Quasi-Static Universal Motions of Homogeneous Monotropic Elastic Rods

By
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A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
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Abstract

This thesis deals with solutions to the dynamic field equations for a director theory of rods. In order to make the analysis feasible, only quasi static motion is considered. Further simplification is obtained by using monotropic symmetry in the constitutive relations. The universal solutions to six different kinds of deformation problems for monotropic elastic rods are given and the monotropic symmetry axis for the different cases are presented. The results are based on homogeneous normal deformations, which excludes transverse shear deformation, and are confined to uniform rods, whose configurations are always straight, circular or helical.

Acknowledgments

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Dedication

To my dearest Parents
Zhong-heng Guo and Dieli Zhang

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CHAPTER 1

Introduction

Rods are the building blocks of structural engineering; they include such structural elements as beams, columns, arches and springs. Rods are slender bodies which are small in two dimensions as compared to the third. The long dimension defines the *axis* of the rod, while the two small dimensions define its *cross-section*; the latter is transverse to the axis. In modern continuum mechanics, rods are modelled as a one-dimensional *directed continuum*. This consists of a mathematical curve which models the rod axis and a field of vectors on the curve, called *directors*, to model the rod cross-section.

Cohen [1] has recently given a number of equilibrium solutions for rods which are nonlinear elastic. These are *universal* in the sense that they hold for all materials in a given class for boundary loads alone. While similar solutions had been given earlier by Whitman & DeSilva [9] and Antman [10] for *isotropic* rods, Cohen's solutions hold for much weaker form of material symmetry, one which he calls *monotropic*. The purpose of this thesis is to extend Cohen's work from statics to dynamics.

A rod *configuration* is given by specifying the vector functions giving the rod axis and its directors as a function of a coordinate along the axis of the rod, say its arc length in some reference configuration. We define the *state* of the rod as the *local configuration* at a particular time, i.e., the configuration at a point of the rod. A rod motion is a time dependent rod configuration. Whereas Cohen's analysis depends upon one independent variable, which is the position coordinate; here the analysis depends upon two independent variables, which are the position coordinate and the time.

To make the analysis feasible, the concept of a *quasi-static* motion is introduced, it being defined as one for which the rod accelerations are zero. And this concept will play a fundamental role in arriving at solutions for *monotropic elastic rods which are universal quasi-static motions.*

Ericksen [8] introduced a *uniform state* as one in which the state at any point of the rod differs from that at any other point by a rotation. He also categorized the set of *uniform* rods as those consisting of rods with straight, circular and helical axes, in which the cross-sections maintain a constant orientation relative to the Frenet-Serret frame.

The axis of the rod is defined by the position vector $\mathbf{r} = \mathbf{r}(S)$ of the curve which models it; here S is a material coordinate along the axis. The additional structure of the rod is described by a non-coplanar set of directors $\mathbf{d}_a = \widetilde{\mathbf{d}_a}(S)$, $a = 1, 2, 3$. The first two directors span the planar cross-section of the rod, while $\mathbf{d}_3$ is a parametric director, left to be identified in the modelling process with some geometric-physical effect: we choose $\mathbf{d}_3$ as the natural tangent $d\mathbf{r}/dS$ to the rod axis. A rod configuration is defined as *normal* if and only if it satisfies the normality condition $\mathbf{d}_a \cdot \mathbf{d}_3 = 0$, $a = 1, 2$, at all points of the rod. The rod may be viewed as built out of planar cross-sections stacked to always be normal to the axis. In this thesis, we only consider the *normal uniform* rod configurations.

We present a system of dynamic field equations for a rod as a consequence of the laws of *linear momentum balance* and *tensor momentum balance.* These two field equations alone are not enough to determine the motion of the rod and the associated stress tensor. To make the problem determinate, we need to introduce constitutive equations. Constitutive equations for any body must satisfy certain invariance principles: they are the *principle of material frame-indifference and the principle of material symmetry.*

The principle of material frame-indifference states that the rod constitutive response is independent of how the rod is positioned in space. Mathematically this means the constitutive equations are invariant under rigid motion of the body. The principle of material symmetry states that the material properties are independent of how the material is positioned in the rod. Cohen [1] introduced the *principle of monotropic symmetry* as a type of material symmetry.

Cohen [1] gave a detailed discussion about the rod deformations which take a straight, homogeneous, monotropic elastic rod into one of four deformations: straight, straight twisted, circular and helical. The universal solutions to these four different kinds of rod deformations were found by solving the equilibrium field and constitutive equations for the statics of rods. Finally, the further simplifications of the universal solutions were found by employing the assumed monotropic symmetry.

The aim of this thesis is to extend Cohen[1]'s work from statics to dynamics, i.e., to find the universal solutions to the dynamics of normal, uniform rods. It is found that generally these will consist of quasi static or accelerationless motion.

An outline of the thesis is as follows. In chapter 2, the dynamic field equations and constitutive equations for rods are presented along with a discussion of their physical interpretation. Universal solution and quasi static motion are defined. Chapter 3 provides a detailed analysis of the consequence of monotropic symmetry for elastic rods. In Chapter 4, the universal solutions for motions from straight to straight and straight twisted configurations are derived. The universal solutions for motions from circular and straight to circular configurations are obtained in Chapter 5. In Chapter 6, we carry out the universal solutions for deformations from helical and straight to helical configurations. In the last chapter, we summarize the results of the previous chapters.

CHAPTER 2

Basic equations

The basic equations, to be adopted here for the mechanical theory of rods, are those which correspond to a direct theory. A *direct theory* of rods is the one developed for a slender material body from a model that consists of a one-dimension base continuum which carries some distribution of mathematical structure to represent the effect of the cross-section [2-8]. Here the rod is modelled by a curve with three directors: two directors are used to model the cross-section, the third director is along the rod axis and constrained to deform with it.

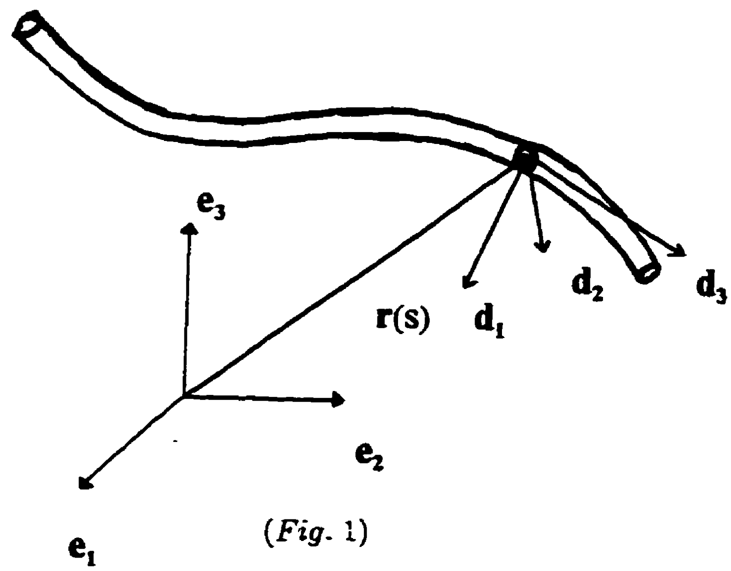
The *current configuration* c_d is specified by vector functions

$$\mathbf{r} = \mathbf{r}\,(S)\,,\ \mathbf{d}_a = \mathbf{d}_a\,(S)\,, \qquad a = 1, 2, 3, \tag{2.1}$$

(2.1) specifies the position vector of rod axis $\mathbf{r}$ and the directors $\mathbf{d}_a$ of any point on c_d are defined by the material coordinate S. In (2.1), $\mathbf{d}_a$, $a = 1, 2, 3$, are a non-coplanar set of directors, the first two of which span the cross-section of the rod, while $\mathbf{d}_3$ may be identified with the placement of the rod axis through the constraint condition

$$\mathbf{d}_3 = \frac{d\mathbf{r}}{dS} = \mathbf{r}', \tag{2.2}$$

where prime denotes differentiation with respect to S. Thus, $\mathbf{d}_3$ is the natural unit tangent to the rod axis.



$(Fig.\ 1)$

It is also necessary to define the *reference configuration* C_D by

$$\mathbf{r}_R = \mathbf{r}_R(S), \qquad \mathbf{d}_{Ra} = \mathbf{d}_{Ra}(S), \qquad a = 1, 2, 3, \tag{2.3}$$

where S is the arc length along C_D and is taken to be a material coordinate in the analysis. It is most convenient to take the reference configuration as the undeformed configuration; then (2.1) corresponds to the deformed configuration. It is usual to choose S to be arc length in the reference configuration and to introduce arc length s in the deformed configuration, but in this thesis, S is regarded as a convected coordinate in the deformed configuration.

In $(Fig.\ 1)$, $\mathbf{e}_a$ is a fixed orthonormal basis of the physical space of vectors and $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$, $\mathbf{e}_a \equiv \mathbf{e}^a$.

The model assumes no transverse shear deformation, which means

$$\mathbf{d}_a \cdot \mathbf{d}_3 = 0, \ a = 1, 2. \tag{2.4}$$

and no cross-sectional shearing, which is given by

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = 0. \tag{2.5}$$

In order to simplify the problem, we always choose

$$\mathbf{d}_{Ra} = \mathbf{e}_a, \; a = 1, 2, 3. \tag{2.6}$$

A motion or time dependent deformation of C_D into c_d is then defined by a smooth mapping

$$s = s(S, t), \tag{2.7}$$

where the *stretch* $\lambda(t) = \partial s / \partial S$ is assumed to be bounded and positive, so that (2.7) has a smooth inverse $S = S(s)$ for each fixed t.

The deformation tensor $\mathbf{F}$ is defined as

$$\mathbf{F} := \mathbf{D}\mathbf{D}_R^{-1} = \mathbf{d}_a \otimes \mathbf{d}_R^a \tag{2.8}$$

where $\mathbf{D}$, the director tensor, characterizes the distribution of directors along the rod axis and is defined as

$$\mathbf{D} := \mathbf{d}_a \otimes \mathbf{e}^a. \tag{2.9}$$

The differential geometry of the rod is characterized by the wryness tensor $\mathbf{W}$ defined as

$$\mathbf{W} := \mathbf{d}'_a \otimes \mathbf{d}^a \tag{2.10}$$

where $\mathbf{d}^a$ is the basis reciprocal to $\mathbf{d}_a$, i.e., $\mathbf{d}_a \cdot \mathbf{d}^b = \delta_a^b$ and Prime denotes differentiation with respect to S.

Note that we have

$$\mathbf{W} = \mathbf{D}'\mathbf{D}^{-1}, \tag{2.11}$$

which implies $\mathbf{W}$ is determined by $\mathbf{D}$. Conversely, $\mathbf{D}$ is determined by $\mathbf{W}$.

In particular, the polar decomposition (Jaunzemis [11]) is available to decompose the deformation tensor $\mathbf{F}$, viz,

$$\mathbf{F} = \mathbf{RU},\ \mathbf{U} = \mathbf{U}^{\mathsf{T}},\ \mathbf{R} = \mathbf{R}^{\mathsf{T}}, \tag{2.12}$$

Where superscript T denotes the transpose operation on a tensor.

This formula describes the decomposition of a general deformation into a rotation, specified by the orthogonal *rotation tensor* $\mathbf{R}$, and a pure deformation specified by the positive, symmetric right *stretch tensor* $\mathbf{U}$.

The dynamic field equations for rods, given by Capriz [12], are

$$\mathbf{n}' + \rho\mathbf{f} = \rho\,\ddot{\mathbf{r}} \tag{2.13}$$

and

$$\mathbf{M}' - \mathbf{N} + (\mathbf{r}' \otimes \mathbf{n})^{\mathsf{T}} + \rho\mathbf{B} = \dot{\mathbf{H}} - \mathbf{H}\mathbf{L}^{\mathsf{T}}, \tag{2.14}$$

with

$$Sk\ \mathbf{N} = \mathbf{0}. \tag{2.15}$$

dot denotes differentiation with respect to time t; the superscript T denotes the transpose of a tensor. Equations (2.13) , (2.14) are statements of the laws of linear momentum balance and tensor momentum balance, respectively. Equation (2.15) is commonly regarded as a "constitutive restriction", it is also equivalent to the angular momentum balance which is obtained from skew part of equation (2.14) . The vector $\mathbf{n}$ is the axial force, the tensor $\mathbf{M}$ is the cross-sectional tensor moment, $\mathbf{N}$ is the internal force-moment tensor, and $\mathbf{f}$ and $\mathbf{B}$ represent body force distributions. $\mathbf{H}$ is the moment of momentum and $\mathbf{L}$ is the velocity gradient, where

$$\mathbf{L} = \dot{\mathbf{F}}\,\mathbf{F}^{-1}, \mathbf{H} = \mathbf{LE}. \tag{2.16}$$

The cross-sectional Euler tensor $\mathbf{E}$ is relative to the current configuration, and can be written as

$$\mathbf{E} = \delta^{ab}\mathbf{d}_a \otimes \mathbf{d}_b. \tag{2.17}$$

(2.17) is not an invariant representation under deformation, since the directors will, in general, stretch and shear, thus deforming off the principal Euler axes. The law of transformation is given by

$$\mathbf{E} = \mathbf{F}\mathbf{E}_R\mathbf{F}^{\mathsf{T}}, \tag{2.18}$$

where $\mathbf{E}_R = \delta^{ab}\mathbf{d}_{Ra}\otimes\mathbf{d}_{Rb}$, the cross-sectional Euler tensor relative to the reference configuration, is constant.

Then $\mathbf{H}$ can be simplified as

$$\mathbf{H} = \mathbf{L}\mathbf{E} = \left(\dot{\mathbf{F}}\,\mathbf{F}^{-1}\right)\left(\mathbf{F}\mathbf{E}_R\mathbf{F}^{\mathsf{T}}\right) = \dot{\mathbf{F}}\,(\mathbf{F}^{-1}\mathbf{F})\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} = \dot{\mathbf{F}}\,\mathbf{I}\mathbf{E}_R\mathbf{F}^{\mathsf{T}} = \dot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}. \tag{2.19}$$

Differentiating $\mathbf{H}$ with respect to time t, we have

$$\dot{\mathbf{H}} = \frac{d}{dt}\left(\dot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}\right) = \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \dot{\mathbf{F}}\dot{\mathbf{E}}_R\,\mathbf{F}^{\mathsf{T}} + \dot{\mathbf{F}}\,\mathbf{E}_R\,\dot{\mathbf{F}}^{\mathsf{T}}. \tag{2.20}$$

Since $\mathbf{E}_R$ is constant in the reference configuration, $\dot{\mathbf{E}}_R = \mathbf{0}$. Substituting $\dot{\mathbf{E}}_R = \mathbf{0}$ into equation (2.20), leads to

$$\dot{\mathbf{H}} = \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \dot{\mathbf{F}}\,\mathbf{E}_R\,\dot{\mathbf{F}}^{\mathsf{T}}. \tag{2.21}$$

Therefore,

$$\begin{aligned}
\dot{\mathbf{H}} - \mathbf{H}\mathbf{L}^{\mathsf{T}} &= \left(\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \dot{\mathbf{F}}\,\mathbf{E}_R\,\dot{\mathbf{F}}^{\mathsf{T}}\right) - \left(\dot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}\right)\left(\dot{\mathbf{F}}\,\mathbf{F}^{-1}\right)^{\mathsf{T}} \qquad (2.22)\\
&= \left(\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \dot{\mathbf{F}}\,\mathbf{E}_R\,\dot{\mathbf{F}}^{\mathsf{T}}\right) - \left(\dot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}\right)\left(\mathbf{F}^{-\mathsf{T}}\,\dot{\mathbf{F}}^{\mathsf{T}}\right)\\
&= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \dot{\mathbf{F}}\,\mathbf{E}_R\,\dot{\mathbf{F}}^{\mathsf{T}} - \dot{\mathbf{F}}\,\mathbf{E}_R\,\dot{\mathbf{F}}^{\mathsf{T}}\\
&= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}.
\end{aligned}$$

Equation (2.14) can now be rewritten as

$$\mathbf{M}' - \mathbf{N} + (\mathbf{r}' \otimes \mathbf{n})^{\mathsf{T}} + \mathbf{T} = \ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} \tag{2.23}$$

where we assume that $\rho\mathbf{B} = \mathbf{T}$, a cross-section tensor, represents the tensor moment of the traction distribution on the edges of the rod. Moreover, we assume the torque resultants of this distribution vanish, (cf. Cohen [1]) so that

$$Sk\ \mathbf{T} = \mathbf{0}. \tag{2.24}$$

In chapters 4, 5 and 6, we will derive the universal solutions to six different kinds of deformation problems by solving their dynamic field equations. For three-dimensional elastostatics, a universal solution is one for which a prescribed deformation may be maintained by *surface tractions alone* for all materials of a prescribed constitutive class. For rods, a *universal solution* will be one that can be maintained by *end loads, loads on the lateral surfaces* and, in some cases, by *special types of body forces.*

In order to make the analysis feasible, we restrict attention to the case of quasi static motion throughout the last three chapters. This implies the acceleration is zero even though the rod is deforming. In another words, we have

$$\mathbf{a} = \ddot{\mathbf{r}} = \mathbf{0}. \tag{2.25}$$

For studying rods it proves very convenient to define a notation for tensor pairs. Hence, relevant to the geometry and deformation, we define the local state $\mathbb{S}$ and the generalized deformation $\mathbb{F}$ by

$$\mathbb{S} := (\mathbf{D}, \mathbf{D}'), \mathbb{F} := (\mathbf{F}, \mathbf{F}'). \tag{2.26}$$

Then

$$\mathbb{S} = \mathbb{F}\mathbb{S}_{\mathbf{R}} \tag{2.27}$$

if the law of composition of these tensor pairs is defined by

$$(\mathbf{D}, \mathbf{D}') = (\mathbf{F}, \mathbf{F}') \, (\mathbf{D_R}, \mathbf{D_R'}) = (\mathbf{F}\mathbf{D}_R, \;\mathbf{F}'\mathbf{D}_R + \mathbf{F}\mathbf{D}'_R) \, . \tag{2.28}$$

Relevant to the stress, like tensors, we define the generalized stress $\mathbb{T}$ by

$$\mathbb{T} := (\mathbf{N}, \mathbf{M}) \tag{2.29}$$

where $\mathbf{N}$ is the internal force-moment tensor force and $\mathbf{M}$ is the cross-sectional tensor moment.

To complete the formulation of the basic equations requires that constitutive relations for rods be specified. A elastic rod is 'defined by the constitutive equation

$$\mathbb{T} := \tilde{\mathbb{T}}\,(\mathbb{F}; S) \, , \tag{2.30}$$

where S represents the material coordinate in the analysis.

Finally, we need to discuss the concept of *homogeneity. The rod is said to be in a homogeneous configuration if and only if*

$$\mathbf{W} = \mathbf{0}. \tag{2.31}$$

This is a geometric condition. The physical notion of a *homogeneous rod* is defined as one with a homogeneous reference configuration in which the *response, density and Euler tensor* are all independent of position. Clearly, in this situation

$$\mathbb{T} := \tilde{\mathbb{T}}\,(\mathbb{F}) \, . \tag{2.32}$$

CHAPTER 3

Monotropic Symmetry

The constitutive equation can be put into a reduced and simplified form by employing the principles of material frame-indifference and material symmetry.

Applying the principle of material frame-indifference to the constitutive law (2.32), we have

$$\tilde{\mathbb{T}}\left(\mathbb{Q}\mathbb{F}\right) = \mathbb{Q}\tilde{\mathbb{T}}\left(\mathbb{F}\right)\mathbb{Q}^{\mathsf{T}} \tag{3.1}$$

for all $\mathbb{F}$ and $\mathbb{Q}$, where

$$\mathbb{F} = \left(\mathbf{F}, \mathbf{F}'\right), \mathbb{Q} := \left(\mathbf{Q}, \mathbf{Q}\right). \tag{3.2}$$

$\mathbf{F}$ is the deformation tensor and $\mathbf{Q}$ is an arbitrary rotation of the current (deformed) rod configuration. Operations between tensor pairs are defined by

$$\mathbb{A}\mathbb{B} = \left(\mathbf{A}_1, \mathbf{A}_2\right)\left(\mathbf{B}_1, \mathbf{B}_2\right) := \left(\mathbf{A}_1\mathbf{B}_1, \mathbf{A}_2\mathbf{B}_2\right), \tag{3.3}$$

where

$$\mathbb{A} = \left(\mathbf{A}_1, \mathbf{A}_2\right), \mathbb{B} = \left(\mathbf{B}_1, \mathbf{B}_2\right). \tag{3.4}$$

We also have

$$\mathbb{A}^{\mathsf{T}} := (\mathbf{A}_1^{\mathsf{T}}, \mathbf{A}_2^{\mathsf{T}}). \tag{3.5}$$

The notation $\mathbb{A}^{\pm}$ is defined as

$$\mathbb{A}^{\pm} := \left(\mathbf{A}_1, \pm\mathbf{A}_2\right). \tag{3.6}$$

We introduce $\mathbb{G}$ as a tensor pair of the elements of the special linear group, i.e., $\det \mathbf{G} = 1$, and $\mathbb{G} := \left(\mathbf{G}, \mathbf{G}\right)$. Define the material symmetry set $\mathcal{G}_R^+$ to consist of the set of elements of the type $\mathbb{G}^+ = \mathbb{G}$ and the material anti-symmetry set $\mathcal{G}_R^-$ to consist of some set of elements of type $\mathbb{G}^-$. The principle of combined material symmetry requires the constitutive relation (2.24) to satisfy the transformation rule

$$\widetilde{\mathbb{T}}(\mathbb{F}) = \widetilde{\mathbb{T}}^{\pm}(\mathbb{F}\mathbb{G}^{\pm}), \tag{3.7}$$

for all admissible $\mathbb{F}$ and $\mathbb{G}^{\pm} \in \mathcal{G}_R^{\pm}$. The transformation $\mathbb{T}^+$, $\mathbb{T}^-$, correspond to the elements in $\mathcal{G}_R^+$, $\mathcal{G}_R^-$, respectively. $\mathcal{G}_R$, the combined material symmetry set relative to $\mathcal{S}_R$, is defined by

$$\mathcal{G}_R := \mathcal{G}_R^+ \cup \mathcal{G}_R^-, \qquad \mathcal{G}_R^+ \cap \mathcal{G}_R^- = \varnothing. \tag{3.8}$$

With (3.3), equation (3.7) yields

$$\widetilde{\mathbb{T}}(\mathbf{F},\mathbf{F}') = \widetilde{\mathbb{T}}(\mathbb{F}) = \widetilde{\mathbb{T}}^+(\mathbb{F}\mathbb{G}^+) = \widetilde{\mathbb{T}}^+((\mathbf{F},\mathbf{F}')(\mathbf{G},\mathbf{G})) = \widetilde{\mathbb{T}}^+(\mathbf{FG},\ \mathbf{F}'\mathbf{G}), \tag{3.9}$$

and

$$\widetilde{\mathbb{T}}(\mathbf{F},\mathbf{F}') = \widetilde{\mathbb{T}}(\mathbb{F}) = \widetilde{\mathbb{T}}^-(\mathbb{F}\mathbb{G}^-) = \widetilde{\mathbb{T}}^-((\mathbf{F},\mathbf{F}')(\mathbf{G},-\mathbf{G})) = \widetilde{\mathbb{T}}^-(\mathbf{FG},-\mathbf{F}'\mathbf{G}). \tag{3.10}$$

Cohen [1] defined a rod point to be materially monotropic if its combined symmetry set $\mathcal{G}_R$ contains at least one of the elements $\mathbb{Q}_e^{\pm}$, where

$$\mathbb{Q}_e := (\mathbf{Q}_e, \mathbf{Q}_e), \tag{3.11}$$

and $\mathbf{Q}_e$ is a rotation of angle π about a fixed axis $\mathbf{e}$ at the rod point, where $\mathbf{e}$ is a unit vector.

We may write

$$\mathbf{Q}_e = -\mathbf{I}_e^{\perp} + \mathbf{e} \otimes \mathbf{e}, \tag{3.12}$$

where $\mathbf{I}_e^{\perp}$ is the unit tensor in the plane orthogonal to $\mathbf{e}$, and $\mathbf{Q}_e^{\mathsf{T}} = \mathbf{Q}_e^{-1}$.

$-\mathbf{Q}_e$ is the reflection or inverse in the plane orthogonal to the direction $\mathbf{e}$. $\mathbb{Q}_e^-$ is called a roto-inversion. For example, if we choose $\mathbf{e}$ to be $\mathbf{e}_3$ in the Euclidean coordinate system, (3.12) becomes

$$\mathbf{Q}_3 = \mathbf{Q}_{e_3} = -\mathbf{I}_3^{\perp} + \mathbf{e}_3 \otimes \mathbf{e}_3 = -\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}, \tag{3.13}$$

and we also have

$$\mathbf{Q}_1 = \mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}, \tag{3.14}$$

and

$$\mathbf{Q}_2 = -\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}. \tag{3.15}$$

Here $\mathbf{E}_{11} = \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{E}_{22} = \mathbf{e}_2 \otimes \mathbf{e}_2$ and $\mathbf{E}_{33} = \mathbf{e}_3 \otimes \mathbf{e}_3$.

Combining the principles of material frame-indifference and the material symmetry, with a single transformation rule (3.7), we have the following

$$\mathbb{Q}^{\mathsf{T}}\widetilde{\mathbb{T}}\,(\mathbb{Q}\mathbb{F})\,\mathbb{Q} \;= \widetilde{\mathbb{T}}\,(\mathbb{F})\,. \tag{3.16}$$

Replacing (3.16) by (3.7) , we obtain

$$\mathbb{Q}^{\mathsf{T}}\widetilde{\mathbb{T}}\,(\mathbb{Q}\mathbb{F})\,\mathbb{Q} = \widetilde{\mathbb{T}}^{\pm}(\mathbb{F}\mathbb{G}^{\pm}), \tag{3.17}$$

and multiplying by $\mathbb{Q}$ on both sides, we get

$$\widetilde{\mathbb{T}}\,(\mathbb{Q}\mathbb{F})\,\mathbb{Q} = \mathbb{Q}\widetilde{\mathbb{T}}^{\pm}(\mathbb{F}\mathbb{G}^{\pm}), \tag{3.18}$$

for all $\mathbb{F}$, $\mathbb{Q}$ and $\mathbb{G}^{\pm}$ in their respective sets of definition. If we can find elements $\mathbb{Q}$ and $\mathbb{G}$ such that the deformation $\mathbb{F}$ satisfies

$$\mathbb{Q}\mathbb{F} = \mathbb{F}\mathbb{G}^{\pm}, \tag{3.19}$$

then

$$\mathbb{T}\mathbb{Q} = \mathbb{Q}\mathbb{T}^{\pm}. \tag{3.20}$$

i.e. $\mathbb{Q}$ and $\mathbb{T}$ either commute or anti-commute.

In particular, for a monotropic rod, if we can chose $\mathbb{Q} = \mathbb{Q}_e$, for the deformation $\mathbb{F}$ satisfying (3.19), then equation (3.19) can be replaced by

$$\mathbb{Q}_e\mathbb{F}=\mathbb{F}\mathbb{Q}_e^{\pm}, \tag{3.21}$$

where

$$\mathbb{G}^{\pm} = \mathbb{Q}_e^{\pm}. \tag{3.22}$$

Then we have $\mathbb{T}$ which satisfies

$$\mathbb{T}\mathbb{Q}_e=\mathbb{Q}_e\mathbb{T}^{\pm}. \tag{3.23}$$

We can rewrite (3.21) explicitly as

$$\mathbf{Q}_e\mathbf{F} = \mathbf{F}\mathbf{Q}_e, \mathbf{Q}_e\mathbf{F}'= \mathbf{F}'\mathbf{Q}_e; \tag{3.24.1}$$

and

$$\mathbf{Q}_e\mathbf{F} = \mathbf{F}\mathbf{Q}_e,\ \mathbf{Q}_e\mathbf{F}'= -\mathbf{F}'\mathbf{Q}_e. \tag{3.24.2}$$

Now consider two different cases for $\mathbb{T} = (\mathbf{N}, \mathbf{M})$.

Case 1:

If we can choose proper $\mathbb{Q} = \mathbb{Q}_e$ which satisfies (3.24.1), then there exists a $\mathbb{T}$ which satisfies

$$\mathbb{T}\mathbb{Q}_e = \mathbb{Q}_e\mathbb{T}^{+}, \tag{3.23.1}$$

where $\mathbb{T}^{+} = (\mathbf{N}, \mathbf{M})$, and

$$(\mathbf{N}, \mathbf{M})\,(\mathbf{Q}_e\mathbf{Q}_e) = (\mathbf{Q}_e\mathbf{Q}_e)\,(\mathbf{N}, \mathbf{M}),$$

$\Longrightarrow$

$$(\mathbf{N}\mathbf{Q}_e, \mathbf{M}\mathbf{Q}_e) = (\mathbf{Q}_e\mathbf{N}, \mathbf{Q}_e\mathbf{M}),$$

$\Longrightarrow$

$$\mathbf{N}\mathbf{Q}_e = \mathbf{Q}_e\mathbf{N},\ \mathbf{M}\mathbf{Q}_e = \mathbf{Q}_e\mathbf{M}.$$

Choosing $\mathbf{Q}_e = \mathbf{Q}_3$, the above equations imply

$$\begin{aligned} N_{13} &= N_{23} = 0, \\ M_{13} &= M_{23} = M_{31} = M_{32} = 0. \end{aligned} \tag{3.25}$$

That is:

$$\begin{aligned} \mathbf{N} &= N_{11}\mathbf{E}_{11} + N_{12}(\mathbf{E}_{12} + \mathbf{E}_{21}) + N_{22}\mathbf{E}_{22} + N_{33}\mathbf{E}_{33}, \\ \mathbf{M} &= M_{11}\mathbf{E}+M_{12}\mathbf{E}_{12} + M_{21}\mathbf{E}_{21} + M_{22}\mathbf{E}_{22} + M_{33}\mathbf{E}_{33}. \end{aligned} \tag{3.26}$$

Case 2:

If we can choose proper $\mathbb{Q} = \mathbb{Q}_e$ satisfying (3.24.2), then there exists a $\mathbb{T}$ which satisfies

$$\mathbb{T}\mathbb{Q}_e = \mathbb{Q}_e\mathbb{T}^-, \tag{3.23.2}$$

where $\mathbb{T}^- = (\mathbf{N}, -\mathbf{M})$, and

$$(\mathbf{N}, \mathbf{M})(\mathbf{Q}_e\mathbf{Q}_e) = (\mathbf{Q}_e\mathbf{Q}_e)(\mathbf{N}, -\mathbf{M}),$$

$\Longrightarrow$

$$(\mathbf{N}\mathbf{Q}_e, \mathbf{M}\mathbf{Q}_e) = (\mathbf{Q}_e\mathbf{N}, -\mathbf{Q}_e\mathbf{M}),$$

$\Longrightarrow$

$$\mathbf{N}\mathbf{Q}_e = \mathbf{Q}_e\mathbf{N},\ \mathbf{M}\mathbf{Q}_e = -\mathbf{Q}_e\mathbf{M}.$$

Choosing $\mathbf{Q}_e = \mathbf{Q}_3$, the above equations imply

$$\begin{aligned} N_{13} &= N_{23} = 0, \\ M_{11} &= M_{12} = M_{21} = M_{22} = M_{33} = 0. \end{aligned} \tag{3.27}$$

That is,

$$\begin{aligned}\mathbf{N} &= N_{11}\mathbf{E}_{11} + N_{12}(\mathbf{E}_{12} + \mathbf{E}_{21}) + N_{22}\mathbf{E}_{22} + N_{33}\mathbf{E}_{33}, \\ \mathbf{M} &= M_{13}\mathbf{E}_{13} + M_{23}\mathbf{E}_{23} + M_{31}\mathbf{E}_{31} + M_{32}\mathbf{E}_{32}.\end{aligned} \tag{3.28}$$

To derive these results, we have put

$$\mathbf{N} = N_{ij}\mathbf{E}_{ij}, \mathbf{M} = M_{ij}\mathbf{E}_{ij}, \tag{3.29}$$

and used (3.13) as well as the relations (cf. Wang [13])

$$\begin{aligned}\mathbf{E}_{ij}\mathbf{E}_{pq} &= (\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_p \otimes \mathbf{e}_q) \\ &= \delta_{jp}(\mathbf{e}_i \otimes \mathbf{e}_q) \\ &= \delta_{jp}\mathbf{E}_{iq}.\end{aligned} \tag{3.30}$$

More generally, if we can chose $\overline{\mathbf{Q}}_{\overline{\mathbf{e}}}$ for $\mathbf{Q}$ in (3.21), where

$$\overline{\mathbf{Q}}_{\overline{\mathbf{e}}} := \mathbf{R}\mathbf{Q}_{\mathbf{e}}\mathbf{R}^T, \tag{3.31}$$

with $\mathbf{R}$ rotation, then (3.21) and (3.23) will be replaced by

$$\overline{\mathbf{Q}}_{\overline{\mathbf{e}}}\mathbb{F} = \mathbb{F}\mathbf{Q}_{\mathbf{e}}^{\pm}, \tag{3.32}$$

and

$$\mathbb{T}\overline{\mathbf{Q}}_{\overline{\mathbf{e}}} = \overline{\mathbf{Q}}_{\overline{\mathbf{e}}}\mathbb{T}^{\pm}, \tag{3.33}$$

where

$$\overline{\mathbf{e}} = \mathbf{R}\mathbf{e} \tag{3.34}$$

and $\overline{\mathbf{Q}}_{\overline{\mathbf{e}}}$ is a rotation of angle π about an axis $\overline{\mathbf{e}}$. Similar results as in (3.26) and (3.28) hold with $\mathbf{E}_{ij}$ replaced by

$$\overline{\mathbf{E}}_{ij} = \overline{\mathbf{e}}_i \otimes \overline{\mathbf{e}}_j \tag{3.35}$$

and

$$\overline{\mathbf{e}_i} = \mathbf{R}\mathbf{e}_i \tag{3.36}$$

CHAPTER 4

Simpler Universal Solutions: Straight⟶Straight or Straight⟶Straight twisted

In this chapter, we derive the Universal Solution for the deformation of an initially straight homogeneous monotropic rod which is deformed into normal configurations of straight untwisted and twisted rods.

Since each of the configurations is itself uniform, it follows that the deformations will always be of the form

$$\begin{aligned} \mathbf{F} &= \mathbf{R}\mathbf{U}_0, \qquad (4.1) \\ \mathbf{F}' &= \mathbf{R}'\mathbf{U}_0 = \mathbf{W}_0\mathbf{R}\mathbf{U}_0 = \mathbf{W}_0\mathbf{F}. \end{aligned}$$

where $\mathbf{U}_0$ is a right stretch tensor, and $\mathbf{F}$ is the deformation tensor.

$$\mathbf{U}_0 = \lambda_{(a)}(t)\,\delta^{ab}\mathbf{e}_a \otimes \mathbf{e}_b, \qquad (4.2)$$

with the principal stretch $\lambda_a(t)$ given by

$$\lambda_a(t) = \left(\mathbf{d}_{(a)} \cdot \mathbf{d}_{(a)}\right)^{\frac{1}{2}}, \qquad (4.3)$$

where (a) is not a dummy index.

$\mathbf{R}$ is a fixed axis rotation which satisfies

$$\mathbf{R}'\mathbf{R}^{\mathsf{T}} = \mathbf{R}^{\mathsf{T}}\mathbf{R}' = \mathbf{W}_0, \qquad (4.4)$$

where $\mathbf{W}_0$, the wryness of the deformed configuration, is a time-dependent skew-symmetric tensor.

The principle of material frame-indifference allows a reduction of the constitutive equations. If we choose $\mathbb{Q} = \mathbb{R}^{\mathsf{T}} = \left(\mathbf{R}^{\mathsf{T}}, \mathbf{R}^{\mathsf{T}}\right)$, $\mathbb{R} := (\mathbf{R}, \mathbf{R})$, with $\mathbf{F}$ defined by (4.1), then equation (3.1),

$$\widetilde{\mathbb{T}}(\mathbb{Q}\mathbb{F}) = \mathbb{Q}\widetilde{\mathbb{T}}(\mathbb{F})\mathbb{Q}^{\mathsf{T}},$$

becomes

$$\widetilde{\mathbb{T}}\left(\mathbb{R}^{\mathsf{T}}\mathbb{F}\right) = \mathbb{R}^{\mathsf{T}}\widetilde{\mathbb{T}}(\mathbb{F})\left(\mathbb{R}^{\mathsf{T}}\right)^{\mathsf{T}}, \tag{4.5}$$

which implies

$$\widetilde{\mathbb{T}}\left(\mathbb{R}^{\mathsf{T}}\mathbb{F}\right) = \mathbb{R}^{\mathsf{T}}\widetilde{\mathbb{T}}(\mathbb{F})\mathbb{R} \tag{4.6}$$

or

$$\widetilde{\mathbb{T}}(\mathbb{F}) = \mathbb{R}\widetilde{\mathbb{T}}\left(\mathbb{R}^{\mathsf{T}}\mathbb{F}\right)\mathbb{R}.^{\mathsf{T}} \tag{4.7}$$

Note that

$$\mathbb{R}^{\mathsf{T}}\mathbb{F} = \left(\mathbf{R}^{\mathsf{T}}, \mathbf{R}^{\mathsf{T}}\right)(\mathbf{F}, \mathbf{F}') = \left(\mathbf{R}^{\mathsf{T}}\mathbf{F}, \mathbf{R}^{\mathsf{T}}\mathbf{F}'\right). \tag{4.8}$$

By using (4.1), we have:

$$\mathbf{R}^{\mathsf{T}}\mathbf{F} = \mathbf{U}_0, \tag{4.9}$$

so

$$\mathbf{R}^{\mathsf{T}}\mathbf{F}' = \mathbf{R}^{\mathsf{T}}\mathbf{W}_0\mathbf{F} = \mathbf{W}_0\mathbf{R}^{\mathsf{T}}\mathbf{F} = \mathbf{W}_0\mathbf{U}_0. \tag{4.10}$$

With (4.8), (4.9), and (4.10), we get the reduced equation:

$$\mathbb{T} = \mathbb{R}\mathbb{T}_0\mathbb{R}^{\mathsf{T}}, \tag{4.11}$$

where

$$\mathbb{T} \;:\; = \widetilde{\mathbb{T}}(\mathbb{F}), \tag{4.12}$$

$$\mathbb{T}_0 \;:\; = \widetilde{\mathbb{T}}\left(\mathbb{R}^{\mathsf{T}}\mathbb{F}\right) = \widetilde{\mathbb{T}}(\mathbf{U}_0, \mathbf{W}_0\mathbf{U}_0) = (\mathbf{N}_0, \mathbf{M}_0).$$

From equation (4.2), we have $\mathbf{U_0}$ is a function of stretch functions $\lambda_a(t)$, $a = 1, 2, 3$. The constitutive relation (4.11) - (4.12) yields

$$\mathbf{N} = \mathbf{N}_0 = \widetilde{\mathbf{N}}(\lambda_a), \;\; \mathbf{M} = \mathbf{M}_0 = \widetilde{\mathbf{M}}(\lambda_a).$$

In the next three chapters, we will derive equations relating to the components of $\mathbf{N}$ and $\mathbf{M}$. If we knew the explicit dependence of $\mathbf{N}$ and $\mathbf{M}$ on $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$, we could in principle solve explicitly for λ_1, λ_2 and λ_3 in terms of t.

4.1 Straight $\longrightarrow$ Straight

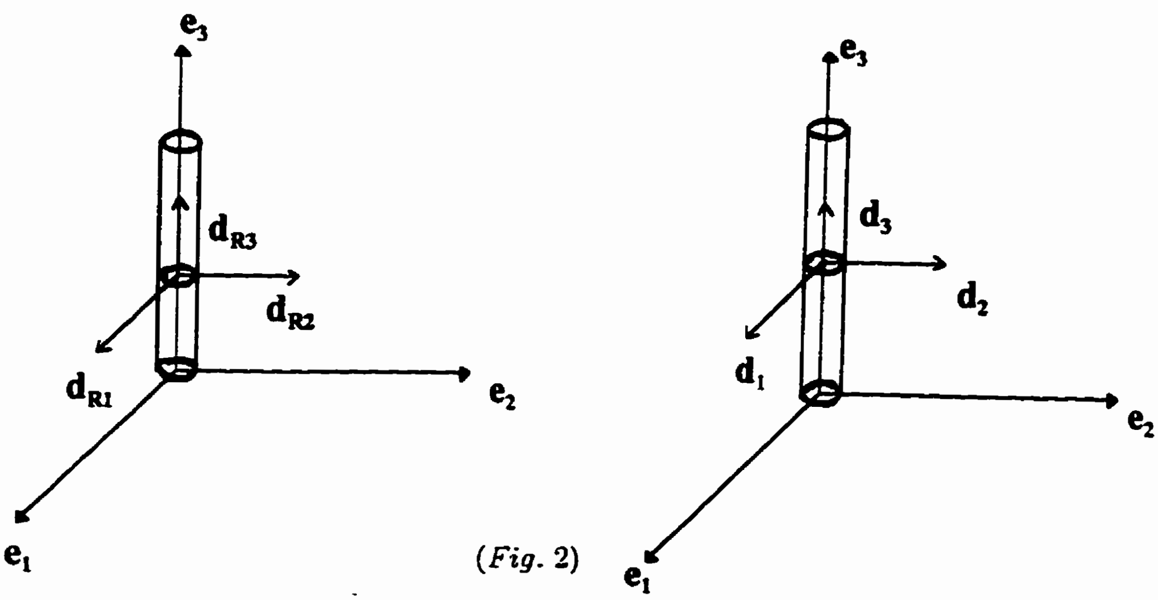
This deformation is the nonlinear analogue of pure extension. The reference configuration is defined as

$$\mathbf{r}_R = S\mathbf{e}_3 \tag{4.13}$$

and the current configuration is defined as

$$\mathbf{r} = s\mathbf{e}_3 = \lambda_3(t)\, S\mathbf{e}_3, \tag{4.14}$$

where $\lambda_3(t) = ds/dS$ is the stretch function in the $\mathbf{e}_3$ direction which deforms the rod axis ($Fig.$ 2).



($Fig.$ 2)

Note:

1) The initial condition is in the reference configuration, in which $t = 0$, $s = S$, $\lambda_a(0) = 1$; the rod is not deformed at this time.

2) In the current configuration, $\mathbf{d}_1 = \lambda_1\mathbf{e}_1$, $\mathbf{d}_2 = \lambda_2\mathbf{e}_2$, $\mathbf{d}_3 = \lambda_3\mathbf{e}_3$, $(\mathbf{d}_1, \mathbf{d}_2)$ spans the cross-section and $\mathbf{d}_3$ is along the tangent to the rod axis. In general, cross-sectional shear will occur unless $\lambda_1 = \lambda_2$. In another words, if $\lambda_1 \neq \lambda_2$, then the cross-section will be deformed.

For the motion being considered in (4.14), the rod cross-sections is not rotated, so we have

$$\mathbf{R} = \mathbf{I},\ \mathbf{W}_0 = \mathbf{0}, \tag{4.15}$$

and

$$\mathbf{F} = \mathbf{R}\mathbf{U}_0 = \mathbf{I}\mathbf{U}_0 = \mathbf{U}_0,\ \mathbf{F}' = \mathbf{W}_0\mathbf{F} = \mathbf{0}. \tag{4.16}$$

By using (4.1) and (4.11), the constitutive relation yields

$$\mathbb{T} = \mathbb{R}\mathbb{T}_0\mathbb{R}^T = \mathbb{I}\mathbb{T}_0\mathbb{I}^T = \mathbb{T}_0, \tag{4.17}$$

where

$$\mathbb{T}_0 = \widetilde{\mathbb{T}}\left(\mathbf{U}_0, \mathbf{W}_0\mathbf{U}_0\right) = \widetilde{\mathbb{T}}\left(\mathbf{U}_0, \mathbf{0}\right) = \left(\mathbf{N}_0, \mathbf{M}_0\right). \tag{4.18}$$

Note that $\mathbf{U}_0$ and hence $\mathbf{N}_0$, $\mathbf{M}_0$ are in general time-dependent.

Since the current configuration is $\mathbf{r} = s\mathbf{e}_3 = \lambda_3(t)\,S\mathbf{e}_3$, the acceleration $\mathbf{a}$ of the rod is

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{\lambda}_3(t)\,S\mathbf{e}_3 \tag{4.19}$$

and

$$\mathbf{r}' = \lambda_3(t)\,\mathbf{e}_3. \tag{4.20}$$

The dynamic field equations (2.13) - (2.15) and (2.22) become

$$\mathbf{n}' = \rho \ddot{\lambda}_3 (t) S\mathbf{e}_3, \tag{4.21}$$

$$\mathbf{M}' - \mathbf{N} + (\lambda_3 (t) \mathbf{e}_3 \otimes \mathbf{n})^{\mathsf{T}} + \mathbf{T} = \ddot{\mathbf{F}} \mathbf{E}_R \mathbf{F}^{\mathsf{T}}, \tag{4.22}$$

$$Sk\mathbf{N} = \mathbf{0}, \ Sk\mathbf{T} = \mathbf{0}, \tag{4.23}$$

with

$$\mathbf{f} = \mathbf{0}. \tag{4.24}$$

The deformation tensor $\mathbf{F}$ is

$$\mathbf{F} = \lambda_1 (t) \mathbf{d}_1 \otimes \mathbf{d}_1 + \lambda_2 (t) \mathbf{d}_2 \otimes \mathbf{d}_2 + \lambda_3 (t) \mathbf{d}_3 \otimes \mathbf{d}_3, \tag{4.25}$$

in this case,

$$\mathbf{F} = \lambda_1 (t) \mathbf{E}_{11} + \lambda_2 (t) \mathbf{E}_{22} + \lambda_3 (t) \mathbf{E}_{33}$$

and

$$\ddot{\mathbf{F}} = \ddot{\lambda}_1 (t) \mathbf{E}_{11} + \ddot{\lambda}_2 (t) \mathbf{E}_{22} + \ddot{\lambda}_3 (t) \mathbf{E}_{33}. \tag{4.26}$$

With

$$\mathbf{E}_R = E_1 \mathbf{E}_{11} + E_2 \mathbf{E}_{22} + E_3 \mathbf{E}_{33}, \tag{4.27}$$

where E_a are constants, $a = 1, 2.3$. From (4.25), (4.26) and (4.27), we have

$$\begin{aligned} & \ddot{\mathbf{F}} \mathbf{E}_R \mathbf{F}^{\mathsf{T}} \\ = \ & \ddot{\lambda}_1 (t) \lambda_1 (t) E_1 \mathbf{E}_{11} + \ddot{\lambda}_2 (t) \lambda_2 (t) E_2 \mathbf{E}_{22} + \ddot{\lambda}_3 (t) \lambda_3 (t) E_3 \mathbf{E}_{33}. \end{aligned} \tag{4.28}$$

where we have used (3.30) simplify the result. $\ddot{\mathbf{F}} \mathbf{E}_R \mathbf{F}^{\mathsf{T}}$ is a symmetric tensor; $\ddot{\mathbf{F}} \mathbf{E}_R \mathbf{F}^{\mathsf{T}} = \left(\ddot{\mathbf{F}} \mathbf{E}_R \mathbf{F}^{\mathsf{T}}\right)^{\mathsf{T}} = \mathbf{F} \mathbf{E}_R^{\mathsf{T}} \ddot{\mathbf{F}}^{\mathsf{T}}$.

The constitutive relation (4.17) yields $\mathbf{N} = \mathbf{R}\mathbf{N}_0\mathbf{R}^\mathsf{T} = \mathbf{I}\mathbf{N}_0\mathbf{I} = \mathbf{N}_0$, and $\mathbf{M} = \mathbf{R}\mathbf{M}_0\mathbf{R}^\mathsf{T} = \mathbf{I}\mathbf{M}_0\mathbf{I} = \mathbf{M}_0$; they are all functions of $\lambda_a(t)$. $\mathbf{M}$ and $\mathbf{N}$ do not depend on the arc length s and hence $\mathbf{M}' = \mathbf{M}_0' = \mathbf{0}$.

In order to make the analysis feasible, we only consider universal quasi-static motion: that is the rod accelerations are zero. Therefore, $\ddot{\lambda}_1(t) = \ddot{\lambda}_2(t) = \ddot{\lambda}_3(t) = 0$ and we have $\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^\mathsf{T} = \mathbf{0}$. The dynamic field equations (4.21) and (4.22) become

$$\mathbf{n}' = \mathbf{0} \tag{4.29}$$

and

$$\mathbf{T} = \mathbf{N}_0 - (\lambda_3(t)\,\mathbf{n}_0 \otimes \mathbf{e}_3). \tag{4.30}$$

Equation (4.29) implies

$$\mathbf{n} = \mathbf{n}_0(t), \tag{4.31}$$

where $\mathbf{n}_0(t)$ is independent of arc length S.

The skew part of (4.30) is

$$\begin{aligned} & Sk\,\mathbf{T} && (4.32)\\ =\;& \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^\mathsf{T}\right) \\ =\;& \frac{1}{2}\left(\left(\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^\mathsf{T} + \mathbf{N}_0 - (\lambda_3(t)\,\mathbf{n}_0 \otimes \mathbf{e}_3)\right) - \left(\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^\mathsf{T} + \mathbf{N}_0 - (\lambda_3(t)\,\mathbf{n}_0 \otimes \mathbf{e}_3)\right)^\mathsf{T}\right) \\ =\;& \frac{1}{2}\left(\mathbf{N}_0 - \mathbf{N}_0^\mathsf{T}\right) + \frac{1}{2}\lambda_3(t)\,(\mathbf{e}_3 \otimes \mathbf{n}_0 - \mathbf{n}_0 \otimes \mathbf{e}_3) \\ =\;& Sk\{\mathbf{N}_0 + \lambda_3(t)\,(\mathbf{e}_3 \otimes \mathbf{n}_0)\}. \end{aligned}$$

Since $Sk\;\mathbf{N} = Sk\;\mathbf{N}_0 = \mathbf{0}$, the constitutive equation

$$Sk\;\mathbf{T} = \mathbf{0}$$

requires

$$\mathbf{n}_0 = n_0\mathbf{e}_3, \tag{4.33}$$

where n_0 is only a function of time t. Replacing $\mathbf{n}_0$ in (4.30) by (4.33), we get

$$\mathbf{T} = \mathbf{N}_0 - (\lambda_3(t)\, n_0 \mathbf{e}_3 \otimes \mathbf{e}_3). \tag{4.34}$$

For a monotropic rod we assume that

$$\mathcal{G}_R^+ = \{(\mathbf{Q}_3, \mathbf{Q}_3), (\mathbf{I}, \mathbf{I})\}$$

and

$$\mathcal{G}_R^- = \{(\mathbf{Q}_3, -\mathbf{Q}_3), (\mathbf{I}, \mathbf{I})\}.$$

In these definitions, $\mathbf{Q}_3$ is a rotation of angle π about $\mathbf{e}_3$ and is given by $\mathbf{Q}_3 = -\mathbf{I}_3^{\perp} + \mathbf{e}_3 \otimes \mathbf{e}_3 = -\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}$.

From the above and (4.25), we have (by (3.30))

$$\begin{aligned}
\mathbf{Q}_3\mathbf{F} &= (-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33})(\lambda_1(t)\,\mathbf{E}_{11} + \lambda_2(t)\,\mathbf{E}_{22} + \lambda_3(t)\,\mathbf{E}_{33}) \\
&= -\lambda_1(t)\,\mathbf{E}_{11} - \lambda_2(t)\,\mathbf{E}_{22} - \lambda_3(t)\,\mathbf{E}_{33}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}\mathbf{Q}_3 &= (\lambda_1(t)\,\mathbf{E}_{11} + \lambda_2(t)\,\mathbf{E}_{22} + \lambda_3(t)\,\mathbf{E}_{33})(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \\
&= -\lambda_1(t)\,\mathbf{E}_{11} - \lambda_2(t)\,\mathbf{E}_{22} - \lambda_3(t)\,\mathbf{E}_{33},
\end{aligned}$$

which implies

$$\mathbf{Q}_3\mathbf{F} = \mathbf{F}\mathbf{Q}_3. \tag{4.35}$$

Note that $\mathbf{F} = \mathbf{R}\mathbf{U}_0 = \mathbf{U}_0$ does not depend on arc length S. Thus, $\mathbf{F}' = \mathbf{0}$ and we have $\mathbf{Q}_3\mathbf{F}' = \mathbf{F}'\mathbf{Q}_3 = \mathbf{0}$. This satisfies (3.24.1) and thus (3.23.1) yields

$$N_{13} = N_{23} = 0,\; M_{13} = M_{23} = M_{31} = M_{32} = 0. \tag{4.36}$$

Since $\mathbf{Q}_3\mathbf{F}' = \mathbf{0} = \mathbf{F}'\mathbf{Q}_3$, we have $\mathbf{Q}_3\mathbf{F}' = -\mathbf{F}'\mathbf{Q}_3$ and with (4.34), equation (3.24.2) is satisfied. (3.23.2) yields

$$\begin{aligned} N_{13} &= N_{23} = 0, \\ M_{11} &= M_{12} = M_{21} = M_{22} = M_{33} = 0. \end{aligned} \tag{4.37}$$

We can see that $\mathcal{G}_R^-$ serves to further reduce the complexity of the boundary loads required to produce the deformation. This also shows the *monotropic symmetry axis* is $\mathbf{d}_3 = \mathbf{R}\mathbf{e}_3 = \mathbf{I}\mathbf{e}_3 = \mathbf{e}_3$.

Checking the other two rotations $\mathbf{Q}_1$ and $\mathbf{Q}_2$, where $\mathbf{Q}_1 = \mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}$ and $\mathbf{Q}_2 = -\mathbf{E}_{11} + \mathbf{E}_{22} + \mathbf{E}_{33}$, we find

$$\mathbf{Q}_1\mathbf{F} = \mathbf{F}\mathbf{Q}_1, \mathbf{Q}_1\mathbf{F}' = \mathbf{F}'\mathbf{Q}_1 \text{ and } \mathbf{Q}_1\mathbf{F} = \mathbf{F}\mathbf{Q}_1, \mathbf{Q}_1\mathbf{F}' = -\mathbf{F}'\mathbf{Q}_1 \tag{4.38}$$

and

$$\mathbf{Q}_2\mathbf{F} = \mathbf{F}\mathbf{Q}_2, \mathbf{Q}_2\mathbf{F}' = \mathbf{F}'\mathbf{Q}_2 \text{ and } \mathbf{Q}_2\mathbf{F} = \mathbf{F}\mathbf{Q}_2, \mathbf{Q}_2\mathbf{F}' = -\mathbf{F}'\mathbf{Q}_2. \tag{4.39}$$

This gives the result that $\mathbf{d}_1 = \mathbf{R}\mathbf{e}_1 = \mathbf{I}\mathbf{e}_1 = \mathbf{e}_1$ and $\mathbf{d}_2 = \mathbf{R}\mathbf{e}_2 = \mathbf{I}\mathbf{e}_2 = \mathbf{e}_2$ are both potential monotropic symmetry axes. Therefore the rod is *orthotropic* with respect to the axes $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$.

From the monotropic symmetry theory and using (4.38) , (3.24) gives the further deduction

$$N_{12} = N_{13} = 0, \; M_{12} = M_{13} = M_{21} = M_{31} = 0 \tag{4.40}$$

and

$$N_{12} = N_{13} = 0, M_{11} = M_{22} = M_{23} = M_{32} = M_{33} = 0. \tag{4.41}$$

Using (4.39), we obtain

$$N_{12} = N_{23} = 0, \; M_{12} = M_{21} = M_{23} = M_{32} = 0 \tag{4.42}$$

and

$$N_{12} = N_{23} = 0, M_{11} = M_{22} = M_{13} = M_{31} = M_{33} = 0. \tag{4.43}$$

Combining all above results, we get $\mathbf{N}$ and $\mathbf{M}$ as

$$\mathbf{N_0} = N_{11}\mathbf{E}_{11} + N_{22}\mathbf{E}_{22} + N_{33}\mathbf{E}_{33} \text{ and } \mathbf{M_0} = \mathbf{0}, \tag{4.44}$$

where N_{11}, N_{22} and N_{33} are time dependent functions through their constitutive dependence on the $\mathbf{U_0}$, cf. (4.18).

$\mathbf{M_0} = \mathbf{0}$ makes physical sense for a rod that is simply being stretched along its axes and in its cross-section.

Replacing (4.33) by (4.44), we have

$$n_0 = \lambda_3^{-1}(t)\, N_{33} \tag{4.45}$$

and

$$\begin{aligned} \mathbf{T} &= \mathbf{N_0} - (\lambda_3(t)\, \mathbf{n_0} \otimes \mathbf{e}_3) \\ &= (N_{11}\mathbf{E}_{11} + N_{22}\mathbf{E}_{22} + N_{33}\mathbf{E}_{33}) - \left(\lambda_3(t) \cdot \lambda_3^{-1}(t)\, N_{33}\mathbf{e}_3 \otimes \mathbf{e}_3\right) \\ &= (N_{11}\mathbf{E}_{11} + N_{22}\mathbf{E}_{22} + N_{33}\mathbf{E}_{33}) - (N_{33}\mathbf{E}_{33}) \\ &= N_{11}\mathbf{E}_{11} + N_{22}\mathbf{E}_{22}. \end{aligned} \tag{4.46}$$

If we assume $\mathbf{T} = \mathbf{0}$, then we have

$$N_{11} = 0, \;\; N_{22} = 0. \tag{4.47}$$

If we knew how N_{11} and N_{22} depend on stretch functions $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$, we could, in principle use equations (4.45) and (4.47) to solve explicitly for λ_1, λ_2 and λ_3 as function of t.

4.2 Straight$\longrightarrow$ Straight twisted

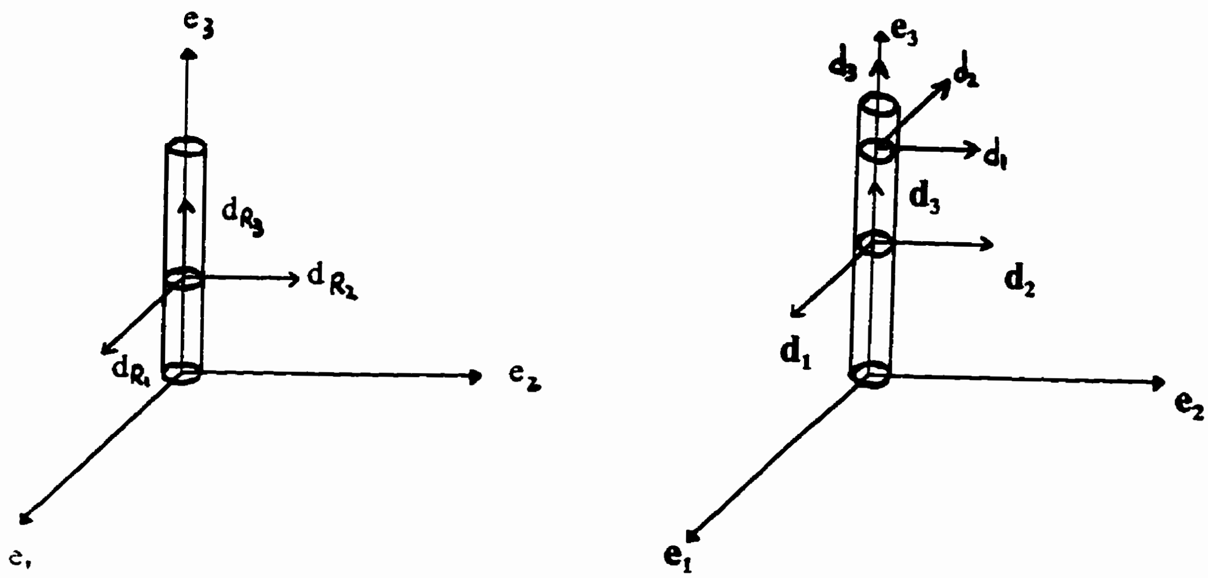
This type of deformation is the nonlinear analogue of torsion; it is the same as the Straight $\longrightarrow$ Straight case except that $\mathbf{W}_0 \neq \mathbf{0}$. Here

$$
\begin{aligned}
\mathbf{R}_3 &= \cos\omega S \ \mathbf{I}_3^{\perp} + \sin\omega t S \ \mathbf{A}_3^{\perp} + \mathbf{E}_{33} \qquad (4.48) \\
&= \cos\omega S \ \mathbf{E}_{11} + \cos\omega S \ \mathbf{E}_{22} + \sin\omega S \ \mathbf{E}_{21} - \sin\omega S \ \mathbf{E}_{12} + \mathbf{E}_{33}
\end{aligned}
$$

and

$$\mathbf{W}_0 = \omega \mathbf{A}_3^{\perp} = \omega\left(\mathbf{E}_{21} - \mathbf{E}_{12}\right). \qquad (4.49)$$

$\mathbf{r}_R = S\mathbf{e}_3$ is the reference configuration, $\mathbf{r} = s\mathbf{e}_3 = \lambda_3(t)\, S\mathbf{e}_3$ is the current configuration $(Fig.\ 3)$, and $\lambda_3(t) = ds/dS$ is the stretch function. The function $\omega = \omega(t)$ measures the twist per unit length in the reference configuration.



$(Fig.\ 3)$

In the current configuration, the directors $\mathbf{d}_1 = \lambda_1 \mathbf{e}_1$, $\mathbf{d}_2 = \lambda_2 \mathbf{e}_2$, $\mathbf{d}_3 = \lambda_3 \mathbf{e}_3$, and $(\mathbf{d}_1, \mathbf{d}_2)$ span the cross-section with $\mathbf{d}_3$ along the tangent of the rod axis.

In the current configuration, $\mathbf{r} = s\mathbf{e}_3 = \lambda_3(t) S\mathbf{e}_3$, the acceleration $\mathbf{a}$ is

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{\lambda}_3(t) S\mathbf{e}_3 \tag{4.50}$$

and

$$\mathbf{r}' = \lambda_3(t)\,\mathbf{e}_3. \tag{4.51}$$

The dynamic field equations (2.13)-(2.15) become

$$\mathbf{n}' = \rho\,\ddot{\lambda}_3(t) S\mathbf{e}_3, \tag{4.52}$$

$$\mathbf{M}' - \mathbf{N} + (\lambda_3(t)\,\mathbf{e}_3 \otimes \mathbf{n})^{\mathsf{T}} + \mathbf{T} = \ddot{\mathbf{F}}\,\mathbf{E}_R \mathbf{F}^{\mathsf{T}}, \tag{4.53}$$

$$Sk\ \mathbf{N} = \mathbf{0},\ \ Sk\ \mathbf{T} = \mathbf{0}, \tag{4.54}$$

with

$$\mathbf{f} = \mathbf{0}. \tag{4.55}$$

Here the deformation tensor $\mathbf{F}$ is

$$\begin{aligned}
\mathbf{F} &= \mathbf{R}_3\mathbf{U}_0 \qquad (4.56)\\
&= \begin{array}{l}(\cos\omega S\mathbf{E}_{11} + \cos\omega S\mathbf{E}_{22}\\ + \sin\omega S\mathbf{E}_{21} - \sin\omega S\mathbf{E}_{12} + \mathbf{E}_{33})\\ \cdot(\lambda_1(t)\,\mathbf{E}_{11} + \lambda_2(t)\,\mathbf{E}_{22} + \lambda_3(t)\,\mathbf{E}_{33})\end{array}\\
&= \lambda_1(t)\cos\omega S\mathbf{E}_{11} + \lambda_2(t)\cos\omega S\mathbf{E}_{22} +\\
&\quad \lambda_1(t)\sin\omega S\mathbf{E}_{21} - \lambda_2(t)\sin\omega S\mathbf{E}_{12} + \lambda_3(t)\,\mathbf{E}_{33},
\end{aligned}$$

and

$$\ddot{\mathbf{F}} = (\ddot{\lambda}_1 \cos\omega S - 2\,\dot{\lambda}_1\ S\,\dot{\omega}\,\sin\omega S - \lambda_1 S\,\ddot{\omega}\,\sin\omega S - \lambda_1 S^2(\dot{\omega})^2\cos\omega S)\mathbf{E}_{11} \tag{4.57}$$

$$+(\ddot{\lambda}_2 \cos\omega S - 2\,\dot{\lambda}_2\, S\,\dot{\omega}\,\sin\omega S - \lambda_2 S\,\ddot{\omega}\,\sin\omega S + \lambda_2 S^2(\dot{\omega})^2\cos\omega S)\mathbf{E}_{22}$$

$$+(\ddot{\lambda}_1 \sin\omega S + 2\,\dot{\lambda}_1\, S\,\dot{\omega}\,\cos\omega S + \lambda_1 S\,\ddot{\omega}\,\cos\omega S - \lambda_1 S^2(\dot{\omega})^2\sin\omega S)\mathbf{E}_{21}$$

$$-(\ddot{\lambda}_2 \sin\omega S + 2\,\dot{\lambda}_2\, S\,\dot{\omega}\,\cos\omega S + \lambda_2 S\,\ddot{\omega}\,\cos\omega S - \lambda_2 S^2(\dot{\omega})^2\sin\omega S)\mathbf{E}_{12}$$

$$+\lambda_3\,(t)\,\mathbf{E}_{33}.$$

From (4.49) and (4.56), we have

$$\begin{aligned}
\mathbf{F}' &= \mathbf{W}_0\mathbf{F} \qquad (4.58)\\
&= -\omega\lambda_1\,(t)\sin\omega S\mathbf{E}_{11} - \omega\lambda_2\,(t)\sin\omega S\mathbf{E}_{22}\\
&\quad +\omega\lambda_1\,(t)\cos\omega S\mathbf{E}_{21} - \omega\lambda_2\,(t)\cos\omega S\mathbf{E}_{12}.
\end{aligned}$$

Furthermore, for the quasi-static motion, we assume $\ddot{\lambda}_1 = \ddot{\lambda}_2 = \ddot{\lambda}_3 = 0$, and $\ddot{\omega} = \dot{\omega} = 0$, then we have $\omega = \omega_0$, is a constant, hence $\ddot{\mathbf{F}}\;\mathbf{E}_R\mathbf{F}^{\mathsf{T}} = \mathbf{0}$. Since $\mathbf{M} = \mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}}$, $\mathbf{M}_0' = \mathbf{0}$, $\mathbf{R}_3' = \mathbf{W}_0\mathbf{R}_3 = \mathbf{R}_3\mathbf{W}_0$ and $\mathbf{W}_0^{\mathsf{T}} = -\mathbf{W}_0$, then

$$\begin{aligned}
\mathbf{M}' &= \mathbf{R}_3'\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}} + \mathbf{R}_3\mathbf{M}_0'\mathbf{R}_3^{\mathsf{T}} + \mathbf{R}_3\mathbf{M}_0\left(\mathbf{R}_3^{\mathsf{T}}\right)' \qquad (4.59)\\
&= \mathbf{W}_0\mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}} + \mathbf{R}_3\mathbf{M}_0\,(\mathbf{W}_0\mathbf{R}_3)^{\mathsf{T}}\\
&= \mathbf{W}_0\mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}} - \mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}}\mathbf{W}_0\\
&= \mathbf{W}_0\mathbf{M} - \mathbf{M}\mathbf{W}_0.
\end{aligned}$$

Then, with the use of (4.59), the dynamic field equations (4.52) and (4.53) become

$$\mathbf{n}' = \mathbf{0} \qquad (4.60)$$

and

$$\mathbf{T} = \mathbf{N} + \mathbf{M}\mathbf{W}_0 - \mathbf{W}_0\mathbf{M} - \lambda_3\,(t)\;\mathbf{n}\otimes\mathbf{e}_3\;. \qquad (4.61)$$

From (4.60), we obtain

$$\mathbf{n} = \mathbf{n}_0\,(t)\,, \qquad (4.62)$$

where $\mathbf{n}_0(t)$ is independent of arc length S.

Since $Sk\ \mathbf{N} = \mathbf{0}$ and

$$\begin{aligned} & -2sym\left(\mathbf{W}_0 sym\left(\mathbf{M}\right)\right) \qquad (4.63) \\ = & -2sym\left(\mathbf{W}_0 \frac{1}{2}\left(\mathbf{M}+\mathbf{M}^{\mathsf{T}}\right)\right) \\ = & -sym\left(\mathbf{W}_0\mathbf{M}+\mathbf{W}_0\mathbf{M}^{T}\right) \\ = & -\frac{1}{2}\left(\left(\mathbf{W}_0\mathbf{M}+\mathbf{W}_0\mathbf{M}^{\mathsf{T}}\right)+\left(\mathbf{W}_0\mathbf{M}+\mathbf{W}_0\mathbf{M}^{\mathsf{T}}\right)^{\mathsf{T}}\right) \\ = & -\frac{1}{2}\left(\mathbf{W}_0\mathbf{M}+\mathbf{W}_0\mathbf{M}^{\mathsf{T}}+\mathbf{M}^{\mathsf{T}}\mathbf{W}_0^{\mathsf{T}}+\mathbf{M}\mathbf{W}_0^{\mathsf{T}}\right) \\ = & -\frac{1}{2}\left(\mathbf{W}_0\mathbf{M}+\mathbf{W}_0\mathbf{M}^{\mathsf{T}}-\mathbf{M}^{\mathsf{T}}\mathbf{W}_0-\mathbf{M}\mathbf{W}_0\right) \\ = & -sym(\mathbf{W}_0\mathbf{M})+sym(\mathbf{M}\mathbf{W}_0) \\ = & \ sym(\mathbf{M}\mathbf{W}_0-\mathbf{W}_0\mathbf{M}). \end{aligned}$$

The constitutive restriction $Sk\ \mathbf{T} = \mathbf{0}$ requires that

$$\mathbf{n} = n_0\mathbf{e}_3, \qquad (4.64)$$

where n_0 depends only upon time. Then the symmetry parts of (4.61) may be written as

$$\mathbf{T} = \mathbf{N} - 2sym\left(\mathbf{W}_0 sym\left(\mathbf{M}\right)\right) - \lambda_3\left(t\right) n_0\mathbf{e}_3 \otimes \mathbf{e}_3 \qquad (4.65)$$

and the skew-symmetry parts of (4.61) may be written as

$$Sk\ \left(\mathbf{W}_0 sk\ \left(\mathbf{M}\right)\right) = \mathbf{0}. \qquad (4.66)$$

For a monotropic rod we assume

$$\mathcal{G}_R^+ = \left\{\left(\mathbf{Q}_3, \mathbf{Q}_3\right), \left(\mathbf{I}, \mathbf{I}\right)\right\} \qquad (4.67)$$

and

$$\mathcal{G}_R^- = \left\{\left(\mathbf{Q}_1, -\mathbf{Q}_1\right), \left(\mathbf{I}, \mathbf{I}\right)\right\} \qquad (4.68)$$

where $\mathbf{Q}_3$ and $\mathbf{Q}_1$ are rotations of angle π about $\mathbf{e}_3$ and $\mathbf{e}_1$ respectively.

We define

$$\begin{aligned}
\overline{\mathbf{Q}}_3 \quad : & = \mathbf{R}_3\mathbf{Q}_3\mathbf{R}_3^T && (4.69)\\
= & \left(\cos\omega_0 S\mathbf{E}_{11} + \cos\omega_0 S\mathbf{E}_{22} + \sin\omega_0 S\mathbf{E}_{21} - \sin\omega_0 S\mathbf{E}_{12} + E_{33}\right)\\
& \cdot\left(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}\right)\\
& \cdot\left(\cos\omega_0 S\mathbf{E}_{11} + \cos\omega_0 S\mathbf{E}_{22} + \sin\omega_0 S\mathbf{E}_{12} - \sin\omega_0 S\mathbf{E}_{21} + E_{33}\right)\\
= & -\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33} = \mathbf{Q}_3.
\end{aligned}$$

From (4.69) and (4.56) in the case of quasi static motion, we have

$$\begin{aligned}
\overline{\mathbf{Q}}_3\mathbf{F} = & -\lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22}\\
& + \lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} - \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} + \lambda_3(t)\,\mathbf{E}_{33}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}\mathbf{Q}_3 = & -\lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22}\\
& + \lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} - \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} + \lambda_3(t)\,\mathbf{E}_{33},
\end{aligned}$$

this implies

$$\overline{\mathbf{Q}}_3\mathbf{F} = \mathbf{F}\mathbf{Q}_3. \tag{4.70}$$

Similarly,

$$\begin{aligned}
\overline{\mathbf{Q}}_3\mathbf{F}' = & \ \omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} + \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22}\\
& + \omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12} - \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}'\mathbf{Q}_3 = & \ \omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} + \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22}\\
& + \omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12} - \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21},
\end{aligned}$$

which imply

$$\overline{\mathbf{Q}}_3\mathbf{F}' = \mathbf{F}'\mathbf{Q}_3. \tag{4.71}$$

This shows the monotropic symmetry axis is $\mathbf{d}_3 = \mathbf{R}_3\mathbf{e}_3 = \mathbf{e}_3$. Since (4.70) and (4.71) satisfy (3.24.1), then we have $\mathbb{T} = \mathbb{T}^+ = (\mathbf{N}, \mathbf{M})$ which satisfies (3.23.1). This gives:

$$N_{13} = N_{23} = 0, M_{13} = M_{23} = M_{31} = M_{32} = 0. \tag{4.72}$$

Now consider $\mathbf{Q} = \mathbf{Q}_1$. We define

$$\begin{aligned} \overline{\mathbf{Q}}_1 \quad : \quad &= \mathbf{R}_3\mathbf{Q}_1\mathbf{R}_3^T \qquad (4.73) \\ &= \cos 2\omega_0 S\mathbf{E}_{11} - \cos 2\omega_0 S\mathbf{E}_{22} + \sin 2\omega_0 S\mathbf{E}_{12} + \sin 2\omega_0 S\mathbf{E}_{21} - \mathbf{E}_{33}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \overline{\mathbf{Q}}_1\mathbf{F} &= \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad + \lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} + \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} - \lambda_3(t)\mathbf{E}_{33} \end{aligned}$$

and

$$\begin{aligned} \mathbf{F}\mathbf{Q}_1 &= \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad + \lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} + \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} - \lambda_3(t)\mathbf{E}_{33} \end{aligned}$$

which imply

$$\overline{\mathbf{Q}}_1\mathbf{F} = \mathbf{F}\mathbf{Q}_1. \tag{4.74}$$

Similarly,

$$\begin{aligned} \overline{\mathbf{Q}}_1\mathbf{F}' &= \omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} - \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22} \\ &\quad - \omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12} - \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21} \end{aligned}$$

and

$$\begin{aligned}\mathbf{F}'\mathbf{Q}_1 &= -\omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11}+\omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22}\\ &\quad +\omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12}+\omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21},\end{aligned}$$

which imply

$$\overline{\mathbf{Q}}_1\mathbf{F}^{\cdot} = -\mathbf{F}'\mathbf{Q}_1. \tag{4.75}$$

This shows the monotropic symmetry axis is $\mathbf{d}_1 = \mathbf{R}_3\mathbf{e}_1 = \mathbf{e}_r$. Since (4.74) and (4.75) satisfy (3.24.2), then we have $\mathbb{T}=\mathbb{T}^- = (\mathbf{N}, -\mathbf{M})$ which satisfies (3.23.2). This gives

$$\begin{aligned}N_{12} &= N_{13} = 0, &(4.76)\\ M_{11} &= M_{22} = M_{33} = M_{23} = M_{32} = 0.\end{aligned}$$

Combining (4.72) and (4.76), we have $\mathbf{N}$ and $\mathbf{M}$ as

$$\mathbf{N} = N_{11}\mathbf{E}_{11} + N_{22}\mathbf{E}_{22} + N_{33}\mathbf{E}_{33},\ \mathbf{M} = M_{12}\mathbf{E}_{12} + M_{21}\mathbf{E}_{21}. \tag{4.77}$$

Furthermore,

$$\begin{aligned}\mathbf{M}' &= \mathbf{M}\mathbf{W}_0 - \mathbf{W}_0\mathbf{M} &(4.78)\\ &= (M_{12}\mathbf{E}_{12} + M_{21}\mathbf{E}_{21})(-\omega_0\mathbf{E}_{12} + \omega_0\mathbf{E}_{21})\\ &\quad -(-\omega_0\mathbf{E}_{12} + \omega_0\mathbf{E}_{21})(M_{12}\mathbf{E}_{12} + M_{21}\mathbf{E}_{21})\\ &= \omega_0(M_{12} + M_{21})\mathbf{E}_{11} - \omega_0(M_{21} + M_{12})\mathbf{E}_{22}.\end{aligned}$$

By using (4.77) and (4.78), (4.67) could be simplified as

$$\mathbf{T} = \{N_{11} + \omega_0(M_{12} + M_{21})\}\mathbf{E}_{11} + \{N_{22} - \omega_0(M_{12} + M_{21})\}\mathbf{E}_{22},\ n_0 = \lambda_3^{-1}N_{33} \tag{4.79}$$

where N_{11}, N_{22}, N_{33}, M_{12} and M_{21} are time dependent functions through their constitutive dependence on the $\mathbf{U}_0$, cf. (4.18).

If we assume $\mathbf{T} = \mathbf{0}$, then we will have

$$N_{11} + \omega_0 \left(M_{12} + M_{21} \right) = 0 \tag{4.80}$$

and

$$N_{22} - \omega_0 \left(M_{12} + M_{21} \right) = 0. \tag{4.81}$$

If we knew how $\mathbf{N}$ and $\mathbf{M}$ depend on stretch functions $\lambda_1 (t)$, $\lambda_2 (t)$ and $\lambda_3 (t)$, we could, in principle use equations (4.79), (4.80) and (4.81) to solve explicitly for λ_1, λ_2 and λ_3 as function of t.

CHAPTER 5

Simpler Universal Solutions: Circular $\longrightarrow$ Circular and Straight$\longrightarrow$ Circular

In this chapter, we derive two deformations, an initially circular rod which is deformed into a circular rod and an initially straight rod which is deformed into a circular rod.

5.1 Circular $\longrightarrow$ Circular

This deformation is the nonlinear analogue of pure expansion in the case of a classical circular ring. The reference configuration is defined as

$$\mathbf{r}_R = \mathbf{e}_r, \tag{5.1}$$

where $\mathbf{e}_r$ is the unit vector of a plane polar coordinate system. The radius of the initial circle is the unit 1.

For the plan polar coordinate system, we have the orthonormal basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as

$$\begin{aligned} \mathbf{e}_r &= \cos\omega_0 S \mathbf{e}_1 + \sin\omega_0 S \mathbf{e}_2, \\ \mathbf{e}_\theta &= -\sin\omega_0 S \mathbf{e}_1 + \cos\omega_0 S \mathbf{e}_2. \end{aligned} \tag{5.2}$$

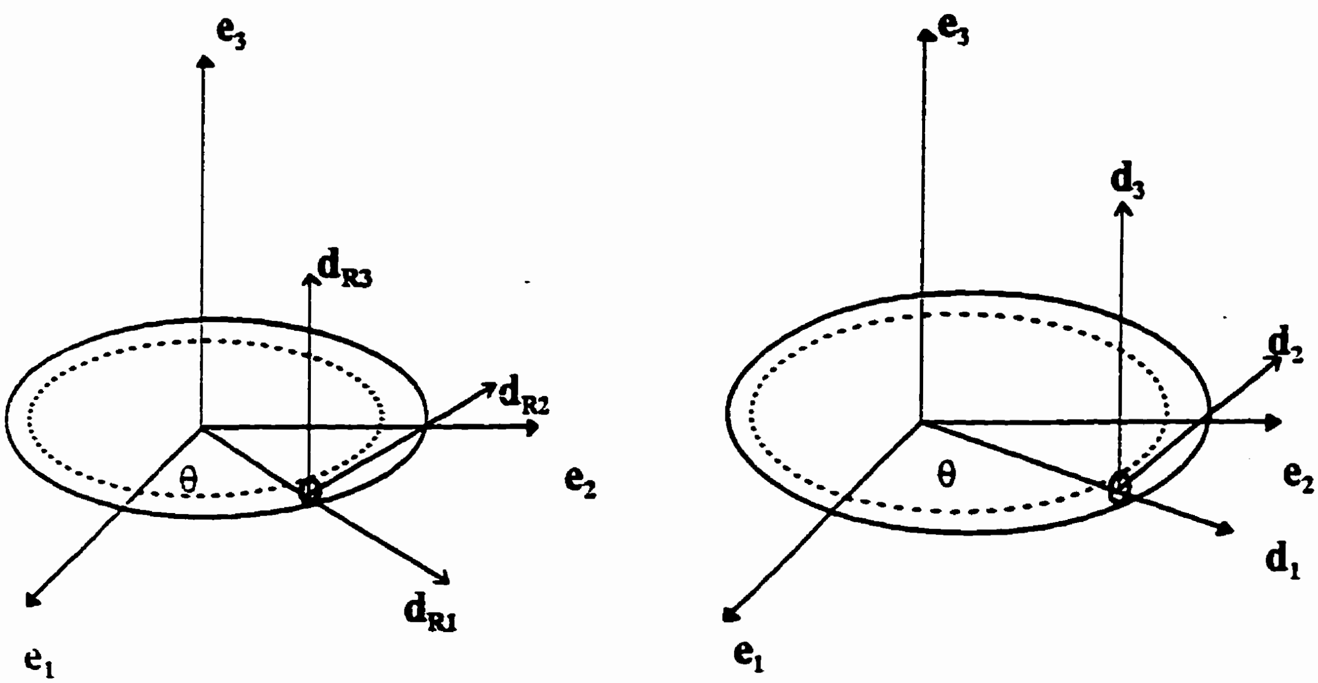
Here ω_0 measures the angle per unit arc length S subtended by a point in the rod and the point $S = 0$. Note the formulas

$$\begin{aligned} \mathbf{e}_r' &= -\omega_0 \sin\omega_0 S \mathbf{e}_1 + \omega_0 \cos\omega_0 S \mathbf{e}_2 = \omega_0\ \mathbf{e}_\theta, \\ \mathbf{e}_\theta' &= -\omega_0 \cos\omega_0 S \mathbf{e}_1 - \omega_0 \sin\omega_0 S \mathbf{e}_2 = -\omega_0 \mathbf{e}_r. \end{aligned} \tag{5.3}$$

The current configuration is defined as

$$\mathbf{r} = a(t)\,\mathbf{e}_r, \tag{5.4}$$

where $a(t)$ is the radius of the circle which varies with time t $(Fig.\ 4)$.



$(Fig.\ 4)$

Note:

1) The initial condition is the reference configuration, at which $t = 0$, $s = S$, $\lambda_a(0) = 1$ and the circular rod is not deformed.

2) The choice of the director is changed for convenience. Now $\mathbf{d}_2 = \mathbf{r}'$ is along the tangent to the rod axis and $(\mathbf{d}_1, \mathbf{d}_3)$ spans the cross-section, where $\mathbf{d}_1 = \lambda_1 \mathbf{e}_r, \mathbf{d}_2 = \lambda_2 \mathbf{e}_\theta$, $\mathbf{d}_3 = \lambda_3 \mathbf{e}_3$ and in general cross-sectional shear will occur unless $\lambda_1 = \lambda_3$.

In this case, the only deformation the rod experiences is extension. Therefore, we have $\mathbf{R} = \mathbf{I}$ and $\mathbf{W}_0 = \mathbf{0}$; the same rotation tensor and wryness tensor as in the previous case (Straight $\longrightarrow$ Straight).

The acceleration $\mathbf{a}$ of the rod is

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{a}(t)\,\mathbf{e}_r \tag{5.5}$$

and

$$\mathbf{r}' = a(t)\,\mathbf{e}_r' = \omega_0 a(t)\ \ \mathbf{e}_\theta = \lambda_2(t)\,\mathbf{e}_\theta, \tag{5.6}$$

with

$$\omega_0 = \lambda_2 / a. \tag{5.7}$$

The dynamic field equations (2.13)- (2.15) become

$$(\mathbf{n} - n_\theta \mathbf{e}_\theta)' = \ \mathbf{0}, \tag{5.8}$$

$$\mathbf{M}' - \mathbf{N} + (\lambda_2(t)\ \ \mathbf{e}_\theta \otimes \mathbf{n})^{\mathsf{T}} + \mathbf{T}\ \ = \ddot{\mathbf{F}}\,\mathbf{E}_R \mathbf{F}^{\mathsf{T}}, \tag{5.9}$$

with

$$Sk\ \mathbf{N} = \mathbf{0},\ \ Sk\ \mathbf{T} = \mathbf{0}, \tag{5.10}$$

and

$$\mathbf{f} = (\ddot{a}\ (t) + n_\theta)\mathbf{e}_r. \tag{5.11}$$

The deformation tensor $\mathbf{F}$ is

$$\mathbf{F} = \mathbf{R}\mathbf{U}_0 = \mathbf{I}\mathbf{U}_0 = \mathbf{U}_0 = \lambda_1(t)\,\mathbf{d}_1 \otimes \mathbf{d}_1 + \lambda_2(t)\,\mathbf{d}_2 \otimes \mathbf{d}_2 + \lambda_3(t)\,\mathbf{d}_3 \otimes \mathbf{d}_3, \tag{5.12}$$

which is

$$\mathbf{F} = \lambda_1(t)\,\mathbf{e}_r \otimes \mathbf{e}_r + \lambda_2(t)\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_3(t)\,\mathbf{e}_3 \otimes \mathbf{e}_3,$$

and

$$\ddot{\mathbf{F}} = \ddot{\lambda}_1\ (t)\,\mathbf{d}_1 \otimes \mathbf{d}_1 + \ \ddot{\lambda}_2\ (t)\,\mathbf{d}_2 \otimes \mathbf{d}_2 + \ \ddot{\lambda}_3\ (t)\,\mathbf{d}_3 \otimes \mathbf{d}_3. \tag{5.13}$$

With $\mathbf{E}_R = E_1 \mathbf{E}_{11} + E_2 \mathbf{E}_{22} + E_3 \mathbf{E}_{33}$,we have

$$\ddot{\mathbf{F}}\ \mathbf{E}_R \mathbf{F}^{\mathsf{T}} \tag{5.14}$$

$$= \ddot{\lambda}_1(t)\,\lambda_1(t)\,E_1\mathbf{d}_1\otimes\mathbf{d}_1 + \ddot{\lambda}_2(t)\,\lambda_2(t)\,E_2\mathbf{d}_2\otimes\mathbf{d}_2 + \ddot{\lambda}_3(t)\,\lambda_3(t)\,E_3\mathbf{d}_3\otimes\mathbf{d}_3$$

$$= \ddot{\lambda}_1(t)\,\lambda_1(t)\,E_1\mathbf{E}_{11} + \ddot{\lambda}_2(t)\,\lambda_2(t)\,E_2\mathbf{E}_{22} + \ddot{\lambda}_3(t)\,\lambda_3(t)\,E_3\mathbf{E}_{33}.$$

$\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}$ is a symmetric tensor, $\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} = \left(\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}\right)^{\mathsf{T}} = \mathbf{F}\mathbf{E}_R^{\mathsf{T}}\,\ddot{\mathbf{F}}^{\mathsf{T}}$.

As with the deformation in the Straight $\longrightarrow$ Straight case, we have the rotation tensor $\mathbf{R} = \mathbf{I}$. Therefore $\mathbf{N} = \mathbf{N}_0$, $\mathbf{M} = \mathbf{M}_0$ are functions which are time-dependent only and $\mathbf{M}' = \mathbf{M}_0' = \mathbf{0}$.

Then (5.9) becomes

$$\mathbf{0} - \mathbf{N}_0 + \left(\lambda_2(t)\ \mathbf{e}_\theta\otimes\mathbf{n}\right)^{\mathsf{T}} + \mathbf{T} = \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}}, \tag{5.15}$$

which implies

$$\mathbf{T} = \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - \left(\lambda_2(t)\ \mathbf{e}_\theta\otimes\mathbf{n}\right)^{\mathsf{T}}, \tag{5.16}$$

this implies

$$\mathbf{T} = \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - \left(\lambda_2(t)\ \mathbf{n}\otimes\mathbf{e}_\theta\right) \tag{5.17}$$

The field equations (5.8) and (5.9), in accordance with (5.10) are satisfied if

$$\mathbf{n} = n_\theta\mathbf{e}_\theta + n_0\ \mathbf{e}_3, \tag{5.18}$$

where n_θ and n_0 are both functions of time t, and

$$\begin{aligned} \mathbf{T} &= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - \left(\lambda_2(t)\cdot\left(n_\theta\mathbf{e}_\theta + n_0\ \mathbf{e}_3\right)\otimes\mathbf{e}_\theta\right) \qquad (5.19)\\ &= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - n_\theta\lambda_2(t)\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta - n_0\lambda_2(t)\,\mathbf{e}_3\otimes\mathbf{e}_\theta. \end{aligned}$$

Since $Sk\ \mathbf{N} = Sk\ \mathbf{N}_0 = \mathbf{0}$ and $sk\ \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} = 0$, $Sk\ \mathbf{T} = \mathbf{0}$ requires

$$n_0 = 0, \tag{5.20}$$

For the special case of quasi-static motion, we have $\ddot{\mathbf{F}}\ \mathbf{E}_R\mathbf{F}^{\mathrm{T}} = \mathbf{0}$. Then (5.19) and (5.18) become

$$\mathbf{T} = \mathbf{N}_0 - n_\theta \lambda_2 (t)\, \mathbf{e}_\theta \otimes \mathbf{e}_\theta \tag{5.21}$$

and

$$\mathbf{n} = \mathbf{n}_0 (t) = n_\theta \mathbf{e}_\theta. \tag{5.22}$$

This is an example of universal motion which is sustained by a special type of body force, one along the normal to the rod.

For a monotropic rod, we assume

$$\mathcal{G}_R^+ = \{(\mathbf{Q}_3, \mathbf{Q}_3), (\mathbf{I}, \mathbf{I})\} \tag{5.23}$$

and

$$\mathcal{G}_R^- = \{(\mathbf{Q}_3, -\mathbf{Q}_3), (\mathbf{I}, \mathbf{I})\}. \tag{5.24}$$

$\mathbf{Q}_3$ is a rotation of angle π about $\mathbf{e}_3$ and is given by $\mathbf{Q}_3 = -\mathbf{I}_3^{\perp} + \mathbf{e}_3 \otimes \mathbf{e}_3 = -\mathbf{E}_{rr} - \mathbf{E}_{\theta\theta} + \mathbf{E}_{33}$.

Since $\mathbf{F} = \mathbf{R}\mathbf{U}_0 = \mathbf{U}_0$, it does not depend on arc length S. Thus, $\mathbf{F}' = \mathbf{0}$ and we have the same equations for monotropic symmetry as in the Straight $\longrightarrow$ Straight case. The monotropic symmetry axis is $\mathbf{d}_3 = \mathbf{R}\mathbf{e}_3 = \mathbf{I}\mathbf{e}_3 = \mathbf{e}_3$; similarly, we have another two monotropic axes, they are $\mathbf{d}_1 = \mathbf{R}\mathbf{e}_r = \mathbf{I}\mathbf{e}_r = \mathbf{e}_r$,and $\mathbf{d}_2 = \mathbf{R}\mathbf{e}_\theta = \mathbf{I}\mathbf{e}_\theta = \mathbf{e}_\theta$. This shows the rod is orthotropic with respect to the axes $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_3$. Equation (3.24) gives all zero terms in $\mathbf{N}$ and $\mathbf{M}$ as

$$\begin{aligned} N_{12} &= N_{13} = N_{23} = 0, \\ M_{12} &= M_{21} = M_{13} = M_{31} = M_{23} = M_{32} = M_{11} = M_{22} = M_{33} = 0. \end{aligned} \tag{5.25}$$

Note that $N_{11} = N_{rr}$, $N_{22} = N_{\theta\theta}$, $N_{12} = N_{21} = N_{r\theta}$, $N_{13} = N_{31} = N_{r3}$ *etc.*, then equation (5.21) can be rewritten as

$$\begin{aligned} N_{r\theta} &= N_{r3} = N_{\theta 3} = 0, \\ M_{r\theta} &= M_{\theta r} = M_{r3} = M_{3r} = M_{\theta 3} = M_{3\theta} = M_{rr} = M_{\theta\theta} = M_{33} = 0. \end{aligned} \tag{5.26}$$

From (5.26), we get $\mathbf{N}_0$ and $\mathbf{M}_0$ as

$$\begin{aligned} \mathbf{N}_0 &= N_{rr}\mathbf{E}_{rr} + N_{\theta\theta}\mathbf{E}_{\theta\theta} + N_{33}\mathbf{E}_{33}, \\ \mathbf{M}_0 &= \mathbf{0}. \end{aligned}$$

Then equations (5.21) and (5.22) may be replaced by

$$\mathbf{T} = N_{rr}\mathbf{E}_{rr} + N_{33}\mathbf{E}_{33} \tag{5.27}$$

and

$$n_\theta = \lambda_2^{-1}(t) N_{\theta\theta}, \tag{5.28}$$

where N_{rr}, $N_{\theta\theta}$ and N_{33} are all time dependent.

If $\mathbf{T} = \mathbf{0}$, we will have

$$N_{rr} = 0, \;\; N_{33} = 0. \tag{5.29}$$

If we knew how $\mathbf{N}$ depends on stretch functions $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$, we could, in principle use equations (5.28) and (5.29) to solve explicitly for λ_1, λ_2 and λ_3 as functions of t.

5.2 Straight $\rightarrow$ Circular

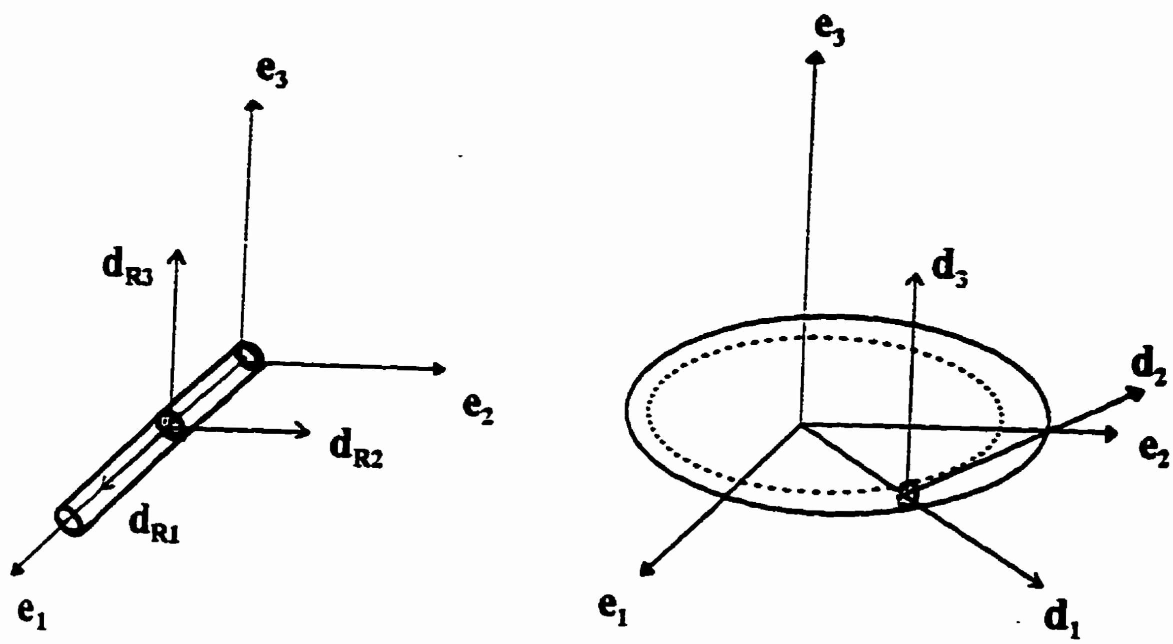
This type of deformation is the nonlinear analogue of pure bending in the case of classical beam theory. The reference configuration is defined as

$$\mathbf{r}_R = S\mathbf{e}_1, \tag{5.30}$$

and the current configuration is defined as

$$\mathbf{r} = a(t)\,\mathbf{R}_3\mathbf{e}_1 = a(t)\,\mathbf{e}_r, \tag{5.31}$$

where $a(t)$ is the radius of the circular rod and varies with time t $(Fig.\ 5)$.



$(Fig.5)$

The orthonormal basis of a plane polar coordinate system is given by

$$\mathbf{e}_r = \cos\omega_0 S\mathbf{e}_1 + \sin\omega_0 S\mathbf{e}_2 \text{ and } \mathbf{e}_\theta = -\sin\omega_0 S\mathbf{e}_1 + \cos\omega_0 S\mathbf{e}_2. \tag{5.32}$$

The rotation and wryness tensors are

$$\begin{aligned} \mathbf{R} &= \mathbf{R}_3 = \cos\omega_0 S\mathbf{I}_3^\perp + \sin\omega_0 S\mathbf{A}_3^\perp + \mathbf{E}_{33} \\ &= \cos\omega_0 S\mathbf{E}_{11} + \cos\omega_0 S\mathbf{E}_{22} + \sin\omega_0 S\mathbf{E}_{21} - \sin\omega_0 S\mathbf{E}_{12} + \mathbf{E}_{33} \end{aligned} \tag{5.33}$$

and

$$\mathbf{W}_0 = \omega_0\mathbf{A}_3^\perp = \omega_0\left(\mathbf{E}_{21} - \mathbf{E}_{12}\right), \tag{5.34}$$

with

$$\omega_0 = \lambda_2(t)\,/\,a. \tag{5.35}$$

The acceleration $\mathbf{a}$ of the rod is

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{a}(t)\,\mathbf{e}_r \tag{5.36}$$

and

$$\mathbf{r}' = a(t)\,\mathbf{e}_r' = \omega_0 a(t)\ \mathbf{e}_\theta = \lambda_2(t)\,\mathbf{e}_\theta. \tag{5.37}$$

In this case, $\mathbf{r}' = \mathbf{d}_2$ is along the tangent of the rod axis and $(\mathbf{d}_1, \mathbf{d}_3)$ spans the rod cross-section.

The deformation tensor $\mathbf{F}$ is

$$\begin{aligned}
\mathbf{F} &= \mathbf{R}_3\mathbf{U}_0 \\
&= \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} + \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22} \\
&\quad +\lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} - \lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} + \lambda_3(t)\,\mathbf{E}_{33},
\end{aligned} \tag{5.38}$$

$$\begin{aligned}
\ddot{\mathbf{F}} &= \ddot{\lambda}_1(t)\cos\omega_0 S\mathbf{E}_{11} + \ddot{\lambda}_2(t)\cos\omega_0 S\mathbf{E}_{22} + \ddot{\lambda}_1(t)\sin\omega_0 S\mathbf{E}_{21} \\
&\quad - \ddot{\lambda}_2(t)\sin\omega_0 S\mathbf{E}_{12} + \lambda_3(t)\,\mathbf{E}_{33}
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
\mathbf{F}' &= \mathbf{W}_0\mathbf{F} \\
&= -\omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} - \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22} \\
&\quad +\omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21} - \omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12}.
\end{aligned} \tag{5.40}$$

Therefore with $\mathbf{E}_R = E_1\mathbf{E}_{11} + E_2\mathbf{E}_{22} + E_3\mathbf{E}_{33}$, we have

$$\begin{aligned}
&\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} \\
&= \left(\lambda_1(t)\,\ddot{\lambda}_1(t)\,E_1\cos^2\omega_0 S + \lambda_2(t)\,\ddot{\lambda}_2(t)\,E_2\sin^2\omega_0 S\right)\mathbf{E}_{11} \\
&\quad + \left(\lambda_1(t)\,\ddot{\lambda}_1(t)\,E_1\sin^2\omega_0 S + \lambda_2(t)\,\ddot{\lambda}_2(t)\,E_2\cos^2\omega_0 S\right)\mathbf{E}_{22}
\end{aligned} \tag{5.41}$$

$$+\left(\left(\lambda_1(t)\ \ddot{\lambda}_1\ (t)\ E_1-\lambda_2(t)\ \ddot{\lambda}_2\ (t)\ E_2\right)\sin\omega_0 S\cos\omega_0 S\right)\mathbf{E}_{12}$$
$$+\left(\left(\lambda_1(t)\ \ddot{\lambda}_1\ (t)\ E_1-\lambda_2(t)\ \ddot{\lambda}_2\ (t)\ E_2\right)\sin\omega_0 S\cos\omega_0 S\right)\mathbf{E}_{21}$$
$$+\lambda_3(t)\ \ddot{\lambda}_3\ (t)\ E_3\mathbf{E}_{33}.$$

The dynamic field equations $(2.13)-(2.15)$ become

$$\mathbf{n}'=\mathbf{0}, \tag{5.42}$$

$$\mathbf{M}'-\mathbf{N}+\left(\lambda_2(t)\ \ \mathbf{e}_\theta\otimes\mathbf{n}\right)^{\mathsf{T}}+\mathbf{T}\ =\ddot{\mathbf{F}}\ \mathbf{E}_R\mathbf{F}^{\mathsf{T}} \tag{5.43}$$

and

$$Sk\ \mathbf{N}=\mathbf{0},\ Sk\ \mathbf{T}=\mathbf{0} \tag{5.44}$$

with

$$\mathbf{f}=\ \ddot{a}\ (t)\,\mathbf{e}_r. \tag{5.45}$$

Since $\mathbf{M}=\mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}}$, $\mathbf{M}_0'=\mathbf{0}$, $\mathbf{R}_3'=\mathbf{W}_0\mathbf{R}_3=\mathbf{R}_3\mathbf{W}_0$ and $\mathbf{W}_0^{\mathsf{T}}=-\mathbf{W}_0$, we have

$$\begin{aligned}\mathbf{M}' &= \mathbf{R}_3'\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}}+\mathbf{R}_3\mathbf{M}_0'\mathbf{R}_3^{\mathsf{T}}+\mathbf{R}_3\mathbf{M}_0\left(\mathbf{R}_3'\right)^{\mathsf{T}} \\ &= \mathbf{W}_0\mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}}+\mathbf{R}_3\mathbf{M}_0\left(\mathbf{W}_0\mathbf{R}_3\right)^{\mathsf{T}} \\ &= \mathbf{W}_0\mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}}-\mathbf{R}_3\mathbf{M}_0\mathbf{R}_3^{\mathsf{T}}\mathbf{W}_0 \\ &= \mathbf{W}_0\mathbf{M}-\mathbf{M}\mathbf{W}_0.\end{aligned} \tag{5.46}$$

Then (5.43) may be replaced by

$$\mathbf{W}_0\mathbf{M}-\mathbf{M}\mathbf{W}_0-\mathbf{N}+\left(\lambda_2(t)\ \ \mathbf{e}_\theta\otimes\mathbf{n}\right)^{\mathsf{T}}+\mathbf{T}\ =\ddot{\mathbf{F}}\ \mathbf{E}_R\mathbf{F}^{\mathsf{T}}. \tag{5.47}$$

Frame-indifference implies for the constraint reaction, provided $\mathbf{n}'=\mathbf{0}$, that

$$\mathbf{n}=\mathbf{R}_3\mathbf{n}_0, \tag{5.48}$$

where $\mathbf{n}_0$ is a vector which does not depend on the arc length S. The field equations (5.42) and (5.43), in accordance with (5.44) and (5.48) are satisfied if

$$\mathbf{n} = n_0\, \mathbf{e}_3 \tag{5.49}$$

where n_0 is a function of time t and

$$\mathbf{T} = \ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} + \mathbf{N} + \mathbf{M}\mathbf{W}_0 - \mathbf{W}_0\mathbf{M} - n_0\lambda_2(t)\;\; \mathbf{e}_3 \otimes \mathbf{e}_\theta. \tag{5.50}$$

Since $\ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}}$, $\mathbf{N}$ and $\mathbf{M}\mathbf{W}_0 - \mathbf{W}_0\mathbf{M}$ are all the symmetric tensors, $Sk\mathbf{T} = \mathbf{0}$ requires

$$n_0 = 0. \tag{5.51}$$

$\mathbf{T}$ may be written as

$$\mathbf{T} = \ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} + \mathbf{N} + \mathbf{M}\mathbf{W}_0 - \mathbf{W}_0\mathbf{M}. \tag{5.52}$$

For the special case of quasi-static motion, we have $\ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} = \mathbf{0}$ and (5.52) becomes

$$\mathbf{T} = \mathbf{N} + \mathbf{M}\mathbf{W}_0 - \mathbf{W}_0\mathbf{M}, \tag{5.53}$$

with

$$n_0 = 0.$$

For a monotropic rod we assume

$$\mathcal{G}_R^+ = \{(\mathbf{Q}_3, \mathbf{Q}_3), (\mathbf{I}, \mathbf{I})\}$$

and

$$\mathcal{G}_R^- = \{(\mathbf{Q}_1, -\mathbf{Q}_1), (\mathbf{I}, \mathbf{I})\}$$

where $\mathbf{Q}_3$ and $\mathbf{Q}_1$ are rotations of angle π about $\mathbf{e}_3$ and $\mathbf{e}_1$ respectively.

We define

$$\begin{aligned}\overline{\mathbf{Q}}_3 \quad : &= \mathbf{R}_3\mathbf{Q}_3\mathbf{R}_3^T \\ &= -\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33} \\ &= \mathbf{Q}_3.\end{aligned} \tag{5.54}$$

Since

$$\begin{aligned}\overline{\mathbf{Q}}_3\mathbf{F} &= -\lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad +\lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} - \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} + \lambda_3(t)\mathbf{E}_{33}\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}\mathbf{Q}_3 &= -\lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad +\lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} - \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} + \lambda_3(t)\mathbf{E}_{33},\end{aligned}$$

we have

$$\overline{\mathbf{Q}}_3\mathbf{F} = \mathbf{F}\mathbf{Q}_3; \tag{5.55}$$

Similarly,

$$\begin{aligned}\overline{\mathbf{Q}}_3\mathbf{F}' &= \omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} + \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22} \\ &\quad +\omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12} - \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21}\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}'\mathbf{Q}_3 &= \omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} + \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22} \\ &\quad +\omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12} - \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21},\end{aligned}$$

imply

$$\overline{\mathbf{Q}}_3\mathbf{F}' = \mathbf{F}'\mathbf{Q}_3. \tag{5.56}$$

This shows that a monotropic symmetry axis is $\mathbf{d}_3 = \mathbf{R}_3\mathbf{e}_3 = \mathbf{e}_3$. Since (5.55) and (5.56) satisfy (3.24.1), equation (3.23.1) gives

$$N_{13} = N_{23} = 0, M_{13} = M_{23} = M_{31} = M_{32} = 0.$$

Note that $N_{11} = N_{rr}, N_{22} = N_{\theta\theta}, N_{12} = N_{r\theta}, N_{13} = N_{r3}$, etc., which means

$$\begin{aligned} N_{r3} &= N_{\theta 3} = 0, \\ M_{r3} &= M_{\theta 3} = M_{3r} = M_{3\theta} = 0. \end{aligned} \tag{5.57}$$

Now we consider $\mathbf{Q} = \mathbf{Q}_1$ and define

$$\begin{aligned} \overline{\mathbf{Q}}_1 &:= \mathbf{R}_3\mathbf{Q}_1\mathbf{R}_3^T \\ &= \cos 2\omega_0 S\mathbf{E}_{11} - \cos 2\omega_0 S\mathbf{E}_{22} + \sin 2\omega_0 S\mathbf{E}_{12} + \sin 2\omega_0 S\mathbf{E}_{21} - \mathbf{E}_{33}. \end{aligned} \tag{5.58}$$

Then we obtain

$$\begin{aligned} \overline{\mathbf{Q}}_1\mathbf{F} &= \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad + \lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} + \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} - \lambda_3(t)\mathbf{E}_{33} \end{aligned}$$

and

$$\begin{aligned} \mathbf{F}\mathbf{Q}_1 &= \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad + \lambda_2(t)\sin\omega_0 S\mathbf{E}_{12} + \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} - \lambda_3(t)\mathbf{E}_{33}, \end{aligned}$$

which imply

$$\overline{\mathbf{Q}}_1\mathbf{F} = \mathbf{F}\mathbf{Q}_1. \tag{5.59}$$

Also,

$$\begin{aligned} \overline{\mathbf{Q}}_1\mathbf{F}' &= \omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} - \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22} \\ &\quad - \omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12} - \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21} \end{aligned}$$

and

$$\begin{aligned}\mathbf{F}'\mathbf{Q}_1 &= -\omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} + \omega_0\lambda_2(t)\sin\omega_0 S\mathbf{E}_{22} \\ &\quad +\omega_0\lambda_2(t)\cos\omega_0 S\mathbf{E}_{12} + \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21},\end{aligned}$$

imply

$$\overline{\mathbf{Q}}_1\mathbf{F}' = -\mathbf{F}'\mathbf{Q}_1. \tag{5.60}$$

This shows that a monotropic symmetry axis is $\mathbf{d}_1 = \mathbf{R}_3\mathbf{e}_1 = \mathbf{e}_r$. Since (5.59) and (5.60) satisfy (3.24.2), therefore equation (3.23.2) gives

$$\begin{aligned}N_{12} &= N_{13} = 0, \\ M_{11} &= M_{22} = M_{33} = M_{23} = M_{32} = 0.\end{aligned}$$

Note that $N_{11} = N_{rr}, N_{22} = N_{\theta\theta}, N_{12} = N_{r\theta}, N_{13} = N_{r3}$, etc., which means

$$\begin{aligned}N_{r\theta} &= N_{r3} = 0, \\ M_{rr} &= M_{\theta\theta} = M_{33} = M_{\theta 3} = M_{3\theta} = 0.\end{aligned} \tag{5.61}$$

cf. Note (2) on page 36. Combining (5.57) and (5.61), we have $\mathbf{N}$ and $\mathbf{M}$ as:

$$\mathbf{N} = N_{rr}\mathbf{E}_{rr} + N_{\theta\theta}\mathbf{E}_{\theta\theta} + N_{33}\mathbf{E}_{33}, \mathbf{M} = M_{r\theta}\mathbf{E}_{r\theta} + M_{\theta r}\mathbf{E}_{\theta r}. \tag{5.62}$$

Therefore,

$$\begin{aligned}\mathbf{M}' &= \mathbf{M}\mathbf{W}_0 - \mathbf{W}_0\mathbf{M} \qquad (5.63) \\ &= (M_{r\theta}\mathbf{E}_{r\theta} + M_{\theta r}\mathbf{E}_{\theta r})(-\omega_0\mathbf{E}_{r\theta} + \omega_0\mathbf{E}_{\theta r}) - (-\omega_0\mathbf{E}_{r\theta} + \omega_0\mathbf{E}_{\theta r})(M_{r\theta}\mathbf{E}_{r\theta} + M_{\theta r}\mathbf{E}_{\theta r}) \\ &= (\omega_0 M_{r\theta}\mathbf{E}_{rr} - \omega_0 M_{\theta r}\mathbf{E}_{\theta\theta}) - (-\omega_0 M_{\theta r}\mathbf{E}_{rr} + \omega_0 M_{r\theta}\mathbf{E}_{\theta\theta}) \\ &= \omega_0 M_{r\theta}\mathbf{E}_{rr} - \omega_0 M_{\theta r}\mathbf{E}_{\theta\theta} + \omega_0 M_{\theta r}\mathbf{E}_{rr} - \omega_0 M_{r\theta}\mathbf{E}_{\theta\theta} \\ &= \omega_0(M_{r\theta} + M_{\theta r})\mathbf{E}_{rr} - \omega_0(M_{\theta r} + M_{r\theta})\mathbf{E}_{\theta\theta}.\end{aligned}$$

Then (5.53) can be written as

$$n_0 = 0, \; \mathbf{T} = \{N_{rr} + \omega_0 (M_{r\theta} + M_{\theta r})\} \mathbf{E}_{rr} + N_{33}\mathbf{E}_{33} \tag{5.64}$$

and

$$\{N_{\theta\theta} - \omega_0 (M_{\theta r} + M_{r\theta})\} \mathbf{E}_{\theta\theta} = 0 \tag{5.65}$$

where N_{rr}, $N_{\theta\theta}$, N_{33}, $M_{r\theta}$ and $M_{\theta r}$ are all functions of time t.

If we assume $\mathbf{T} = \mathbf{0}$, then we will have the following equations:

$$N_{rr} + \omega_0 (M_{r\theta} + M_{\theta r}) = 0 \tag{5.66}$$

and

$$N_{33} = 0. \tag{5.67}$$

If we knew how $\mathbf{N}$ and $\mathbf{M}$ depend on stretch functions $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$, we could, in principle use equations (5.65), (5.66) and (5.27) to solve explicitly for λ_1, λ_2 and λ_3 as functions of t.

CHAPTER 6

Simpler Universal Solutions: Helical $\longrightarrow$ Helical and Straight$\longrightarrow$ Helical

In this chapter, we derive two deformations, an initially helical rod which is deformed into a helical rod and an initially straight rod which is deformed into a helical rod.

6.1 Helical$\longrightarrow$ Helical

This type of deformation is similar to a combination of Straight $\longrightarrow$ Straight Twisted and Straight $\longrightarrow$ Circular. The reference configuration is defined as

$$\mathbf{r}_R = a_0\mathbf{e}_r + \left(b_0 \big/ \sqrt{a_0^2 + b_0^2}\right) S\mathbf{e}_3 \tag{6.1}$$

with

$$c_0 := \sqrt{a_0^2 + b_0^2},$$

where a_0 is the radius of the initial helix and b_0 is the pitch; both of them are constant.

The current configuration is defined as

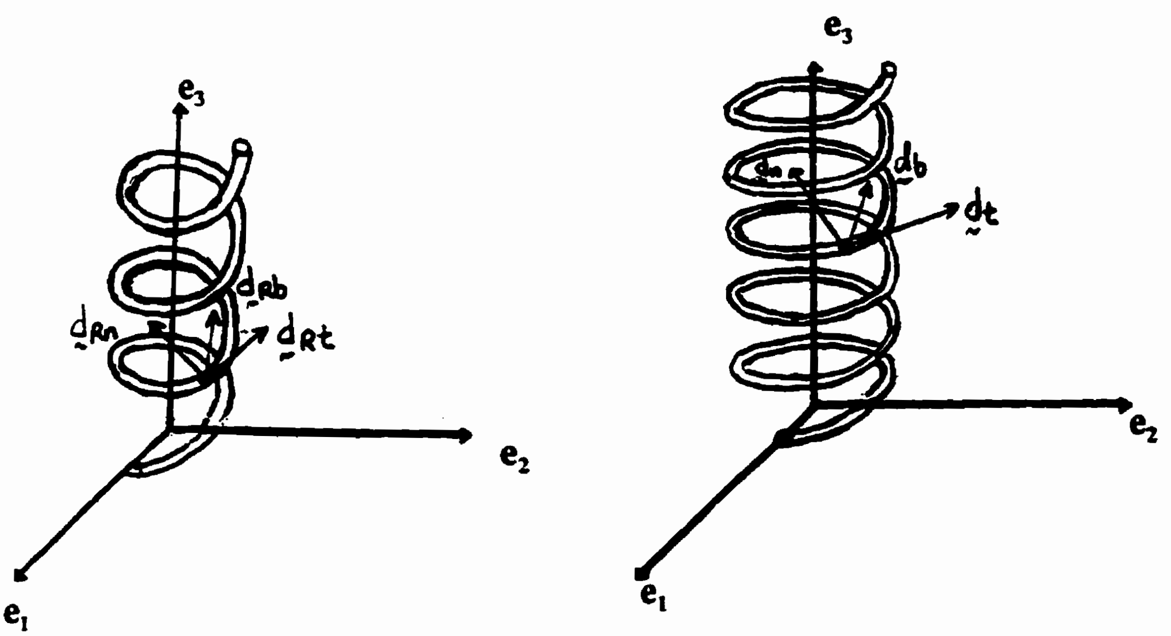
$$\begin{aligned} \mathbf{r} &= a(t)\,\mathbf{e}_r + \left(b(t) \big/ \sqrt{a^2(t) + b^2(t)}\right) s\mathbf{e}_3 \\ & a\mathbf{e}_r + \left(b\lambda_3 \big/ \sqrt{a^2 + b^2}\right) S\mathbf{e}_3 \end{aligned} \tag{6.2}$$

with

$$c := \sqrt{a^2 + b^2}.$$

The radius a and pitch b of the helical rod in the current configuration are both functions of time t $(Fig.\ 6)$.

Note in this case that $\mathbf{d}_n = \mathbf{n}$, $\mathbf{d}_b = \mathbf{b}$ and $\mathbf{d}_t = \mathbf{t}$, where $(\mathbf{n}, \mathbf{b}, \mathbf{t})$ is the Frenet-Serret basis, are the unit principal normal, binormal and tangent, respectively in the reference configuration, dependent on initial arc length S. Since we'll assume the deformation of the directors is only in length, the notation $(\mathbf{d}_n, \mathbf{d}_b, \mathbf{d}_t)$ refers to reference directors throughout this chapter. The deformation is completely specified by the stretch functions $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$. So $(\lambda_1(t)\,\mathbf{d}_n, \lambda_2(t)\,\mathbf{d}_b, \lambda_3(t)\,\mathbf{d}_t)$ are the directors in the current configuration corresponding to $(\mathbf{d}_n, \mathbf{d}_b, \mathbf{d}_t)$ in the reference configuration respectively. Here $(\mathbf{d}_n, \mathbf{d}_b)$ spans the cross-section and $\mathbf{d}_t$ is along the natural tangent to the rod axis. $\lambda_3(t) = ds/dS$. In this thesis, we assume $a(t) = \lambda_3(t)\,a_0$ and $b(t) = \lambda_3(t)\,b_0$, the special motion, therefore, $c(t) = \lambda_3(t)\,c_0$.



$(Fig.\ 6)$

The relationship between $(\mathbf{d}_n, \mathbf{d}_b, \mathbf{d}_t)$ and the fixed orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is

$$\mathbf{d}_n = -\cos\omega_0 S\mathbf{e}_1 - \sin\omega_0 S\mathbf{e}_2, \tag{6.3}$$

$$\mathbf{d}_b = -(b/c)\cos\omega_0 S\mathbf{e}_2 + (b/c)\sin\omega_0 S\mathbf{e}_1 + (a/c)\,\mathbf{e}_3,$$

$$\mathbf{d}_t = (a/c)\cos\omega_0 S\mathbf{e}_2 - (a/c)\sin\omega_0 S\mathbf{e}_1 + (b/c)\,\mathbf{e}_3.$$

As we mentioned before, we have $a(t) = \lambda_3(t)\,a_0$, $b(t) = \lambda_3(t)\,b_0$, and $c(t) = \lambda_3(t)\,c_0$, therefore, $b/c = b_0/c_0 = const$, and $a/c = a_0/c_0 = const$. We also assume that, for the quasi static motion, $\omega_0 = \lambda_3/c = const$, then we have $\dot{\mathbf{d}}_n = \mathbf{0}$, $\dot{\mathbf{d}}_b = \mathbf{0}$, and $\dot{\mathbf{d}}_t = \mathbf{0}$. This indicates that the orientation of $\mathbf{d}_n$, $\mathbf{d}_b$ and $\mathbf{d}_t$ won't change when time t changes.

The relationship between the orthonormal basis of a plane polar coordinate system $(\mathbf{e}_r, \mathbf{e}_\theta)$ and the fixed orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ is

$$\begin{aligned} \mathbf{e}_r &= \cos\omega_0 S\mathbf{e}_1 + \sin\omega_0 S\mathbf{e}_2, \\ \mathbf{e}_\theta &= -\sin\omega_0 S\mathbf{e}_1 + \cos\omega_0 S\mathbf{e}_2. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4), we obtain the relationship between $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3)$ and $(\mathbf{d}_n, \mathbf{d}_b, \mathbf{d}_t)$ as follows

$$\begin{aligned} \mathbf{e}_r &= -\mathbf{d}_n, \\ \mathbf{e}_\theta &= -(b/c)\,\mathbf{d}_b + (a/c)\,\mathbf{d}_t, \\ \mathbf{e}_3 &= (a/c)\,\mathbf{d}_b + (b/c)\,\mathbf{d}_t. \end{aligned} \tag{6.5}$$

Then the current configuration $\mathbf{r}$ becomes

$$\begin{aligned} \mathbf{r} &= a\mathbf{e}_r + \left(b\lambda_3/\sqrt{a^2+b^2}\right) S\mathbf{e}_3 \\ &= -a\mathbf{d}_n + \left(ab/c^2\right)\lambda_3 S\mathbf{d}_b + \left(b^2/c^2\right)\lambda_3 S\mathbf{d}_t \\ &= -a\mathbf{d}_n + b\overline{\kappa} S\mathbf{d}_b + b\overline{\tau} S\mathbf{d}_t \\ &= -a\mathbf{d}_n + bS\left(\overline{\kappa}\mathbf{d}_b + \overline{\tau}\mathbf{d}_t\right) \end{aligned} \tag{6.6}$$

where the curvature κ and the torsion τ are given by $\kappa = a/c^2$, $\tau = b/c^2$. Since we have $\omega_0 = \lambda_3/c = const$, then $\overline{\kappa} = \lambda_3\kappa = (\lambda_3/c)(a/c) = \omega_0(a/c) = const$ and $\overline{\tau} = \lambda_3\tau = (\lambda_3/c)(b/c) = \omega_0(b/c) = const$, therefore, we have $\dot{\overline{\kappa}} = 0$ and $\dot{\overline{\tau}} = 0$.

From equation (6.6), we notice that $\mathbf{r} = a\mathbf{e}_r + \left(b\lambda_3/\sqrt{a^2+b^2}\right) S\mathbf{e}_3 = \lambda_3(a_0\mathbf{e}_r + \left(b_0/\sqrt{a_0^2+b_0^2}\right) S\mathbf{e}_3) = \lambda_3\mathbf{r}_R$, we call this kind of motion *central expansion*.

From (6.6), we have the acceleration $\mathbf{a}$ of the rod as

$$\begin{aligned}\mathbf{a} \;=\; \ddot{\mathbf{r}} = \; & -\ddot{a}\,\mathbf{d}_n - 2\,\dot{a}\dot{\mathbf{d}}_n - a\,\ddot{\mathbf{d}}_n + (\ddot{b}\,S\bar{\kappa} + 2\,\dot{b}\,S\,\dot{\bar{\kappa}} + bS\,\ddot{\bar{\kappa}})\mathbf{d}_b \\ & + (2\,\dot{b}\,S\bar{\kappa} + 2bS\,\dot{\bar{\kappa}})\,\dot{\mathbf{d}}_b + bS\bar{\kappa}\,\ddot{\mathbf{d}}_b + (\ddot{b}\,S\bar{\tau} + 2\,\dot{b}\,S\,\dot{\bar{\tau}} + bS\,\ddot{\bar{\tau}})\mathbf{d}_t \\ & + (2\,\dot{b}\,S\bar{\tau} + 2bS\,\dot{\bar{\tau}})\,\dot{\mathbf{d}}_t + bS\bar{\tau}\,\ddot{\mathbf{d}}_t,\end{aligned} \tag{6.7}$$

and

$$\mathbf{r}' = \lambda_3\,(t)\,\mathbf{d}_t. \tag{6.8}$$

Since we have $\dot{\mathbf{d}}_n = \mathbf{0}$, $\dot{\mathbf{d}}_b = \mathbf{0}$, $\dot{\mathbf{d}}_t = \mathbf{0}$ and $\dot{\bar{\kappa}} = 0$, $\dot{\bar{\tau}} = 0$, then equation (6.7) can be simplified as

$$\mathbf{a} = \ddot{\mathbf{r}} = -\,\ddot{a}\,\mathbf{d}_n + \ddot{b}\,S\bar{\kappa}\mathbf{d}_b + \ddot{b}\,S\bar{\tau}\mathbf{d}_t.$$

The dynamic field equations (2.13) − (2.15) become

$$\mathbf{n}' = \mathbf{0}, \tag{6.9}$$

$$\mathbf{M}' - \mathbf{N} + (\lambda_3\,(t)\,\mathbf{d}_t \;\otimes \mathbf{n})^{\mathsf{T}} + \mathbf{T} \;= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} \tag{6.10}$$

with

$$Sk\;\mathbf{N} = \mathbf{0},\;\; Sk\;\mathbf{T} = \mathbf{0}, \tag{6.11}$$

and provided

$$\mathbf{f} = -\,\ddot{a}\,(t)\,\mathbf{d}_n,\;\; \ddot{b}\,(t) = 0. \tag{6.12}$$

The deformation tensor $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{R}\mathbf{U}_0 = \mathbf{I}\mathbf{U}_0 = \mathbf{U}_0, \tag{6.13}$$

since $\mathbf{R} = \mathbf{I}$, $\mathbf{F}$ may be written as

$$\mathbf{F} = \lambda_1(t)\, \mathbf{d}_n \otimes \mathbf{d}_n + \lambda_2(t)\, \mathbf{d}_b \otimes \mathbf{d}_b + \lambda_3(t)\, \mathbf{d}_t \otimes \mathbf{d}_t,$$

and then

$$\ddot{\mathbf{F}} = \ddot{\lambda}_1(t)\, \mathbf{d}_n \otimes \mathbf{d}_n + \ddot{\lambda}_2(t)\, \mathbf{d}_b \otimes \mathbf{d}_b + \ddot{\lambda}_3(t)\, \mathbf{d}_t \otimes \mathbf{d}_t. \tag{6.14}$$

With $\mathbf{E}_R = E_n \mathbf{E}_{nn} + E_b \mathbf{E}_{bb} + E_t \mathbf{E}_{tt}$, we have

$$\begin{aligned} & \ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} \\ = \; & \ddot{\lambda}_1(t)\, \lambda_1(t)\, E_n \mathbf{E}_{nn} + \ddot{\lambda}_2(t)\, \lambda_2(t)\, E_b \mathbf{E}_{bb} + \ddot{\lambda}_3(t)\, \lambda_3(t)\, E_t \mathbf{E}_{tt}. \end{aligned} \tag{6.15}$$

$\ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}}$ is a symmetric tensor and $\ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} = \left(\ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}}\right)^{\mathsf{T}} = \mathbf{F} \mathbf{E}_R^{\mathsf{T}}\, \ddot{\mathbf{F}}^{\mathsf{T}}$.

As in the previous cases, we have $\mathbf{R} = \mathbf{I}$, so $\mathbf{N} = \mathbf{N}_0, \mathbf{M} = \mathbf{M}_0$ are functions of time t and independent of the arc length S. Thus $\mathbf{M}' = \mathbf{M}_0' = \mathbf{0}$.

Then (6.10) becomes

$$\mathbf{0} - \mathbf{N} + (\lambda_3(t)\, \mathbf{d}_t \otimes \mathbf{n})^{\mathsf{T}} + \mathbf{T} = \ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} \tag{6.16}$$

which implies

$$\begin{aligned} \mathbf{T} &= \ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - (\lambda_3(t)\, \mathbf{d}_t \otimes \mathbf{n})^{\mathsf{T}} \\ &= \ddot{\mathbf{F}}\, \mathbf{E}_R \mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - (\lambda_3(t)\; \mathbf{n} \otimes \mathbf{d}_t). \end{aligned} \tag{6.17}$$

The field equations (6.9) and (6.10), in accordance with (6.11) are satisfied if

$$\begin{aligned} \mathbf{n} &= n_0\, \mathbf{e}_3 \\ &= n_0\left((a/c)\, \mathbf{d}_b + (b/c)\, \mathbf{d}_t\right) \\ &= n_0\,(a/c)\, \mathbf{d}_b + n_0\,(b/c)\, \mathbf{d}_t, \end{aligned} \tag{6.18}$$

where n_0 depends only on time t.

Putting (6.18) into (6.17), we get

$$
\begin{aligned}
\mathbf{T} &= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - \{\lambda_3(t)\cdot[n_0(a/c)\,\mathbf{d}_b + n_0(b/c)\,\mathbf{d}_t]\otimes\mathbf{e}_t\} \qquad (6.19)\\
&= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N}_0 - n_0\lambda_3(t)\,[(a/c)\,\mathbf{E}_{bt} + (b/c)\,\mathbf{E}_{tt}]\,.
\end{aligned}
$$

Since $Sk\ \mathbf{N} = Sk\ \mathbf{N}_0 = \mathbf{0}$ and $\ddot{\mathbf{F}}\ \mathbf{E}_R\mathbf{F}^{\mathsf{T}}$ is a symmetric tensor, $Sk\ \mathbf{T} = \mathbf{0}$ requires

$$n_0 = 0. \qquad (6.20)$$

For the special case of quasi-static motion, we have $\ddot{\mathbf{F}}\ \mathbf{E}_R\mathbf{F}^{\mathsf{T}} = \mathbf{0}$. Then (6.19) and (6.20) become

$$\mathbf{T} = \mathbf{N}_0, \qquad (6.21)$$

and

$$\mathbf{n}_0 = n_0\mathbf{e}_3 = \mathbf{0}. \qquad (6.22)$$

For a monotropic rod, we assume that

$$\mathcal{G}_R^+ = \{(\mathbf{Q}_3, \mathbf{Q}_3)\,, (\mathbf{I}, \mathbf{I})\}\,, \qquad (6.23)$$

and

$$\mathcal{G}_R^- = \{(\mathbf{Q}_3, -\mathbf{Q}_3)\,, (\mathbf{I}, \mathbf{I})\}\,. \qquad (6.24)$$

In these definitions, $\mathbf{Q}_3$ is a rotation of angle π about $\mathbf{t}$ and is given by $\mathbf{Q}_3 = -\mathbf{I}_t^{\perp} + \mathbf{d}_t \otimes \mathbf{d}_t = -\mathbf{E}_{nn} - \mathbf{E}_{bb} + \mathbf{E}_{tt}$.

Since $\mathbf{F} = \mathbf{R}\mathbf{U}_0 = \mathbf{I}\mathbf{U}_0 = \mathbf{U}_0$, it does not depend on arc length S. Thus, $\mathbf{F}' = \mathbf{0}$, and we have the same equations for monotropic symmetry as in the Straight $\longrightarrow$ Straight case. The monotropic symmetry axes are $\mathbf{d}_n = \mathbf{R}\mathbf{n} = \mathbf{I}\mathbf{n} = \mathbf{n}$, $\mathbf{d}_b = \mathbf{R}\mathbf{b} = \mathbf{I}\mathbf{b} = \mathbf{b}$ and

$\mathbf{d}_t = \mathbf{Rt} = \mathbf{It} = \mathbf{t}$; this indicates the rod is orthotropic with respect to the axes $\mathbf{n}, \mathbf{b}$ and $\mathbf{t}$.

Equation (3.23) gives all the zero terms in $\mathbf{N}$ and $\mathbf{M}$ as

$$
\begin{aligned}
N_{12} &= N_{13} = N_{23} = 0, \\
M_{12} &= M_{21} = M_{13} = M_{31} = M_{23} = M_{32} = M_{11} = M_{22} = M_{33} = 0.
\end{aligned} \tag{6.25}
$$

Note that $N_{11} = N_{nn}, N_{22} = N_{bb}, N_{12} = N_{nb}, N_{13} = N_{31} = N_{nt}$, etc.. (6.25) becomes

$$
\begin{aligned}
N_{nb} &= N_{nt} = N_{bt} = 0, \\
M_{nb} &= M_{bn} = M_{nt} = M_{tn} = M_{bt} = M_{tb} = M_{nn} = M_{bb} = M_{tt} = 0.
\end{aligned} \tag{6.26}
$$

Finally, we have $\mathbf{N}_0$ and $\mathbf{M}_0$ given as

$$
\begin{aligned}
\mathbf{N}_0 &= N_{nn}\mathbf{E}_{nn} + N_{bb}\mathbf{E}_{bb} + N_{tt}\mathbf{E}_{tt}, \\
\mathbf{M}_0 &= \mathbf{0}.
\end{aligned} \tag{6.27}
$$

Replacing (6.21) by (6.27), we get

$$
n_0 = 0, \ \mathbf{T} = \mathbf{N}_0 = N_{nn}\mathbf{E}_{nn} + N_{bb}\mathbf{E}_{bb} + N_{tt}\mathbf{E}_{tt} \tag{6.28}
$$

where N_{nn}, N_{bb} and N_{tt} are all functions of time t.

If $\mathbf{T} = \mathbf{0}$, we will have the following equations:

$$
N_{nn} = 0, \ N_{bb} = 0, \ N_{tt} = 0. \tag{6.29}
$$

If we knew how $\mathbf{N}$ depends on stretch functions $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$, we could, in principle use equations (6.28) and (6.29) to solve explicitly for λ_1, λ_2 and λ_3 as functions of t.

6.2 Straight $\longrightarrow$ Helical

Compared with the previous deformations considered, this deformation is the most interesting and difficult one. The reference configuration is defined as

$$\mathbf{r}_R = S\mathbf{e}_3. \tag{6.30}$$

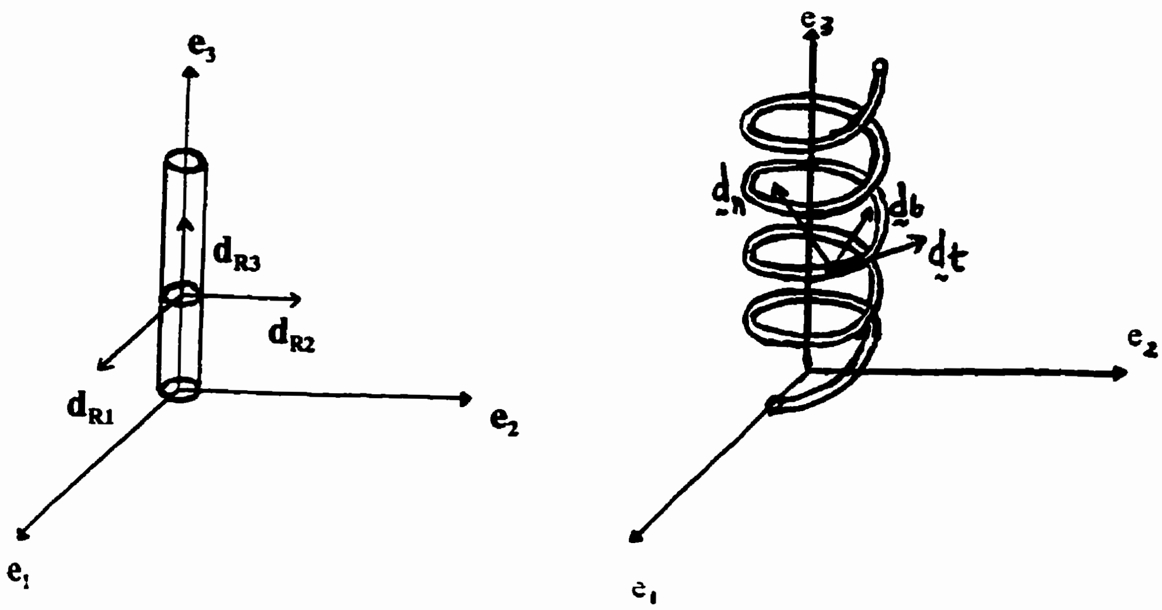
The current configuration is defined as

$$\mathbf{r} = a(t)\,\mathbf{R}_3\mathbf{e}_1 + b(t)\,\omega_0 S\mathbf{e}_3, \tag{6.31}$$

where $a(t)$ and $b(t)$ are the radius and pitch of the helix respectively. $\omega_0 = \lambda_3/c$, where $c = \sqrt{a^2+b^2}$. The curvature κ and torsion τ of the axis are given by: $\kappa = a/c^2$ and $\tau = b/c^2\,(Fig.\ 7)$ .In this thesis, we only consider the case in which ω_0 is a constant.

The rotation $\mathbf{R}_3$ is given by

$$\begin{aligned}\mathbf{R}_3 &= \cos\omega_0 S\mathbf{I}_3^{\perp} + \sin\omega_0 S\mathbf{A}_3^{\perp} + \mathbf{E}_{33}\\ &= \cos\omega_0 S\mathbf{E}_{11} + \cos\omega_0 S\mathbf{E}_{22} + \sin\omega_0 S\mathbf{E}_{21} - \sin\omega_0 S\mathbf{E}_{12} + \mathbf{E}_{33}.\end{aligned}$$



$(Fig.7)$

The directors spanning the cross-section are defined by

$$\mathbf{d}_a = \mathbf{R}_3\mathbf{R}_1\mathbf{e}_a, \tag{6.32}$$

where $\mathbf{R}_1$ is defined as

$$\mathbf{R}_1 := (b/c)\,\mathbf{I}_1^{\perp} + (a/c)\,\mathbf{A}_1^{\perp} + \mathbf{E}_{11} \tag{6.33}$$

and $\mathbf{I}_1^{\perp} := \mathbf{E}_{22} + \mathbf{E}_{33}$, $\mathbf{A}_1^{\perp} := \mathbf{E}_{23} - \mathbf{E}_{32}$. Note that in this example, both $\mathbf{R}_3$ and $\mathbf{R}_1$ are assumed to be time independent.

Substituting for $\mathbf{R}_3$ and $\mathbf{R}_1$ in (6.32) gives

$$\begin{aligned}
\mathbf{d}_a &= \mathbf{R}_3\mathbf{R}_1\mathbf{e}_a \\
&= [\cos\omega_0 S\,(\mathbf{E}_{11} + \mathbf{E}_{22}) + \sin\omega_0 S\mathbf{E}_{21} - \sin\omega_0 S\mathbf{E}_{12} + \mathbf{E}_{33}] \\
&\quad \cdot [(b/c)\,\mathbf{E}_{22} + (b/c)\,\mathbf{E}_{33} + (a/c)\,\mathbf{E}_{23} - (a/c)\,\mathbf{E}_{32} + \mathbf{E}_{11}]\,\mathbf{e}\alpha \\
&= \begin{bmatrix} \cos\omega_0 S\mathbf{E}_{11} + (b/c)\cos\omega_0 S\mathbf{E}_{22} + (a/c)\cos\omega_0 S\mathbf{E}_{23} + \sin\omega_0 S\mathbf{E}_{21} \\ -(b/c)\sin\omega_0 S\mathbf{E}_{12} - (a/c)\sin\omega_0 S\mathbf{E}_{13} + (b/c)\,\mathbf{E}_{33} - (a/c)\,\mathbf{E}_{32} \end{bmatrix} \mathbf{e}_a.
\end{aligned} \tag{6.34}$$

Therefore,

$$\begin{aligned}
\mathbf{d}_1 &= \cos\omega_0 S\mathbf{e}_1 + \sin\omega_0 S\mathbf{e}_2, \\
\mathbf{d}_2 &= (b/c)\cos\omega_0 S\mathbf{e}_2 - (b/c)\sin\omega_0 S\mathbf{e}_1 - (a/c)\,\mathbf{e}_3, \\
\mathbf{d}_3 &= (a/c)\cos\omega_0 S\mathbf{e}_2 - (a/c)\sin\omega_0 S\mathbf{e}_1 + (b/c)\,\mathbf{e}_3.
\end{aligned} \tag{6.35}$$

We define $\mathbf{d}_n = -\mathbf{d}_1, \mathbf{d}_b = -\mathbf{d}_2$ and $\mathbf{d}_t = \mathbf{d}_3$, where $(\mathbf{d}_n, \mathbf{d}_b, \mathbf{d}_t)$, the Frenet-Serret basis, are the unit principal normal, binormal and tangent respectively. Here, $(\mathbf{d}_n, \mathbf{d}_b)$ spans the cross-section and $\mathbf{d}_t$ is along the tangent to the rod axis.

$$\begin{aligned}
\mathbf{d}_n &= -\cos\omega_0 S\mathbf{e}_1 - \sin\omega_0 S\mathbf{e}_2, \\
\mathbf{d}_b &= -(b/c)\cos\omega_0 S\mathbf{e}_2 + (b/c)\sin\omega_0 S\mathbf{e}_1 + (a/c)\,\mathbf{e}_3, \\
\mathbf{d}_t &= (a/c)\cos\omega_0 S\mathbf{e}_2 - (a/c)\sin\omega_0 S\mathbf{e}_1 + (b/c)\,\mathbf{e}_3.
\end{aligned} \tag{6.36}$$

The directors

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{R}_3\mathbf{e}_1 = [\cos\omega_0 S\,(\mathbf{E}_{11}+\mathbf{E}_{22}) + \sin\omega_0 S\,(\mathbf{E}_{21}-\mathbf{E}_{12}) + \mathbf{E}_{33}]\,\mathbf{e}_1 \qquad (6.37)\\
&= \cos\omega_0 S\mathbf{e}_1 + \sin\omega_0 S\mathbf{e}_2,\\
\mathbf{e}_\theta &= \mathbf{R}_3\mathbf{e}_2 = [\cos\omega_0 S\,(\mathbf{E}_{11}+\mathbf{E}_{22}) + \sin\omega_0 S\,(\mathbf{E}_{21}-\mathbf{E}_{12}) + \mathbf{E}_{33}]\,\mathbf{e}_2\\
&= \cos\omega_0 S\mathbf{e}_2 - \sin\omega_0 S\mathbf{e}_1
\end{aligned}$$

comprise the orthonormal basis of a plane polar coordinate system $(\mathbf{e}_r, \mathbf{e}_\theta)$ in which $\theta = S/a$.

From (6.36), we have

$$\begin{aligned}
\mathbf{d}_t &= (a/c)\,(-\sin\omega_0 S\mathbf{e}_1 + \cos\omega_0 S\mathbf{e}_2) + (b/c)\,\mathbf{e}_3 \qquad (6.38)\\
&= (a/c)\,\mathbf{e}_\theta + (b/c)\,\mathbf{e}_3,\\
\mathbf{d}_b &= (b/c)\,(\sin\omega_0 S\mathbf{e}_1 - \cos\omega_0 S\mathbf{e}_2) + (a/c)\,\mathbf{e}_3\\
&= -(b/c)\,\mathbf{e}_\theta + (a/c)\,\mathbf{e}_3
\end{aligned}$$

and

$$\sin\omega_0 S\mathbf{e}_1 - \cos\omega_0 S\mathbf{e}_2 = (c/b)\,(\mathbf{d}_b - (a/c)\,\mathbf{e}_3)\,. \qquad (6.39)$$

Replacing (6.38) by (6.39) gives

$$\begin{aligned}
\mathbf{d}_t &= -(a/c)\,(\sin\omega_0 S\mathbf{e}_1 - \cos\omega_0 S\mathbf{e}_2) + (b/c)\,\mathbf{e}_3\\
&= -(a/c)\,(c/b)\,[\mathbf{d}_b - (a/c)\,\mathbf{e}_3] + (b/c)\,\mathbf{e}_3\\
&= -(a/b)\,\mathbf{d}_b + (a^2/bc)\,\mathbf{e}_3 + (ab^2/bc)\,\mathbf{e}_3 = -(a/b)\,\mathbf{d}_b + (c/b)\,\mathbf{e}_3.
\end{aligned}$$

Therefore,

$$\mathbf{e}_3 = (b/c)\,[\mathbf{d}_t + (a/b)\,\mathbf{d}_b] = (b/c)\,\mathbf{d}_t + (a/c)\,\mathbf{d}_b. \qquad (6.40)$$

By replacing $\mathbf{e}_r$ and $\mathbf{e}_3$ by $\mathbf{d}_n, \mathbf{d}_b$ and $\mathbf{d}_t$, we obtain the current configuration

$$\mathbf{r} = a\mathbf{e}_r + \left(b\lambda_3/\sqrt{a^2+b^2}\right) S\mathbf{e}_3 = -a\mathbf{d}_n + bS\left(\overline{\kappa}\mathbf{d}_b + \overline{\tau}\mathbf{d}_t\right) \tag{6.41}$$

and

$$\begin{aligned} \mathbf{r}' &= a\omega_0\mathbf{e}_\theta + \lambda_3 (b/c)\,\mathbf{e}_3 \\ &= \lambda_3\left[(a/c)\,\mathbf{e}_\theta + (b/c)\,\mathbf{e}_3\right] \\ &= \lambda_3\mathbf{d}_t. \end{aligned} \tag{6.42}$$

The deformation tensor $\mathbf{F}$ is given by

$$\begin{aligned} \mathbf{F} &= \mathbf{R}\mathbf{U}_0 = \mathbf{R}_3\mathbf{R}_1\mathbf{U}_0 \\ &= \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} + \lambda_2(t)(b/c)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad +\lambda_3(t)(a/c)\cos\omega_0 S\mathbf{E}_{23} + \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} - \lambda_2(t)(b/c)\sin\omega_0 S\mathbf{E}_{12} \\ &\quad -\lambda_3(t)(a/c)\sin\omega_0 S\mathbf{E}_{13} + \lambda_3(t)(b/c)\,\mathbf{E}_{33} - \lambda_2(t)(a/c)\,\mathbf{E}_{32}. \end{aligned} \tag{6.43}$$

As we indicated before, we have $a/c = const$ and $b/c = const$, then

$$\begin{aligned} \ddot{\mathbf{F}} &= \ddot{\lambda}_1(t)\cos\omega_0 S\mathbf{E}_{11} + \ddot{\lambda}_2(t)(b/c)\cos\omega_0 S\mathbf{E}_{22} \\ &\quad + \ddot{\lambda}_3(t)(a/c)\cos\omega_0 S\mathbf{E}_{23} + \ddot{\lambda}_1(t)\sin\omega_0 S\mathbf{E}_{21} - \ddot{\lambda}_2(t)(b/c)\sin\omega_0 S\mathbf{E}_{12} \\ &\quad - \ddot{\lambda}_3(t)(a/c)\sin\omega_0 S\mathbf{E}_{13} + \ddot{\lambda}_3(t)(b/c)\,\mathbf{E}_{33} - \ddot{\lambda}_2(t)(a/c)\,\mathbf{E}_{32} \end{aligned} \tag{6.44}$$

and

$$\begin{aligned} \mathbf{F}' &= \mathbf{W}_0\mathbf{F} \\ &= \omega_0\lambda_1(t)\cos\omega_0 S\mathbf{E}_{21} - \omega_0\lambda_2(t)(b/c)\sin\omega_0 S\mathbf{E}_{22} \\ &\quad -\omega_0\lambda_3(t)(a/c)\sin\omega_0 S\mathbf{E}_{23} - \omega_0\lambda_2(t)(b/c)\cos\omega_0 S\mathbf{E}_{12} \\ &\quad -\omega_0\lambda_3(t)(a/c)\cos\omega_0 S\mathbf{E}_{13} - \omega_0\lambda_1(t)\sin\omega_0 S\mathbf{E}_{11} \end{aligned} \tag{6.45}$$

where

$$\mathbf{W}_0 = \omega_0(\mathbf{E}_{21} - \mathbf{E}_{12}).$$

The dynamic field equations (2.13) − (2.15) become

$$\mathbf{n}' = \mathbf{0}, \tag{6.46}$$

$$\mathbf{M}' - \mathbf{N} + (\lambda_3(t)\ \mathbf{d}_t \otimes \mathbf{n})^{\mathsf{T}} + \mathbf{T} = \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} \tag{6.47}$$

and

$$Sk\ \mathbf{N} = \mathbf{0},\ \ Sk\ \mathbf{T} = \mathbf{0} \tag{6.48}$$

with

$$\mathbf{f} = -\ddot{a}(t)\,\mathbf{d}_n,\ \ \ddot{b}(t) = 0. \tag{6.49}$$

Frame-indifference implies for the constraint reaction that

$$\mathbf{n} = \mathbf{R}\mathbf{n}_0, \tag{6.50}$$

where $\mathbf{n}_0$ is a vector which does not vary with the arc length S. The field equations (6.46) and (6.47), in accordance with (6.48), are satisfied if

$$\mathbf{n} = n_0\ \mathbf{e}_3, \tag{6.51}$$

where n_0 depends only on time t and

$$\begin{aligned}\mathbf{T} &= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N} - \mathbf{M}' - (n_0\lambda_3(t)\,\mathbf{e}_3 \otimes \mathbf{d}_t) \qquad (6.52)\\ &= \ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} + \mathbf{N} - \mathbf{M}' - (n_0\,(b/c)\,\lambda_3(t)\,\mathbf{d}_t \otimes \mathbf{d}_t + n_0\,(a/c)\,\lambda_3(t)\,\mathbf{d}_b \otimes \mathbf{d}_t)\,.\end{aligned}$$

For the special case of quasi-static motion, we have $\ddot{\mathbf{F}}\,\mathbf{E}_R\mathbf{F}^{\mathsf{T}} = \mathbf{0}$. Then (6.52) becomes

$$\begin{aligned}\mathbf{T} &= \mathbf{N} - \mathbf{M}' - (n_0\,(b/c)\,\lambda_3(t)\,\mathbf{d}_t \otimes \mathbf{d}_t + n_0\,(a/c)\,\lambda_3(t)\,\mathbf{d}_b \otimes \mathbf{d}_t) \qquad (6.53)\\ &= \mathbf{N} - \mathbf{M}' - (cn_0\overline{\tau}\mathbf{E}_{tt} + cn_0\overline{\kappa}\mathbf{E}_{bt})\end{aligned}$$

where $\overline{\tau} := \lambda_3 \tau = \omega_0 (b/c) = const$ and $\overline{\kappa} := \lambda_3 \kappa = \omega_0 (a/c) = const.$

For a monotropic rod we assume

$$\mathcal{G}_R^- = \{(\mathbf{Q}_1, -\mathbf{Q}_1), (\mathbf{I}, \mathbf{I})\}$$

where $\mathbf{Q}_1$ is the rotation of angle π about $\mathbf{e}_1$.

We define

$$\begin{aligned} \overline{\mathbf{Q}}_1 \quad &: \quad = \mathbf{R}\mathbf{Q}_1\mathbf{R}^\mathsf{T} = \mathbf{R}_3\mathbf{R}_1\mathbf{Q}_1\mathbf{R}_1^\mathsf{T}\mathbf{R}_3^\mathsf{T} \\ &= \cos(2\omega_0 S)\,\mathbf{E}_{11} - \cos(2\omega_0 S)\,\mathbf{E}_{22} + \sin(2\omega_0 S)\,\mathbf{E}_{12} + \sin(2\omega_0 S)\,\mathbf{E}_{21} - \mathbf{E}_{33}. \end{aligned} \tag{6.54}$$

We have

$$\begin{aligned} \overline{\mathbf{Q}}_1\mathbf{F} \quad = \quad & \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} + \lambda_2(t)(b/c)\sin\omega_0 S\mathbf{E}_{12} + \lambda_3(t)(a/c)\sin\omega_0 S\mathbf{E}_{13} \\ & -\lambda_2(t)(b/c)\cos\omega_0 S\mathbf{E}_{22} - \lambda_3(t)(a/c)\cos\omega_0 S\mathbf{E}_{23} + \lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} \\ & +\lambda_2(t)(a/c)\,\mathbf{E}_{32} - \lambda_3(t)(b/c)\,\mathbf{E}_{33}. \end{aligned}$$

and

$$\begin{aligned} \mathbf{F}\mathbf{Q}_1 \quad = \quad & \lambda_1(t)\cos\omega_0 S\mathbf{E}_{11} - \lambda_2(t)(b/c)\cos\omega_0 S\mathbf{E}_{22} - \lambda_3(t)(a/c)\cos\omega_0 S\mathbf{E}_{23} \\ & +\lambda_1(t)\sin\omega_0 S\mathbf{E}_{21} + \lambda_2(t)(b/c)\sin\omega_0 S\mathbf{E}_{12} + \lambda_3(t)(a/c)\sin\omega_0 S\mathbf{E}_{13} \\ & -\lambda_3(t)(b/c)\,\mathbf{E}_{33} + \lambda_2(t)(a/c)\,\mathbf{E}_{32} \end{aligned}$$

implying

$$\overline{\mathbf{Q}}_1\mathbf{F} = \mathbf{F}\mathbf{Q}_1. \tag{6.55}$$

We also have

$$\begin{aligned} \overline{\mathbf{Q}}_1\mathbf{F}' \quad = \quad & \lambda_1(t)\,\omega_0\sin\omega_0 S\mathbf{E}_{11} - \lambda_2(t)\,\omega_0(b/c)\cos\omega_0 S\mathbf{E}_{12} \\ & -\lambda_3(t)\,\omega_0(a/c)\cos\omega_0 S\mathbf{E}_{13} - \lambda_1(t)\,\omega_0\cos\omega_0 S\mathbf{E}_{21} \\ & -\lambda_2(t)\,\omega_0(b/c)\sin\omega_0 S\mathbf{E}_{22} - \lambda_3(t)\,\omega_0(a/c)\sin\omega_0 S\mathbf{E}_{23}, \end{aligned}$$

and

$$\begin{aligned}\mathbf{F}'\mathbf{Q}_1 &= \lambda_1(t)\,\omega_0\cos\omega_0 S\mathbf{E}_{21} + \lambda_2(t)\,\omega_0\,(b/c)\sin\omega_0 S\mathbf{E}_{22} \\ &\quad +\lambda_3(t)\,\omega_0\,(a/c)\sin\omega_0 S\mathbf{E}_{23} + \lambda_2(t)\,\omega_0\,(b/c)\cos\omega_0 S\mathbf{E}_{12} \\ &\quad +\lambda_3(t)\,\omega_0\,(a/c)\cos\omega_0 S\mathbf{E}_{13} - \lambda_1(t)\,\omega_0\sin\omega_0 S\mathbf{E}_{11}\end{aligned}$$

implying

$$\overline{\mathbf{Q}}_1\mathbf{F}' = -\,\mathbf{F}'\mathbf{Q}_1. \tag{6.56}$$

This shows the monotropic symmetry axis is $\overline{\mathbf{e}_1} = \mathbf{R}\mathbf{e}_1 = \mathbf{d}_n$. Since (6.55) and (6.56) satisfy (3.24.2), equation (3.23.2) gives

$$\begin{aligned}N_{12} &= N_{13} = 0, \\ M_{11} &= M_{22} = M_{33} = M_{23} = M_{32} = 0.\end{aligned}$$

$\mathbf{N}$ and $\mathbf{M}$ may be written as

$$\begin{aligned}\mathbf{N} &= N_{nn}\mathbf{E}_{nn} + N_{bb}\mathbf{E}_{bb} + N_{tt}\mathbf{E}_{tt} + N_{bt}\left(\mathbf{E}_{bt} + \mathbf{E}_{tb}\right), \\ \mathbf{M} &= M_{nb}\mathbf{E}_{nb} + M_{nt}\mathbf{E}_{nt} + M_{bn}\mathbf{E}_{bn} + M_{tn}\mathbf{E}_{tn}.\end{aligned} \tag{6.57}$$

Note that $N_{11} = N_{nn}, N_{22} = N_{bb}, N_{33} = N_{tt}, N_{12} = N_{nb}, N_{13} = N_{nt}$, etc.. This gives

$$\begin{aligned}N_{nb} &= N_{nt} = 0, \\ M_{nn} &= M_{bb} = M_{tt} = M_{bt} = M_{tb} = 0.\end{aligned} \tag{6.58}$$

Since

$$\begin{aligned}\mathbf{M} &= M_{nb}\mathbf{E}_{nb} + M_{nt}\mathbf{E}_{nt} + M_{bn}\mathbf{E}_{bn} + M_{tn}\mathbf{E}_{tn} \\ &= M_{nb}\mathbf{d}_n \otimes \mathbf{d}_b + M_{nt}\mathbf{d}_n \otimes \mathbf{d}_t + M_{bn}\mathbf{d}_b \otimes \mathbf{d}_n + M_{tn}\mathbf{d}_t \otimes \mathbf{d}_n,\end{aligned}$$

then

$$\mathbf{M}' = M_{nb}\left(\mathbf{d}_n' \otimes \mathbf{d}_b + \mathbf{d}_n \otimes \mathbf{d}_b'\right) + M_{nt}\left(\mathbf{d}_n' \otimes \mathbf{d}_t + \mathbf{d}_n \otimes \mathbf{d}_t'\right) \\ + M_{bn}\left(\mathbf{d}_b' \otimes \mathbf{d}_n + \mathbf{d}_b \otimes \mathbf{d}_n'\right) + M_{tn}\left(\mathbf{d}_t' \otimes \mathbf{d}_n + \mathbf{d}_t \otimes \mathbf{d}_n'\right) \tag{6.59}$$

From (6.36), we have

$$\begin{aligned} \mathbf{d}_n' &= \omega_0 \sin \omega_0 S \mathbf{e}_1 - \omega_0 \cos \omega_0 S \mathbf{e}_2 \\ &= \omega_0 \left(\sin \omega_0 S \mathbf{e}_1 - \cos \omega_0 S \mathbf{e}_2\right) \\ &= \omega_0 \mathbf{e}_\theta, \end{aligned} \tag{6.60}$$

and

$$\begin{aligned} \mathbf{d}_b &= (b/c)(-\mathbf{e}_\theta) + (a/c)\mathbf{e}_3, \\ \mathbf{d}_t &= -(a/c)(-\mathbf{e}_\theta) + (b/c)\mathbf{e}_3. \end{aligned} \tag{6.61}$$

From (6.61), we get

$$\mathbf{e}_3 = (c/b)\mathbf{d}_t + (a/b)(-\mathbf{e}_\theta).$$

Putting this into (6.38), we obtain

$$\begin{aligned} \mathbf{d}_b &= (b/c)\left(\sin \omega_0 S \mathbf{e}_1 - \cos \omega_0 S \mathbf{e}_2\right) \\ &\quad + (a/c)\left[(c/b)\mathbf{d}_t + (a/b)\left(\sin \omega_0 S \mathbf{e}_1 - \cos \omega_0 S \mathbf{e}_2\right)\right] \\ &= \left[(b/c) + \left(a^2/bc\right)\right]\left(\sin \omega_0 S \mathbf{e}_1 - \cos \omega_0 S \mathbf{e}_2\right) + (a/b)\mathbf{d}_t \\ &= (c/b)\left(\sin \omega_0 S \mathbf{e}_1 - \cos \omega_0 S \mathbf{e}_2\right) + (a/b)\mathbf{d}_t \\ &= (c/b)(-\mathbf{e}_\theta) + (a/b)\mathbf{d}_t. \end{aligned} \tag{6.62}$$

Then

$$\sin \omega_0 S \mathbf{e}_1 - \cos \omega_0 S \mathbf{e}_2 = (b/c)\left[\mathbf{d}_b - (a/b)\mathbf{d}_t\right] = (b/c)\mathbf{d}_b - (a/c)\mathbf{d}_t. \tag{6.63}$$

Replacing (6.60) by (6.63) gives

$$\mathbf{d}_n' = \omega_0 (\sin \omega_0 S \mathbf{e}_1 - \cos \omega_0 S \mathbf{e}_2) = (b/c)\,\omega_0 \mathbf{d}_b - (a/c)\,\omega_0 \mathbf{d}_t = \overline{\tau}\mathbf{d}_b - \overline{\kappa}\mathbf{d}_t. \tag{6.64}$$

From (6.38) , we also have

$$\begin{aligned} \mathbf{d}_b' &= (b/c)\,\omega_0 \cos \omega_0 S \mathbf{e}_1 + (b/c)\,\omega_0 \sin \omega_0 S \mathbf{e}_2 + 0\mathbf{e}_3 \\ &= (b/c)\,\omega_0 (\cos \omega_0 S \mathbf{e}_1 + \sin \omega_0 S \mathbf{e}_2) \\ &= (b/c)\,\omega_0 (-\mathbf{d}_n) = -\overline{\tau}\mathbf{d}_n \end{aligned} \tag{6.65}$$

and

$$\begin{aligned} \mathbf{d}_t' &= -(a/c)\,\omega_0 \cos \omega_0 S \mathbf{e}_1 - (a/c)\,\omega_0 \sin \omega_0 S \mathbf{e}_2 + 0\mathbf{e}_3 \\ &= -(a/c)\,\omega_0 (\cos \omega_0 S \mathbf{e}_1 + \sin \omega_0 S \mathbf{e}_2) \\ &= -(a/c)\,\omega_0 (-\mathbf{d}_n) = \overline{\kappa}\mathbf{d}_n. \end{aligned} \tag{6.66}$$

Replacing $\mathbf{M}'$ in equation (6.59) by (6.64) , (6.65) and (6.66) , we get

$$\begin{aligned} \mathbf{M}' &= M_{nb} \left[(\overline{\tau}\mathbf{d}_b - \overline{\kappa}\mathbf{d}_t) \otimes \mathbf{d}_b + \mathbf{d}_n \otimes (-\overline{\tau}\mathbf{d}_n) \right] \\ &\quad + M_{nt} \left[(\overline{\tau}\mathbf{d}_b - \overline{\kappa}\mathbf{d}_t) \otimes \mathbf{d}_t + \mathbf{d}_n \otimes (\overline{\kappa}\mathbf{d}_n) \right] \\ &\quad + M_{bn} \left[(-\overline{\tau}\mathbf{d}_n) \otimes \mathbf{d}_n + \mathbf{d}_b \otimes (\overline{\tau}\mathbf{d}_b - \overline{\kappa}\mathbf{d}_t) \right] \\ &\quad + M_{tn} \left[(\overline{\kappa}\mathbf{d}_n) \otimes \mathbf{d}_n + \mathbf{d}_t \otimes (\overline{\tau}\mathbf{d}_b - \overline{\kappa}\mathbf{d}_t) \right] \\ &= \overline{\tau} M_{nb} \mathbf{E}_{bb} - \overline{\kappa} M_{nb} \mathbf{E}_{tb} - \overline{\tau} M_{nb} \mathbf{E}_{nn} + \overline{\tau} M_{nt} \mathbf{E}_{bt} \\ &\quad - \overline{\kappa} M_{nt} \mathbf{E}_{tt} + \overline{\kappa} M_{nt} \mathbf{E}_{nn} - \overline{\tau} M_{bn} \mathbf{E}_{nn} + \overline{\tau} M_{bn} \mathbf{E}_{bb} \\ &\quad - \overline{\kappa} M_{bn} \mathbf{E}_{bt} + \overline{\kappa} M_{tn} \mathbf{E}_{nn} + \overline{\tau} M_{tn} \mathbf{E}_{tb} - \overline{\kappa} M_{tn} \mathbf{E}_{tt} \\ &= (-\overline{\tau} M_{nb} + \overline{\kappa} M_{nt} - \overline{\tau} M_{bn} + \overline{\kappa} M_{tn})\, \mathbf{E}_{nn} \\ &\quad + (\overline{\tau} M_{nb} + \overline{\tau} M_{bn})\, \mathbf{E}_{bb} + (-\overline{\kappa} M_{nb} + \overline{\tau} M_{tn})\, \mathbf{E}_{tb} \\ &\quad + (\overline{\tau} M_{nt} - \overline{\kappa} M_{bn})\, \mathbf{E}_{bt} + (-\overline{\kappa} M_{nt} - \overline{\kappa} M_{tn})\, \mathbf{E}_{tt}. \end{aligned} \tag{6.67}$$

Here we define $M_{(nt)} = \frac{1}{2}(M_{nt} + M_{tn})$ as the symmetry part of M_{nt} and $M_{[nt]} = \frac{1}{2}(M_{nt} - M_{tn})$ as the skew-symmetry part of M_{nt}. Then (6.67) can be simplified as

$$\begin{aligned} \mathbf{M}' &= \left(2\overline{\kappa} M_{(nt)} - 2\overline{\tau} M_{(nb)}\right) \mathbf{E}_{nn} \\ &\quad + 2\overline{\tau} M_{(nb)} \mathbf{E}_{bb} + \left(-\overline{\kappa} M_{nb} + \overline{\tau} M_{tn}\right) \mathbf{E}_{tb} \\ &\quad + \left(\overline{\tau} M_{nt} - \overline{\kappa} M_{bn}\right) \mathbf{E}_{bt} - 2\overline{\kappa} M_{(nt)} \mathbf{E}_{tt}. \end{aligned} \tag{6.68}$$

Combining (6.58) and (6.68) together, we may solve equation (6.53) for $\mathbf{T}$ as

$$\begin{aligned} \mathbf{T} &= \mathbf{N} - \mathbf{M}' - \left(\mathbf{n} \otimes \mathbf{r}'\right) \\ &= \left(N_{nn} - 2\overline{\kappa} M_{(nt)} + 2\overline{\tau} M_{(nb)}\right) \mathbf{E}_{nn} + \left(N_{bb} - 2\overline{\tau} M_{(nb)}\right) \mathbf{E}_{bb} \\ &\quad + \left(N_{bt} + \overline{\kappa} M_{nb} - \overline{\tau} M_{tn}\right) \mathbf{E}_{tb} + \left(N_{bt} - \overline{\tau} M_{nt} + \overline{\kappa} M_{bn} - c n_0 \overline{\kappa}\right) \mathbf{E}_{bt} \\ &\quad + \left(N_{tt} + 2\overline{\kappa} M_{(nt)} - c n_0 \overline{\tau}\right) \mathbf{E}_{tt}. \end{aligned} \tag{6.69}$$

Then we will have the following equations:

$$N_{bt} + \overline{\kappa} M_{nb} - \overline{\tau} M_{tn} = 0, \tag{6.70}$$

$$N_{bt} - \overline{\tau} M_{nt} + \overline{\kappa} M_{bn} - c n_0 \overline{\kappa} = 0, \tag{6.71}$$

and

$$N_{tt} + 2\overline{\kappa} M_{(nt)} - c n_0 \overline{\tau} = 0. \tag{6.72}$$

Subtracting (6.71) from (6.70) gives

$$\overline{\kappa} M_{nb} - \overline{\tau} M_{tn} + \overline{\tau} M_{nt} - \overline{\kappa} M_{bn} + c n_0 \overline{\kappa} = 0,$$

this implies

$$\begin{aligned} c n_0 \overline{\kappa} &= -\overline{\kappa} M_{nb} + \overline{\tau} M_{tn} - \overline{\tau} M_{nt} + \overline{\kappa} M_{bn} \\ &= 2\overline{\tau} M_{[tn]} + 2\overline{\kappa} M_{[bn]}. \end{aligned}$$

And further, it implies

$$n_0 = \frac{2}{c\overline{\kappa}}\left(\overline{\tau}M_{[tn]} + \overline{\kappa}M_{[bn]}\right) = \frac{2}{c\kappa}\left(\tau M_{[tn]} + \kappa M_{[bn]}\right). \tag{6.73}$$

Adding equations (6.70) and (6.71) gives

$$2N_{bt} + \overline{\kappa}M_{nb} - \overline{\tau}M_{tn} - \overline{\tau}M_{nt} + \overline{\kappa}M_{bn} - cn_0\overline{\kappa} = 0,$$

implying

$$2N_{bt} + 2\overline{\kappa}M_{(nb)} - 2\overline{\tau}M_{(nt)} - cn_0\overline{\kappa} = 0. \tag{6.74}$$

Finally, we could write a consequence of the equations of dynamics as

$$\mathbf{n} = \mathbf{n}_0 = n_0\mathbf{e}_3,\ n_0 = 2\left(\tau M_{[tn]} + \kappa M_{[bn]}\right)/c\kappa, \tag{6.75}$$

$$\mathbf{T} = \left(N_{nn} - 2\overline{\kappa}M_{(nt)} + 2\overline{\tau}M_{(nb)}\right)\mathbf{E}_{nn} + \left(N_{bb} - 2\overline{\tau}M_{(nb)}\right)\mathbf{E}_{bb} \tag{6.76}$$

and

$$N_{tt} + 2\overline{\kappa}M_{(nt)} - cn_0\overline{\tau} = 0,\ 2N_{bt} + 2\overline{\kappa}M_{(nb)} - 2\overline{\tau}M_{(nt)} - cn_0\overline{\kappa} = 0. \tag{6.77}$$

Equation $(6.75)_2$ is a consequence of angular momentum balance. In (6.76) and (6.77), $\overline{\kappa} := \lambda_3\kappa$, $\overline{\tau} := \lambda_3\tau$.

If $\mathbf{T} = \mathbf{0}$, we have the following equations:

$$N_{nn} - 2\overline{\kappa}M_{(nt)} + 2\overline{\tau}M_{(nb)} = 0, \tag{6.78}$$

and

$$N_{bb} - 2\overline{\tau}M_{(nb)} = 0. \tag{6.79}$$

If we knew how $\mathbf{N}$ and $\mathbf{M}$ depend on stretch functions $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$, we could, in principle use equations $(6.75)_2$, (6.77) (6.78) and (6.79) to solve explicitly for λ_1, λ_2 and λ_3 as functions of t.

CHAPTER 7

Summary

This chapter summarizes the results obtained from previous three chapters: Chapter 4, Chapter 5 and Chapter 6.The summary is in the form of the 6 tables:

	Straight $\longrightarrow$ Straight
Reference Configuration	$\mathbf{r}_R = S\mathbf{e}_3$
Current Configuration	$\mathbf{r} = \lambda_3(t)\, S\mathbf{e}_3$
the directors which span the cross-section	$(\mathbf{d}_1, \mathbf{d}_2)$ $\mathbf{d}_1 = \mathbf{e}_1,\ \mathbf{d}_2 = \mathbf{e}_2$
the director which is along the tangent to the rod axis	$\mathbf{d}_3 = \mathbf{r}' = \mathbf{e}_3$
the axial force $\mathbf{n} = n_0\mathbf{e}_3$	$n_0 = \lambda_3^{-1}(t)\, N_{33}$
body force distribution $\mathbf{f}$	$\mathbf{f} = \mathbf{0}$
end loads	0
the rotation tensor $\mathbf{R}$	$\mathbf{R} = \mathbf{I}$
Monotropic axes	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
cross-sectional tensor $\mathbf{T}$	$\mathbf{T} = N_{11}\mathbf{E}_{11} + N_{22}\mathbf{E}_{22}$

(*Table*.1 *Straight* $\longrightarrow$ *Straight*)

	Straight $\longrightarrow$ Straight twisted
Reference Configuration	$\mathbf{r}_R = S\mathbf{e}_3$
Current Configuration	$\mathbf{r} = \lambda_3(t)\, S\mathbf{e}_3$
the directors which span the cross-section	$(\mathbf{d}_1, \mathbf{d}_2)$ $\mathbf{d}_1 = \mathbf{e}_1,\ \mathbf{d}_2 = \mathbf{e}_2$
the director which is along the tangent to the rod axis	$\mathbf{d}_3 = \mathbf{r}' = \mathbf{e}_3$
the axial force $\mathbf{n} = n_0\mathbf{e}_3$	$n_0 = \lambda_3^{-1}(t)\, N_{33}$
body force distribution $\mathbf{f}$	$\mathbf{f} = \mathbf{0}$
end loads	M_{12}, M_{21}
the rotation tensor $\mathbf{R}$	$\mathbf{R} = \mathbf{R}_3 = \cos\omega_0 S\mathbf{I}_3^{\perp} + \sin\omega_0 S\mathbf{A}_3^{\perp} + \mathbf{E}_{33}$
Monotropic axes	$\overline{\mathbf{e}}_1 = \mathbf{e}_r,\ \overline{\mathbf{e}}_3 = \mathbf{e}_3$
cross-sectional tensor $\mathbf{T}$	$\mathbf{T} = \{N_{11} + \omega_0(M_{12} + M_{21})\}\,\mathbf{E}_{11}$ $+ \{N_{22} - \omega_0(M_{12} + M_{21})\}\,\mathbf{E}_{22}$

(*Table.2 Straight* $\longrightarrow$ *StraightTwisted*)

	Circular $\longrightarrow$ Circular
Reference Configuration	$\mathbf{r}_R = \mathbf{e}_r$
Current Configuration	$\mathbf{r} = a(t)\,\mathbf{e}_r$
the directors which span the cross-section	$(\mathbf{d}_1, \mathbf{d}_3)$ $\mathbf{d}_1 = \mathbf{e}_r,\ \mathbf{d}_3 = \mathbf{e}_3$
the director which is along the tangent to the rod axis	$\mathbf{d}_2 = \mathbf{r}' = \mathbf{e}_\theta$
the axial force $\mathbf{n} = n_\theta \mathbf{e}_\theta + n_0 \mathbf{e}_3$	$n_\theta = \lambda_2^{-1}(t) N_{\theta\theta}, n_0 = 0$
body force distribution $\mathbf{f}$	$\mathbf{f} = (\ \ddot{n}_\theta + a(t))\mathbf{e}_r$
end loads	0
the rotation tensor $\mathbf{R}$	$\mathbf{R} = \mathbf{I}$
Monotropic axes	$\mathbf{e}_r,\ \mathbf{e}_\theta,\ \mathbf{e}_3$
cross-sectional tensor $\mathbf{T}$	$\mathbf{T} = N_{rr}\mathbf{E}_{rr} + N_{33}\mathbf{E}_{33}$

(Table.3 Circular $\longrightarrow$ Circular)

	Straight $\longrightarrow$ Circular
Reference Configuration	$\mathbf{r}_R = S\mathbf{e}_1$
Current Configuration	$\mathbf{r} = a(t)\,\mathbf{e}_r$
the directors which span the cross-section	$(\mathbf{d}_1, \mathbf{d}_3)$, $\mathbf{d}_1 = \mathbf{e}_r,\ \mathbf{d}_3 = \mathbf{e}_3$
the director which is along the tangent to the rod axis	$\mathbf{d}_2 = \mathbf{r}' = \mathbf{e}_\theta$
the axial force $\mathbf{n} = n_0\mathbf{e}_3$	$n_0 = 0$
body force distribution $\mathbf{f}$	$\mathbf{f} = \ddot{a}\,(t)\,\mathbf{e}_r$
end loads	$M_{r\theta}, M_{\theta r}$
the rotation tensor $\mathbf{R}$	$\mathbf{R} = \mathbf{R}_3 = \cos\omega_0 S\mathbf{I}_3^{\perp} + \sin\omega_0 S\mathbf{A}_3^{\perp} + \mathbf{E}_{33}$
Monotropic axes	$\bar{\mathbf{e}}_1 = \mathbf{e}_r,\ \bar{\mathbf{e}}_3 = \mathbf{e}_3$
cross-sectional tensor $\mathbf{T}$	$\mathbf{T} = \{N_{rr} + \omega_0\,(M_{r\theta} + M_{\theta r})\}\,\mathbf{E}_{rr} + N_{33}\mathbf{E}_{33}$ and $\{N_{\theta\theta} - \omega_0\,(M_{\theta r} + M_{r\theta})\}\,\mathbf{E}_{\theta\theta} = 0$

(Table.4 Straight $\longrightarrow$ Circular)

	Helical $\longrightarrow$ Helical
Reference Configuration	$\mathbf{r}_R = a_0\mathbf{e}_r + \frac{\lambda_3(t)b_0}{\sqrt{a_0^2+b_0^2}}S\mathbf{e}_3$
Current Configuration	$\mathbf{r} = a\left(t\right)\mathbf{e}_r + \frac{\lambda_3(t)b(t)}{\sqrt{a^2(t)+b^2(t)}}S\mathbf{e}_3$
the directors which span the cross-section	$\left(\mathbf{d}_n, \mathbf{d}_b\right), \mathbf{d}_n = -\cos\omega_0 S\mathbf{e}_1 - \sin\omega_0 S\mathbf{e}_2$ and $\mathbf{d}_b = -\left(b/c\right)\cos\omega_0 S\mathbf{e}_2$ $+\left(b/c\right)\sin\omega_0 S\mathbf{e}_1 + \left(a/c\right)\mathbf{e}_3$
the director which is along the tangent to the rod axis	$\mathbf{d}_t = \mathbf{r}' = \left(a/c\right)\cos\omega_0 S\mathbf{e}_2$ $-\left(a/c\right)\sin\omega_0 S\mathbf{e}_1 + \left(b/c\right)\mathbf{e}_3$
the axial force $\mathbf{n} = n_0\mathbf{e}_3$	$n_0 = 0$
body force distribution $\mathbf{f}$	$\mathbf{f} = -\ddot{a}\left(t\right)\mathbf{d}_n, \ddot{b}\left(t\right) = 0$
end loads	0
the rotation tensor $\mathbf{R}$	$\mathbf{R} = \mathbf{I}$
Monotropic axes	$\mathbf{d}_n, \mathbf{d}_b, \mathbf{d}_t$
cross-sectional tensor $\mathbf{T}$	$\mathbf{T} = N_{nn}\mathbf{E}_{nn} + N_{bb}\mathbf{E}_{bb} + N_{tt}\mathbf{E}_{tt}$

(*Table*.5 *Helical* $\longrightarrow$ *Helical*)

	Straight $\longrightarrow$ Helical
Reference Configuration	$\mathbf{r}_R = S\mathbf{e}_3$
Current Configuration	$\mathbf{r} = a\,(t)\,\mathbf{e}_r + \frac{\lambda_3(t)b(t)}{\sqrt{a^2(t)+b^2(t)}}S\mathbf{e}_3$
the directors which span the cross-section	$(\mathbf{d}_n, \mathbf{d}_b)\,, \mathbf{d}_n = -\cos\omega_0 S\mathbf{e}_1 - \sin\omega_0 S\mathbf{e}_2$ and $\mathbf{d}_b = -(b/c)\cos\omega_0 S\mathbf{e}_2 + (b/c)\sin\omega_0 S\mathbf{e}_1 + (a/c)\,\mathbf{e}_3$
the director which is along the tangent to the rod axis	$\mathbf{d}_t = \mathbf{r}' = (a/c)\cos\omega_0 S\mathbf{e}_2 - (a/c)\sin\omega_0 S\mathbf{e}_1 + (b/c)\,\mathbf{e}_3$
the axial force $\mathbf{n} = n_0\mathbf{e}_3$	$n_0 = 2/c\kappa\left(\tau M_{[tn]} + \kappa M_{[bn]}\right)$
body force distribution $\mathbf{f}$	$\mathbf{f} = -\ddot{a}\,(t)\,\mathbf{d}_n,\ \ddot{b}\,(t) = 0$
end loads	$M_{(nt)}, M_{(nb)}, M_{[tn]}, M_{[bn]}$
the rotation tensor $\mathbf{R}$	$\mathbf{R} = \mathbf{R}_3\mathbf{R}_1 = \cos\omega_0 S\mathbf{E}_{11} + \sin\omega_0 S\mathbf{E}_{21} - (\mathbf{b}/\mathbf{c})\sin\omega_0 S\mathbf{E}_{12} - (\mathbf{a}/\mathbf{c})\sin\omega_0 S\mathbf{E}_{13} + (b/c)\cos\omega_0 S\mathbf{E}_{22} - (\mathbf{a}/\mathbf{c})\,\mathbf{E}_{32} + (a/c)\cos\omega_0 S\mathbf{E}_{23} + (\mathbf{b}/\mathbf{c})\,\mathbf{E}_{33}$
Monotropic axes	$\bar{\mathbf{e}}_1 = \mathbf{d}_n$
cross-sectional tensor $\mathbf{T}$	$\mathbf{T} = \left(N_{nn} - 2\overline{\kappa}M_{(nt)} + 2\overline{\tau}M_{(nb)}\right)\mathbf{E}_{nn} + \left(N_{bb} - 2\overline{\tau}M_{(nb)}\right)\mathbf{E}_{bb}$ with $N_{tt} + 2\overline{\kappa}M_{(nt)} - cn_0\overline{\tau} = 0$ and $2N_{bt} + 2\overline{\kappa}M_{(nb)} - 2\overline{\tau}M_{(nt)} - cn_0\overline{\kappa} = 0$

(*Table.6 Straight* $\longrightarrow$ *Helical*)

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