar{x}) + \sum_{k \in K} p_k \bar{z}_k h_k(\bar{x})] = 0 \tag{63}$$

or
$$v^t [\sum_{i \in M} w_i \nabla \bar{y}_i g_i(\bar{x}) + \sum_{k \in K} p_k \nabla \bar{z}_k h_k(\bar{x})] = 0 \tag{64}$$

Premultiplying (45) by v^t and using (63) we have

$$\tau v^t \nabla f(\bar{x}) - v^t \nabla^2 [\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})] v = 0 \tag{65}$$

We now claim that

$$\tau > 0 \tag{66}$$

Otherwise, from (65) we have $v^t \nabla^2 [\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})] v = 0$. This along with the hypothesis $\nabla^2 [\bar{y}_0 f(\bar{x}) - \bar{y}^t g(\bar{x}) - \bar{z}^t h(\bar{x})]$ is positive or negative definite yields $v = 0$. Therefore, from (45) we have

$$\nabla [\sum_{i \in M} w_i \bar{y}_i g_i(\bar{x}) + \sum_{k \in K} p_k \bar{z}_k h_k(\bar{x})] = 0$$

that is
$$\sum_{i \in M} w_i \nabla \bar{y}_i g_i(\bar{x}) + \sum_{k \in K} p_k \nabla \bar{z}_k h_k(\bar{x}) = 0 \tag{67}$$

For all $i \in M$ and all $k \in K$, using the linear independence of the vectors $\nabla \bar{y}_i g_i(\bar{x})$ and $\nabla \bar{z}_k h_k(\bar{x})$, we obtain that from (67) $w_i = 0$, $\forall\ i \in M$,

$p_k = 0$, $\forall\, k \in K$, which along with $v = 0$ yield $s_0 = 0$ and $s_i = 0\ \forall\, i \in M$.

Thus,

$$\tau = 0 \Rightarrow v = 0,\ \ w = 0,\ \ p = 0,\ \ s_0 = 0,\ \ s = 0$$

a contradiction to (58). Hence (66) holds.

Using (60) in (65), with $\overline{y}_0 \geq 0$ we obtain

$$(\overline{y}_0 v^t)\ \nabla^2[\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})]\ (\overline{y}_0 v) = 0 \tag{68}$$

By hypothesis, $\nabla^2[\overline{y}_0 f(\overline{x}) - \overline{y}^t g(\overline{x}) - \overline{z}^t h(\overline{x})]$ is positive or negative definite, therefore, from (68) we have

$$\overline{y}_0 v = 0 \qquad \text{with } \overline{y}_0 v \geq 0 \tag{69}$$

We now claim that $\overline{y}_0 > 0$, otherwise from (52) we obtain

$\sum_{i\in M} \nabla \overline{y}_i g_i(\overline{x}) + \sum_{k\in K} \nabla \overline{z}_k h_k(\overline{x}) = 0$, which contradicts the fact that, $\forall\, i \in M$ and $\forall\, k \in K$ $\nabla \overline{y}_i g_i(\overline{x})$ and $\nabla \overline{z}_k h_k(\overline{x})$ are linearly independent. Hence $\overline{y}_0 > 0$ and, therefore, from (69) we have $v = 0$. This, in conjunction with (45) yields,

$$\tau \nabla f(\overline{x}) = \sum_{i\in M} w_i \nabla \overline{y}_i g_i(\overline{x}) + \sum_{k\in K} p_k \nabla \overline{z}_k h_k(\overline{x})\ . \tag{70}$$

From (52) we have

$$\overline{y}_0 \nabla f(\overline{x}) = \sum_{i\in M} \nabla \overline{y}_i g_i(\overline{x}) + \sum_{k\in K} \nabla \overline{z}_k h_k(\overline{x})\ . \tag{71}$$

From (70) and (71) we get

$$\sum_{i\in M} (\tau - \overline{y}_0 w_i) \nabla \overline{y}_i g_i(\overline{x}) + \sum_{k\in K} (\tau - \overline{y}_0 p_k) \nabla \overline{z}_k h_k(\overline{x}) = 0\ . \tag{72}$$

Using the hypothesis that for all $i \in M$ and for all $k \in K$ the vectors $\nabla \overline{y}_i g_i(\overline{x})$ and $\nabla \overline{z}_k h_k(\overline{x})$ are linearly independent, we get from (72)

$$\tau = \bar{y}_0 w_i \qquad \forall\ i \in M \tag{73}$$

and

$$\tau = \bar{y}_0 p_k \qquad \forall\ k \in K \tag{74}$$

Using (73) with $v = 0$ in (47), and using (74) with $v = 0$ in (48), respectively, we obtain

$$g_i(\bar{x}) \geq 0 \qquad \forall\ i \in M$$

and

$$h_k(\bar{x}) = 0 \qquad \forall\ k \in K$$

which shows that $\bar{x}$ is (PE)-feasible. Also, the values of the two objectives at (PE)-feasible $\bar{x}$ and (DE-2)-feasible $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z})$ are equal, therefore, using Corollary 4.1 and the hypothesis of the theorem we see that $\bar{x}$ is (PE)-optimal.

5. CONCLUDING REMARKS.

(i) Fritz John sufficient optimality criteria have been established for differentiable functions under generalized convexity assumptions. The objective function is assumed to be quasiconvex ([61], 3f, p.116) and the constraint functions are assumed to be quasi-concave/strict psuedoconcave in various forms. Under these generalized assumptions various duality theorems have been proved for Mond-Weir duality. These results are generalizations of results of Bector and Bector [15], Mangasarian [54], Bector and Grover [9], Bector and Gulati [11], Singh [69], Skarpness and Sposito [73] and Bector, Bector and Klassen [10].

(ii) Similar results can also be established by assuming quasiinvexity [31] of the objective function and quasiinvexity or strict pseudo-invexity of the constraints.

(iii) If we replace (15) by the inequality $(x - \bar{x})^t[\bar{y}_0 \nabla f(\bar{x}) - \nabla(\bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x}))] \geq 0$ for $x \in \bar{X}^0$ where, as in [54], $\bar{X}^0$ is the closure of X^0 and make slight modifications in the statement [54] of Theorem 3.1, we can still prove it following same pattern of proof.

(iv) As in Mond and Weir [58] we can have a general dual

(DEG) maximize $f(u)$

subject to

$$\nabla[y_0 f(u) - y^t g(u) - z^t h(u)] = 0$$

$$\sum_{i \in I_\alpha} y_i g_i(u) + \sum_{j \in J_\alpha} z_j h_j(u) \leq 0, \qquad \alpha = 1, 2, \ldots, r$$

$$y \geq 0$$

where, different notations are same as in ([58], p.267). Assuming quasiconvexity assumption of f and strict pseudoconcavity assumption on g_s with corresponding $y_s > 0$ and quasiconcavity

of
$$\sum_{\substack{i \in I_\alpha \\ i \neq s}} y_i g_i + \sum_{j \in J_\alpha} z_j h_j \,, \qquad \alpha = 1, 2, \ldots, r\,,$$

we can prove Theorems 1-5, with appropriate modifications in their statements, in an analogous manner.

CHAPTER 4

A DUALITY MODEL FOR A GENERALISED MINMAX PROGRAM

The purpose of the present chapter is to consider a generalized minmax programming problem, in which several functions are to be optimized simultaneously and the overall objective is to minimize (maximize) the largest (smallest) of the objectives, and, under weaker convexity assumptions,

(i) prove the Fritz John type sufficient optimality conditions,

(ii) introduce a Mond and Weir type [24] dual program for the minmax programming problem and, using the Fritz John type conditions prove duality theorems,

(iii) apply the results proved to define a form of duality for a generalized fractional programming problem.

Such problems have numerous applications, as given in Chapter 1.

1. FRITZ JOHN TYPE OPTIMALITY Conditions

We now consider the following nonlinear minmax program as the primal problem (P) whose Fritz John type optimality conditions and duality we want to discuss.

(P) $$\underset{x \in S}{\text{Minimize}} \quad \underset{1 \leq i \leq p}{\text{Maximum}} \quad \{f_i(x)\} \tag{1}$$

where,

(i) $S = \{x \in R^n;\ h_k(x) \leq 0,\ \ k = 1, 2, \ldots, m\}$ is nonempty and compact,

(ii) f_i $(1 \leq i \leq p)$, h_k $(1 \leq k \leq m)$ are real valued and differentiable functions.

We have the following problem (EP) from (P).

(EP) $$\underset{x,q}{\text{Minimize}} \quad q \tag{2}$$

subject to

$$f_i(x) \leq q \qquad i = 1, 2, \ldots, p \tag{3}$$

$$h_k(x) \leq 0 \qquad k = 1, 2, \ldots, m \tag{4}$$

$$x \in R^n, \quad q \in R \tag{5}$$

(EP) is equivalent to (P) in the sense of the following Lemmas 1 and 2.

Lemma 1. If x is P-feasible then there exists a $q \in R$ such that (x, q) is EP-feasible and if (x, q) is EP-feasible then x is P-feasible.

Lemma 2. $\hat{x}$ is optimal to (P) with the corrresponding optimal objective value $\hat{q}$ iff $(\hat{x}, \hat{q})$ is optimal to (EP) with the corresponding optimal objective value $\hat{q}$.

Before we introduce a dual (D) to (EP) (and hence a dual to (P) we state and prove the following Fritz John type optimality theorems.

Theorem 1 (Fritz John Necessary Optimality). Let $\bar{x} \in S$ be P-optimal with corresponding optimal value of P-objective equal to $\bar{q}$. The there exist $\bar{y}_0 \in R$, $\bar{y} \in R^p$, $\bar{z} \in R^k$ such that $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{q})$ satisfies

$$\nabla[y^t f(x) + z^t h(x)] = 0 \tag{6}$$

$$(\forall i = 1,2,\ldots,p) \qquad y_i(f_i(x) - q) = 0 \tag{7}$$

$$(\forall k = 1,2,\ldots,) \qquad z_k h_k(x) = 0 \tag{8}$$

$$(\forall i = 1,2,\ldots,p) \qquad f_i(x) - q \leq 0 \tag{9}$$

$$(\forall k = 1,2,\ldots,m) \qquad h_k(x) \leq 0 \tag{10}$$

$$\sum_{i=1}^{p} y_i = y_0 \tag{11}$$

$$y_0, y, z \geq 0 \tag{12}$$

Proof. Follows from Lemma 2 and Mangasarian [54, Theorem 11.3.1, p.170].

Theorem 2 (Fritz John Sufficient Optimality). Let $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{q})$ satisfy conditions (6) - (12). If at $\bar{x}$, $\bar{y}^t f(\cdot)$ is quasiconvex and $\bar{z}^t h(\cdot)$ is strictly pseudoconvex, with respect to S, then $\bar{x}$ is P-optimal with the optimal P-objective value $\bar{q}$.

Proof. If $\overline{x} \epsilon S$ (with $\overline{q}$ as the corresponding value of the P-objective) is not P-optimal, then let $x^* \neq \overline{x}, \epsilon S$ (with q^* as the corresponding value of the P-objective) be such that

$$q^* < \overline{q} \tag{13}$$

Since $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z}, \overline{q})$ satisfy conditions (6) - (12), (12) and (13) using $\overline{y}_0 \geq 0$ yield

$$\overline{y}_0 \, q^* \leqq \overline{y}_0 \, \overline{q} \tag{14}$$

From (11) and (14) we obtain

$$(\sum_{i=1}^{p} \overline{y}_i) q^* \leqq (\sum_{i=1}^{p} \overline{y}_i) \overline{q} \; . \tag{15}$$

(12), (3), (7) and (15) gives

$$\overline{y}^{\,t} f(x^*) \leqq \overline{y}^{\,t} f(\overline{x}) \; . \tag{16}$$

Using the hypothesis that $\overline{y}^{\,t} f(\cdot)$ is quasiconvex at $\overline{x}$, we have, from (16),

$$(x^* - \overline{x})^t \, \nabla(\overline{y}^{\,t} f(\overline{x})) \leqq 0 \; . \tag{17}$$

Using (12), (4) and (8) we have

$$\overline{z}^{\,t} h(x^*) \leq 0 = \overline{z}^{\,t} h(\overline{x}) \; . \tag{18}$$

Using the hypothesis that $\overline{z}^{\,t} h(\cdot)$ is strictly pseudoconvex at $\overline{x}$ with respect to S, from (18) we have

$$(x^* - \overline{x})^t \, \nabla(\overline{z}^{\,t} h(\overline{x})) < 0 \; . \tag{19}$$

From (19) and (6) we have

$$(x^*-\overline{x})^t \nabla(\overline{y}^t f(\overline{x})) > 0 \tag{20}$$

(17) and (20) contradict each other. Hence the result.

2. Duality

In this section we consider the following Mond and Weir [58] type dual (D) and, under weaker convexity assumptions on $y^t f(\cdot)$ and $z^t h(\cdot)$, establish different duality theorems relating the primal problem (P) and the dual problem (D).

(D) Maximize η

subject to

$$\nabla[y^t f(u) + z^t h(u)] = 0 \tag{21}$$

$$y^t f(u) \geqq y_0 \eta \tag{22}$$

$$z^t h(u) \geqq 0 \tag{23}$$

$$\sum_{i=1}^{p} y_i = y_0 \tag{24}$$

$$\eta \in R,\ y_0 \in R,\ y \in R^p,\ z \in R^m,\ (y_0, y, z) \geq 0 \tag{25}$$

We now prove the following theorems relating (P) and (D).

Theorem 3 (Weak Duality). Let x be P-feasible and (u, y_0, y, z, η) be D-feasible. If, for all feasible solutions (x, u, y_0, y, z, η), $y^t f(\cdot)$ is quasiconvex and $z^t h(\cdot)$ is strictly pseudoconvex, then Inf (P) $\geq$ Sup (D).

Proof. For P-feasible $\bar{x}$ (and hence EP-feasible $(\bar{x}, \bar{q})$ and D-feasible $(\bar{u}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$), we let

$$\bar{q} < \bar{\eta} \tag{26}$$

and show that this leads to a contradiction. From (26) we have using $y_0 \geq 0$

$$\bar{y}_0 \bar{q} \leq \bar{y}_0 \bar{\eta} \tag{27}$$

Using (24), (3) and (22) in (27) we obtain

$$\bar{y}^t f(\bar{x}) \leq \bar{y}^t f(\bar{u}) \tag{28}$$

Since $\bar{y}^t f(\cdot)$ is quasiconvex for all (x, u, y_0, y, z, η), (28) yields

$$(\bar{x} - \bar{u})^t \nabla(\bar{y}^t f(\bar{u})) \leq 0 \tag{29}$$

Using (23), (25) and (4) we have

$$\bar{z}^t h(x) \leq \bar{z}^t h(u) \tag{30}$$

Since, for all feasible solutions (x, u, y_0, y, z, η), $\bar{z}^t h(\cdot)$ is strictly pseudoconvex, (30) yields

$$(x-u)^t \nabla(\bar{z}^t h(u)) < 0 \tag{31}$$

From (21) we have for all feasible (x, u, y_0, y, z, η)

$$(x-u)^t \nabla[y^t f(u) + z^t h(u)] = 0 \tag{32}$$

Now (31) and (32) give for all (x, u, y_0, y, z, η)

$$(x-u)^t \nabla(y^t f(u)) > 0 , \tag{33}$$

which contradicts (29). Hence the result.

Corollary 1. Let $\bar{x}$ be P-feasible (and hence $(\bar{x}, \bar{q})$ be EP-feasible) and $(\bar{u}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ be D-feasible such that $\bar{q} = \bar{\eta}$. Let hypothesis of Theorem 3 hold. Then $\bar{x}$ is P-optimal with the corresponding optimal P-objective value $\bar{q}$ and $(\bar{u}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ is D-optimal with the corresponding optimal D-objective value $\bar{\eta}$.

Theorem 4 (Direct Duality). Let $\bar{x}$ be a local or global optimum of (P). Then there exist $\bar{y}_0 \in R$, $\bar{y} \in R^p$, $\bar{z} \in R^m$, $\bar{\eta} \in R$ such that $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ is D-feasible and the corresponding objective values of (P) and (D) are equal. If the hypothesis of Theorem 3 is also satisfied, then $\bar{x}$ and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ are global optima for (P) and (D), respectively.

Proof. Since $\bar{x}$ is P-optimal, $(\bar{x}, \bar{q})$ is EP-optimal and, by Theorem 1, there exists $\bar{y}_0 \in R$, $\bar{y} \in R^p$, $\bar{z} \in R^m$ such that

$$\nabla[\bar{y}^{t} f(\bar{x}) + \bar{z}^{t} h(\bar{x})] = 0 \tag{34}$$

$(\forall i = 1,2,\ldots,p)$ $$\bar{y}_i\,[f_i(\bar{x}) - \bar{q}] = 0\,, \tag{35}$$

$(\forall k = 1,2,\ldots,m)$ $$\bar{z}_k h_k(\bar{x}) = 0\,, \tag{36}$$

$(\forall i = 1,2,\ldots,p)$ $$f_i(\bar{x}) - \bar{q} \le 0\,, \tag{37}$$

$(\forall k = 1,2,\ldots,m)$ $$h_k(\bar{x}) \le 0\,, \tag{38}$$

$$\sum_{i=1}^{p} \bar{y}_i = \bar{y}_0\,, \tag{39}$$

and $$(\bar{y}_0, \bar{y}, \bar{z}) \ge 0\,, \tag{40}$$

(35) and (36) yield, respectively

$$\bar{y}^{t} f(\bar{x}) \ge \bar{y}_0\, \bar{q} \tag{41}$$

and $$\bar{z}^{t} h(\bar{x}) \ge 0 \tag{42}$$

Comparing (34), (41), (42), (39) and (40) with (21) - (25), we see that $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{q})$ is D-feasible with $\bar{q} = \bar{\eta}$.

Theorem 3 and Corollary 1 yield that $\bar{x}$ is a global optimum to (P) with corresponding P-objective value $\bar{q}$ and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ is a global optimum to (D) with corresponding D-objective value $\bar{\eta}$.

Theorem 5 (Strict Converse Duality) Let $\bar{x}$ be P-optimal with $\bar{q}$ as optimal value and let the hypothesis of Theorem 3 hold. If

$(\bar{u}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ is D-optimal and If, for all feasible solutions (x, u, y_0, y, z, η), $\bar{y}^t f(\cdot)$ is quasiconvex and $\bar{z}^t h(\cdot)$ is strictly pseudoconvex, then $\bar{u} = \bar{x}$, that is, $\bar{u}$ is the optimal solution of (P).

Proof. We assume that $\bar{u} \neq \bar{x}$ and exhibit a contradiction. Since $\bar{x}$ is an optimal solution to (P), therefore, $(\bar{x}, \bar{q})$ is EP-optimal. Hence by Theorem 1 there exists (y_0^*, y^*, z^*) such that $(\bar{x}, y_0^*, y^*, z^*, \bar{q})$ is D-optimal. Since $(\bar{u}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ is also D-optimal, it follows that

$$\bar{q} = \bar{\eta} .$$

This implies for $\bar{y}_0 \geq 0$

$$\bar{y}_0 \, \bar{q} = \bar{y}_0 \, \bar{\eta} \quad . \tag{43}$$

As in Theorem 3, we obtain from (43)

$$\bar{y}^t f(\bar{x}) = \bar{y}^t f(\bar{u}) \quad . \tag{44}$$

Since $\bar{y}^t f(\cdot)$ is quasoconvex for all (x, u, y_0, y, z, η), (44) yields

$$(\bar{x}-\bar{u})^t \, \nabla(\bar{y}^t f(\bar{u})) \leqq 0 \quad . \tag{45}$$

Using the hypothesis that $\bar{z}^t h(\cdot)$ is strictly pseudoconvex, we obtain on the lines of Theorem 3,

$$(\bar{x}-\bar{u})^t \, \nabla(\bar{z}^t h(\bar{u})) < 0 \, , \tag{46}$$

and hence

$$(\bar{x}-\bar{u})^t \, \nabla(\bar{y}^t f(\bar{u})) > 0 \, . \tag{47}$$

(45) and (47) contradict each other. Hence the result follows.

Theorem 6 (Converse Duality). Let $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ be a local or global optimum of (D) and let, for $i=1,2,\ldots,p$, $f_i \in C^2$ and, for $k=1,2,\ldots,m$, $h_k \in C^2$. If, for (D),

(i) the vectors $\nabla \bar{z}_k h_k(\bar{x})$ are linearly independent for all $k=1,2,\ldots,m$,

the vector $\nabla(\bar{y}^t f(\bar{x})) \neq 0$, and

(ii) the $(n \times n)$-matrix $\nabla^2[\bar{y}^t f(\bar{x}) + \bar{z}^t h(\bar{x})]$ is positive or negative definite,

then $(\bar{x}, \bar{\eta})$ is EP-optimal and, hence, $\bar{x}$ is P-optimal with $\bar{\eta} = \bar{q}$ as the optimal P-objective value. If the hypothesis of Theorem 3 holds then $\bar{x}$ is a global optimum for (P) and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{z}, \bar{\eta})$ is global optimum for (D).

Proof. We claim that $\bar{y}_0 > 0$. If $\bar{y}_0 = 0$, then, from (24), $\sum_{i=1}^{p} \bar{y}_i = 0$. This, along with (25), yields $\bar{y}_i = 0$ for all $i=1,2,\ldots,p$ and hence (21) gives $\nabla(\bar{z}^t h(\bar{x})) = 0$, that is, $\sum_{k=1}^{m} \nabla(\bar{z}_k h_k(\bar{x})) = 0$, which contradicts the hypothesis that for $k=1,2,\ldots,m$ the vectors $\nabla \bar{z}_k h_k(\bar{x})$ are linearly independent. Hence

$$\bar{y}_0 > 0 \tag{48}$$

Since $(\overline{x}, \overline{y}_0, \overline{y}, \overline{z}, \overline{\eta})$ is (D)-optimal, there exist $\tau \in R$, $\nu \in R^n$, $\omega \in R$, $s \in R$, $s_0 \in R$, $\delta_0 \in R$, $\delta \in R^p$ and $\mu \in R^m$ such that

$$\nabla^2 [\overline{y}^t f(\overline{x}) + \overline{z}^t h(\overline{x})]\, \nu + \omega\, \nabla(\overline{y}^t f(\overline{x})) + s\, \nabla(\overline{z}^t h(\overline{x})) = 0 \tag{49}$$

$$\tau - \omega\, y_0 = 0 \tag{50}$$

$$(\forall i = 1,2,\ldots,p) \qquad \nu^t \nabla f_i(\overline{x}) + \omega\, f_i(x) + \delta_i + s_0 = 0 \tag{51}$$

$$(\forall k = 1,2,\ldots,m) \qquad \nu^t\, \nabla\, h_k(\overline{x}) + s\, h_k(\overline{x}) + \mu_k = 0 \tag{52}$$

$$\omega \overline{\eta} + s_0 - \delta_0 = 0 \tag{53}$$

$$\nu^t\, \nabla\, [\overline{y}^t f(\overline{x}) + \overline{z}^t h(\overline{x})] = 0 \tag{54}$$

$$\omega\, (\overline{y}^t f(\overline{x}) - \overline{y}_0\, \overline{\eta}) = 0 \tag{55}$$

$$s\, \overline{z}^t\, h(\overline{x}) = 0 \tag{56}$$

$$\overline{y}^t f(\overline{x}) - \overline{y}_0\, \overline{\eta} \geq 0 \tag{57}$$

$$\overline{z}^t h(\overline{x}) \geq 0 \tag{58}$$

$$s_0\Big(\sum_{i=1}^{p} \overline{y}_i - \overline{y}_0\Big) = 0 \tag{59}$$

$$\delta_0\, \overline{y}_0 = 0 \tag{60}$$

$$(\forall i = 1,2,\ldots,p) \qquad \delta_i\, \overline{y}_i = 0 \tag{61}$$

$$(\forall k = 1,2,\ldots,m) \qquad \mu_k\, \overline{z}_k = 0 \tag{62}$$

$$\sum_{i=1}^{p} \overline{y}_i = \overline{y}_0 \tag{63}$$

$$(\tau,\ \nu,\ \omega,\ s_0,\ s,\ \delta_0,\ \delta,\ \mu) \neq 0 \tag{64}$$

$$(\tau,\ \omega,\ s,\ \delta_0,\ \delta,\ \mu) \geq 0 \tag{65}$$

$$(\overline{y}_0,\ \overline{y},\ \overline{z}) \geq 0 \tag{66}$$

(48) and (60) yield
$$\delta_0 = 0 \tag{67}$$

(53) and (67) yield
$$s_0 = -\omega\overline{\eta} \tag{68}$$

Multiplying (51) by $\overline{y}_i$ and using (61) and (68) we have

$$(\forall i = 1,2,\ldots,p) \qquad \nu^t\ \nabla(\overline{y}_i f_i(\overline{x}) + \omega(\overline{y}_i f_i(\overline{x}) - \overline{\eta}\ \overline{y}_i) = 0$$

or
$$\nu^t\ \nabla(\overline{y}^{\,t} f(\overline{x})) + \omega(\overline{y}^{\,t} f(\overline{x}) - \overline{\eta}\ \overline{y}_0) = 0 \tag{69}$$

Using (55) in (69) we obtain

$$\nu^t\ \nabla(\overline{y}^{\,t} f(\overline{x})) = 0 \tag{70}$$

Multiplying (52) by $\overline{z}_k$ for all $k = 1,2,\ldots,m$ and using (56) and (62) we have

$$\nu^t\ \nabla(\overline{z}^{\,t} h(\overline{x})) = 0 \tag{71}$$

(70) and (71) yield

$$\nu^t\ [\omega\nabla(\overline{y}^{\,t} f(\overline{x})) + s\ \nabla(\overline{x}^{\,t} h(\overline{x}))] = 0 \tag{72}$$

Premultiplying (49) by ν^t and using (72) gives

$$\nu^t\nabla^2[\overline{y}^{\,t} f(\overline{x}) + \overline{z}^{\,t} h(\overline{x})]\nu = 0 \tag{73}$$

Using the hypothesis that the matrix $\nabla^2[\bar{y}^t f(\bar{x}) + \bar{z}^t h(\bar{x})]$ is positive or negative definite in (73), we get

$$\nu = 0 \tag{74}$$

We now claim that

$$\tau > 0 \tag{75}$$

If not then $\tau = 0 \Rightarrow \omega\bar{y}_0 = 0$ from (50) $\Rightarrow \omega = 0$ since $\bar{y}_0 > 0$. This in turn, yields from (68) that $s_0 = 0$, from (51) that $\delta = 0$ and from (49) that $s \sum_{k=1}^{m} \nabla\bar{z}_k h_k(\bar{x}) = 0$, which, on using the hypothesis that the vectors $\nabla\bar{z}_k h_k(\bar{x})$ for all $k = 1,2,\dots,m$ are linearly independent, yields $s = 0$. Using $\nu = 0$, $s = 0$ in (52), we get $\mu = 0$. Therefore $\tau = 0$ yields $(\tau, \omega, s, \delta_0, \delta, \mu) = 0$, a contradiction to (64). Hence $\tau > 0$. This, in turn, with (50) and (48) results in

$$\omega > 0 \tag{76}$$

We now claim that $s \neq 0$ because, if $s = 0$ we obtain from (49) and (74) that $\omega \nabla(\bar{y}^t f(\bar{x})) = 0$ which, in conjunction with the hypothesis that $\nabla(\bar{y}^t f(\bar{x})) \neq 0$, yields $\omega = 0$, a contradiction to (76). Hence

$$s > 0\,. \tag{77}$$

(76) along with (51), (74) and (68) yields

$$(\forall i = 1,2,\ldots,p) \qquad f_i(\bar{x}) \leq \bar{\eta} \tag{78}$$

(77) along with (52) and (74) yields

$$(\forall k = 1,2,\ldots,m) \qquad h_k(\bar{x}) \leq 0 \tag{79}$$

It follows from (3), (78) and (79) that $\bar{x}$ is P-feasible with $\bar{\eta} = \bar{q}$. By Corollary 1, we get rest of the theorem.

3. APPLICATIONS

In the present section we consider an application of the results proved in the previous sections to generalized fractional programming.

Generalized Fractional Programming. A generalized fractional program as considered by Crouzeix, Ferland and Schaible [10], Jagannathan and Schaible [19], Chandra, Craven and Mond [6] and Bector, Chandra and Bector [4] is as follows:

$$\text{(GFP)} \qquad v^* = \underset{x \in S}{\text{Min}} \ \underset{1 \leq i \leq p}{\text{Max}} \ [\varphi_i(x)/\Psi_i(x)] \tag{80}$$

where,

(A1). S is as in (P),

(A2). φ_i, Ψ_i, $i = 1, 2, \ldots, p$ and h_k, $k = 1, 2, \ldots, m$ are differentiable functions on R^n,

(A3). $\Psi_i(x) > 0$, for $i = 1, 2, \ldots, p$ and $x \in S$,

(A4). If Ψ_i for $i = 1, 2, \ldots, p$, is not affine, then $\varphi_i(x) \geq 0$ for $i = 1, 2, \ldots, p$ and $x \in S$.

If we take φ_i, $-\Psi_i$ $i = 1, 2, \ldots, p$, and h_k $k = 1, 2, \ldots, m$; as convex functions and let $f_i \equiv \varphi_i/\Psi_i$, then for $y \in R^p$, $z \in R^m$ with $(y, z) \geq 0$, $\sum_{i=1}^{p} y_i f_i(x) + \sum_{i=1}^{m} z_k h_k(x)$ i.e. $y^t f(x) + z^t h(x)$ are neither convex nor concave, or neither generalized convex nor generalized concave, as assumed in Theorems 2-6. Therefore the results proved in the present paper are not directly applicable to (GFP) on the lines of (D). However, by using a result of Crouzeix, Ferland and Schaible [11] we relate (GFP) to a parametric problem on which we can apply the results proved in the present paper and thus can obtain dual problem (GFD) on the lines of (D).

Crouzeix, Ferland and Schaible [11] considered the following minmax nonlinear parametric programming problem in parameter v

$$(GFP_v) \qquad F(v) = \min_{x \in S} \ \max_{1 \leq i \leq p} \ [\varphi_i(x) - v\Psi_i(x)] \tag{81}$$

and extended a result of Jagannathan [17] and Dinkelbach [13] as follows.

Lemma 3. If (GFP) has an optimal solution $x_v{}^*$ (hereafter denoted by x^*) with optimal value of the (P)-objective as v^*, then $F(v^*) = 0$ and, conversely, if $F(v^*) = 0$, then (GFP) and (GFP_{v^*}) have the same optimal solution set.

Now if we take

$$f_i(x) = \varphi_i(x) - v\Psi_i(x) \tag{82}$$

then, on the lines of (D) we get the following dual.

(GFD) Maximize η

subject to

$$\nabla[\, y^t\varphi(u) - v\, y^t\Psi(u) + z^t h(u)] = 0$$

$$y^t\varphi(u) - v\, y^t\Psi(u) \geq y_0\eta$$

$$z^t h(u) \geq 0$$

$$\sum_{i=1}^{p} y_i = y_0$$

$\eta \in R,\ y_0 \in R,\ v \in R,\ y \in R^p,\ z \in R^m,\ (y_0, y, z) \geq 0$

We now see that, under assumptions (A1) - (A5) on the functions involved, duality theorems similar to Theorems 3-5 relating (GFP) to (GFD) exist.

4. CONCLUDING REMARKS.

In the present paper we considered a generalized minmax (maxmin) programming problem, proved the Fritz John type sufficient optimality conditions and, using Fritz John type optimality conditions, discussed duality under weaker convexity assumptions. As an application of the results duality for a generalized fractional programming problem is obtained. These results can be extended further to more general concepts of (generalized) invexity [8] , (generalized) ρ-convexity [30], and (generalised) semi-local convexity [21] of the functions involved in (P), for continuous programs and programming problems in complex and Banach spaces.

CHAPTER - 5

SECOND ORDER DUALITY FOR A QUASIBONVEX PROGRAMMING PROBLEM

The purpose of the present chapter is to introduce a second order dual problem for (P) given by (1. 1) , on the lines of Mond and Weir [58] and Bector and Bector [14], and, under generalised bonvexity assumptions, prove Weak, Direct and Strict Converse duality theorems using **Fritz John** [54] conditions. We introduced bonvex (BX) and generalized bonvex functions in Chapter 1. The motivation for naming them as (generalized) BX functions is that they are bidifferentiable and behave like (generalized) convex functions. The class of BX functions is included in the class of Pseudobonvex (PBX) and quasibonvex (QBX) functions and the class of strictly bonvex (SBX) functions is included in the class of strictly pseudobonvex (SPBX) functions; the sum of a number of BX functions is a BX function.

1. DUALITY

Primal Problem. We consider the following primal problem :

$$\text{(P)} \qquad \underset{x \in X}{\text{Minimize}} \quad f(x) \tag{1}$$

where

(i) X^0 is an open set of R^n,

(ii) $M = \{1, 2, \ldots, m\}$,

(iii) $f : X^0 \to R$ and $g : X^0 \to R^m$ are differentiable functions, and

(iv) $$X = \{x;\ x \in X^0,\ g(x) \geq 0\} \tag{2}$$

$$= \{x;\ x \in X^0,\ g_i(x) \geq 0,\ i \in M\} \tag{3}$$

Dual Problem. We now introduce two duality models (D-1) and (D-2) and prove duality theorems relating them to (P).

$$\text{(D-1)} \qquad \text{Maximize} \quad f(u) - 1/2\, p^t \nabla^2 f(u)\, p \tag{4}$$

subject to

$$\nabla[y_0 f(u) - \sum_{i \in M} y_i g_i(u)] + \nabla^2[y_0 f(u) - \sum_{i \in M} y_i g_i(u)]\, p = 0 \tag{5}$$

$$(\forall\ i \in M) \qquad y_i g_i(u) - 1/2\, p^t \nabla^2 (y_i g_i(u))\, p \leq 0 \tag{6}$$

$$(\forall\ i \in M) \qquad y_0, y_i \geq 0 \tag{7}$$

(D-2) Maximize $f(u) - 1/2\, p^t \nabla^2 f(u)\, p$ (8)

subject to

$$\nabla[y_0 f(u) - y^t g(u)] + \nabla^2[y_0 f(u) - y^t g(u)]\, p = 0 \tag{9}$$

$$y^t g(u) - 1/2\, p^t \nabla^2 (y^t g(u))\, p \leqq 0 \tag{10}$$

$$y_0 \in R, \quad y \in R^m, \qquad y_0, y \geqq 0 \tag{11}$$

Theorem 1 (Weak Duality). If, for all P-feasible solutions x and D-feasible (u, y_0, y, p) f is QBX at u with respect to X and

(a) for (D-1): $\forall\, i \in M$ and $i \neq s$, g_i is QBV but for $i = s$, g_s is SPBV, $y_s > 0$,

(b) for (D-2): $y^t g$ is SPCV at u with respect to X,

then

$$\text{Infimum (P)} \geq \text{Supremum (DE)}$$

Proof. If possible let $f(x) < f(u) - 1/2\, p^t \nabla^2 f(u)\, p$. Since f is QBX at u with respect to X, therefore, using (6) we have

$$(x - u)^t[\nabla f(u) + \nabla^2 f(u)\, p] \leqq 0$$

or using $y_0 \geqq 0$

$$(x - u)^t[\nabla(y_0 f(u) + \nabla^2(y_0 f(u))\, p] \leqq 0 \tag{18}$$

(a) Using (13), (9) and (12) we have,

$$\forall\, i \in M \qquad y_i g_i(x) \geqq 0 \geqq (y_i g(u) - 1/2\, p^t \nabla^2 (y_i g_i(u))\, p \tag{19}$$

Since for $i \in M$, $i \neq s$ g_i is QBV, therefore, $y_i g_i$ is also QBV for $i \in M$, $i \neq s$ and $y_i \geqq 0$. Hence (19) yields

$$i \in M, \ i \neq s \quad (x - u)^t[\nabla(y_i g_i(u)) + \nabla^2(y_i g_i(u))\, p] \geqq 0 \tag{20}$$

Similarly for $i = s \in M$ g is SPBV with $y_s > 0$, therefore, (19) yields

$$(x - u)^t[\nabla(y_i g_i(u)) + \nabla^2(y_i g_i(u))\, p] > 0 \tag{21}$$

(20) and (21) yield

$$(x - u)^t[\nabla(\sum_{i \in M} y_i g_i(u)) + \nabla^2(\sum_{i \in M} y_i g_i(u))\, p] > 0 \tag{22}$$

(11) and (22) yield

$$(x - u)^t[\nabla(y_0 f(u)) + \nabla^2(y_0 f(u))\, p] > 0 \tag{23}$$

(23) contradicts (18). Hence the result follows.

(b) Using (8), (17) and (16) we get

$$y^t g(x) \geqq 0 \geqq y^t g(u) - 1/2\, p^t \nabla^2(y^t g(u))\, p \tag{24}$$

Since $y^t g$ is SPCV at u with respect to X, (24) yields

$$(x - u)^t[\nabla(y^t g(u)) + \nabla^2(y^t g(u))\, p] > 0 \tag{25}$$

(15) and (25) yield (23) which contradicts (18). Hence the result.

Corollary 1. Let $\overline{x}$ be (P)-feasible and $(\overline{u}, \overline{y_0}, \overline{y}, \overline{p})$ be (D)-feasible such that $f(\overline{x}) = f(\overline{u}) - 1/2\, \overline{p}^t \nabla^2 f(u)\, \overline{p}$. Let the hypothesis of Theorem 1 hold. Then $\overline{x}$ is a global optimum for (P) and $(\overline{u}, \overline{y_0}, \overline{y}, \overline{p})$ is a global optimum for (D) with corresponding optimal objective values equal to $f(\overline{x})$ and $f(\overline{u}) - 1/2\, \overline{p}^t \nabla^2 f(u)\, \overline{p}$.

THeorem 2 (Direct Duality). Let $\bar{x}$ be a local or global optimum of (P). Then there exist $\bar{y}_0 \in R$ and $\bar{y} \in R^m$ such that $(\bar{x}, \bar{y}_0, \bar{y}, \bar{p} = 0)$ is D-feasible and the corresponding objective values of (P) and (D) are equal. If, also, the hypotheses of Theorem 1 are satisfied then $\bar{x}$ and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{p} = 0)$ are respectively global optima for (P) and (D).

Proof. Since $\bar{x}$ is a local or global optimum of (P), therefore, there exist $\bar{y}_0 \in R$, $\bar{y} \in R^m$ [54] such that

$$\bar{y}_0 \nabla f(\bar{x}) - \bar{y}_0 \nabla g(\bar{x}) = 0$$

$$\bar{y}^t g(\bar{x}) = 0$$

$$\bar{y}_0 \, , \bar{y} \geq 0$$

This implies that $(\bar{x}, \bar{y}_0, \bar{y}, \bar{p} = 0)$ is D-feasible and the corresponding values of (P) and (D) are equal. Since the hypotheses of Theorem 1 hold, therefore, by Corollary 1, $\bar{x}$ is global optimum for (P) and $(\bar{x}, \bar{y}_0, \bar{y}, \bar{p} = 0)$ is global optimum for (D).

Theorem 3 (Strict Converse Duality). Let (P) have an optimal solution and let the hypotheses of Theorem 1 hold. If $(\bar{u}, \bar{y}_0, \bar{y}, \bar{p})$ is an optimal solution of (D), f is QBX at $\bar{u}$ with respect to X, and

(a) for (D-1) g_i is QBV $\forall$ i $\in$ M and $i \neq s$, and g_s is SPBV with $\bar{y}_s > 0$ for $i = s \in M$, at $\bar{u}$ with respect to X, or

(b) for (D-2) $\bar{y}^t g$ is SPBV at $\bar{u}$ with respect to X,

then $\bar{u} = \bar{x}$, that is, $\bar{u}$ is an optimal solutions of (P).

Proof. We assume $\bar{u} \neq \bar{x}$ and exhibit a contradiction. Since $\bar{x}$ is P-optimal, Theorem 2 yields that there exist $\bar{y}_0 \in R$, $\bar{y} \in R^m$ such that $(\bar{x}, y_0, y, p = 0)$ is D-feasible and D-optimal. This implies

$$f(\bar{x}) = f(\bar{u}) - 1/2\, \bar{p}^t \nabla^2 f(\bar{u})\, \bar{p} \tag{26}$$

Since $(\bar{u}, \bar{y}_0, \bar{y}, \bar{p})$ is D-optimal, it is D-feasible also.

(a) Hence, from (11), we have

$$\nabla[\bar{y}_0 f(\bar{u}) - \sum_{i \in M} \bar{y}_i g_i(\bar{u})] + \nabla^2[\bar{y}_0 f(\bar{u}) - \sum_{i \in M} \bar{y}_i g_i(\bar{u})]\, p = 0 \tag{27}$$

Using the hypothesis that at $\bar{u}$ with respect to X, g_i is QBV $\forall\, i \in M$ and $i \neq s$, and g_s is SPBV with $\bar{y}_s > 0$ for $i = s \in M$, at $\bar{u}$ with respect to X, we obtain, as in Theorem 1,

$$(\bar{x} - \bar{u})^t[\nabla \sum_{i \in M} \bar{y}_i g_i(\bar{u})] + \nabla^2 \sum_{i \in M} \bar{y}_i g_i(\bar{u})\, \bar{p}] > 0 \tag{28}$$

(27) and (28) yield

$$(\bar{x} - \bar{u})^t[\nabla \bar{y}_0 f(\bar{u}) + \nabla^2 \bar{y}_0 f(\bar{u})\, \bar{p}] > 0 \tag{29}$$

Since f is QBX at $\bar{u}$ with respect to X, (26) yields

$$(\bar{x} - \bar{u})^t\, [\, \nabla f(\bar{u}) + \nabla^2 f(\bar{u})\, \bar{p}\,] \leq 0$$

or using $\bar{y}_0 \geq 0$

$$(\bar{x} - \bar{u})^t[\nabla \bar{y}_0 f(\bar{u}) + \nabla^2 \bar{y}_0 f(\bar{u})\, \bar{p}\,] \leq 0 \tag{30}$$

(29) and (30) contradict each other. Hence the result.

(b) Hence, from (15) we have

$$\nabla[\bar{y}_0 f(\bar{u}) - \bar{y}^t g(\bar{u})] + \nabla^2[\bar{y}_0 f(\bar{u}) - \bar{y}^t g(\bar{u})]\, \bar{p} = 0 \tag{31}$$

Using (17), (8) and (16) we have

$$\bar{y}^t g(\bar{x}) \geq 0 \geq \bar{y}^t g(\bar{u}) - 1/2\, \bar{p}^t \nabla^2 g(\bar{u}))\, \bar{p}$$

Since $\bar{y}^t g$ is SPBV at $\bar{u}$ with respect to X,

$$(\bar{x} - \bar{u})^t [\nabla \bar{y}^t g(u) + \nabla^2 \bar{y}^t g(u)\, p] > 0 \tag{32}$$

As in part **(a)**, from (32) and (27) we get that (29) and (26) give (30) which leads to a contradiction.

Remark. It may be pointed out here that in Theorems 2 and 3 we have not assumed any constraints qualifications. This is because we have used Fritz John conditions in their proofs.

2. CONCLUDING REMARKS.

(i) The Mond-Weir type second order dual program is introduced and using Fritz John conditions , Weak, Direct and Strict Converse duality theorems are proved, under generalized bonvexity (generalized boncavity) assumptions.

(ii) The results can be easily extended to higher order dual programs as is done in Mond and Weir [19] and for (generalised) binvex i.e. second order invex functions.

CHAPTER 6

DUALITY FOR A MULTIOBJECTIVE PSEUDOLINEAR PROGRAMMING PROBLEM

1. PRIMAL PROBLEM.

We now consider the following pseudolinear multiobjective program (P) as the *primal problem* .

(P) V-minimize $(f_1(x), f_2(x), \ldots, f_p(x))$ (1)

subject to

$$g_i(x) \leqq 0, \qquad (i = 1, 2, \ldots, m). \qquad (2)$$

Here,

(i) the symbol "V-minimize" stands for vector minimization, minimality being taken in terms of "efficient points" or "Pareto optimal solutions" as defined below,

(ii) $f : R^n \to R^p$ with $f = (f_1, f_2, \ldots, f_p)$ and the function f_i , $i = 1, 2, \ldots, p$, is pseudolinear with respect to the kernel function K_i ($K_i(x, y) > 0$),

(iii) $g : R^n \to R^m$ with $g = (g_1, g_2, \ldots, g_m)$ and the function g_j , $j=1, 2, \ldots, m$, is pseudolinear with respect to the kernel function G ($G(x, y) > 0$).

The purpose of the present chapter is to study Mond-Weir [58] type duality for the above multiobjective programming problems in which the constraints are pseudolinear with respect to the same proportional (kernel) function and the objective functions are pseudolinear with different proportional (kernel) functions. The above problem was considered by Bector et al [18]. The main difference between the problem (1), (2) and the problem considered by Bector et al [18] is that in Bector et al [18] the kernel functions K_i's of the corresponding functions f_i's in the primal objective are assumed to be the same (i.e. $K_i = K$) where as in the present work we assume those kernels K_i's to be different .

In the next section we shall use the following lemma which is easy to prove.

Lemma 1. Let $\mu \in R^m$, $\mu \geqq 0$ and let each function g_j for $j = 1, 2, \ldots, m$, be pseudolinear with respect to the kernel function G. Then the function $\mu^T g$ is a pseudolinear function with respect to G.

Let X denote the set of all feasible solutions of (P).

Definition. A point $\tilde{x} \in X$ is said to be an efficient (*Pareto Optimal*) point of (P) if there does not exist any $x \in X$ such that

$$f(x) \leq f(\tilde{x}). \tag{3}$$

Remark 1. For a vector maximization problem an efficient solution is defined similarly.

To eliminate the anomalies, if any, on the points in the constraint set of (P) we assume that the constraint functions of (P) satisfy the following constraint qualification given by Kanniappan [11].

Assumption 1 (*Constraint Qualification*). For (P) we assume the following Slater's type constraint qualification **(CQ)** (Kanniappan [11]) : Let $\tilde{x}$ be efficient to **(P)**. Then there exists $x^i \in X$ for each $i = 1, 2, \ldots, p$ such that $g_k(x^i) < 0$ for $k = 1, 2, \ldots, m$, and $f_j(x^i) < f_j(\tilde{x})$ for $j \neq i$.

1. OPTIMALITY CONDITIONS. For a pseudolinear programming problem Choo and Chew [6] proved the following necessary and sufficient conditions.

Theorem 1 (Necessary Optimality Conditions). Let x_* be P - efficient. Then there exist $\lambda \in R^p$ and $\mu \in R^m$ such that

$$\nabla[\lambda^t f(x^*) + \mu^t g(x_*)] = 0 \tag{4}$$

$$\mu^t g(x_*) = 0 \tag{5}$$

$$g(x_*) \leqq 0 \tag{6}$$

$$\lambda > 0 , \ \mu \geqq 0 \tag{7}$$

Theorem 2 (Sufficient Optimality Conditions). Suppose

(i) x_* satisfies (4) - (7),

(ii) the function f_i , $i = 1, 2, \ldots, p$, is pseudolinear at x_* with respect to the kernel function K_i (i.e. $K_i(x, y) > 0$), and

(iii) the function g_j , $j = 1, 2, \ldots, m$, is pseudolinear at x_* with respect to the kernel function G (i.e. $G(x, y) > 0$).

Then x_* is (P) - efficient (*Pareto Optimal*).

2. DUALITY

In the present section we introduce the Mond - Weir [58] type dual problem (D) to (P).

(D) V-maximize $(f_1(u), f_2(u), \ldots, f_p(u))$ (8)

subject to

$$\nabla[\lambda^t f(u) + \mu^t g(u)] = 0 \tag{9}$$

$$\mu^t g(u) \geqq 0 \tag{10}$$

$$\lambda > 0, \quad \mu \geqq 0 \tag{11}$$

Theorem 3 (Weak Duality) . Suppose

(i) x is P-feasible and (u, λ, μ) is D-feasible,

(ii) for all feasible solutions (x, u, λ, μ) the function f_i, $i = 1, 2, \ldots, p$ is pseudolinear with respect to the kernel function K_i ($K_i(x, y) > 0$), and

(iii) the function g_j, $j = 1, 2, \ldots, m$, is pseudolinear with respect to the kernel function G (i.e. $G(x, y) > 0$).

Then $f(x) \leq f(u)$.

Proof. If possible let $f(x) \leq f(u)$.

$$\Rightarrow \qquad f_i(x) \leqq f_i(u) \quad \text{for all } i = 1, 2, \ldots, m, \tag{12}$$

$$f_k(x) < f_k(u) \quad \text{for some } i = k. \tag{13}$$

The pseudolinearity of f_i, $i = 1, 2, \ldots, m$, (12) and (13) yield

$$(x - u)^t \nabla f_i(u) \leqq 0 \qquad \text{for all } i = 1, 2, \ldots, m \tag{14}$$

and
$$(x - u)^t \nabla f_k(u) < 0 . \qquad \text{for some } i = k \tag{15}$$

From (11), (14) and (15) we have

$$(x - u)^t \nabla(\lambda^T f(u)) < 0 . \tag{16}$$

(2), (11) and (10) give $\qquad \mu^t g(x) \leqq \mu^t g(u)$. $\qquad$ (17)

(17), hypothesis (iii) of the theorem and Lemma 2.1 give

$$(x - u)^t \nabla(\mu^t g(u)) \leqq 0 . \tag{18}$$

From (9) and (18) we obtain

$$(x - u)^T \nabla(\lambda^t f(u)) \geqq 0 . \tag{19}$$

(16) and (19) contradict each other. Hence the result.

Corollary 1. Suppose

(i) $\tilde{x}$ is P - feasible and $(\tilde{u}, \tilde{\lambda}, \tilde{\mu})$ is D - feasible, with $f(\tilde{x}) = f(\tilde{u})$ and

(ii) the hypotheses of Theorem 3 hold.

Then $\tilde{x}$ is P-efficient and $(\tilde{u}, \tilde{\lambda}, \tilde{\mu})$ is D-efficient.

Proof. If possible let $\tilde{x}$ be not P-efficient. Then there exists a P-feasible x^0 such that

$$f(x^0) \leq f(\tilde{x}) . \tag{20}$$

But
$$f(\tilde{x}) = f(\tilde{u}) . \tag{21}$$

Therefore, (20) and (21) give

$$f(x^0) \leq f(\tilde{u}) .$$

which contradicts the conclusion of Theorem 3. Hence $\tilde{x}$ is P-efficient. We can similarly prove that $(\tilde{u}, \tilde{\lambda}, \tilde{\mu})$ is D-efficient.

Theorem 4 (Direct Duality) . Suppose $\tilde{x}$ is P-efficient. Then there exist $\tilde{\lambda} \in R^p$, $\tilde{\mu} \in R^m$ ($\tilde{\lambda} > 0$, $\tilde{\mu} \geq 0$) such that $(\tilde{x}, \tilde{\lambda}, \tilde{\mu})$ is D-feasible and the corresponding objective values are equal. Furthermore, if the hypotheses of Theorem 3 are satisfied, then $(\tilde{x}, \tilde{\lambda}, \tilde{\mu})$ is D-efficient.

Proof. Since $\tilde{x}$ is P-efficient, there exist, by Theorem 1 $\lambda \in R^p$, $\tilde{\mu} \in R^m$ ($\tilde{\lambda} > 0$, $\tilde{\mu} \geq 0$) such that $(\tilde{x}, \tilde{\lambda}, \tilde{\mu})$ satisfies

$$\nabla[\tilde{\lambda}^t f(\tilde{x}) + \tilde{\mu}^t g(\tilde{x})] = 0 \tag{22}$$

$$\mu^t g(\tilde{x}) = 0 \tag{23}$$

$$g(\tilde{x}) \leq 0 \tag{24}$$

$$\lambda > 0, \tilde{\mu} \geq 0 \tag{25}$$

(22), (23) and (25) imply that $(\tilde{x}, \tilde{\lambda}, \tilde{\mu})$ is D-feasible. Equality of the objectives follows from the fact that each of them is equal to $f(\tilde{x})$. If the hypotheses of Theorem 3 are satisfied, using the equality of the two objectives and Corollary 1 we get $(\tilde{x}, \tilde{\lambda}, \tilde{\mu})$ is D-efficient.

Theorem 5 (Converse Duality). Suppose $(\tilde{u}, \tilde{\lambda}, \tilde{\mu})$ is D-efficient at which the Assumption 1 type **(CQ)** holds for dual constraints. If

(i) $\{\nabla(f_i(\tilde{u})),\ i = 1, 2, \ldots, p\}$ are linearly independent, and

(ii) the matrix $\nabla^2(\tilde{\lambda}^t f(\tilde{u}) + \tilde{\mu}^t h(\tilde{u}))$ is positive or negative definite at $(\tilde{u}, \tilde{\lambda}, \tilde{\mu})$

Then $\tilde{u}$ is feasible for (P).

If in addition,

(iii) Theorem 3 (the Weak Duality Theorem) holds,

then $\tilde{u}$ is P-efficient.

Proof. Since $(\tilde{u}, \tilde{\lambda}, \tilde{\mu})$ is D-efficient, we rewrite (D) in the form of (P) and use necessary optimality conditions of [71]. Therefore there exist $\eta \in R^n$, $\delta \in R^p_+$, $\rho \in R$, $\omega \in R^m_+$ and $\sigma \in R^p_+$, $(\delta, \rho, \omega, \sigma) \geq 0$ such that

$$\nabla \left[\sum_{i=1}^{p} \delta_i f_i(\tilde{u}) + \rho \tilde{\mu}^t g(\tilde{u}) \right] - \nabla^2 [\, \tilde{\lambda}^t f(\tilde{u}) + \tilde{\mu}^t g(\tilde{u})]\, \eta = 0 \tag{26}$$

$$\eta^t(\nabla(f_i(\tilde{u})) + \sigma_i = 0 \qquad (i = 1, 2, \ldots, p), \tag{27}$$

$$\rho g_j(\tilde{u}) - \eta^t \nabla(g_j(\tilde{u})) + \omega_j = 0 \qquad (j = 1, 2, \ldots, m), \tag{28}$$

$$\nabla[\, \tilde{\lambda}^t f(\tilde{u}) + \tilde{\mu}^t g(\tilde{u})] = 0 \tag{29}$$

$$\tilde{\mu}^t g(\tilde{u})) \geqq 0 \tag{30}$$

$$\nabla^2 [\, \tilde{\lambda}^t f(\tilde{u}) + \tilde{\mu}^t g(\tilde{u})]\, \eta = 0 \tag{31}$$

$$\eta(\tilde{\mu}^T g(\tilde{u})) = 0, \tag{32}$$

$$\sigma^T \tilde{\lambda} = 0, \quad \omega^t \tilde{\mu} = 0, \tag{33}$$

$$\tilde{\lambda} > 0, \quad \tilde{\mu} \geq 0 \tag{34}$$

Since $\tilde{\lambda} > 0$,(33) yields $\sigma = 0$ (35)

(27) and (35) give $\eta^T(\nabla(f_i(\tilde{u})) = 0$ (36)

Multiplying (28) by $\tilde{\mu}_j$ and summing over j and then using $\omega^t \tilde{\mu} = 0$ from (33) we get

$$\eta^T\Big(\nabla\, (\tilde{\mu}^t\, g(\tilde{u}))\Big) - \rho\Big(\, \tilde{\mu}^t\, g(\tilde{u})\Big) = 0, \tag{37}$$

From (37) and (32) we have, $\eta^T\Big(\nabla\, (\tilde{\mu}^t\, g(\tilde{u}))\Big) = 0$ (38)

(36) and (38) results in

$$\nabla \Big[\sum_{i=1}^{p} \delta_i f_i(\tilde{u}) + \rho \tilde{\mu}^t g(\tilde{u})\Big] = 0 \tag{39}$$

Pre-multiplying (26) by $\boldsymbol{\eta}^T$ and using (39) we get

$$\eta^t\, \nabla^2 [\, \tilde{\lambda}^t f(\tilde{u}) + \tilde{\mu}^t g(\tilde{u})]\, \eta = 0$$

which by hypothesis (ii) of the theorem implies

$$\eta = 0 \tag{40}$$

Next, in (26), we substitute from (29) for $\nabla(\tilde{\mu}^t g(\tilde{u}))$ and from (40) for η to get

$$\sum_{i=1}^{p} (\delta_i - \rho\tilde{\lambda}_i)\, \nabla(f_i(\tilde{u})) = 0$$

Since, by hypothesis (i) of the theorem, the vectors $\{\nabla(f_i(\tilde{u})),\ i = 1, 2, \ldots, p\}$ are linearly independent, the above equations give

$$\delta_i = \rho\tilde{\lambda}_i \qquad \text{for all } i = 1, 2, \ldots, p \qquad (41)$$

We now claim that $\rho > 0$. Because if $\rho = 0$ then from (41) this means that $\delta_i = 0$ and from (40) $\omega = 0$ and this along with (36), contradicts $(\delta, \rho, \omega, \sigma) \geq 0$. Thus $\rho > 0$ as claimed. This also gives $\delta_i = \lambda_i \rho > 0$ for all i. Therefore, (40) gives

$$\rho g_j(\tilde{u}) + \omega_j = 0 \qquad \text{for all } j=1,2, \ldots, m$$

i.e.

$$g_j(\tilde{u}) = -(\omega_j/\rho) \leq 0 \qquad \text{for all } j=1,2, \ldots, m.$$

Thus $\tilde{u}$ is feasible for (P).

Since the P-objective at P-feasible $\tilde{u}$ is equal to the D-objective at D-feasible $(\tilde{u}, \tilde{\lambda}, \tilde{\mu})$, the result follows by using hypothesis (iii) of the theorem and Corollary 1.

4. CONCLUDING REMARKS.

Under the the pseudolinearity assumptions on the constraint and the objective functions, various duality theorems have been proved for Mond-Weir [58] type duality for a multiobjective programming problem. These results can probably be further generalized under appropriate assumptions of more generalized convexity and will be the topics of further research.

REFERENCES

1. Ashton, D. J. and Atkins, D. R., "Multicriteria Programming for Financial Planning", Journal of the Operational Research Society . 30 (1979) pp.259-270
2. Avriel, M., Diewert, W.E., Schaible, S., and Ziemba, W.T.,"Introduction to Concave and Generalized Concave Functions", in, Generalized Concavity in Optimization and Economics, (Ed.) S. Schaible and W.T. Ziemba, Academic Press (1981) pp.21-50.
3. Barrodale, I., "Best rational approximation and strict quasiconvexity", SIAM Journal of Numerical Analysis 10 (1973) pp. 8-12
4. Bazaraa, M. and Shetty, C. M., Nonlinear Programming: Theory and Algorithms, John Wiley and Sons, New York (1979)
5. BECTOR, C.R., Mathematical Analysis of some Nonlinear Programming Problem, Ph.D. Thesis, Department of Mathematics, I.I.T. Kanpur (India) (1968).
6. BECTOR , C.R., "Duality in Linear Fractional Programming", Utilitas Mathematica, 4 (1973) 155-168.
7. Bector, C.R., "Duality in Nonlinear Fractional Programming", Zeitschrift für Operations Research 17, pp. 183-193 (1973).
8. Bector, C. R., "Weaker Convex Programming", VIII Int. Symp. on Mathematical Programming, Stanford Univ. August 27-31 (1973) pp. 10-11 (Abstract)

9. Bector, C.R. and Grover, T.R.,"On a Sufficient Optimality Theorem of Mangasarian in Nonlinear Programming",Cahiers du Centre d' Etudes de Recherche Operationnelle, Vol. ·16, No. 1, 1974,

10. Bector, C.R., Bector, Meena K. and Klassen, J.E., "Duality for a Nonlinear Programming Problem", Utilitas Mathematica, Vol. 11, pp. 87-99, 1977.

11. Bector, C.R. and Gulati, T.R.,"Sufficient Optimality Conditions in Presence of Quasi Convex Equality Constraints", Mathematische Operationsforschung und Statistik Series Optimization, Vol. 8, No.2, pp. 181-184, 1977.

12. Bector, C.R., Bector, M.K., and Klassen J.E., "Duality for a Nonlinear Programming Problem", Utilitas Mathematica 11, pp. 87-99 (1977).

13. Bector, C. R., and Chandra, S., "(Generalized)-Bonvex Functions and Second Order Duality in Mathematical Programming", Res. Rep. 85-2, Dept. of Acturial and Management Sciences, Jan. (1985).

14. Bector, C. R. and Bector, Meena K., "On Various Duality Theorems for Second Order Duality in Nonlinear Programming", Cahiers du C.E.R.O. , Volume 28, NO. 4, 1986.

15. Bector, C.R. and Bector, Meena K.,"On Various Duality Theorems in Nonlinear Programming", Journal of Optimization Theory and Applications, 53 (1987), 509-515.

16. Bector, C.R., Chandra, S. and Meena K. Bector,"Sufficient Optimality Conditions and Duality for a Quasi Convex Programming Problem", Journal of Optimization Theory and Applications, 59(1988), 209-221.

17. Bector, C.R., Chandra, S. and Meena K. Bector,"Generalized Fractional Programming Duality : A Parametric Approach", Jour. of Opt. Theory and Appl., 60 (1989), 245-262.

18. Bector, C.R., Chandra, S. and Durgaprasad, M.V.,"Duality in Pseudo-linear Multiobjective Programming", Asia-Pacific J. Operl. Res. Soc., 5 (1988), 150-159.

19. Bector, C. R. and Suneja," Duality in Nonldifferentiable Generalized Fractional Programming", Asia-Pacific Journal of Operational Research, 5 (1988), pp. 134-139.

20. Bector, C. R. and Singh, C.,"Beevex Functions", Research Report No. 2, Dept. of Maths., St. Lawrence University, Canton, New York (1988).

21. Ben Tal, A. and Ben Israel, A., "F--Convex Functions; Properties and Applications", in, Generalized Concavity in Optimization and Economics, (Ed.) S. Schaible and W. T. Ziemba, Academic Press (1981) pp. 301-334.

22. Bhatt, S.K. and Misra, S.K.,"Sufficient Optimality Criteria in Nonlinear Programming in the Presence of Convex Equality and Inequality Constraints", Zeitschrift für Operations Research, Ser. A, Vol. 19, No. 3, pp. 101-105, 1975.

23. Bitran, G.R.,"Duality for Nonlinear Multicriterion Optimization Problems", Journal of Optimization Theory and Applications, 35 (1981) 367-401.

24. Brumelle, S., "Duality for Multiobjective Convex Programs", Mathematics of Operations Research, 6 (1981) 159-172.

25. Chandra, S., Craven, B.D. and Mond, B., "Generalized Fractional Programming Duality: A Ratio Game Approach", Research Report No. 23, Department of Mathematics, University of Melbourne, Australia (1983).

26. Charnes, A. and Cooper, W. W. , "Goal Programming and Multi-objective Optimization (Part I)", European Journal of Operational Research 1 (1977) pp. 39-54.

27. Charnes, A. and Cox, L. and Lane, M. , "A Note on the Redesigning of a Rate Structure for Allocation of State Funds to Educational Institutions", Working Paper 70-49 of Project GUM, University of Texas at Austin (Austin, Texas, 1970).

28. Chew, K.L. AND Choo, E.U., "Pseudolinearity and Efficiency", Mathematical Programming, 28 (1984) 226-239.

29. Craven, B.D.; Mathematical Programming and Control Theory, Chapman and Hall, London 1978.

30. Craven, B.D., "Strong Vector Minimization and Duality", Zeitschrift fur Angewandte Mathematik und Mechanik, 60 (1980) 1-5.

31. Craven, B.D., "Invex Functions and Constrained Local Minima", Bulletin of Australian Mathematical Society 24, pp. 357-366 (1981).

32. Crouzeix, J.P., "A Duality Framework in Quasi-convex Programming", in Schaible, S. and Ziemba, W.T. (eds.). Generalized Concavity in Optimization and Economics, Academic Press, New York, pp. 207-225 (1981).

33. Crouzeix, J.P., Ferland, J.A. and Schaible, S., "Duality in Generalized Fractional Programming", Mathematical Programming 27, pp.342-354 (1983).

34. Crouzeix, J.P., Ferland, J.A. and Schaible, S., "An Algorithm for Generalized Fractioal Programs", Journal of Optimization Theory and Applications 47 (1985).

35. Crouzeix, J.P., Ferland, J.A. and Schaible, S.,"A Note on'An Algorithm for Generalized Fractional Programs'", Journal of Optimization Theory and Applications 47, (1985).

36. Dinkelbach, W. (1967), "On Nonlinear Fractional Programming", Management Science 13, pp. 492-498.

37. Ferland, J.A. and Potvin, Jean-Yves, "Generalized Fractional Programming: Algorithms and Numerical Experimentation", European Journal of Operational Reseaarach 20, pp. 92-101 (1985).

38. Geoffrion A.M., "Proper Efficiency and the Theory of Vector Maximization", Journal of Mathematical Analysis and Applications 22 (1968) 618-630.

39. Gol'stein, E.G.; Theory of Convex Programming; Translation of Mathematical Monographs 36, American Mathematical Society, Providence, Rhode Island, U.S.A. 1972.

40. Hanson, M.A., "On Sufficiency of Kuhn-Tucker Conditions", Journal of Mathematical Analysis and Applications 80, pp. 12.

41. Hanson, M. A., "Bounds for Functionally Convex Optimal Control Problems", Jour. Math. Anal. Appl. 8 (1964) pp. 84-89.

42. Ivanov, E.H. and Neshe, R., "Some Results on Dual Vector Optimization Problems", Math. Operations Forschung Statisik, Ser. Optimization, 16 (1985) 505-517.

43. Jagannathan, R., "On Some Properties of Programming Problems in Parametric Form Pertaining to Fractional Programming", Management Sceince 12, pp. 609-615 (1966).

44. Jagannathan, R., "Duality for Nonlinear Fractional Programs", Zeitschrift für Operations Research 17, pp. 1-3(1973).

45. Jagannathan, R. and Schaible, S., "Duality in Generalized Fractional Programming via Farkas' Lemma, Journal of Optimization Theory and Applications, 41, pp. 417-424 (1983).

46. Kanniappan, P., "Necessary Conditions for Optimality of Nondifferentiable Convex Multiobjective Programming", Journal of Optimization Theory and Applications, 40(1983) 167 - 174.

47. Kaul, R.N. and Kaur, Surjeet,"Generalizations of Convex and Related Functions", European Journal of Operational Research 9, 269-377 (1982).

48. Kornbluth, J.S.H. , "A Survey of Goal Programming", OMEGA 1 (1973) pp. 193-205

49. Kawasaki, H.,"A Duality Theorem in Multiobjective Nonlinear Programming", Mathematics of Operations Research, 7 (1982) 95-110.

50. Kuhn, H.W. and Tucker A.W., "Nonlinear Programming", in J. Neyman (ed.), Proceedings of the Second Berkeley Symposium in Mathematical Statatistics. and Probability, University of California Press, Berkeley, California, pp. 487-492,1951.

51. Lai, H.C. and Ho, C.P., " Duality Theorem of Nondifferentiable Convex Multiobjective Programming", Journal of Optimization Theory and Applications, 50(1986) 407 - 419.

52. Mahajan, D. G. and Vartak, M. N.,"Generalization of Some Duality Theorems in Nonlinear Programming",Math. Prog. 12 (1977) pp. 293-317.

53. Mangasarian, O.L. and Fromovitz, S.,"The Fritz John Necessary Optimality Conditions in the Presence of Equality and Inequality Constaints", Journal of Mathematical Analysis and Applications, Vol. 17, pp. 37-47, 1967.

54. Mangasarian, O.L., Nonlinear Programming, McGraw-Hill, New York, 1969.

55. Mangasarian, O. L., "Second and Higher-Order Duality in Nonlinear Programming", Jour. Math. Anal. Appl. 51 (1975) pp. 607-620.

56 Martos, B., "Quasi-Convexity and Quasi-Monotonicity in Nonlinear Programming", Studia Scientarium Mathematics Hungarica 2, (1967), pp. 265-273.

57. Mond, B. "Second Order Duality in Nonlinear Programming", Opsearch 11, (1974), pp. 90-99.

58. Mond, B. and Weir, T.,"Generalized Concavity and Duality", Generalized Concavity in Optimization and Economics, edited by S. Schaible and W.T. Ziemba, Academic Press, New York, pp. 263-280, 1981.

59. Mond, B. and Weir, T.,"Generalized Convexity and Higher Order Duality", Pure Mathematics Research Report No. 81-16 (1981), Department of Mathematics, La Trobe University, Australia

60. Passy, U. and Keslassy, A., "Pseudo Duality and Duality for Explicitly Quasiconvex Functions", Mimeograph Series No. 249. Faculty of Industrial Engineering and Management, Technion (Haifa, Israel, 1979)

61. Ponstein, J.,"Seven Kinds of Convexity", SIAM Review, Vol. 9, (1967) pp. 115-119,.

62. Rubinshtein, G. S., "Duality in Mathematical Programming and Some Problems of Convex Analysis", (Englich translation) Russian Mathematical Surveys 25 (1970) pp. 171-200

63. Schaible, S., "Parameter-free Convex Equivalent and Dual Programs of Fractional Programming Problems", Zeitschrift für Operations Research 18, pp. 187-196 (1974).

64. Schaible, S. "Fractional Programming I. Duality", Management Science, 22, pp. 858-867 (1976).

65. Shaible, S., Analyse und Anwendungen von Quotiontenprogrammen (Hain-Verlag, Meisenheim, 1978)

66. Schaible, S., "A Survey of Fractional Programming", in Schaible, S. and Ziemba, W.T. (eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, pp. 417-440,1981.

67. Schonfeld, P., "Some Duality Theorems for the Nonlinear Vector Maximum Problem", Unternehmensforschung 14 (1970) 51-63.

68. Schroeder, R.G., "Linear Programming Solutions to Ratio Games", Operations Research 18, pp. 300-305 (1970).

69. Singh, C., "Sufficient Optimality Criteria in Nonlinear Programming for Generalized Equality-Inequality Constraints", Journal of Optimization Theory and Applications, Vol. 22, No. 4, pp. 631-635, 1977.

70. Singh, C., "A Class of Multiple-Criteria Fractional Programming Problems", Journal of Mathematical Analysis and Applications, 115 (1986) 202-212.

71. Singh, C., "Optimality Conditions in Multiobjective Differentiable Programming", Journal of Optimization Theory and Applications, 53 (1987), 115-123.

72. Singh, C., "Duality Theory in Multiobjective Differentiable Programming," Journal of Information and Optimization Sciences, 9 (1988), 231-240.

73. Skarpness, B. and Sposito, V.A.,"A Modified Fritz John Optimality Criterion", Journal of Optimization Theory and Applications, Vol. 31, No. 1, pp. 113-115, 1980.

74. Tanino, T. and Sawaragi, Y., "Duality in Multiobjective Programming", Journal of Optimization Theory and Applications, 27 (1979) 509-529.

75. Vial, J.P.,"Strong and Weak Convexity of Sets and Functions", Mathematics of Operations Research 8, pp. 231-259 (1983).

76. von Neumann, J., "A Model of General Economic Equilibrium", Review of Economic Studies 13 (1945) pp. 1-9.

77. Vogt, R., "A Corporate Strategy for Realizing Equal Employment Opportunity", Behavioral and Social Accounting, to appear.

78. Weir, T., "Proper Efficiency and Duality for Vector Valued Optimization Problems", Department of Mathematics, Royal Military College, Duntroon (Australia) Research Report No. 12 (1985).

78. Weir, T., "A Dual for a Multiple Objective Programming Problem," Journal of Information and Optimization Sciences, 7 (1986), 261-269.

80. Wolfe, P., " A Duality Theorem for Nonlinear Programming", Quart. of Appl. Math., 19 (1961), pp. 239-244.

81. Zang, I., " Cocavifiability of C^2-Functions : A Unified Exposition", Generalized Concavity in Optimization and Economics, edited by S. Schaible and W.T. Ziemba, Academic Press, New York, pp. 131-152, 1981.

82. Zangwill, W. I., Nonlinear Programming : A Unified Approach, Prentice-Hall, Inc., Englewood Cliffs, N. J. (U.S.A.)