LINEAR PROGRAMMING AND QUADRATIC PROGRAMMING APPROACH FOR GRADUATION IN FUZZY ENVIRONMENT

BY

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Linear Programming and Quadratic Programming Approach for Graduation in Fuzzy Environment

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

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TO MY WIFE

ABSTRACT

Decision making under uncertainty has become a key issue in the present alternative way of thinking. There is an emerging interest in the use of new techniques to draw definite conclusions from imprecise or vague information in order to take competitive advantage. A critical challenge in decision-making process is not only to find a suitable method to measure and quantify the uncertainty involved in the problem under consideration but also its successful applications.

In the present thesis, we consider a graduation problem with imprecise observed values data and imprecise combination of fit and smoothness. The problem is first formulated, solved and analyzed as a fuzzy linear program. Next, a finite iteration technique is developed to solve a fuzzy quadratic programming problem. Significance of this model can be hopefully seen in the light of usage of quadratic program in the field of Finance, Economics, Structural Engineering and Actuarial Sciences under uncertainty. Furthermore, the graduation problem is revisited using fuzzy quadratic programming model and solutions are obtained both under crisp and fuzzy environment. The results so obtained are shown to be better than the results obtained by using fuzzy linear programming, and the results obtained by Schuette using crisp linear programming. The methods introduced in the present thesis, offer an opportunity to view a graduation problem from a different prospective.

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Chapter 1

INTRODUCTION

Uncertainty is one of the main and most important issues that have to be addressed by modern management systems. The main subjects of modern analysis are characterized by a number of general features that make them particularly difficult for existing methods. These features are: complexity, dynamics and uncertainty. In certain cases the presence of uncertainty makes the traditional approaches insufficient [35]. When it comes time to make a sound decision on an uncertain problem, it is important for the decision makers to consider and evaluate the uncertainty involved in it and its surroundings. Uncertainty may result from many sources: imprecise/vague knowledge regarding future conditions, inaccurate data, forecasting errors, subjective influences or existence of external uncontrollable disturbances. For decision making under uncertainty, one should, normally, develop an active approach rather than ignore it.

Classical set theory based on two-valued logic defines a set as a collection of objects with well-defined 'crisp' boundaries. An element either belongs to the set or does not belong to the set, that is, its membership is either 1 or 0. To deal with the sets with imprecise boundaries, Lotfi A. Zadeh [49] in 1965 introduced fuzzy set theory. The membership function in a fuzzy set, unlike that of a 'crisp' set is not a matter of being either true or false, but a matter of degree of truth/belief. In general, degrees of membership in fuzzy sets are expressed by values in [0, 1]. The extreme values 0 and 1 in the interval [0, 1] represent total non-belongingness and total belongingness respectively.

This makes, crisp sets, a special case of fuzzy sets, for which only two grades of memberships are allowed. Thus we can say 'crisp is fuzzy', with a membership of either 1 or 0.

Probability theory is the traditional theory describing and measuring the phenomenon of uncertainty. It is assumed that the probability theory can be used in every situation of uncertainty [22]. Since both fuzzy set theory and probability theory deal with uncertainty, most of the time former is confused with later. But, fuzziness is only one aspect of uncertainty. It is the vagueness or ambiguity found in the definition of a concept or meaning of terms. The probability generally relates to randomly occurring events that are clearly defined and may contain the uncertainty of randomness.

Fuzzy logic is basically a multivalued logic that allows intermediate values to be defined between conventional evaluations like yes/no, true/false, tall/very tall, etc. Fuzzy reasoning and logic have the ability to express the amount of ambiguity in human thinking and subjectivity in a comparatively undistorted manner. Hence, fuzzy logic techniques find their major applications in areas such as control, pattern recognition, quantitative analysis, inference, and in information retrieval.

Fuzzy systems are being used in various consumer products e.g. washing machines, air conditioners, camcorders, auto-focus cameras, system of traffic light controlling, and subways trains [34]. The NASA space agency is engaged in applying fuzzy logic for complex docking-maneuvers.

1.1 Fuzzy Set Theory

In this section we introduce some of the basic concepts and terminology of fuzzy set theory. Theory of fuzzy sets is basically a theory of graded concepts [51].

Fuzzy Set

Let X be a classical set of objects, called the universe, whose generic elements are denoted by x. The membership in a crisp subset of X is viewed as characteristic function μ_A from X to {0, 1} such that:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

where {0, 1} is called a valuation set [23].

If the valuation set is allowed to be the real interval [0, 1], A is called a fuzzy set proposed by Zadeh [50]. $\mu_A(x)$ is the degree of membership of x in A. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. Therefore, A is completely characterized by the set of ordered pairs:

$$A = \{(x, \mu_A(x)) / x \in X\}$$

where $\mu_A(x)$ maps X to the membership space [0, 1]. Elements with zero degree of membership are usually not listed. If Sup $\mu_A(x) = 1$, $\forall\, x \in R$, then the fuzzy set A is called a normal fuzzy set in R. A fuzzy set that is not normal is called subnormal fuzzy set.

α – Level Set or α – Cut

One of the most important concepts of fuzzy sets is the concept of an α-cut or α-level set. An α-cut denoted by A_α is the crisp set of elements x in R whose degree of belongings to the fuzzy set A is at least $\alpha \in [0, 1]$. This means

$$A_\alpha = \{x \in R \mid \mu_A(x) \geq \alpha, \alpha \in [0, 1]\}$$

that is, the α-cut or α-level set of a fuzzy set is the crisp set A_α that contains all elements of the universal set $X \in R$ whose membership grades in A are greater than or equal to the specified value of α, $\alpha \in [0, 1]$.

Support of a Fuzzy Set

The support of a fuzzy set A is a set S(A) such that $x \in S(A) \Leftrightarrow \mu_A(x) > 0$. If $\mu_A(x)$ is constant over S(A), then A is non-fuzzy.

Intersection of Fuzzy Sets

Intersection of two fuzzy sets A and B is a fuzzy set C denoted by $C = A \cap B$, whose membership function is related to those of A and B by

$$\mu_C(x) = \min [\mu_A(x), \mu_B(x)], \ \forall x \in X$$

1.2 Algebraic Operations on Fuzzy Sets

In addition to the set theoretic operations, we can also define a number of other ways of forming combinations of fuzzy sets and relating them to one another. Here we present some more important operations among those:

1. Algebraic product of two fuzzy sets A and B, is A(·)B, whose membership function is

$$\mu_{A(\cdot)B}(x) = \mu_A(x) (\cdot) \mu_B(x), \qquad \forall\, x \in X$$

2. The algebraic sum of A and B is A + B whose membership function is defined as

$$\mu_{(A+B)}(x) = \mu_A(x) (+) \mu_B(x), \qquad \forall\, x \in X$$

provided $\mu_A(x) (+) \mu_B(x) \leq 1$, $\forall\, x \in X$

1.3 Convexity

The notion of convexity can be extended to fuzzy sets in such a way as to preserve many of the properties that it has in case of crisp sets. In what follows, we assume that the set X is the n-dimensional space R^n. We now have the following two equivalent definitions of convexity of a fuzzy set.

A fuzzy set A is convex if and only if every set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ for all $\alpha \in [0, 1]$ is a convex set.

The second definition of convexity of a fuzzy set is as follows:

A fuzzy set A is said to be a convex set if

$$\mu\,(\lambda x_1 + (1-\lambda)x_2) \geq \min\,(\mu(x_1), \mu(x_2)), \quad x_1, x_2 \in X, \ \lambda \in [0, 1].$$

1.4 Fuzzy Arithmetic

The first definition of a fuzzy set allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.

An ordinary number 'a' can be characterized by using the notation of membership function as,

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

A fuzzy number A is a fuzzy set on the real line R,

that possesses the following properties:

(1) A is a normal, convex fuzzy set on R,

(2) The α-level set A_α must be a closed interval for every $\alpha \in [0, 1]$,

(3) The support of A, $S(A) = \{x \mid \mu_A(x) > 0\}$, must be bounded.

Fuzzy arithmetic is based on two properties of fuzzy numbers:

- Each fuzzy set and thus, each fuzzy number can be fully and uniquely represented by its α-level sets.
- α-level sets of each fuzzy numbers are closed intervals of real numbers for all $\alpha \in [0, 1]$.

These properties enable us to define an arithmetic operation on fuzzy numbers in terms of arithmetic operations on their α-level sets (i.e. arithmetic operations on closed intervals).

1.5 Fuzzy Arithmetic Based on Operations on Closed Intervals

A fuzzy number can be characterized by an interval of confidence at level α,

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$$

which has the property

$$\alpha \leq \alpha' \Rightarrow A_{\alpha'} \subset A_\alpha$$

Let $A = [a, b] \in R$ and $B = [c, d] \in R$ be two fuzzy numbers then we define the arithmetic operations on them as

Addition $A + B = [a + c, b + d]$

Subtraction $A - B = [a - d, b - c]$

Multiplication $AB = [\min (ac, ad, bc, bd), \max (ac, ad, bc, bd)]$

Inverse of A $A^{-1} = [\min (1/a, 1/b), \max (1/a, 1/b)]$

Division $A/B = [\min (a/c, a/d, b/c, b/d), \max (a/c, a/d, b/c, b/d)]$

Minimum ($\wedge$) $A \wedge B = [a \wedge c, b \wedge d]$

Maximum ($\vee$) $A \vee B = [a \vee c, b \vee d]$

Let A and B be two fuzzy numbers, $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ be the α-level set of A, and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-level set of B.

Let * denote any of the arithmetic operations $+, -, ., /, \wedge$ and $\vee$ on fuzzy numbers.

Then, we define a fuzzy set A * B in R, by defining its α-level sets $(A * B)_\alpha$ as

$$(A * B)_\alpha = A_\alpha * B_\alpha \quad \text{for any } \alpha \in [0, 1]$$

Since $(A * B)_\alpha$ is a closed interval for each $\alpha \in [0, 1]$ and A and B are fuzzy numbers, A * B is also a fuzzy number.

The multiplication of fuzzy number $A \subset R$ by an ordinary number $k \in R^+$ can also be defined as

$$(k * A_\alpha) = k (\cdot) A_\alpha = [ka_1^{(\alpha)}, ka_2^{(\alpha)}]$$

or equivalently, $\mu_{k.A}(x) = \mu_A(x/k) \quad \forall\ x \in R$

1.6 Graduation

According to Miller ([30], page 6), the problem of graduation can be described as "*the problem of graduation is a mathematical problem in which we are asked to estimate, or secure a representation of, the series of true rates of mortality that is assumed to have given rise to the irregular series of observed probabilities.*"

In the present thesis, we consider the problem of graduation as a general case. We obtain a sequence of observed values in which we suspect that there exists a strong relationship among the elements of the sequence of observed values. In order to predict future occurrence of the series, the process of graduation is applied to obtain a proper representation of the basic pattern, which the observed values under consideration are believed to follow.

The process of graduation is defined by Andrew and Nesbitt ([1], page 2) as "*an effort to represent a physical phenomenon by a systematic revision of some observations of that phenomenon*".

The above definition suggests that a model-building process takes place in the problem of graduation. It also suggests that we should have some preliminary information about the model and a set of observed values. The definition also indicates that the observed values can be revised to improve the model under consideration as a representative of the underlying phenomenon.

Several methods have been developed by which the graduation of an observed series may be accomplished and the problem of graduation can be solved. These methods are classified by Miller [30] as follows.

The Graphic Method

In this method, the observed values are suitably plotted on graph paper and among them a smooth, continuous curve is drawn as the basis of the graduated series. Grouping of the data is an essential part of this method, which is followed by plotting of the observed values together with some indication of their relative weights if this information is available. At the end, graduated values are read from the diagram and adjusted to improve smoothness and fit.

The Interpolation Method

In this method, the data are combined into groups and the graduated series is obtained by interpolation between points determined as representative of the groups. Since graduation involves the replacement of an irregular observed series by a regular smooth series consistent with the trend of the observed values, clearly the interpolation method of graduation includes more than interpolation alone.

The Adjusted-Average Method

In this process, each term of the graduated series is a weighted average of a fixed number of terms of the observed series to which it is central. It involves two sets of graduation formulas – linear compound formulas and summation formulas.

The Difference-Equation Method

In this method, the graduated series is determined by a difference equation derived from an analytic measure of the relative emphasis to be placed upon fit and smoothness. Professor E. T. Whittaker (please see [30], page 34) enunciated the principles of the difference-equation method in a paper published in 1919. Later, Robert Henderson (please see [30], page 34) developed a practical process for employing the method to make a numerical graduation. For these reasons, difference-equation formulas are also referred to as Whittaker-Henderson formulas. Other difference equations involving differences of other orders were derived by modifying the measure of smoothness, therefore, there is a family of Type A formulas and a set of difference-equation formulas known as Whittaker-Henderson Type B formulas.

Graduation by Mathematical Formula

Under this method graduated series is represented by a mathematical curve fitted to data. There are a large variety of curves, which may be used in representing different types of statistical data. They range from the simple straight to the family of frequency curves developed by Karl Pearson (please see [30], page 42) and to the curve systems of Gram-Charlier, Poisson and Fourier (please see [30], page 42). The curves of commonest use and maximum interest to the actuary in treating mortality rates are Gompertz' [15] and Makeham's Curves (please see [30], page 42), which were developed in the search for a mathematical law of mortality.

1.7 Linear Programming

It is a mathematical method of allocating scarce resources to achieve an objective, such as maximizing profit [24] or minimizing cost. Linear Programming approach is a mathematical representation of real world decision situations that consists of a linear objective function and linear resource constraints. Once the problem has been identified, the goals of management established, and the applicability of the linear programming determined, the next step in solving an unstructured, real world problem is the formulation of a mathematical model. This entails three major steps:

- Identification of decision variables (the quantity of the activity in question).
- The development of an objective function that is a linear relationship of the solution variables, and
- The determination of system constraints, which are also linear relationships of the decision variables, which reflect the limited resources of the problem.

Decision Variables

In each problem, decision variables, which denote a level of activity or quantity produced, are defined. For a general model, n decision variables are defined as

x_j = quantity of activity j, where $j = 1, 2, \ldots, n$.

Objective Function

The objective function represents the sum total of the contribution of each decision variable in the model towards an objective. It is represented as

maximize or minimize $f_0(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_jx_j + \ldots + c_nx_n$

where

$f_0(x_1, x_2, \ldots, x_n)$ = the total value of the objective function

c_j = the contribution per unit of activity j ($j = 1, 2, \ldots, n$)

System Constraints

The constraints of a linear programming model represent the limited availability of resources in the problem. Let the amount of each of m resources available be defined as b_i (for i = 1, 2, ..., m). We also define a_{ij} as the amount of resource i consumed per unit of activity j (j = 1, 2, ..., n). Thus, the constraints can be written as

$$g_i(x_1, x_2, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \ (\leq, =, \geq) \ b_i, \quad i = 1, 2, \ldots, m$$

Non-negativity

$$x_1, x_2, \ldots, x_n \geq 0.$$

Therefore a linear programming problem is

maximize or minimize $f_0(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_jx_j + \ldots + c_nx_n$

subject to

$$g_i(x_1, x_2, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \ (\leq, =, \geq) \ b_i, \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n.$$

A general optimization problem can be written as

maximize $f_0(x)$

subject to

$$g_i(x) \leq b_i \qquad i = 1, 2, \ldots, m$$

where

$x = (x_1, x_2, \ldots, x_n) \in R^n$, n-dimensional real space,

$f_0 : R^n \rightarrow R$, the set of reals, and

$g_i : R^n \rightarrow R$, $i = 1, 2, \ldots, m$.

If, $f_0(x) = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots + c_nx_n$

and $g_i(x) = a_{i1}x_1 + a_{i2}x_2 + \ldots\ldots + a_{in}x_n$

then, the nonlinear programming problem is a linear programming problem.

1.8 Fuzzy Linear Programming

Most of the time, due to incomplete or forecasted information the input data for c_j's, b_i's and a_{ij}'s, and/or the objective function and/or inequalities are imprecise. With these fuzzy/imprecise data the above problem is called fuzzy linear programming problem. Thus a fuzzy linear programming problem is not uniquely defined. The fuzzy problem depends upon the type of fuzziness present and specified by the decision-maker. Fuzzy linear programming problem can be broadly classified as:

Linear Programming Problem with fuzzy resources or fuzzy inequalities and crisp objective function.

Linear Programming Problem with fuzzy resources or fuzzy inequalities and fuzzy objective function.

Linear Programming Problem with fuzzy resources and fuzzy coefficients.

Two major fuzzy linear programming models as given in Zimmermann [51] are:

(i) Symmetric (ii) Non-symmetric.

The symmetric models are based on the definition of fuzzy decision proposed by Bellman and Zadeh [5]. It is assumed [5] that the objective function and constraints are imprecise and can be represented by fuzzy sets and the decision is the confluence of the fuzzy objective function and fuzzy constraint.

The non-symmetric models [5] are based on the following two approaches:

(i) The determination of the fuzzy set decision.

(ii) The determination of a crisp maximizing decision by aggregating the objective function, after appropriate transformations with the constraints.

Thus, in a general format, a fuzzy linear problem (FLPP) can be written as:

(FLPP) Maximize $z = f_0(x)$

subject to

$$f_i(x) \underset{\sim}{\leq} d_i, \qquad i = 1,2, \ldots, k$$

$$g_i(x) \leq b_i \qquad i = k+1, k+2, \ldots, m$$

$$x \geq 0$$

where '$\underset{\sim}{\leq}$' is called the 'fuzzy less than or equal to' , or 'essentially less than or equal to', f_0 , f_i and g_i, $i = 1,2, \ldots, m$ are linear functions and $x \in R^n$.

1.9 Zimmerman's Approach — Symmetric Model

In this approach, on the lines of Zimmermann [51], the goals and the constraints are represented by fuzzy sets and we assume that the decision maker can establish an aspiration level z for the value of the objective function he/she wants to achieve.

Therefore, as proposed by Zimmermann [51], we consider the following format of the symmetric fuzzy linear programming problem (SFLP)

(SFLP) Find x such that

$$f_0(x) \underset{\sim}{\geq} z$$

$$f_i(x) \underset{\sim}{\leq} d_i \qquad i = 1, 2, \ldots, k$$

$$g_i(x) \leq b_i \qquad i = k+1, k+2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

where $f_0, f_i, i = 1, 2, \ldots, k$, and g_i, $i = 1, 2, \ldots, m$ are linear functions.

Also, $\gtrsim$ is the fuzzified version of $\geq$ and represents 'essentially greater than or equal to' and $\lesssim$ represents 'essentially less than equal to'.

Then the problem is interpreted as:

- Make a decision $x \geq 0$ such that at x
- the value of the objective function $f_0(x)$ 'essentially greater than or equal to' the predetermined aspiration level z, and
- the constraints $f_i(x) \lesssim d_i$, $i = 1, 2, \ldots, k$ are satisfied in fuzzy sense, and the constraints $g_i(x) \leq b_i$, $i = k+1, k+2, \ldots, m$ are crisply satisfied.

Then the equation (SFLP) is equivalent to

(EFLP) Find x such that

$$-f_0(x) \lesssim -z$$

$$f_i(x) \lesssim d_i \qquad i = 1, 2, \ldots, k$$

$$g_i(x) \leq b_i \qquad i = k+1, k+2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

where each of the fuzzy constrains, $-f_0(x) \lesssim -z$, and $f_i(x) \lesssim d_i$, $i = 1, 2, \ldots, k$ represents a fuzzy set whose membership function is $\mu_i(x)$, $i = 0, 1, 2, \ldots, k$, is interpreted as the degree to which x satisfies the fuzzy constraints $-f_0(x) \lesssim -z$, and

$f_i(x) \lesssim d_i$, i = 1, 2, ..., k. Then, following Zimmermann [51], we write a symmetric fuzzy programming problem as follows:

(EFLP-1) Find an x that satisfies

$$f_0(x) \lesssim z_0 \tag{1.9.1}$$

$$f_i(x) \lesssim d_i \qquad i = 1, 2, \ldots, k \tag{1.9.2}$$

$$g_i(x) \le b_i \qquad i = k+1, k+2, \ldots, m \tag{1.9.3}$$

$$x_j \ge 0 \qquad j = 1, 2, \ldots, n \tag{1.9.4}$$

z_0 is called the aspiration level of $f_0(x)$ and is given some pre-assigned value. Let, $q_0 > 0$, and $q_i > 0$, (i = 1, 2, . . . , k), be subjectively chosen constants of admissible violations such that q_0 is associated with (1.9.1), and q_i (i = 1, 2, . . . , k) are associated with the i-th linear constraint of (1.9.2). We assume that the membership functions of $\mu_i(x)$, i = 0, 1, 2, ..., k, are linearly decreasing over the 'tolerance level' q_i. Now, on the lines of Zimmerman [51], we define the membership function corresponding to (1.9.1) and (1.9.2), as follows.

Corresponding to $f_0(x)$ membership function $\mu_0(x)$ for objective function is written as

$$\mu_0(x) = \begin{cases} 1 & \text{if} \quad f_0(x) \le z_0 \\ 1 - \dfrac{f_0(x) - z_0}{q_0} & \text{if} \quad z_0 \le f_0(x) \le z_0 + q_0 \\ 0 & \text{if} \quad f_0(x) \le z_0 + q_0 \end{cases} \tag{1.9.5}$$

Corresponding to i = 1, 2, ..., k, the membership function is

$$\mu_i(x) = \begin{cases} 1 & \text{if} \quad f_i(x) \le d_i \\ 1 - \dfrac{f_i(x) - d_i}{q_i} & \text{if} \quad d_i \le f_i(x) \le d_i + q_i \\ 0 & \text{if} \quad d_i + q_i \le f_i(x) \end{cases} \qquad (1.9.6)$$

Once the membership functions are known, then a solution that belongs to the intersection of the fuzzy sets of objective function (1.9.1), constraints (1.9.2), and satisfies the crisp constraints (1.9.3) and (1.9.4) is a solution to (EFLP-1). Suppose that $\mu_D(x)$ is the membership function of the fuzzy set 'decision' of the model. Then,

$$\mu_D(x) = \text{Min}\,(\mu_0(x), \mu_1(x), \mu_2(x), \mu_3(x), \ldots, \ldots, \mu_k(x))$$

Since, we are interested in a large value of $\mu_D(x)$, therefore, following Zimmermann [51], we want to obtain the maximum value of $\mu_D(x)$. Thus, our interest is to

$$\text{maximize } \mu_D(x) = \min\,[\mu_0(x), \mu_1(x), \mu_2(x), \mu_3(x), \ldots, \ldots, \mu_k(x)]$$

subject to the constraints of (1.9.3) and (1.9.4)

Now, along the lines of Zimmermann [51], replacing $\mu_D(x)$ by λ, and using (1.9.5) and (1.9.6) respectively for $\mu_0(x)$, $\mu_i(x)$, $i = 1, 2, \ldots, k$, we have the following problem;

(EFLP-2) Max λ

subject to

$$f_0(x) + q_0\,\lambda \le z_0 + q_0$$

$$f_i(x) + q_i\,\lambda \le q_i + d_i \qquad i = 1, 2, \ldots, k,$$

$$g_i(x) \le b_i \qquad i = k+1, k+2, \ldots, m$$

$$0 \leq \lambda \leq 1$$

$$x_j \geq 0 \qquad\qquad j = 1, 2, \ldots, n$$

It is observed that (ENLP-2) is a crisp optimization problem whose optimal solution, if it exists, provides a solution to (SFLP).

Remark 1.9.1. If in (SFLP), we replace

$$f_0(x) \underset{\sim}{\leq} z_0 \qquad \text{by} \qquad f_0(x) \lesssim z_0$$

that is, if we replace the requirement 'essentially less than or equal to' denoted by '$\underset{\sim}{\leq}$', by the requirement 'desired to be less than or equal to' denoted by '$\lesssim$', then, we take the membership function $\mu_0(x)$ corresponding to $f_0(x)$ as follows:

$$\mu_0(x) = \begin{cases} 1 & \text{if} \quad f_0(x) \leq z_0 - q_0 \\ 1 - \dfrac{f_0(x) - (z_0 - q_0)}{q_0} & \text{if} \quad z_0 - q_0 \leq f_0(x) \leq z_0 \\ 0 & \text{if} \quad z_0 \leq f_0(x) \end{cases}$$

In this case corrosponding to (EFLP-1) we have the following (EFLP-2).

(EFLP-2) Max λ

subject to

$$f_0(x) + q_0 \lambda \leq z_0$$

$$f_i(x) + q_i \lambda \leq q_i + d_i \qquad i = 1, 2, \ldots, k,$$

$$g_i(x) \le b_i \qquad i = k+1, k+2, \ldots, m$$

$$0 \le \lambda \le 1$$

$$x_j \ge 0 \qquad j = 1, 2, \ldots, n$$

1.10 Zimmerman's Approach — Non-Symmetric Model

On the lines of Zimmermann [51], we now consider the following non–symmetric fuzzy optimization problem (NSFLP) with crisp objective function and a mixture of fuzzy and crisp constraints.

(NSFLP) Min $f(x)$ (1.10.1)

subject to

$$f_i(x) \lesssim d_i \qquad i = 1, 2, \ldots, k \tag{1.10.2}$$

$$g_i(x) \le b_i \qquad i = k+1, k+2, \ldots, m \tag{1.10.3}$$

$$x_j \ge 0 \qquad j = 1, 2, \ldots, n \tag{1.10.4}$$

As suggested by Zimmermann [51], we compute the membership function corresponding to the objective function (1.10.1) with the help of the following two crisp optimization programs (COP–1) and (COP–2).

(COP–1) Min $f(x)$

subject to

$$f_i(x) \le d_i \qquad i = 1, 2, \ldots, k$$

$$g_i(x) \le b_i \qquad i = k+1, k+2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

Let the minimum value of the objective function $f(x)$ be f_0.

(COP–2) Min $f(x)$

subject to

$$f_i(x) \leq d_i + q_i \qquad i = 1, 2, \ldots, k$$

$$g_i(x) \leq b_i \qquad i = k+1, k+2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

Let the minimum value of $f(x)$ be f_1.

Then, on the lines of Zimmermann [51], the membership function corresponding to the objective function of (NSFLP) is defined as follows.

$$\mu_0(x) = \begin{cases} 1 & \text{if } f(x) \leq f_0 \\ 1 - \dfrac{f(x) - f_0}{f_1 - f_0} & \text{if } f_0 \leq f(x) \leq f_1 \\ 0 & \text{if } f(x) \geq f_1 \end{cases}$$

The equivalent crisp programming problem corresponding to (NSFLP) is as follows.

Max x_{n+1}

subject to

$$f(x) + (f_1 - f_0)\, x_{n+1} \leq f_1$$

$$f_i(x) + q_i\, x_{n+1} \leq q_i + d_i \qquad i = 1, 2, \ldots, k,$$

$$g_i(x) \leq b_i \qquad i = k+1, k+2, \ldots, m$$

$$x_{n+1} \leq 1$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, (n+1)$$

which is similar to (EFLP-1), and therefore, can be solved on the lines of the method suggested for solving (EFLP-1).

1.11 Quadratic Programming Theory

Quadratic programming is a special type of nonlinear programming in which the objective function is quadratic and the constraints are linear. The standard form of a quadratic programming is as follows.

$$\text{(QP)} \qquad \text{Min} \quad z = \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} x_j h_{jk} x_k \qquad (1.11.1)$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, 2, \ldots, m \qquad (1.11.2)$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n. \qquad (1.11.3)$$

In matrix and vector form the same problem is written as:

$$\text{Min} \quad z = c^t x + \frac{1}{2} x^t H x \qquad (1.11.4)$$

$$\text{subject to} \quad A x \leq b \qquad (1.11.5)$$

$$x \geq 0 \qquad (1.11.6)$$

where x is the n-component column vector for decision variables; c is the n-component column vector of objective function coefficients for the linear terms; H is the $n \times n$ symmetric matrix of twice the objective coefficients for the quadratic and interactive terms; z is the objective function to be minimized; A is the $m \times n$ matrix of constraint-equation coefficients, and b is the m-component column vector of the right hand side coefficients.

In order to obtain the global minimum of the objective function, the objective function and the constraint set must be either convex or pseudo-convex. Since the constraints are linear, the constraint set is a convex set. If H is a positive semi-definite (or positive definite) matrix then z is a convex (or strictly convex) function. Several papers (for example see, [2], [3], [29], [32], [41], [42], [43], [44], [45], and [48]) deal with the theory and finding a solution of the above (QP).

1.12 Organization of the Thesis

In the present thesis, we model a number of problems from a variety of areas under fuzzy environment. Also, we discuss the methods to obtain their solutions and interpretation to the solutions.

Chapter 1 provides an introduction to the concepts and problems considered in this thesis. Chapter 2 deals with the literature review of the related work done by other researchers. In Chapter 3, a linear programming approach to the problem of graduation is presented under both crisp and fuzzy environment. In Chapter 4, a finite iteration technique for solving fuzzy quadratic programming problem is developed. In Chapter 5, we use the fuzzy quadratic programming approach to address the graduation problem under more generalized criteria. At the end of the Chapter 5, results of graduation problem, obtained using linear programming approach and quadratic programming approach both under crisp and fuzzy environment, are compared. Finally, the conclusion and the discussion on the contributions made by the thesis, along with some recommendations for further research, are given in Chapter 6.

Chapter 2

LITERATURE SURVEY

This chapter provides a survey of the literature dealing with Graduation Problem, Fuzzy Quadratic Programming Problem, and other concepts considered in this thesis. The purpose of this chapter is to review the developments, and to identify the status of existing literature in these areas.

2.1 Review of Literature on Graduation Problem

Several methods have been developed (for example see, [8], [9], [37], [39], and [46]) by which the graduation of an observed series may be accomplished and the problem of graduation can be solved.

Broffitt [8] developed a method for determining which smoothness terms to include in the objective function assuming that the graduator has pre-specified a polynomial model, which represents the graduated values under ideal or ultimate smoothness. Brooks et. al. [9] demonstrated Cross-Validatory graduation method that is applied to the choice of parameters that control the degree of smoothing in generalized Whittaker-Henderson graduation [47]. This approach is then compared with the Bayesian method with the help of an example. Taylor [39] presented a paper to place Whittaker-Henderson graduation in a Bayesian context and showed that this determines in a precise manner the extent to which goodness-of-fit should be traded off against smoothness in the Whittaker-Henderson loss function. Verrall [46] showed that the Whittaker graduation is equivalent to a dynamic regression analysis, in which one of the parameter in allowed to vary stochastically. It also suggests an automatic method of estimating the smoothing parameter, which is, at present, chosen subjectively by the graduator. At the end, an example is presented to support the theory.

A common method of actuarial graduation is the difference-equation method as described in London's monograph [26]. In this method of graduation, graduated values v_x (where $x = 1, 2, \ldots, n$), are sought corresponding to a given set of observed values u_x and non-negative weights w_x that minimize the quantity $F + hS$, where $F = \sum_{x=1}^{n} w_x(u_x - v_x)^2$ and $S = \sum_{x=1}^{n-z} (\Delta^z v_x)^2$. F is an expression that measures the degree of fit (or rather, lack of fit) of the graduated values to the observed values, and S is an expression that measures the degree of smoothness (or rather, lack of smoothness) of the graduated values. The order of the forward differences used in measure of smoothness is denoted by z. The values of z commonly used are $z = 2$, $z = 3$ and $z = 4$. The choice of z implies that a polynomial of degree $z - 1$ is being fitted to the observed values. The Δ^z, and hence S will be zero if the graduated values lie exactly on the curve of a polynomial of degree $z - 1$. The parameter h is a non-negative constant that indicates the emphasis assigned to the smoothness of the graduated values relative to how well they fit the observed values. The larger the value of h, the smaller S will be and the smoother will be the graduated values. When h approaches 0, v_x approaches u_x, and fit is emphasized over smoothness.

The method is called the difference-equation method because the values v_x for which the minimum of $F + hS$ is achieved can be shown to satisfy the difference equation

$$w_x v_x + h\delta^{2z} v_x = w_x u_x$$

where δ denotes the central difference operator [37]. This same equation can be found in matrix-vector form in London's monograph [26, p. 56]. Whittaker-Henderson Type B graduation formulas have variable weights and is the more general case of Whittaker-Henderson Type A graduation where $w_x = 1$ for all x.

In the difference-equation method, choices must be made in the objective function, F + hS, for the measures of fit and smoothness. Usually, the measure of fit is the weighted sum of the squares of the deviations, $u_x - v_x$, of the observed values from the graduated values and the measure of smoothness is the sum of the squares of the z-th difference of the graduated values.

Schuette [37] developed a linear programming approach to graduation problem. In this paper, the Whittaker-Henderson Type B method of graduation, in which the weighted sum of the squares of the deviations of graduated values from observed values plus a parameter times the sum of the squares of the z-th differences of the graduated values is minimized using absolute values instead of squares. The end problem is then expressed as a linear programming problem as follows:

$$\text{Minimize } F + hS = \sum_{x=1}^{n} w_x (D_x + E_x) + h\sum_{x=1}^{n-z} (R_x + T_x)$$

subject to constraints

$$\Delta^z (E_x - D_x) + R_x - T_x = \Delta^z u_x, \qquad x = 1, 2, \ldots, n - z,$$

$$D_x \geq 0, E_x \geq 0, R_x \geq 0, \text{ and } T_x \geq 0, \text{ for all appropriate values of } x$$

where

F = measure of degree of fit,

S = measure of degree of smoothness,

u_x = set of observed values,

v_x = set of graduated values,

w_x = non-negative weights,

z = order of forward difference/degree of polynomial used as a standard of smoothness,

$\Delta^z v_x$ = z^{th} difference of the sequence v_x,

E_x, D_x, R_x, T_x = deviational variables,

h = a real number parameter that controls the relative emphasis given to F and S.

Two examples are presented at the end to demonstrate the method and some difficulties are expressed in regards to computational feasibility.

2.2 Review of Literature on Quadratic Programming Problem

Various methods are available in the literature (for example see, [3], [29], [32], [36], [40], [41], [42], [43], [44], [45], and [48]) to solve a quadratic programming problem under crisp environment. Most of the available methods of solving a quadratic programming in crisp environment use simplex tableaus in one or the other form. Van de Panne [43] presented a method to maximize a linear objective function subject to a quadratic and a number of linear constraints. This method presented in [43] differs from general convex programming methods by terminating in a finite number of iterations. Bela Martos [29] developed a method to solve a quadratic programming with quasiconvex objective function. This method is different from other methods, as it doesn't assume the convexity of the objective function. Paper begins with the characterization of quadratic functions that are quasiconvex in the nonnegative orthant. Van de Panne [43] proposed a method for finding the global optimum of a general quadratic programming problem. He developed this method based on the formulation of the problem as a multiparametric convex quadratic programming problem. This results in the formulation of a number of sub problems, which are general quadratic programming problems of a smaller size. A numerical example is worked at the end in detail.

In the literature (for example see [3], [29], [32], [42], [43], and [48]) a classical quadratic programming problem is stated as follows:

(P-1) Minimize $z = p^T x + \frac{1}{2} x^T C x$ (2.2.1)

subject to

$$A x \leq b \qquad (2.2.2)$$

$$x \geq 0 \qquad (2.2.3)$$

where each of p and $x \in R^n$, C is a symmetric $n \times n$ matrix, A is an $m \times n$ matrix and $b \in R^m$. We also assume that the feasible solution set of the constraints is bounded.

Further, we assume that the quadratic objective function is pseudoconvex. Several methods for solving such a problem are available in the literature ((for example see [3], [29], [32], [41], [42], [43], and [48]). Van de Panne [43] considered the following problem.

(VP) Maximize $c^T x$

subject to

$$p^T x + \frac{1}{2} x^T C x \le \beta$$

$$A x \le b$$

$$x \ge 0$$

where $\beta \in R$ and is known in advance, $c \in R^n$. Other notations in the problem (VP) are same as in (P–1). Van de Panne [43] developed the following two-phase method to solve (VP) in a finite number of steps.

Phase 1. In (VP), ignore the quadratic constraint and solve the following ordinary linear programming problem (LP), assuming that (LP) has an optimal solution.

(LP) Maximize $c^T x$

subject to

$$A x \le b$$

$$x \ge 0$$

If the optimal solution x^0 of (LP) satisfies the quadratic constraint, that is if

$$p^T x^0 + \frac{1}{2} x^{0T} C x^0 \le \beta$$

we obviously have found the optimal solution of the original problem. If however,

$$p^T x^0 + \frac{1}{2} x^{0T} C x^0 > \beta$$

we go to Phase 2 of the method.

Phase 2. In this phase, we add the following linear constraint [43],

$$c^T x \geq \lambda$$

where λ is a parameter which is given different values in the course of the computations assuming that $\lambda^0 = c^T x^0$ is the value of the objective function in (LP) at its optimal solution x^0. Now, we consider the following quadratic programming problem (QP).

$$\text{(QP)} \qquad \text{Minimize } z = p^T x + \frac{1}{2} x^T C x$$

subject to

$$A x \leq b$$
$$c^T x \geq \lambda$$
$$x \geq 0$$

Van de Panne [43] solved (QP) by decreasing λ parametrically from λ^0 to lower values. According to Van de Panne [43], Phase 2 can terminate in one of the two ways.

1. It may terminate when for a certain value of λ, say λ^*, the objective function has become equal to β. In this case an optimal solution to (VP) has been found.
2. It may terminate when for a certain value of λ, say λ^*, the constraint $c^T x \geq \lambda$ ceases to be binding for the optimal solution of the problem (QP) with $p^T x + \frac{1}{2} x^T C x$ being still larger than β. This means that for no value of λ a solution exists giving a minimum value of the objective $p^T x + \frac{1}{2} x^T C x$ less than or equal to β. In this case, in (VP), the quadratic constraint is incompatible with the linear constraints and no feasible solution to (VP) exists.

2.3 Summary of the Thesis

The results and methods proved in this thesis are contained in Chapter 3, Chapter 4, Chapter 5 and Chapter 6. We summarize them as follows:

Chapter 3 A Linear Programming Approach to Graduation Under Crisp and Fuzzy Environment

The purpose of the present chapter is to extend the results proved by Schuette [37] further by solving the graduation problem under fuzzy conditions. Advantage of using fuzzy mathematics is that it gives decision-maker flexibility and quantifies the certain type of uncertainty involved in the problem in question.

Chapter 4 A Finite Iteration Technique for a Fuzzy Quadratic Programming Problem

In this chapter, we develop a finite iteration technique to solve a fuzzy quadratic programming problem with single quadratic objective function and a number of linear constraints. The quadratic programming problem has a lots of applications in the field of economics, finance, statistics, and structural engineering. Due to such a vast practical importance of quadratic programming, a large number of papers have been published in past 35 years. All the methods available to solve fuzzy quadratic programming problems are very lengthy and require high level of knowledge in the field of mathematics. Therefore, method proposed in this chapter, is an attempt to provide an easy tool to address this kind of problems.

Chapter 5 A Quadratic Programming Approach to Graduation Under Crisp and Fuzzy Environment

In this chapter, we sharpen the graduation problem discussed in Chapter 3 by developing a quadratic programming approach both under crisp and fuzzy environment. To do so, we also use the approach developed in Chapter 4.

Chapter 6 Conclusion, Contribution and Recommendations

In this chapter, we present the contributions and conclusions, along with some recommendations for further research on the problems considered in this dissertation.

Chapter 3

A LINEAR PROGRAMMING APPROACH TO GRADUATION UNDER CRISP AND FUZZY ENVIRONMENT

In the present chapter, we consider the graduation problem formulated as a linear programming problem, both under crisp and fuzzy environment. First, we obtain the solution of linear programming problem, as discussed by Schuette [37], under crisp environment. Then, we formulate and solve the problem in a fuzzy environment, and compare the results obtained both under crisp and fuzzy environment.

3.1 Introduction

Schuette [37], in his paper, considered the graduation problem using absolute values for both F and S, which Schuette solved as a crisp linear programming problem. The numerical example used by Schuette [37] is solved using linear programming approach under crisp and fuzzy environment in the present chapter.

3.2 Linear Programming Formulation Under Crisp Environment

On the lines of Schuette [37], we present the linear programming formulation of graduation problem under crisp environment.

Assumption

For this model, it is assumed that all the data are known with certainty.

Objective

Objective in this problem is to minimize the combination of fit and smoothness to obtain improved graduated values.

General Formulation

The key component in formulation of graduation problem is the method to deal with absolute values. This problem can be dealt by taking advantage of the fact that any function can be separated into its positive and negative parts [37].

As we know that for the function $|f(x)|$,

if we set $f(x) = (D_x - E_x)$

where $D_x \geq 0$ and $E_x \geq 0$

then

$|f(x)| = (D_x + E_x)$

Therefore, $f(x)$ may be replaced by $D_x - E_x$ and $|f(x)|$ by $D_x + E_x$. The only condition is that D_x and E_x must be nonnegative and they must not be positive simultaneously.

For the graduation problem [37]

$$v_x - u_x = D_x - E_x \tag{3.2.1}$$

with deviational variables $D_x \geq 0$ and $E_x \geq 0$ for $x = 1, 2, \ldots, n$, and let

$$\Delta^z v_x = R_x - T_x \tag{3.2.2}$$

with deviational variables $R_x \geq 0$ and $T_x \geq 0$ for $x = 1, 2, \ldots, n - z$.

Also

$$v_x = u_x + D_x - E_x, \tag{3.2.3}$$

yields

$$\Delta^z v_x = \Delta^z(u_x + D_x - E_x) = R_x - T_x \tag{3.2.4}$$

Thus, under crisp environment, we have to find the values of D_x, E_x, R_x and T_x with the help of the following crisp linear programming problem.

$$\text{(CLP) Minimize } (F + hS) = \sum_{x=1}^{n} w_x (D_x + E_x) + h \sum_{x=1}^{n-z} (R_x + T_x) \tag{3.2.5}$$

subject to constraints

$$\Delta^z (E_x - D_x) + R_x - T_x = \Delta^z u_x, \quad x = 1, 2, \ldots, n-z, \tag{3.2.6}$$

$$D_x \geq 0,\ E_x \geq 0,\ R_x \geq 0, \text{ and } T_x \geq 0, \text{ for all appropriate values of } x. \tag{3.2.7}$$

which is of the type of a standard linear programming problem. The problem involves $2n + 2(n - z)$ variables and $(n - z)$ constraints and it is important to point out here that in (CLP) at most $(n - z)$ of the D_x and E_x can be positive in the optimal solution.

3.3 Numerical Example Under Crisp Environment

We illustrate this method through the numerical example given by Miller [30, page 39], which is explored and formulated by Schuette [37] using a crisp linear programming approach. The data consists of nineteen observed values and nineteen corresponding weights. The values for u_x for $x = 1, 2, \ldots, 19$, are 34, 24, 31, 40, 30, 49, 48, 48, 67, 58, 67, 75, 76, 76, 102, 100, 101, 115, and 134. The values for $w_x = 1, 2, \ldots, 19$ are 3, 5, 8, 10, 15, 20, 23, 20, 15, 13, 11, 10, 9, 9, 7, 5, 5, 3, and 1.

For this example, the parameter $h = 10$, and $z = 2$. Then the problem, under crisp environment, can be formulated as follows.

Minimize $z = (F + hS) =$

$$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) + 20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) + 11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15}) + 5(D_{16} + E_{16}) + 5(D_{17} + E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 + R_{10} + R_{11} + R_{12} + R_{13} + R_{14} + R_{15} + R_{16} + R_{17} + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13} + T_{14} + T_{15} + T_{16} + T_{17})$$

subject to the following constraints:

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 = 17$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 = 2$$

$$-E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 = 19$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 = 29$$

$$-E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 = 20$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 = 1$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 = 19$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 = 28$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 = 18$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} = 1$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} = 7$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} = 1$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} = 26$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} = 28$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} = 3$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} = 13$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} = 5$$

Non-negativity constraints:

$D_x \geq 0$, $E_x \geq 0$, $x = 1, 2, \ldots, 19.$

$R_x \geq 0$, $T_x \geq 0$, $x = 1, 2, \ldots, 17.$

3.4 Results

On solving the above problem, Schuette [37] obtain the following Table 3.1.

Table 3.1 Results of crisp linear program problem

Variable	Value	Variable	Value	Variable	Value	Variable	Value
D_2	1.6667	D_{14}	9	E_9	12	R_{13}	1.5
D_5	11.6667	D_{17}	6.5	E_{12}	2	T_5	4.3333
D_8	1	E_1	13.6667	E_{15}	9.5		
D_{10}	3	E_4	3.6667	E_{19}	11.5		
D_{13}	3	E_6	2	R_7	5		

3.5 Interpretation of the Results

Results given in the above table summarize the solution of the crisp problem as discussed by Schuette [37]. Since D_x and E_x are the deviational variables for $v_x - u_x$, value of each of these variables will imply as how close are the initial values to the graduated ones. Following the lines of Schuette ([37], page 415), in the solution, we should have at least z (=2) values of D_x and E_x that have 0 value. In the solution of crisp linear programming problem, we have $D_3 = E_3 = 0$, $D_7 = E_7 = 0$, $D_{11} = E_{11} = 0$, $D_{16} = E_{16} = 0$ and $D_{18} = E_{18} = 0$.

At the same time, we should have at most (Schuette [37], page 415) $n - z$ (=17) variables D_x and E_x that have positive values. From Table 3.1, we have 14 variables that have positive values. These values are $D_2 = 1.6667$, $D_5 = 11.6667$, $D_8 = 1$, $D_{10} = 3$, $D_{13} = 3$, $D_{14} = 9$, $D_{17} = 6.5$, $E_1 = 13.6667$, $E_4 = 3.6667$, $E_6 = 2$, $E_9 = 12$, $E_{12} = 2$, $E_{15} = 9.5$, $E_{19} = 11.5$, $R_7 = 5$, $R_{13} = 1.5$ and $T_5 = 4.333$. R_x and T_x represent the deviational variables of smoothness function. Clearly, $R_7 = 5$, $R_{13} = 1.5$ and $T_5 = 4.3333$ which indicates that only three variables require additional smoothing. The minimum value of objective function, which represents the minimization of the sum of fit and smoothness, is 886.8333.

On the lines of Schuette [37], Table 1 in Appendix 2, depicts the graduated values and the measures of the fit and smoothness obtained by solving the graduation problem using linear programming approach for $z = 2$ and different values of h. In the same fashion, Table 3 in Appendix 2, represents the graduated values and the measures of the fit and smoothness obtained by solving the graduation problem using linear programming approach for $z = 3$ for different values of h. However, the results obtained by Schuette as shown in Table 1 and 3 are under crisp environment.

3.6 Formulation Under Fuzzy Environment

In general, most of the time, due to incomplete or forecasted information the input data are imprecise. Any vagueness or impreciseness in data of observed values might lead to an inappropriate interpretation of the underlying law, which would in turn completely defeat the purpose of graduation process. The problems of impreciseness in data of observed values and their revision to improve the model are handled effectively by taking advantage of fuzzy set theory ([5], [49], [51] and [52]).

1. **Imprecise objective function limit levels.** The management provides an upper bound of the estimation of the total value of combination of fit and smoothness represented by objective function z_0. Value of objective function is desired to be below this upper bound. A tolerance that defines the dispersion of this value may be given in the form of fraction of z_0.
2. **Imprecise observed values u_x.** Since collection of data of observed values are rarely accurate to the exact number of units, the management can provide a tolerance level in form of a fraction of imprecisely known observed values, that provides a range above and below the observed values in which the actual value is likely to occur.

We now formulate the problem under the following additional assumptions.

Additional Assumptions

(i) The total value of objective function is desired to stay below a given limit.

(ii) The observed value data is known imprecisely.

Objective

The objective of the model is the 'desire' that the combination of fit and smoothness stay below or equal to the aspiration level, which is given some pre-assigned value keeping in view the imprecise data for observed values.

Additional Notation

Let,

z_0 = the aspiration level for the objective and is given some pre-assigned value,

q_0 = the subjectively chosen value of admissible violations corresponding to z_0 ,

q_x = tolerance level associated with imprecisely known observed values u_x, for all x,

μ_0 = membership function associated with imprecisely known objective function z_0,

μ_{xL} = membership function corresponding to lower side of the constraint associated with imprecisely known observed values u_x, for all appropriate values of x,

μ_{xU} = membership function corresponding to upper side of the constraint associated with imprecisely known observed values u_x, for all appropriate values of x.

All other variables and symbols have the same meaning as in crisp formulation.

Formulation of Graduation Problem Under Fuzzy Environments

Using Zimmerman's notation [51], in a fuzzy environment, the crisp constraints

$$\Delta^z (E_x - D_x) + R_x - T_x = \Delta^z u_x, \quad x = 1, 2, \ldots, n - z, \tag{3.6.1}$$

can be replaced by

$$\Delta^z (E_x - D_x) + R_x - T_x \underset{\sim}{=} \Delta^z u_x, \quad x = 1, 2, \ldots, n - z, \tag{3.6.2}$$

which are further replaced by

$$\Delta^z (E_x - D_x) + R_x - T_x \underset{\sim}{\geq} \Delta^z u_x, \quad x = 1, 2, \ldots, n - z, \tag{3.6.3}$$

$$\Delta^z (E_x - D_x) + R_x - T_x \underset{\sim}{\leq} \Delta^z u_x, \quad x = 1, 2, \ldots, n - z, \tag{3.6.4}$$

The notation '$\underset{\sim}{\geq} \Delta^z u_x$' (or $\underset{\sim}{\leq} \Delta^z u_x$ respectively) means that the corresponding fuzzy constraint is 'essentially $\geq \Delta^z u_x$' (or essentially $\leq \Delta^z u_x$, respectively), for all x [51].

We denote by μ_{xL} and μ_{xU}, the membership functions corresponding to (3.6.3) and (3.6.4) respectively.

Using Zimmerman's approach [51], in a fuzzy environment, the objective function, which is the total value of combination of fit and smoothness, can be written as

$$\sum_{x=1}^{n} w_x(D_x + E_x) + h\sum_{x=1}^{n-z}(R_x + T_x) \mathrel{\tilde{\leq}} z_0 \qquad (3.6.5)$$

with μ_0 as the corresponding membership function for the objective function (3.6.5), where '$\tilde{\leq}$ z_0' means that the corresponding membership function is 'desired to be less than or equal to z_0'.

Then, under fuzzy environments, our crisp linear programming problem (CLP) becomes the following fuzzy linear programming problem, denoted by (FLP)

(FLP) Find D_x, E_x, R_x and T_x for all appropriate values of x, we have

for the fuzzy objective

$$\sum_{x=1}^{n} w_x(D_x + E_x) + h\sum_{x=1}^{n-z}(R_x + T_x) \mathrel{\tilde{\leq}} z_0 \qquad (3.6.6)$$

and for the fuzzy constraints with corresponding membership functions μ_{xL} and μ_{xU}

$$\Delta^z(E_x - D_x) + R_x - T_x \mathrel{\underset{\sim}{\geq}} \Delta^z u_x, \qquad x = 1, 2, \ldots, n - z \qquad (3.6.7)$$

$$\Delta^z(E_x - D_x) + R_x - T_x \mathrel{\underset{\sim}{\leq}} \Delta^z u_x, \qquad x = 1, 2, \ldots, n - z \qquad (3.6.8)$$

The non-negativity constraints are written as

$$D_x \geq 0, E_x \geq 0, \qquad x = 1, 2, \ldots, n \qquad (3.6.9)$$

and $$R_x \geq 0,\ T_x \geq 0, \qquad x = 1, 2, \ldots, n - z \qquad (3.6.10)$$

The graduation problem under fuzzy environment now is equivalent to obtain a solution satisfying the fuzzy sets given by (3.6.6), (3.6.7), (3.6.8), (3.6.9) and (3.6.10).

Membership Functions

Following Zimmermann [51], below we define the membership functions, μ_0 for the fuzzy objective (3.6.6), and μ_{xL} and μ_{xU} for the fuzzy constraints (3.6.7) and (3.6.8), respectively.

For the sake of simplicity, we denote $\sum_{x=1}^{n} w_x(D_x + E_x) + h\sum_{x=1}^{n-z}(R_x + T_x)$ by f_0, $\Delta^z(E_x - D_x) + R_x - T_x$ by f_x, and $\Delta^z u_x$ by d_x.

Then, if f_0 is desired to be lower than z_0 and $q_0 > 0$ be the subjectively chosen value of admissible violation corresponding to z_0, then the membership function μ_0 for objective function is written as

$$\mu_0 = \begin{cases} 1 & \text{if} \quad f_0 \le z_0 - q_0 \\ 1 - \dfrac{f_0 - (z_0 - q_0)}{q_0} & \text{if} \quad z_0 - q_0 \le f_0 \le z_0 \\ 0 & \text{if} \quad z_0 \le f_0 \end{cases}$$

Similarly, the membership functions for fuzzy constraints (3.6.7) and (3.6.8) is obtained as below.

Let $q_{xL} > 0$, and $q_{xU} > 0$ be the subjectively chosen constants of admissible violations associated with constraints (3.6.7) and (3.6.8) respectively. Then, following Zimmermann [51, 52],

μ_{xL}, the membership functions for the lower side of the fuzzy region of the fuzzy constraints (3.6.7) are taken as

$$\mu_{xL} = \begin{cases} 1 & \text{if} \quad f_x \geq d_x \\ 1 - \dfrac{(d_x - f_x)}{q_x} & \text{if} \quad d_x - q_x \leq f_x \leq d_x \\ 0 & \text{if} \quad d_x - q_x \geq f_x \end{cases}$$

and μ_{xU}, the membership functions for the upper side of the fuzzy region of the fuzzy constraints (3.6.8) are taken as

$$\mu_{xU} = \begin{cases} 1 & \text{if} \quad f_x \leq d_x \\ 1 - \dfrac{(f_x - d_x)}{q_x} & \text{if} \quad d_x \leq f_x \leq d_x + q_x \\ 0 & \text{if} \quad d_x + q_x \leq f_x \end{cases}$$

Once the membership functions are obtained, we get a solution to (FLP) by finding the intersection of the fuzzy sets given by (3.6.6), (3.6.7) and (3.6.8), to get to a decision. Then μ_D the membership function of decision D satisfying (3.6.6), (3.6.7) and (3.6.8) is

$$\mu_D = \min(\mu_0, \mu_{1L}, \mu_{2L}, \ldots, \mu_{(n-z)L}, \mu_{1U}, \mu_{2U}, \ldots, \mu_{(n-z)U})$$

Since, we are interested in large value of μ_D over (3.6.9) and (3.6.10), therefore, following Zimmermann [51], we obtain

$$\max \ \mu_D = \min(\mu_0, \mu_{1L}, \mu_{2L}, \ldots, \mu_{(n-z)L}, \mu_{1U}, \mu_{2U}, \ldots, \mu_{(n-z)U})$$

subject to the constraints (3.6.9) and (3.6.10).

Replacing μ_D by λ , we have the following problem (LP) along the lines of Zimmermann [51] ;

(LP) $\max\ \lambda$

subject to

$$\mu_0 \geq \lambda$$

$$\mu_{xL} \geq \lambda \qquad x = 1, 2, \ldots, n-z,$$

$$\mu_{xU} \geq \lambda \qquad x = 1, 2, \ldots, n-z,$$

and crisp constraints (3.6.9) and (3.6.10)

It is observed that (LP) is a crisp linear program whose optimal solution provides a solution to (FLP).

In view of the membership functions μ_0, μ_{xL} and μ_{xU}, $x = 1, 2, \ldots, n-z$; the (LP) can be restated as

$$\max\ \lambda$$

subject to

$$f_0 + \lambda q_0 \leq z_0$$

$$f_x - \lambda q_x \geq d_x - q_x \qquad x = 1, 2, \ldots, n-z,$$

$$f_x + \lambda q_x \leq d_x + q_x \qquad x = 1, 2, \ldots, n-z,$$

$$0 \leq \lambda \leq 1$$

and $D_x \geq 0,\ E_x \geq 0, \quad x = 1, 2, \ldots, n$

$$R_x \geq 0,\ T_x \geq 0, \quad x = 1, 2, \ldots, n-z$$

Identifying $f_0 = \sum_{x=1}^{n} w_x (D_x + E_x) + h \sum_{x=1}^{n-z} (R_x + T_x)$,

$$f_x = \Delta^z (E_x - D_x) + R_x - T_x,$$

and $d_x = \Delta^z u_x$,

we can rewrite (LP) as

(LP-1) max λ

subject to

$$\sum_{x=1}^{n} w_x (D_x + E_x) + h\sum_{x=1}^{n-z} (R_x + T_x) + \lambda q_0 \le z_0$$

$$\Delta^z (E_x - D_x) + R_x - T_x - \lambda q_x \ge \Delta^z u_x - q_x \qquad x = 1, 2, \ldots, n-z,$$

$$\Delta^z (E_x - D_x) + R_x - T_x + \lambda q_x \le \Delta^z u_x + q_x \qquad x = 1, 2, \ldots, n-z,$$

$$0 \le \lambda \le 1$$

and

$$D_x \ge 0, E_x \ge 0, \qquad x = 1, 2, \ldots, n$$

$$R_x \ge 0, \ T_x \ge 0, \qquad x = 1, 2, \ldots, n-z.$$

Thus, we see that we obtain a solution to (FLP) by solving (LP-1) which is a crisp linear programming problem.

3.7 Numerical Example Under Fuzzy Environment

Below we write a fuzzified format of (CLP). In this example we assume a tolerance level of approximately 30% for observed values and 0.25% in total objective function. Therefore z_0 is 886.8333 and q_0 is 2.217. For the observed value constraints, the tolerances are $q_1 = 5.1$, $q_2 = .6$, $q_3 = 5.7$, $q_4 = 8.7$, $q_5 = 6$, $q_6 = .3$, $q_7 = 5.7$, $q_8 = 8.4$, $q_9 = 5.4$, $q_{10} = .3$, $q_{11} = 2.1$, $q_{12} = .3$, $q_{13} = 7.8$, $q_{14} = 8.4$, $q_{15} = .9$, $q_{16} = 3.9$, $q_{17} = 1.5$, where as the rest of the data is same as in crisp problem presented by Schuette [37]. In view of (FLP) following is fuzzy version of the above problem.

(NP-1)

$$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) +$$

$$20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) +$$

$$11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15})$$

$$+ 5(D_{16} + E_{16}) + 5(D_{17}+ E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1 + R_2 +$$

$$R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 + R_{10} + R_{11} + R_{12} + R_{13} + R_{14} + R_{15} +$$

$$R_{16} + R_{17} + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} +$$

$$T_{13} + T_{14} + T_{15} + T_{16} + T_{17}) \lesseqgtr 886.8333$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 \underset{\sim}{=} 17$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 \underset{\sim}{=} 2$$

$$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 \underset{\sim}{=} 19$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 \underset{\sim}{=} 29$$

$$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 \underset{\sim}{=} 20$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 \underset{\sim}{=} 1$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 \underset{\sim}{=} 19$$

$$- E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 \underset{\sim}{=} 28$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 \underset{\sim}{=} 18$$

$$- E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} \underset{\sim}{=} 1$$

$$- E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} \underset{\sim}{=} 7$$

$$- E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} \underset{\sim}{=} 1$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} \underset{\sim}{=} 26$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} \underset{\sim}{=} 28$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} \underset{\sim}{=} 3$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} \underset{\sim}{=} 13$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} \underset{\sim}{=} 5$$

Non-negativity constraints:

$D_x \geq 0$, $E_x \geq 0$, $x = 1, 2, \ldots, 19.$

$R_x \geq 0$, $T_x \geq 0$, $x = 1, 2, \ldots, 17.$

Replacing each fuzzy equality with two fuzzy inequalities, we obtain

(NP-2) $$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) + 20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) + 11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15}) + 5(D_{16} + E_{16}) + 5(D_{17} + E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 + R_{10} + R_{11} + R_{12} + R_{13} + R_{14} + R_{15} + R_{16} + R_{17} + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13} + T_{14} + T_{15} + T_{16} + T_{17}) \overset{\sim}{\leq} 886.8333$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 \underset{\sim}{\geq} 17$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 \underset{\sim}{\leq} 17$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 \underset{\sim}{\geq} 2$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 \underset{\sim}{\leq} 2$$

$$-E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 \gtrsim 19$$

$$-E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 \lesssim 19$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 \gtrsim 29$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 \lesssim 29$$

$$-E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 \gtrsim 20$$

$$-E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 \lesssim 20$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 \gtrsim 1$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 \lesssim 1$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 \gtrsim 19$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 \lesssim 19$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 \gtrsim 28$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 \lesssim 28$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 \gtrsim 18$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 \lesssim 18$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} \gtrsim 1$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} \lesssim 1$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} \gtrsim 7$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} \lesssim 7$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} \underset{\sim}{\geq} 1$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} \underset{\sim}{\leq} 1$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} \underset{\sim}{\geq} 26$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} \underset{\sim}{\leq} 26$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} \underset{\sim}{\geq} 28$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} \underset{\sim}{\leq} 28$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} \underset{\sim}{\geq} 3$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} \underset{\sim}{\leq} 3$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} \underset{\sim}{\geq} 13$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} \underset{\sim}{\leq} 13$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} \underset{\sim}{\geq} 5$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} \underset{\sim}{\leq} 5$$

Non-negativity constraints:

$D_x \geq 0$, $E_x \geq 0$, $x = 1, 2, \ldots, 19$.

$R_x \geq 0$, $T_x \geq 0$, $x = 1, 2, \ldots, 17$.

Then the crisp equivalent of this problem can be written as

(NP-3) maximize λ

subject to the following constraints:

$$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) + 20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) + 11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15}) + 5(D_{16} + E_{16}) + 5(D_{17}+ E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 + R_{10} + R_{11} + R_{12} + R_{13} + R_{14} + R_{15} + R_{16} + R_{17} + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13} + T_{14} + T_{15} + T_{16} + T_{17}) + 2.217\lambda \leq 886.8333$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 - 5.1\lambda \geq 11.9$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 + 5.1\lambda \leq 22.1$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 - .6\lambda \geq 1.4$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 + .6\lambda \leq 2.6$$

$$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 - 5.7\lambda \geq 13.3$$

$$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 + 5.7\lambda \leq 24.7$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 - 8.7\lambda \geq 20.3$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 + 8.7\lambda \leq 37.7$$

$$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 - 6\lambda \geq 14$$

$$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 + 6\lambda \leq 26$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 - .3\lambda \geq .7$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 + .3\lambda \leq 1.3$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 - 5.7\lambda \geq 13.3$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 + 5.7\lambda \le 24.7$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 - 8.4\lambda \ge 19.6$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 + 8.4\lambda \le 36.4$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 - 5.4\lambda \ge 12.6$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 + 5.4\lambda \le 23.4$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} - .3\lambda \ge .7$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} + .3\lambda \le 1.3$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} - 2.1\lambda \ge 4.9$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} + 2.1\lambda \le 9.1$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} - .3\lambda \ge .7$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} + .3\lambda \le 1.3$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} - 7.8\lambda \ge 18.2$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} + 7.8\lambda \le 33.8$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} - 8.4\lambda \ge 19.6$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} + 8.4\lambda \le 36.4$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} - .9\lambda \ge 2.1$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} + .9\lambda \le 3.9$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} - 3.9\lambda \ge 9.1$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} + 3.9\lambda \le 16.9$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} - 1.5\lambda \ge 3.5$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} + 1.5\lambda \le 6.5$$

$$\lambda \le 1$$

Non-negativity constraints:

$D_x \geq 0$, $E_x \geq 0$, $x = 1, 2, \ldots, 19.$

$R_x \geq 0$, $T_x \geq 0$, $x = 1, 2, \ldots, 17.$

$\lambda \geq 0$

3.8 Results

The optimal solution to (NP-3) is as described in the following table.

Table 3.2 Results of fuzzy linear program problem

Variable	Value	Variable	Value	Variable	Value	Variable	Value
D_2	1.666	D_{14}	9.0982	E_9	12.0501	R_{13}	1.3381
D_5	11.6367	D_{17}	6.4864	E_{12}	1.9582	T_5	4.3131
D_8	1.0021	E_1	13.6326	E_{15}	9.4802	λ	.9930
D_{10}	2.9561	E_4	3.6618	E_{19}	11.476		
D_{13}	3.0689	E_6	2.0042	R_7	4.906		

3.9 Interpretation of the Results

Results given in the above Table 3.2 summarizes the solution of the linear version of the fuzzy problem. In the solution of the linear fuzzy programming problem, we have $D_3 = E_3 = 0$, $D_7 = E_7 = 0$, $D_{11} = E_{11} = 0$, $D_{16} = E_{16} = 0$ and $D_{18} = E_{18} = 0$. At the same time, we have 14 variables D_x and E_x that have positive value. These values are $D_2 = 1.666$, $D_5 = 11.6367$, $D_8 = 1.0021$, $D_{10} = 2.9561$, $D_{13} = 3.0689$, $D_{14} = 9.0982$, $D_{17} = 6.4864$, $E_1 = 13.6326$, $E_4 = 3.6618$, $E_6 = 2.0042$, $E_9 = 12.0501$, $E_{12} = 1.9582$,

Table 3.3 Value of λ Corresponding to Observed Value Tolerance and Objective Function Tolerance

Observed Value Tolerance	Objective Function Tolerance						
	0.25%	0.5%	1%	2%	3%	4%	5%
10%	0.9794	0.9596	0.9224	0.8560	0.7980	0.7435	0.6954
15%	0.9862	0.9727	0.9469	0.8991	0.8550	0.8116	0.7723
20%	0.9896	0.9794	0.9596	0.9224	0.8868	0.8512	0.8183
25%	0.9917	0.9835	0.9674	0.9369	0.9071	0.8770	0.8488
30%	0.9930	0.9862	0.9727	0.9469	0.9212	0.8952	0.8705

Table 3.4 Value of Objective Function (Membership Function) Corresponding to Observed Value Tolerance and Objective Function Tolerance

Observed Value Tolerance	Objective Function Tolerance						
	0.25%	0.5%	1%	2%	3%	4%	5%
10%	884.66	882.58	878.65	871.65	865.60	860.46	856.00
	(.9803)	(.9592)	(.9228)	(.8559)	(.7981)	(.7435)	(.6954)
15%	884.65	882.52	878.44	870.88	864.09	858.05	852.59
	(.9848)	(.9728)	(.9465)	(.8993)	(.8549)	(.8115)	(.7723)
20%	884.64	882.49	878.32	870.47	863.24	856.64	850.55
	(.9893)	(.9795)	(.9600)	(.9224)	(.8868)	(.8512)	(.8183)
25%	884.63	882.47	878.25	870.21	862.70	855.73	849.20
	(.9938)	(.9841)	(.9679)	(.9371)	(.9071)	(.8769)	(.8487)
30%	884.63	882.46	878.21	870.04	862.32	855.08	848.23
	(.9938)	(.9863)	(.9724)	(.9466)	(.9214)	(.8952)	(.8706)

E_{15} = 9.4802, E_{19} = 11.476, R_7 = 4.906, R_{13} = 1.3381 and T_5 = 4.3131. R_x and T_x represent the deviational variables of smoothness function. The minimum value of the objective function, which represents the minimization of the sum of fit and smoothness to obtain better graduated values is 884.63 and level of satisfaction λ, is .9930.

Table 3.3 and 3.4 show the behavior of the value of λ and the objective function respectively, corresponding to the changes in tolerance levels q_x, of 10%, 20%, 30%, 40% and 50% for imprecisely known observed values, and of 0.25%, 0.5%, 1%, 2%, 3%, 4%, and 5% tolerance levels q_0 for objective function.

3.10 Discussion of the Solution in View of Table 3.3 and 3.4

Table 3.3 shows different values of λ for various tolerance levels for the imprecisely known observed values u_x and desired levels of objective function. Also, Table 3.4 shows different values of objective function i.e. combination of fit and smoothness for various tolerance levels for the imprecisely known crisp objective function value and imprecisely known observed values u_x. Note that in this formulation the membership function λ is used to express the degree of certainty of the solution with respect to fuzzy parameters, objective function which represents the combination of fit and smoothness, and the imprecisely known observed values for u_x [51]. From Table 3.3, it is observed that with the increase in the tolerance level for desired level of objective function, the value of λ decreases. This shows that the smaller the value of membership grade λ, the smaller is the support for the solution and hence, lower the degree of certainty of solution. On the other hand, it is observed that with increase in tolerance limits for imprecisely known u_x, the value of λ increases. This shows that the larger the value of membership grade λ, the larger is the support for the solution. In Table 3.4, the numbers in the brackets represent the value of the membership function corresponding to the value of objective function at the optimal solution given in Table 3.2. From Tables 3.3 and 3.4 we observe that the value of the membership function is, as expected, either greater than or equal to the value of λ. It can therefore be concluded that fuzzy programming does not provide just another solution; instead it produces a solution corresponding to the pre-specified tolerance levels of constraints with an associated degree of one's belief in the solution. Graphs 3.3 and 3.4 in Appendix 1 reinforce the above observation.

In the above observation, the relationship between objective function (which represents the combination of fit and smoothness) and the range of observed values u_x's are investigated for possible values of membership grade between 0 and 1. Such an examination is useful in order to provide the decision makers with sufficient information on the implication of the choice of a membership grade prior to the final choice determined by them. Another advantage of fuzzy programming is that it admits imprecise data. This feature is particularly useful for the situation when the management in an organization is not able to specify precisely the combination of fit and smoothness limit, but is rather able to provide lower and upper bounds, with a specified tolerance level above or below these bounds. Thus, fuzzy programming produces most satisfactory solution within a pre-specified interval, whereas conventional crisp set theory constraints only permit only one solution either to belong (membership grade 1) or not to belong (membership grade 0) to the set $\{0, 1\}$.

Chapter 4

A FINITE ITERATION TECHNIQUE FOR A FUZZY QUADRATIC PROGRAMMING PROBLEM

In this chapter we consider two problems, one under symmetric fuzzy environment, and the second under non–symmetric fuzzy environment, such that each problem has a single quadratic objective function and a number of linear constraints. Each of the two fuzzy problems is converted into a crisp programming problem that has a linear objective function with linear constraints, and has one quadratic constraint. To solve such a problem, we suggest a finite step method that uses linear programming and parametric quadratic programming. Furthermore, we present a numerical example to demonstrate the method developed.

4.1 Introduction

Since Zadeh [49] introduced the concept of fuzzy set theory, a number of researchers have exhibited their interest in the topic of fuzzy mathematical programming (for example see [4], [16], [23], and [51]). However, in contrast with the vast literature available on modeling and solution procedures for a linear program in a fuzzy environment, the studies in quadratic programming under fuzzy environment and its solution are rather scarce. In the present paper we consider both symmetric fuzzy and non–symmetric fuzzy quadratic programming problems and transform each of them to a crisp programming problem of the type presented in (VP). At the end we consider a numerical example to demonstrate the method for the solution of a symmetric problem. It will be observed that the non–symmetric problem could be solved similarly with slight modifications.

4.2 Symmetric Fuzzy Quadratic Programming

Corresponding to (P–1) as described in Chapter 2, we now consider the following symmetric fuzzy version (P–2) on the lines of Zimmermann [51].

(P–2) Find a solution x^* that satisfies:

$$f(x) = p^T x + \frac{1}{2} x^T C x \lesssim z_0 \qquad (4.2.1)$$

$$\sum_{j=1}^{n} g_{ij} x_j \lesssim d_i \qquad i = 1, 2, \ldots, k \qquad (4.2.2)$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = k+1, \ldots, m \qquad (4.2.3)$$

$$x \geq 0 \qquad (4.2.4)$$

where the fuzzy inequality '$\lesssim$' denotes 'essentially less than or equal to' [51], and z_0, called the aspiration level, is given some pre–assigned value. Let, $q_0 > 0$, and $q_i > 0$, $(i = 1, 2, \ldots, k)$, be subjectively chosen constants of admissible violations such that q_0 is associated with (4.2.1), and q_i $(i = 1, 2, \ldots, k)$ are associated with the i-th linear constraint (4.2.2). Now, on the lines of Zimmerman [51], we define the membership function corresponding to (4.2.1) and (4.2.2), as follows.

$$\mu_0(x) = \begin{cases} 1 & \text{if } f(x) \le z_0 \\ 1 - \dfrac{f(x) - z_0}{q_0} & \text{if } z_0 \le f(x) \leqq z_0 + q_0, \quad i = 1, 2, \ldots, k, \\ 0 & \text{if } f(x) \geqq z_0 + q_0, \quad i = 1, 2, \ldots, k. \end{cases}$$

and

$$\mu_i(x) = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} g_{ij} x_j \le d_i, \quad i = 1,2,\ldots,k \\ 1 - \dfrac{\sum_{j=1}^{n} g_{ij} x_j - d_i}{q_i} & \text{if } d_i \le \sum_{j=1}^{n} g_{ij} x_j \le d_i + q_i \quad i = 1,2,\ldots,k, \\ 0 & \text{if } \sum_{j=1}^{n} g_{ij} x_j \ge d_i + q_i \quad i = 1,2,\ldots,k. \end{cases}$$

4.3 The Equivalent Crisp Problem

On the lines of Zimmermann [51], the solution to the problem (P–1) is obtained by solving the following problem (P–3).

(P–3)

$$\text{Maximize} \quad \underset{i \ge 0,1,2,\ldots,k}{\text{Minimize}} \ \mu_i(x)$$

subject to (4.2.3), and (4.2.4).

Now following Schmitendorf [36], (P–3), and Zimmermann [51] a solution to (P–3) is obtained by solving the following problem (P–4).

(P–4) Maximize x_{n+1}

subject to

$$\mu_i(x) - x_{n+1} \geq 0, \qquad i = 0, 1, 2, \ldots, k$$

$$0 \leq x_{n+1} \leq 1$$

and (4.2.3), and (4.2.4).

From above, using the expressions for $\mu_i(x)$ for $i = 0, 1, 2, \ldots, k$, and using (4.2.3) and (4.2.4), we obtain (P-4) as follows.

(P–4) Maximize x_{n+1}

subject to

$$\left(\frac{1}{q_0}\right)\left[p^T x + \frac{1}{2} x^T C x\right] + x_{n+1} \leq 1 + \left(\frac{z_0}{q_0}\right)$$

$$\left(\frac{1}{q_i}\right)\left[\sum_{j=1}^{n} g_{ij} x_j\right] + x_{n+1} \leq 1 + \left(\frac{d_i}{q_i}\right), \qquad i = 1, 2, \ldots, k$$

$$x_{n+1} \leq 1$$

$$\sum_{j=1}^{n} a_{ij}\, x_j \leq b_i \qquad i = k+1, \ldots, m$$

$$x,\ x_{n+1} \geq 0$$

Rewriting (P–4), we obtain (P–5) as follows.

(P–5) Maximize x_{n+1}

subject to

$$p^T x + \frac{1}{2} x^T C x + q_0\, x_{n+1} \leq q_0 + z_0 \tag{4.3.1}$$

$$\sum_{j=1}^{n} g_{ij}x_j + q_i x_{n+1} \leq q_i + d_i, \quad i = 1, 2, \ldots, k \tag{4.3.2}$$

$$x_{n+1} \leq 1 \tag{4.3.3}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = k+1, \ldots, m \tag{4.3.4}$$

$$x, x_{n+1} \geq 0 \tag{4.3.5}$$

In (P–5), objective and the constraints (4.3.2) – (4.3.5) are linear. However, the constraint (4.3.1) is quadratic. Therefore, (P–5) is of the type of the problem (VP).

4.4 Non – Symmetric Fuzzy Quadratic Programming Problem

We now consider the following the following non–symmetric fuzzy quadratic programming problem (NFP).

(NFP) Minimize $f(x) = p^T x + \frac{1}{2} x^T C x$

subject to

$$\sum_{j=1}^{n} g_{ij} x_j \lesssim d_i \qquad i = 1, 2, \ldots, k$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = k+1, \ldots, m$$

$$x \geq 0$$

As suggested by Zimmermann [51], we compute the membership function corresponding to the quadratic objective function with the help of the following two crisp quadratic programs (CP–1) and (CP–2).

(CP–1) $\quad$ Minimize $f(x) = p^T x + \frac{1}{2} x^T C x$

subject to

$$\sum_{j=1}^{n} g_{ij} x_j \le d_i \qquad i = 1, 2, \dots, k$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = k+1, \dots, m$$

$$x \ge 0$$

Let the minimum value of the objective function $f(x)$ be f_0 .

(CP–2) $\quad$ Minimize $f(x) = p^T x + \frac{1}{2} x^T C x$

subject to

$$\sum_{j=1}^{n} g_{ij} x_j \le d_i + q_i \qquad i = 1, 2, \dots, k$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = k+1, \dots, m$$

$$x \ge 0$$

Let the minimum value of $f(x)$ be f_1.

Then, on the lines of Zimmermann [51], the membership function corresponding to the quadratic objective function of (NFP) is defined as follows.

$$\mu_0(x) = \begin{cases} 1 & \text{if } f(x) \le f_0 \\ 1 - \dfrac{f(x) - f_0}{f_1 - f_0} & \text{if } f_0 \le f(x) \le f_1 \\ 0 & \text{if } f(x) \ge f_1 \end{cases}$$

Now, the equivalent crisp programming problem corresponding to (NFP) is as follows.

$$\text{Maximize } x_{n+1}$$

subject to

$$p^T x + \frac{1}{2} x^T C x + (f_1 - f_0) x_{n+1} \le f_1$$

$$\sum_{j=1}^{n} g_{ij} x_j + q_i x_{n+1} \le q_i + d_i, \quad i = 1, 2, \ldots, k$$

$$x_{n+1} \le 1$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad i = k+1, \ldots, m$$

$$x, x_{n+1} \ge 0$$

which is similar to (P–5), and therefore, can be solved on the lines of the Two–Phase method suggested for solving (P–5).

4.5 Numerical Example

We now solve a numerical example for the following fuzzy symmetric quadratic programming problem (FSQP) using the method described above.

(FSQP) $$2x_1 + 1x_2 + 4x_1^2 + 4x_1 x_2 + 2x_2^2 \lesssim 51.88$$

$$4x_1 + 5x_2 \gtrsim 20$$

$$5x_1 + 4x_2 \gtrsim 20$$

$$1x_1 + 1x_2 \lesssim 30$$

$$x_1, x_2 \ge 0.$$

Let $q_0 = 2.12$, $q_1 = 2$, $q_2 = 1$, $q_3 = 3$.

Then, on the lines of (P–5), the crisp equivalent of this problem is

maximize x_3

subject to

$$
\begin{aligned}
2\,x_1 + 1\,x_2 + 4\,x_1^2 + 4\,x_1\,x_2 + 2\,x_2^2 + 2.12\;x_3 &\le 54 \\
4\,x_1 + 5\,x_2 - 2\;x_3 &\ge 18 \\
5\,x_1 + 4\,x_2 - 1\;x_3 &\ge 19 \\
1\,x_1 + 1\,x_2 + 3\;x_3 &\le 33 \\
x_3 &\le 1 \\
x_1\,,\; x_2\,, x_3 &\ge 0\,.
\end{aligned}
$$

This problem is similar to (VP) with linear objective function, exactly one quadratic constraint and three linear constraint. Therefore, we solve it in a finite number of steps using the Two Phase method as outlined above for solving (VP).

In Phase 1, the linear programming problem is as follows.

Maximize x_3

subject to

$$
\begin{aligned}
4\,x_1 + 5\,x_2 - 2\;x_3 &\ge 18 \\
5\,x_1 + 4\,x_2 - 1\;x_3 &\ge 19 \\
1\,x_1 + 1\,x_2 + 3\;x_3 &\le 33 \\
x_3 &\le 1 \\
x_1\,,\; x_2\,, x_3 &\ge 0\,.
\end{aligned}
$$

Its optimal solution is $x_1 = 2.2222$, $x_2 = 2.2222$, $x_3 = 1$.

Since $2x_1 + 1x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12x_3 = 58.17$ at $x_1 = 2.2222$, $x_2 = 2.2222$ and $x_3 = 1$, therefore the constraint $2x_1 + 1x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12x_3 \leq 54$ is violated. Hence we go to Phase 2. In this phase we solve the following quadratic programming problem parametrically.

$$\text{Minimize} \quad 2x_1 + 1x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12x_3$$

subject to

$$\begin{aligned} 4x_1 + 5x_2 - 2x_3 &\geq 18 \\ 5x_1 + 4x_2 - 1x_3 &\geq 19 \\ 1x_1 + 1x_2 + 3x_3 &\leq 33 \\ x_3 &\leq 1 \\ x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

From this problem, by solving a series of quadratic programs parametrically, we obtain the final form of the quadratic programming as follows.

$$\text{Minimize} \quad 2x_1 + 1x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12x_3$$

subject to

$$\begin{aligned} 4x_1 + 5x_2 - 2x_3 &\geq 18 \\ 5x_1 + 4x_2 - 1x_3 &\geq 19 \\ 1x_1 + 1x_2 + 3x_3 &\leq 33 \end{aligned}$$

$$x_3 \le 1$$

$$x_3 \ge .860633$$

$$x_1 ,\ x_2 , x_3 \ge 0 .$$

The optimal solution to this problem is $x_1 = .99$, $x_2 = 3.73$, $x_3 = .860633$, and the minimum value of the objective function is $= 54$.

Thus, the solution that solves the (FSQP) is

$$x_1 = .99 , \quad x_2 = 3.73 ,$$

and the level of satisfaction of this solution is given by $x_3 = .860633$.

4.6 Conclusion

In the present chapter, we consider a symmetric fuzzy quadratic programming problem. Solution to this problem is obtained in a finite number of steps by solving an optimization problem in which one constraint is quadratic, other constraints and the objective function are linear. Also, it is shown that the non-symmetric fuzzy quadratic programming problem can also be solved in a finite number of steps by using a similar technique.

Chapter 5

A QUADRATIC PROGRAMMING APPROACH TO GRADUATION UNDER CRISP AND FUZZY ENVIRONMENT

The Whittaker-Henderson Type B method of graduation consists of minimizing the weighted sum of the squares of the deviations of graduated values from observed values plus a parameter times the sum of the squares of the z-th differences of the graduated values. In Chapter 3, this method is modified by using absolute values instead of squares for the weighed sum of the deviations of graduated values from observed values. The resulting problem is expressed as a linear programming problem and is solved both under crisp and fuzzy environment. In present Chapter, we develop quadratic programming approach to graduation using absolute values method for fit measure and traditional sum of the squares method to measure the smoothness. In order to capture uncertainty factor in observed values data, fuzzy quadratic programming technique is applied. At the end, same numerical example as used by Schuette [37], is given to demonstrate fuzzy approach.

5.1 Introduction

A common method of actuarial graduation is the difference-equation method [26]. In this method of graduation, graduated values v_x. (where $x = 1, 2, \ldots, n$), are sought corresponding to a given set of observed values u_x and non-negative weights w_x that minimize the quantity $F + hS$, where $F = \sum_{x=1}^{n} w_x(u_x - v_x)^2$ and $S = \sum_{x=1}^{n-z} (\Delta^z v_x)^2$.

F is an expression that measures the degree of fit of the graduated values to the observed values, and S is an expression that measures the degree of smoothness of the graduated values.

As shown in Chapter 3, a method of coping with the absolute-value function is available in linear programming. Indeed, a method for coping with both the absolute value function and quadratic function is available in quadratic programming.

A second and undoubtedly more important reason why methods based on minimizing sum of squares have been favored in graduation is the preeminence of the principle of least squares in statistical theory, which in turn can be traced to the normal distribution. The traditional squared criteria in the fit measure are appropriate whenever the error random variable (the deviation of the observed values from the true underlying values) is normally distributed. If the distribution of this error random variable is not normal, and thus generates more 'outliers' than would a normal distribution, the squared criterion is too sensitive to these outliers. The method using absolute values should be less influenced by the outliers and thus is considered to be robust estimation procedure [26].

The problem of the outliers pertains more to the fit measure than the smoothness measure. Hence this chapter will be devoted to the task of adapting quadratic programming to the graduation problem so that, in the case of the fit measure, absolute values may be employed in place of squares, and in the case of the smoothness measure, the traditional sum of squares will be maintained.

5.2 Quadratic Programming Formulation of A Graduation Problem Under Crisp Environment

Following the notation used in Chapter 3 for the graduation problem, we have

$$v_x - u_x = D_x - E_x \tag{5.2.1}$$

therefore, $|v_x - u_x| = D_x + E_x$

with deviational variables $D_x \geq 0$ and $E_x \geq 0$ for $x = 1, 2, \ldots, n$, and let

$$\Delta^z v_x = R_x - T_x \tag{5.2.2}$$

with deviational variables $R_x \geq 0$ and $T_x \geq 0$ for $x = 1, 2, \ldots, n - z$.

Then

$$v_x = u_x + D_x - E_x, \tag{5.2.3}$$

and

$$\Delta^z v_x = \Delta^z(u_x + D_x - E_x) = R_x - T_x \tag{5.2.4}$$

Also, $|v_x - u_x| = D_x + E_x$

and $(R_x - T_x)^2 = R_x^2 - 2\, R_x T_x + T_x^2 = R_x^2 + T_x^2$, since at least one of R_x and T_x must be equal to zero.

Thus, under crisp environment, we have to find the values of D_x, E_x, R_x and T_x with the help of the following crisp quadratic programming problem.

(CQP) Minimize $(F + hS) = \sum_{x=1}^{n} w_x (D_x + E_x) + h \sum_{x=1}^{n-z} (R_x^2 + T_x^2)$ (5.2.5)

subject to constraints

$\Delta^z (E_x - D_x) + R_x - T_x = \Delta^z u_x,\ x = 1, 2, \ldots, n - z,$ (5.2.6)

$D_x \geq 0, E_x \geq 0,\ R_x \geq 0$, and $T_x \geq 0$, for all appropriate values of x. (5.2.7)

Constraint equations (5.2.6) are obtained by rearranging equations (5.2.4). Since, the operator Δ^z is linear and the variables appear linearly in all terms in equations given by (5.2.6), therefore, the problem has linear constraints. Also, the Hessian matrix in (5.2.5) is positive definite, therefore, (F + hS) is strictly convex. This yields that a local minimum of strictly convex function (F + hS) over the constraint set, determined by linear constraints given by (5.2.6), and (5.2.7), is a global minimum also. Furthermore, the function (F + hS) being strictly convex has a unique global minimum over the constraint set. It may also be observed that the problem involves (4n – 2z) variables and (n – z) constraints.

5.3 Formulation of Graduation Problem As A Quadratic Programming Problem Under Fuzzy Environment

Fuzziness or vagueness present in the observed value data can influence the graduated values that might distort the whole underlying phenomenon. In order to deal with this kind of problem, we can take advantage of fuzzy set theory ([5], [49], [51] and [52]). The resultant fuzzy quadratic programming problem then can be solved by the method as demonstrated in Chapter 4.

Using Zimmerman's notation [51], in a fuzzy environment, the crisp constraints

$$\Delta^z (E_x - D_x) + R_x - T_x = \Delta^z u_x, \; x = 1, 2, \ldots, n - z, \qquad (5.3.1)$$

can be replaced by

$$\Delta^z (E_x - D_x) + R_x - T_x \;\underset{\sim}{=}\; \Delta^z u_x, \; x = 1, 2, \ldots, n - z, \qquad (5.3.2)$$

which are further replaced by

$$\Delta^z (E_x - D_x) + R_x - T_x \ \underset{\sim}{\geq}\ \Delta^z u_x,\ x = 1, 2, \ldots, n - z, \qquad (5.3.3)$$

and

$$\Delta^z (E_x - D_x) + R_x - T_x \ \underset{\sim}{\leq}\ \Delta^z u_x,\ x = 1, 2, \ldots, n - z, \qquad (5.3.4)$$

The notation '$\underset{\sim}{\geq}\ \Delta^z u_x$' (or $\underset{\sim}{\leq}\ \Delta^z u_x$ respectively) means that the corresponding fuzzy constraint is 'essentially $\geq \Delta^z u_x$'(or essentially $\leq \Delta^z u_x$, respectively) for all x [51]. We denote by μ_{xL} and μ_{xU}, the membership functions corresponding to (5.3.3) and (5.3.4) respectively.

Using Zimmerman's approach [51], in a fuzzy environment, the objective function, which is the total value of combination of fit and smoothness, can be written as

$$\sum_{x=1}^{n} w_x (D_x + E_x) + h \sum_{x=1}^{n-z} (R_x^2 + T_x^2) \ \lesssim\ z_0 \qquad (5.3.5)$$

with μ_0 as the corresponding membership function for the objective function (5.3.5), where '$\lesssim z_0$' means that the corresponding membership function is 'desired to be less than or equal to z_0'.

Then, under fuzzy environments, our crisp quadratic programming problem (CQP) becomes the following fuzzy quadratic programming problem, denoted by (FQP).

(FQP) Find D_x, E_x, R_x and T_x for all appropriate values of x, we have

for the fuzzy objective function

$$\sum_{x=1}^{n} w_x (D_x + E_x) + h \sum_{x=1}^{n-z} (R_x^2 + T_x^2) \ \lesssim\ z_0 \qquad (5.3.6)$$

and for the fuzzy constraints with corresponding membership functions μ_{xL} and μ_{xU}

$$\Delta^z (E_x - D_x) + R_x - T_x \ \underset{\sim}{\geq}\ \Delta^z u_x,\ x = 1, 2, \ldots, n - z \qquad (5.3.7)$$

$$\Delta^z (E_x - D_x) + R_x - T_x \ \underset{\sim}{\leq}\ \Delta^z u_x,\ x = 1, 2, \ldots, n - z \qquad (5.3.8)$$

The non-negativity constraints are written as

$$D_x \geq 0,\ E_x \geq 0, \qquad x = 1, 2, \ldots, n \qquad (5.3.9)$$

and

$$R_x \geq 0,\ T_x \geq 0, \qquad x = 1, 2, \ldots, n-z \qquad (5.3.10)$$

The graduation problem under fuzzy environment now is equivalent to obtain a solution satisfying the fuzzy sets given by (5.3.6), (5.3.7), (5.3.8), (5.3.9) and (5.3.10).

Membership Functions

Following Zimmermann [51], below we define the membership functions, μ_0 for the fuzzy objective (5.3.6), and μ_{xL} and μ_{xU} for the fuzzy constraints (5.3.7) and (5.3.8), respectively.

For the sake of simplicity, we denote $\sum_{x=1}^{n} w_x(D_x + E_x) + h\sum_{x=1}^{n-z}(R_x^2 + T_x^2)$ by f_0, $\Delta^z(E_x - D_x) + R_x - T_x$ by f_x, and $\Delta^z u_x$ by d_x.

Then, if f_0 is desired to be lower than z_0 and $q_0 > 0$ be the subjectively chosen value of admissible violation corresponding to z_0, then the membership function μ_0 for objective function is written as

$$\mu_0 = \begin{cases} 1 & \text{if} \quad f_0 \leq z_0 - q_0 \\ 1 - \dfrac{f_0 - (z_0 - q_0)}{q_0} & \text{if} \quad z_0 - q_0 \leq f_0 \leq z_0 \\ 0 & \text{if} \quad z_0 \leq f_0 \end{cases}$$

Similarly, the membership functions for fuzzy constraints (5.3.7) and (5.3.8) is obtained as below.

Let $q_{xL} > 0$, and $q_{xU} > 0$ be the subjectively chosen constants of admissible violations associated with constraints (5.3.7) and (5.3.8) respectively. Then, following Zimmermann [51, 52],

μ_{xL}, the membership functions for the lower side of the fuzzy region of the fuzzy constraints (5.3.7) are taken as

$$\mu_{xL} = \begin{cases} 1 & \text{if} \quad f_x \geq d_x \\ 1 - \dfrac{(d_x - f_x)}{q_x} & \text{if} \quad d_x - q_x \leq f_x \leq d_x \\ 0 & \text{if} \quad d_x - q_x \geq f_x \end{cases}$$

and μ_{xU}, the membership functions for the upper side of the fuzzy region of the fuzzy constraints (5.3.8) are taken as

$$\mu_{xU} = \begin{cases} 1 & \text{if} \quad f_x \leq d_x \\ 1 - \dfrac{(f_x - d_x)}{q_x} & \text{if} \quad d_x \leq f_x \leq d_x + q_x \\ 0 & \text{if} \quad d_x + q_x \leq f_x \end{cases}$$

Once the membership functions are obtained, we get a solution to (FQP) by finding the intersection of the fuzzy sets given (5.3.6), (5.3.7) and (5.3.8), to get to a decision. Then μ_D the membership function of decision D satisfying (5.3.6), (5.3.7) and (5.3.8) is

$$\mu_D = \min(\mu_0, \mu_{1L}, \mu_{2L}, \ldots, \mu_{(n-z)L}, \mu_{1U}, \mu_{2U}, \ldots, \mu_{(n-z)U})$$

Since, we are interested in large value of μ_D over (5.3.9) and (5.3.10), therefore, following Zimmermann [51], we obtain

$$\max \ \mu_D = \min (\mu_0, \mu_{1L}, \mu_{2L}, \ldots, \mu_{(n-z)L}, \mu_{1U}, \mu_{2U}, \ldots, \mu_{(n-z)U})$$

subject to the constraints (5.3.9) and (5.3.10).

Replacing μ_D by λ, we have the following problem (EFQP) along the lines of Zimmermann [51] ;

(EFQP) $\quad \max \lambda$

subject to

$$\mu_0 \geq \lambda$$

$$\mu_{xL} \geq \lambda \qquad x = 1, 2, \ldots, n - z,$$

$$\mu_{xU} \geq \lambda \qquad x = 1, 2, \ldots, n - z,$$

and crisp constraints (5.3.9) and (5.3.10)

It is observed that (EFQP) is a crisp linear program whose optimal solution provides a solution to (FQP).

In view of the membership functions μ_0, μ_{xL} and μ_{xU}. $x = 1, 2, \ldots, n - z$, the (EFQP) can be restated as

$\max \lambda$

subject to

$$f_0 + \lambda q_0 \leq z_0$$

$$f_x - \lambda q_x \geq d_x - q_x \qquad x = 1, 2, \ldots, n - z,$$

$$f_x + \lambda q_x \leq d_x + q_x \qquad x = 1, 2, \ldots, n - z,$$

$$0 \leq \lambda \leq 1$$

and $$D_x \geq 0, E_x \geq 0, \qquad x = 1, 2, \ldots, n$$

$$R_x \geq 0, \; T_x \geq 0, \qquad x = 1, 2, \ldots, n-z$$

Identifying $f_0 = \sum_{x=1}^{n} w_x (D_x + E_x) + h \sum_{x=1}^{n-z} (R_x^2 + T_x^2)$,

$F_x = \Delta^z (E_x - D_x) + R_x - T_x$,

and $d_x = \Delta^z u_x$,

we can rewrite (EFQP) as

(EFQP-1) $\max \lambda$

subject to

$$\sum_{x=1}^{n} w_x (D_x + E_x) + h \sum_{x=1}^{n-z} (R_x^2 + T_x^2) + \lambda q_0 \leq z_0$$

$$\Delta^z (E_x - D_x) + R_x - T_x - \lambda q_x \geq \Delta^z u_x - q_x \qquad x = 1, 2, \ldots, n-z,$$

$$\Delta^z (E_x - D_x) + R_x - T_x + \lambda q_x \leq \Delta^z u_x + q_x \qquad x = 1, 2, \ldots, n-z,$$

$$0 \leq \lambda \leq 1$$

and $$D_x \geq 0, E_x \geq 0, \qquad x = 1, 2, \ldots, n$$

$$R_x \geq 0, \; T_x \geq 0, \qquad x = 1, 2, \ldots, n-z.$$

Now the solution of the (EFQP-1) can be obtained by technique developed and demonstrated in Chapter 4.

5.4 Numerical Example of A Graduation Problem As A Quadratic Programming Problem Under Crisp Environment

The data used in this example is that used by Schuette [37] in his paper in Example I, which was taken from the monograph by Miller [30]. The data consists of nineteen ungraduated values and nineteen corresponding weights. The values for u_x, $x = 1, 2, \ldots, 19$, are: 34, 24, 31, 40, 30, 49, 48, 48, 67, 58, 67, 75, 76, 76, 102, 100, 101, 115, and 134. The values for $w_x = 1, 2, \ldots, 19$ are 3, 5, 8, 10, 15, 20, 23, 20, 15, 13, 11, 10, 9, 9, 7, 5, 5, 3, and 1. For this example, the parameter $h = 10$, and $z = 2$. On the lines of technique developed in Chapter 4, the problem can be formulated as follows and is equivalent to (P-1).

(P-1) Minimize $z = (F + hS) =$

$$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) + 20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) + 11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15}) + 5(D_{16} + E_{16}) + 5(D_{17}+ E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1^2 + R_2^2 + R_3^2 + R_4^2 + R_5^2 + R_6^2 + R_7^2 + R_8^2 + R_9^2 + R_{10}^2 + R_{11}^2 + R_{12}^2 + R_{13}^2 + R_{14}^2 + R_{15}^2 + R_{16}^2 + R_{17}^2 + T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2 + T_6^2 + T_7^2 + T_8^2 + T_9^2 + T_{10}^2 + T_{11}^2 + T_{12}^2 + T_{13}^2 + T_{14}^2 + T_{15}^2 + T_{16}^2 + T_{17}^2)$$

subject to the following constraints:

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 = 17$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 = 2$$

$$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 = 19$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 = 29$$

$$-E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 = 20$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 = 1$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 = 19$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 = 28$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 = 18$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} = 1$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} = 7$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} = 1$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} = 26$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} = 28$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} = 3$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} = 13$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} = 5$$

Non-negativity constraints:

$$D_x \geq 0, E_x \geq 0, \; x = 1, 2, \ldots, 19.$$

$$R_x \geq 0, T_x \geq 0, \; x = 1, 2, \ldots, 17.$$

5.5 Results

Solving the above problem (P-1) under crisp environment, we obtain the following results as described in the Table 5.1 below.

Table 5.1 Results of crisp quadratic program problem

Variable	Value	Variable	Value	Variable	Value	Variable	Value
E_1	11.70	D_2	2.58	R_2	.05	R_{12}	.64
E_4	4.53	D_5	9.75	R_4	.06	R_{13}	.62
E_6	4.91	D_8	3.98	R_6	.07	R_{14}	.16
E_9	10.38	D_{10}	3.52	R_7	.66	R_{15}	.04
E_{12}	2.24	D_{13}	2.73	R_8	.25	R_{16}	.17
E_{15}	9.42	D_{14}	9.35	R_9	.59	R_{17}	.05
E_{16}	.03	D_{17}	6.40	R_{10}	.28	T_3	.19
E_{19}	11.35	R_1	.15	R_{11}	.21	T_5	.43

5.6 Interpretation of the Results

In Table 5.1, D_x and E_x represent the deviation between graduated value (v_x) and observed value (u_x). E_1 = 11.70 represent the deviation of first observed value form its graduated value. Similarly, E_4 = 4.53, E_6 = 4.91, E_9 = 10.38, E_{12} = 2.24, E_{15} = 9.42, E_{16} = .03, E_{19} = 11.35, D_2 = 2.58, D_5 = 9.75, D_8 = 3.98, D_{10} = 3.52, D_{13} = 2.73, D_{14} = 9.35 and D_{17} = 6.4 represent the deviation between graduated and observed values for 4th, 6th, 9th, 12th, 15th, 16th, 19th, 2nd, 5th, 8th, 10th, 13th, 14th and 17th variable respectively. R_2 = .05, R_4 = .06, R_6 = .07, R_7 = .66, R_8 = .25, R_9 = .59, R_{10} = .28, R_{11} = .21, R_{12} = .64, R_{13} = .62, R_{14} = .16, R_{15} = .04, R_{16} = .17, R_{17} = .05, T_3 = .19, and T_5 = .43 are the deviations in the second difference of the sequence v_x for 1st, 2nd, 4th, 6th, 7th, 8th, 9th, 10th, 11th, 12th, 13th, 14th, 15th, 16th, 17th, 3rd and 5th variable respectively. Value of the objective function, which represents the minimization of sums of fit and smoothness, is 880.03.

Clearly, value of the objective function (i.e. sum of the fit and smoothness) obtained through Table 3.1 (section 3.4) for the same graduation problem formulated as a linear program under crisp environment has been improved from 886.8333 to 880.03.

Table 2 in Appendix 2 depicts the graduated values obtained by solving the graduation problem using quadratic programming approach for $z = 2$ and different values of h. In the same fashion, Table 4 in Appendix 2 represents the graduated value obtained using $z = 3$ for different values of h.

Thus we can observe that quadratic program approach has been able to help improve the graduated values and the measure of the fit and smoothness of the graduation problem.

5.7 Numerical Example of A Graduation Problem As A Quadratic Programming Problem Under Fuzzy Environment

Any vagueness or impreciseness in data of observed values might lead to an inappropriate interpretation of the underlying law, which would in turn completely defeat the purpose of graduation process. As described before, problems of impreciseness in data can be handled effectively by taking advantage of fuzzy set theory ([5], [49] and [51]).

Now we write the fuzzified format of the quadratic program problem using (EFQP-1). In this example we assume a tolerance level of approximately 30% for observed values and 0.25% in total objective function. Therefore z_0 is 880.03 and q_0 is 2.2. For the observed value constraints, the tolerances are $q_1 = 5.1$, $q_2 = .6$, $q_3 = 5.7$, $q_4 = 8.7$, $q_5 = 6$, $q_6 = .3$, $q_7 = 5.7$, $q_8 = 8.4$, $q_9 = 5.4$, $q_{10} = .3$, $q_{11} = 2.1$, $q_{12} = .3$,

$q_{13} = 7.8$, $q_{14} = 8.4$, $q_{15} = .9$, $q_{16} = 3.9$, $q_{17} = 1.5$, where as the rest of the data is same as in the crisp problem (P-1). We have the fuzzy version of above problem as:

(VP)

$$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) + 20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) + 11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15}) + 5(D_{16} + E_{16}) + 5(D_{17}+ E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1^2 + R_2^2 + R_3^2 + R_4^2 + R_5^2 + R_6^2 + R_7^2 + R_8^2 + R_9^2 + R_{10}^2 + R_{11}^2 + R_{12}^2 + R_{13}^2 + R_{14}^2 + R_{15}^2 + R_{16}^2 + R_{17}^2 + T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2 + T_6^2 + T_7^2 + T_8^2 + T_9^2 + T_{10}^2 + T_{11}^2 + T_{12}^2 + T_{13}^2 + T_{14}^2 + T_{15}^2 + T_{16}^2 + T_{17}^2) \lesssim 880.03$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 \underset{\sim}{=} 17$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 \underset{\sim}{=} 2$$

$$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 \underset{\sim}{=} 19$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 \underset{\sim}{=} 29$$

$$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 \underset{\sim}{=} 20$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 \underset{\sim}{=} 1$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 \underset{\sim}{=} 19$$

$$- E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 \underset{\sim}{=} 28$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 \underset{\sim}{=} 18$$

$$- E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} \underset{\sim}{=} 1$$

$$-E_{13}+2E_{12}-E_{11}+D_{13}-2D_{12}+D_{11}-R_{11}+T_{11} \stackrel{\sim}{=} 7$$

$$-E_{14}+2E_{13}-E_{12}+D_{14}-2D_{13}+D_{12}-R_{12}+T_{12} \stackrel{\sim}{=} 1$$

$$E_{15}-2E_{14}+E_{13}-D_{15}+2D_{14}-D_{13}+R_{13}-T_{13} \stackrel{\sim}{=} 26$$

$$-E_{16}+2E_{15}-E_{14}+D_{16}-2D_{15}+D_{14}-R_{14}+T_{14} \stackrel{\sim}{=} 28$$

$$E_{17}-2E_{16}+E_{15}-D_{17}+2D_{16}-D_{15}+R_{15}-T_{15} \stackrel{\sim}{=} 3$$

$$E_{18}-2E_{17}+E_{16}-D_{18}+2D_{17}-D_{16}+R_{16}-T_{16} \stackrel{\sim}{=} 13$$

$$E_{19}-2E_{18}+E_{17}-D_{19}+2D_{18}-D_{17}+R_{17}-T_{17} \stackrel{\sim}{=} 5$$

Non-negativity constraints:

$D_x \geq 0$, $E_x \geq 0$, $x = 1, 2, \ldots, 19$.

$R_x \geq 0$, $T_x \geq 0$, $x = 1, 2, \ldots, 17$.

Replacing each fuzzy equality with two inequalities, we obtain

$$3(D_1+E_1)+5(D_2+E_2)+8(D_3+E_3)+10(D_4+E_4)+15(D_5+E_5)+20(D_6+E_6)+23(D_7+E_7)+20(D_8+E_8)+15(D_9+E_9)+13(D_{10}+E_{10})+11(D_{11}+E_{11})+10(D_{12}+E_{12})+9(D_{13}+E_{13})+9(D_{14}+E_{14})+7(D_{15}+E_{15})+5(D_{16}+E_{16})+5(D_{17}+E_{17})+3(D_{18}+E_{18})+1(D_{19}+E_{19})+10(R_1^2+R_2^2+R_3^2+R_4^2+R_5^2+R_6^2+R_7^2+R_8^2+R_9^2+R_{10}^2+R_{11}^2+R_{12}^2+R_{13}^2+R_{14}^2+R_{15}^2+R_{16}^2+R_{17}^2+T_1^2+T_2^2+T_3^2+T_4^2+T_5^2+T_6^2+T_7^2+T_8^2+T_9^2+T_{10}^2+T_{11}^2+T_{12}^2+T_{13}^2+T_{14}^2+T_{15}^2+T_{16}^2+T_{17}^2)+2.2\,\lambda \leq 880.03$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 \geq 17$$

$$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 \leq 17$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 \geq 2$$

$$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 \leq 2$$

$$-E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 \geq 19$$

$$-E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 \leq 19$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 \geq 29$$

$$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 \leq 29$$

$$-E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 \geq 20$$

$$-E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 \leq 20$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 \geq 1$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 \leq 1$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 \geq 19$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 \leq 19$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 \geq 28$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 \leq 28$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 \geq 18$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 \leq 18$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} \geqq 1$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} \leqq 1$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} \geqq 7$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} \leqq 7$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} \geqq 1$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} \leqq 1$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} \geqq 26$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} \leqq 26$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} \geqq 28$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} \leqq 28$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} \geqq 3$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} \leqq 3$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} \geqq 13$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} \leqq 13$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} \geqq 5$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} \leqq 5$$

Non-negativity constraints:

$D_x \geq 0$, $E_x \geq 0$, $x = 1, 2, \ldots, 19.$

$R_x \geq 0$, $T_x \geq 0$, $x = 1, 2, \ldots, 17.$

Now we write the crisp version of (VP) as follows.

Maximize λ

subject to the following constraints:

$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) + 20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) + 11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15}) + 5(D_{16} + E_{16}) + 5(D_{17}+ E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1^2 + R_2^2 + R_3^2 + R_4^2 + R_5^2 + R_6^2 + R_7^2 + R_8^2 + R_9^2 + R_{10}^2 + R_{11}^2 + R_{12}^2 + R_{13}^2 + R_{14}^2 + R_{15}^2 + R_{16}^2 + R_{17}^2 + T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2 + T_6^2 + T_7^2 + T_8^2 + T_9^2 + T_{10}^2 + T_{11}^2 + T_{12}^2 + T_{13}^2 + T_{14}^2 + T_{15}^2 + T_{16}^2 + T_{17}^2) + 2.2\,\lambda \leq 880.03$

$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 - 5.1\lambda \geq 11.9$

$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 + 5.1\lambda \leq 22.1$

$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 - .6\lambda \geq 1.4$

$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 + .6\lambda \leq 2.6$

$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 - 5.7\lambda \geq 13.3$

$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 + 5.7\lambda \leq 24.7$

$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 - 8.7\lambda \geq 20.3$

$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 + 8.7\lambda \leq 37.7$

$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 - 6\lambda \geq 14$

$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 + 6\lambda \leq 26$

$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 - .3\lambda \geq .7$

$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 + .3\lambda \leq 1.3$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 - 5.7\lambda \geq 13.3$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 + 5.7\lambda \leq 24.7$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 - 8.4\lambda \geq 19.6$$

$$-E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 + 8.4\lambda \leq 36.4$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 - 5.4\lambda \geq 12.6$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 + 5.4\lambda \leq 23.4$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} - .3\lambda \geq .7$$

$$-E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} + .3\lambda \leq 1.3$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} - 2.1\lambda \geq 4.9$$

$$-E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} + 2.1\lambda \leq 9.1$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} - .3\lambda \geq .7$$

$$-E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} + .3\lambda \leq 1.3$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} - 7.8\lambda \geq 18.2$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} + 7.8\lambda \leq 33.8$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} - 8.4\lambda \geq 19.6$$

$$-E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} + 8.4\lambda \leq 36.4$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} - .9\lambda \geq 2.1$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} + .9\lambda \leq 3.9$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} - 3.9\lambda \geq 9.1$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} + 3.9\lambda \leq 16.9$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} - 1.5\lambda \geq 3.5$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} + 1.5\lambda \leq 6.5$$

$\lambda \leq 1$

Non-negativity constraints:

$D_x \geq 0, E_x \geq 0, \; x = 1, 2, \ldots, 19.$

$R_x \geq 0, T_x \geq 0, \; x = 1, 2, \ldots, 17.$

$\lambda \geq 0$

In crisp version of problem (VP), we ignore the quadratic constraint and solve the following ordinary linear programming problem (LP). In this case we have the (LP) version of (VP) as:

(LP) Maximize λ

subject to the following constraints:

$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 - 5.1\lambda \geq 11.9$

$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 + 5.1\lambda \leq 22.1$

$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 - .6\lambda \geq 1.4$

$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 + .6\lambda \leq 2.6$

$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 - 5.7\lambda \geq 13.3$

$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 + 5.7\lambda \leq 24.7$

$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 - 8.7\lambda \geq 20.3$

$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 + 8.7\lambda \leq 37.7$

$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 - 6\lambda \geq 14$

$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 + 6\lambda \leq 26$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 - .3\lambda \geq .7$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 + .3\lambda \leq 1.3$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 - 5.7\lambda \geq 13.3$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 + 5.7\lambda \leq 24.7$$

$$- E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 - 8.4\lambda \geq 19.6$$

$$- E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 + 8.4\lambda \leq 36.4$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 - 5.4\lambda \geq 12.6$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 + 5.4\lambda \leq 23.4$$

$$- E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} - .3\lambda \geq .7$$

$$- E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} + .3\lambda \leq 1.3$$

$$- E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} - 2.1\lambda \geq 4.9$$

$$- E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} + 2.1\lambda \leq 9.1$$

$$- E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} - .3\lambda \geq .7$$

$$- E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} + .3\lambda \leq 1.3$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} - 7.8\lambda \geq 18.2$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} + 7.8\lambda \leq 33.8$$

$$- E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} - 8.4\lambda \geq 19.6$$

$$- E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} + 8.4\lambda \leq 36.4$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} - .9\lambda \geq 2.1$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} + .9\lambda \leq 3.9$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} - 3.9\lambda \geq 9.1$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} + 3.9\lambda \leq 16.9$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} - 1.5\lambda \geq 3.5$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} + 1.5\lambda \leq 6.5$$

$$\lambda \leq 1$$

Non-negativity constraints:

$D_x \geq 0$, $E_x \geq 0$, $x = 1, 2, \ldots, 19.$

$R_x \geq 0$, $T_x \geq 0$, $x = 1, 2, \ldots, 17.$

$\lambda \geq 0$

5.8 Results

Solving the above problem, we obtain the following Table 5.2 representing the optimal solution of the problem (LP).

Table 5.2 Results of the problem (LP)

Variable	Value	Variable	Value	Variable	Value	Variable	Value
E_1	12	E_9	12.6667	E_{18}	5.6667	R_{13}	5.3333
E_3	5	E_{12}	5	E_{19}	16.3333	T_{10}	3.3333
E_4	12	E_{13}	3	D_{10}	2.6667	λ	1
E_6	13	E_{15}	17.6667	R_4	4		
E_7	6	E_{16}	7.3333	R_7	0.3333		

Substituting the values obtained in Table 5.2 in the quadratic constraint, which was ignored, we obtain the value of the quadratic constraint as 1648.20. We observe that the quadratic constraint is violated. Therefore, according to the method developed in Chapter 4, we need to go to the Phase 2 of the solution method.

In the second Phase, we solve the quadratic programming problem (QP) parametrically as follows.

(QP) Minimize

$3(D_1 + E_1) + 5(D_2 + E_2) + 8(D_3 + E_3) + 10(D_4 + E_4) + 15(D_5 + E_5) + 20(D_6 + E_6) + 23(D_7 + E_7) + 20(D_8 + E_8) + 15(D_9 + E_9) + 13(D_{10} + E_{10}) + 11(D_{11} + E_{11}) + 10(D_{12} + E_{12}) + 9(D_{13} + E_{13}) + 9(D_{14} + E_{14}) + 7(D_{15} + E_{15}) + 5(D_{16} + E_{16}) + 5(D_{17}+ E_{17}) + 3(D_{18} + E_{18}) + 1(D_{19} + E_{19}) + 10(R_1^2 + R_2^2 + R_3^2 + R_4^2 + R_5^2 + R_6^2 + R_7^2 + R_8^2 + R_9^2 + R_{10}^2 + R_{11}^2 + R_{12}^2 + R_{13}^2 + R_{14}^2 + R_{15}^2 + R_{16}^2 + R_{17}^2 + T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2 + T_6^2 + T_7^2 + T_8^2 + T_9^2 + T_{10}^2 + T_{11}^2 + T_{12}^2 + T_{13}^2 + T_{14}^2 + T_{15}^2 + T_{16}^2 + T_{17}^2) + 2.2\,\lambda$

subject to the following constraints:

$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 - 5.1\lambda \geq 11.9$

$E_3 - 2E_2 + E_1 - D_3 + 2D_2 - D_1 + R_1 - T_1 + 5.1\lambda \leq 22.1$

$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 - .6\lambda \geq 1.4$

$E_4 - 2E_3 + E_2 - D_4 + 2D_3 - D_2 + R_2 - T_2 + .6\lambda \leq 2.6$

$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 - 5.7\lambda \geq 13.3$

$- E_5 + 2E_4 - E_3 + D_5 - 2D_4 + D_3 - R_3 + T_3 + 5.7\lambda \leq 24.7$

$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 - 8.7\lambda \geq 20.3$

$E_6 - 2E_5 + E_4 - D_6 + 2D_5 - D_4 + R_4 - T_4 + 8.7\lambda \leq 37.7$

$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 - 6\lambda \geq 14$

$- E_7 + 2E_6 - E_5 + D_7 - 2D_6 + D_5 - R_5 + T_5 + 6\lambda \leq 26$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 - .3\lambda \geq .7$$

$$E_8 - 2E_7 + E_6 - D_8 + 2D_7 - D_6 + R_6 - T_6 + .3\lambda \leq 1.3$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 - 5.7\lambda \geq 13.3$$

$$E_9 - 2E_8 + E_7 - D_9 + 2D_8 - D_7 + R_7 - T_7 + 5.7\lambda \leq 24.7$$

$$- E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 - 8.4\lambda \geq 19.6$$

$$- E_{10} + 2E_9 - E_8 + D_{10} - 2D_9 + D_8 - R_8 + T_8 + 8.4\lambda \leq 36.4$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 - 5.4\lambda \geq 12.6$$

$$E_{11} - 2E_{10} + E_9 - D_{11} + 2D_{10} - D_9 + R_9 - T_9 + 5.4\lambda \leq 23.4$$

$$- E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} - .3\lambda \geq .7$$

$$- E_{12} + 2E_{11} - E_{10} + D_{12} - 2D_{11} + D_{10} - R_{10} + T_{10} + .3\lambda \leq 1.3$$

$$- E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} - 2.1\lambda \geq 4.9$$

$$- E_{13} + 2E_{12} - E_{11} + D_{13} - 2D_{12} + D_{11} - R_{11} + T_{11} + 2.1\lambda \leq 9.1$$

$$- E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} - .3\lambda \geq .7$$

$$- E_{14} + 2E_{13} - E_{12} + D_{14} - 2D_{13} + D_{12} - R_{12} + T_{12} + .3\lambda \leq 1.3$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} - 7.8\lambda \geq 18.2$$

$$E_{15} - 2E_{14} + E_{13} - D_{15} + 2D_{14} - D_{13} + R_{13} - T_{13} + 7.8\lambda \leq 33.8$$

$$- E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} - 8.4\lambda \geq 19.6$$

$$- E_{16} + 2E_{15} - E_{14} + D_{16} - 2D_{15} + D_{14} - R_{14} + T_{14} + 8.4\lambda \leq 36.4$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} - .9\lambda \geq 2.1$$

$$E_{17} - 2E_{16} + E_{15} - D_{17} + 2D_{16} - D_{15} + R_{15} - T_{15} + .9\lambda \leq 3.9$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} - 3.9\lambda \geq 9.1$$

$$E_{18} - 2E_{17} + E_{16} - D_{18} + 2D_{17} - D_{16} + R_{16} - T_{16} + 3.9\lambda \leq 16.9$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} - 1.5\lambda \geq 3.5$$

$$E_{19} - 2E_{18} + E_{17} - D_{19} + 2D_{18} - D_{17} + R_{17} - T_{17} + 1.5\lambda \leq 6.5$$

$$\lambda \leq 1$$

$$\lambda \geq 1$$

Non-negativity constraints:

$D_x \geq 0, E_x \geq 0, \; x = 1, 2, \ldots, 19.$

$R_x \geq 0, T_x \geq 0, \; x = 1, 2, \ldots, 17.$

$\lambda \geq 0$

5.9 Results

By solving a series of quadratic programs parametrically we obtain the following Table 5.3 representing the solution of the problem (QP).

Table 5.3 Results of the problem (QP)

Variable	Value	Variable	Value	Variable	Value	Variable	Value
E_1	11.69	D_2	2.57	R_2	.05	R_{12}	.62
E_4	4.51	D_5	9.75	R_4	.06	R_{13}	.61
E_6	4.89	D_8	3.96	R_6	.06	R_{14}	.14
E_9	10.41	D_{10}	3.50	R_7	.65	R_{15}	.03
E_{12}	2.24	D_{13}	2.71	R_8	.23	R_{16}	.16
E_{15}	9.49	D_{14}	9.29	R_9	.57	R_{17}	.05
E_{16}	.08	D_{17}	6.37	R_{10}	.25	T_3	.20
E_{19}	11.31	R_1	.15	R_{11}	.19	T_5	.44
						λ	.9949

The value of objective function, which represents the combination of fit and smoothness is 877.84 and the level of satisfaction, λ, of this solution is .9949 .

Deviation between the graduated value v_x and observed value u_x is the absolute value of the difference between deviational variables D_x and E_x. Therefore, $E_1 = 11.69$ indicates the absolute difference between graduated value v_1 and observed value u_1, where not both v_1 and u_1 can not be positive simultaneously. Though, they can assume value zero at the same time. Similar interpretation would stand true for any values of D_x and E_x. Hence, $E_4 = 4.51$, $E_6 = 4.89$, $E_9 = 10.41$, $E_{12} = 2.24$, $E_{15} = 9.49$, $E_{16} = .08$, $E_{19} = 11.31$, $D_2 = 2.57$, $D_5 = 9.75$, $D_8 = 3.96$, $D_{10} = 3.50$, $D_{13} = 2.71$, $D_{14} = 9.29$, and $D_{17} = 6.37$ means the absolute difference between graduated value and observed value for 4^{th}, 6^{th}, 9^{th}, 12^{th}, 15^{th} , 16^{th}, 19^{th}, 2^{nd}, 5^{th}, 8^{th}, 10^{th}, 13^{th}, 14^{th}, and 17^{th} term in the series respectively. Whereas, $R_1 = .15$, $R_2 = .05$, $R_4 = .06$, $R_6 = .06$, $R_7 = .65$, $R_8 = .23$, $R_9 = .57$, $R_{10} = .25$, $R_{11} = .19$, $R_{12} = .62$, $R_{13} = .61$, $R_{14} = .14$, $R_{15} = .03$, $R_{16} = .16$, $R_{17} = .05$, $T_3 = .20$ and $T_5 = .44$ are the deviations in the second difference of the sequence v_x for 1^{st}, 2^{nd}, 4^{th}, 6^{th}, 7^{th}, 8^{th}, 9^{th}, 10^{th}, 11^{th}, 12^{th}, 13^{th}, 14^{th}, 15^{th}, 16^{th}, 17^{th}, 3^{rd} and 5^{th} term respectively. The value of λ, which stands for the level of satisfaction for solution obtained in Table 5.3 is .9949.

The following Table 5.4 and Figure 5.5 show the behavior of the value of λ corresponding to changes in tolerance levels, of 10%, 20%, 30%, 40% and 50% for imprecisely known observed values u_x's, and of 0.25%, 0.5%, 1%, 2%, 3%, 4%, and 5% tolerance levels q_0 for objective function, which represents the combination of fit and smoothness.

Table 5.4 Value of λ Corresponding to Observed Value Tolerance and Objective Function Tolerance

Observed Value Tolerance	Objective Function Tolerance						
	0.25%	0.5%	1%	2%	3%	4%	5%
10%	0.9848	0.9697	0.93949	0.87953	0.82164	0.76847	0.7193
15%	0.98985	0.9796	0.95877	0.91587	0.88245	0.8304	0.79027
20%	0.99237	0.9846	0.96874	0.93535	0.900067	0.8660	0.8200
25%	0.99388	0.98765	0.9748	0.9475	0.91866	0.88926	0.86067
30%	0.9949	0.98969	.97892	0.9558	0.9311	0.90563	0.8807

Table 5.5 Value of Objective Function (Membership Function) Corresponding to Observed Value Tolerance and Objective Function Tolerance

Observed Value Tolerance	Objective Function Tolerance						
	0.25%	0.5%	1%	2%	3%	4%	5%
10%	877.86	875.76	871.76	864.55	858.17	852.98	848.38
	(.9864)	(.9705)	(.9398)	(.8795)	(.8280)	(.7685)	(.7193)
15%	877.85	875.72	871.59	863.91	856.99	850.80	845.26
	(.9909)	(.9795)	(.9591)	(.9159)	(.8727)	(.8304)	(.7902)
20%	877.84	875.70	871.51	863.57	856.27	849.55	843.95
	(.9955)	(.9841)	(.9682)	(.9352)	(.9000)	(.8659)	(.8200)
25%	877.84	875.68	871.45	863.35	855.78	848.73	842.16
	(.9955)	(.9886)	(.9750)	(.9477)	(.9185)	(.8892)	(.8606)
30%	877.84	875.68	871.42	863.21	855.45	848.15	841.28
	(.9955)	(.9886)	(.9784)	(.9557)	(.9310)	(.9057)	(.8806)

5.10 Discussion of the Solution in View of Table 5.4 and Table 5.5

Table 5.4 shows different values of λ for various tolerance levels for the imprecisely known observed values and desired levels of objective function. Also, Table 5.5 shows different values of objective function for various tolerance levels for the imprecisely known crisp objective function value and imprecisely known observed values u_x. Note that in this formulation the membership function λ is used to express the degree of certainty of the solution with respect to fuzzy parameters, objective function which represents the combination of fit and smoothness and imprecisely known observed values for u_x [43]. From Table 5.4, it is observed that with the increase in the tolerance level for desired level of objective function, the value of λ decreases. This shows that the smaller the value of membership grade λ, the smaller is the support for the solution and hence, lower the degree of certainty of solution. On the other hand, it is observed that with increase in tolerance limits for imprecisely known u_x, the value of λ increases. This shows that the larger the value of membership grade λ, the larger is the support for the solution. In Table 5.5, the numbers in the brackets represent the value of the membership function corresponding to the value of objective function at the optimal solution given in Table 5.3. From Tables 5.4 and 5.5 we observe that the value of the membership function is, as expected, either greater than or equal to the value of λ. It can therefore be concluded that fuzzy programming does not provide just another crisp solution; instead it produces the optimum solution corresponding to the pre-specified tolerance levels of constraints. The above observation is also seems to be clear from Graphs 5.4 and Graphs 5.5 in Appendix 1.

In the above examination, the relationship between objective function (which represents the combination of fit and smoothness) and the range of observed values u_x's are investigated for possible values of membership grade between 0 and 1. Such an examination is useful in order to provide the decision makers with sufficient information on the implication of the choice of the membership grade prior to making final decision. Fuzzy programming is a suitable method to admit imprecise data. Especially, when the management or the decision makers are unable to specify precisely the combination of fit and smoothness level, but are rather able to provide lower and upper bounds, with respect to some pre-assigned aspiration level, taken as representing imprecision in setting of such bounds. As already stated, fuzzy set theory permits the partial belonging of an element to a fuzzy set characterized by a membership function that takes values in the interval [0, 1]. Thus, fuzzy programming produces most satisfactory solution within a pre specified interval, whereas a conventional crisp set theory constraint only permits an element either to belong (membership grade 1) or not to belong (membership grade 0) to the set {0, 1}.

5.11 Comparison and Discussion of the Results In View of Graphs and Tables in Appendix 1 and Appendix 2

On the lines of Schuette [37], we draw Table 1 in Appendix 2 which depicts the graduated values and measures of fit and smoothness obtained by using linear programming approach to graduation for $z = 2$ and various values of h. Similarly, Table 2 in Appendix 2 is the representation of values obtained by utilizing quadratic programming approach to graduation for $z = 2$ and various values of h. By analyzing the Graph 1 in Appendix 2, which corresponds to Table 1 and Table 2, it is observed that the quadratic programming improves the graduated values and measure of fit and smoothness. Graph 2 in Appendix 2 presents a comparison of graduated values and measures of fit and smoothness obtained in Table 1, Table 2 and by Schuette [37, page 423, Table 1]. It is clear from Graph 2 that the quadratic programming approach has not only been able to improve the values but also the smoothness. Table 2 and Table 4 in Appendix 2 present the graduated values and measures of fit and smoothness utilizing linear programming approach and quadratic programming approach respectively for $z = 3$ and various values of h. Graph 4 in Appendix 2 presents a comparison of graduated values and measure of fit and smoothness obtained in Table 3, Table 4 and by Schuette [37, page 424, Table 2]. Again, the quadratic programming approach gives smoother and more improved values.

Table 5 and Table 6 in Appendix 2 compare the values of measures of fit and smoothness, graduated values and level of satisfaction using linear programming approach and quadratic programming approach under fuzzy environment respectively. This comparison is done by taking $z = 2$, $h = 10$, $q_x = 30\%$ (which is the tolerance level for all the constraints) and changing the tolerance level of objective function i.e. measures of fit and smoothness.

As it is clear from Table 5 and Table 6 (Appendix 2) that the quadratic programming approach help improve the level of satisfaction, graduated values and the measures of fit and smoothness. This conclusion becomes more clear by looking at Graph 5 and Graph 6 in Appendix 2.

Chapter 6

CONCLUSION, CONTRIBUTION AND RECOMMENDATIONS

In the present chapter, we state the contributions and conclusions of this dissertation. Finally, we give some recommendations for further research on the problems considered in this dissertation.

6.1 Conclusion and Contribution

In the present dissertation, an important problem in the field of Actuarial Science, i.e. graduation problem addressed by Schuette [37], has been revisited. We have modeled graduation problem as linear program with deviational variables for observed and graduated values, under fuzzy environment. Also, the problem is modeled as quadratic program by incorporating quadratic objective function as compared to linear objective function proposed by Schuette [37]. Three most significant contributions of this thesis are

1. Graduation problem considered by Schuette [37] is formulated and solved under fuzzy environment to capture the impreciseness present in the data set of observed values in Chapter 3.
2. We propose a finite iteration technique for solving fuzzy quadratic programming problems in Chapter 4.
3. Graduation problem is formulated and solved as a quadratic program both under crisp and fuzzy environment in Chapter 5. This improves the results of Chapter 3.

The graduation problem with imprecise observed values data under both crisp and fuzzy environments is considered in Chapter 3. Under crisp environment the problem was formulated on the lines of formulation proposed by Schuette [37]. However, one underlying assumption in the above model, and most of the models in the literature is that data used is deterministically known. But data set of observed values rarely-if-ever turns out to be crisply correct. Therefore, the models based on precise knowledge of observed values have little practical applications. We deal with such a problem through fuzzy logic approach. Under fuzzy environment, the problem is formulated as fuzzy linear program.

In chapter 4 we propose a new approach to solve fuzzy quadratic programming problem. Although, there are lots of techniques available in the literature but there is hardly one that is easy to use. This method can be helpful for the managers to take appropriated decisions taking vagueness of the data into account.

Chapter 5 presents the graduation problem as considered by Schuette [37] as a quadratic programming problem. It is observed that the results obtained improve the ones obtained by applying linear programming technique in the Chapter 3, and the results obtained by other researchers using other techniques.

It is suggested that the methods presented in this dissertation are computationally effective and useful for determining the optimal solution to the problems discussed in Chapter 3, 4 and 5.

6.2 Applications and Recommendations for Future Research

The technique presented in this dissertation for fuzzy quadratic programming can be utilized and extended in portfolio selection process, where market rate of return is most of the time fuzzy. A typical portfolio selection problem will contain following components:

- Risk factor measured by variance of the portfolio, which is a quadratic function
- Expected return measured by a linear function
- Total fund available, which represents the resource availability
- Upper and lower limit for investing in a particular type of security.

Further, the methods introduced in the present thesis, offer an opportunity to view a graduation problem from a different prospective. In the present thesis, we discuss and solve the graduation problem using linear programming and quadratic programming approaches as a symmetric case. However, the non-symmetric problem can be solved utilizing the same approaches with appropriate modification.

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APPENDIX 1

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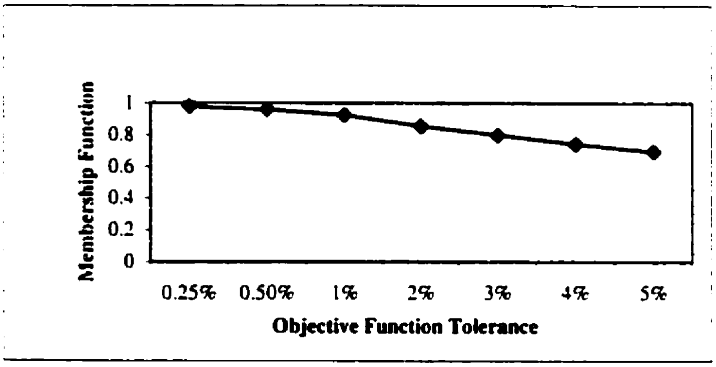
Graphs 3.3

Depicting

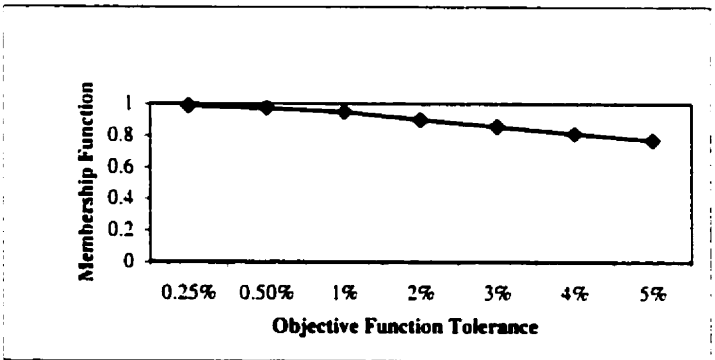
Value of λ Corresponding to Observed Value and Objective Function Tolerance

(as per Table 3.3)

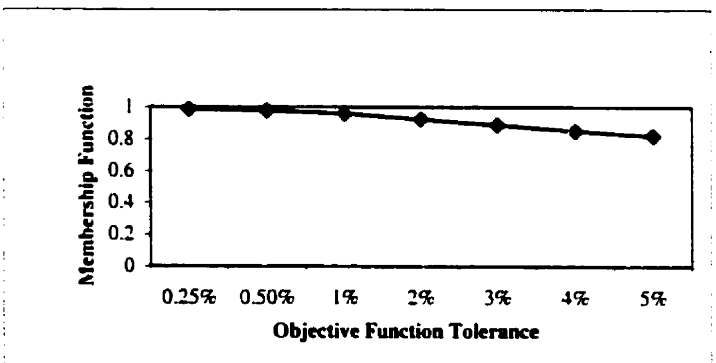
Constraint Tolerance 10%



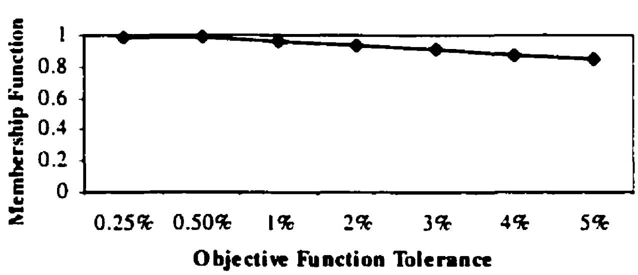
Constraint Tolerance 15%



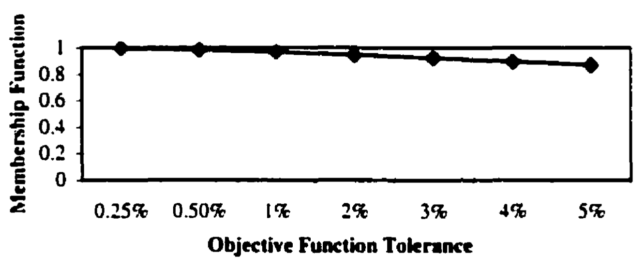
Constraint Tolerance 20%



Constraint Tolerance 25%



Constraint Tolerance 30%



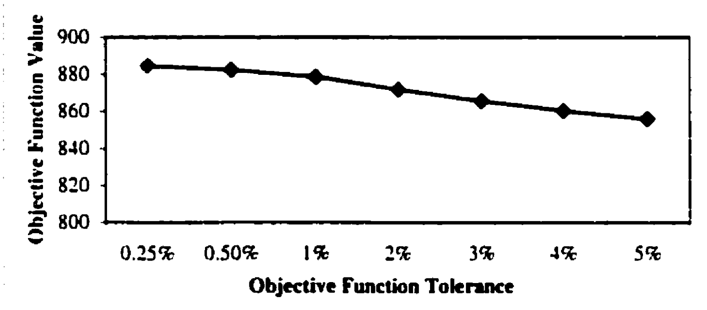
Graphs 3.4

Depicting

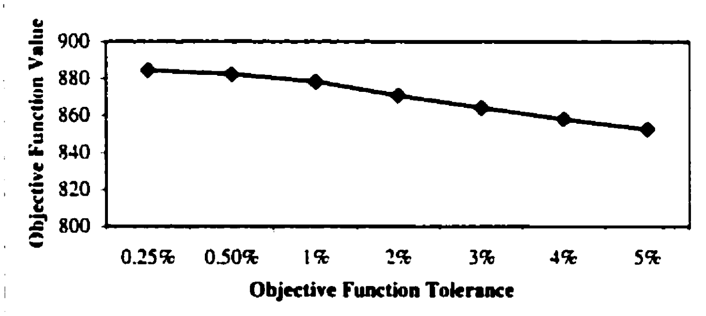
Value of Objective Function Corresponding to Constraints and Objective Function Tolerance

(as per Table 3.4)

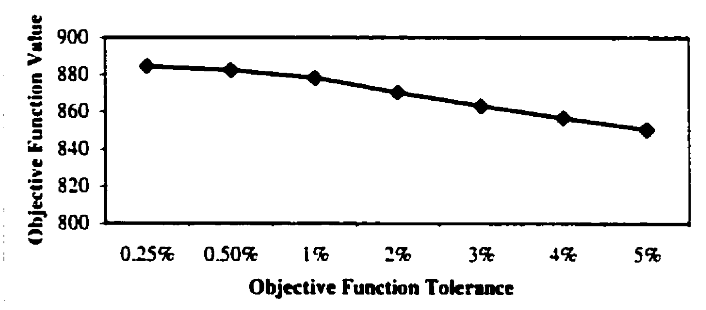
Constraint Tolerance 10%



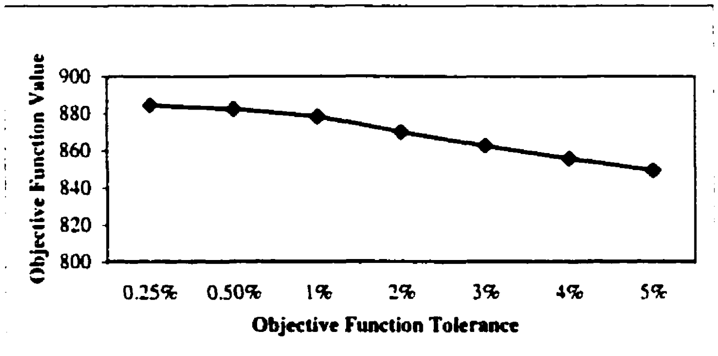
Constraint Tolerance 15%



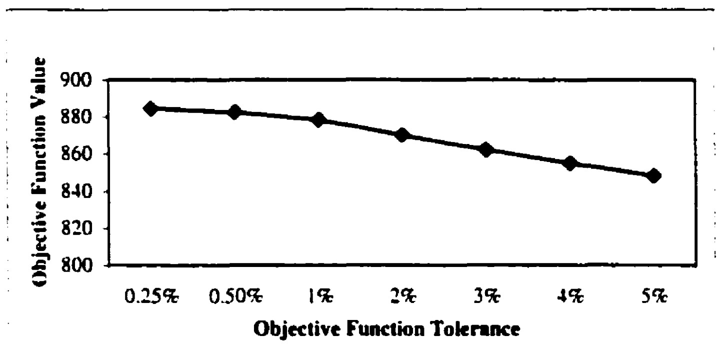
Constraint Tolerance 20%



Constraint Tolerance 25%



Constraint Tolerance 30%



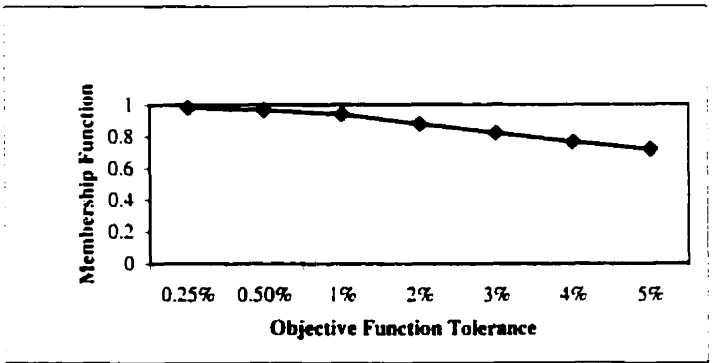
Graphs 5.4

Depicting

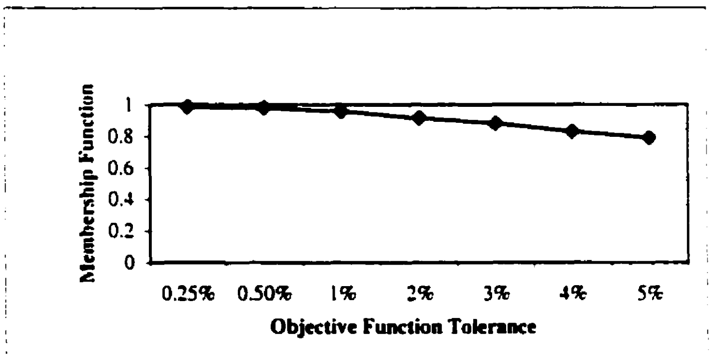
Value of λ Corresponding to Observed Value and Objective Function Tolerance

(as per Table 5.4)

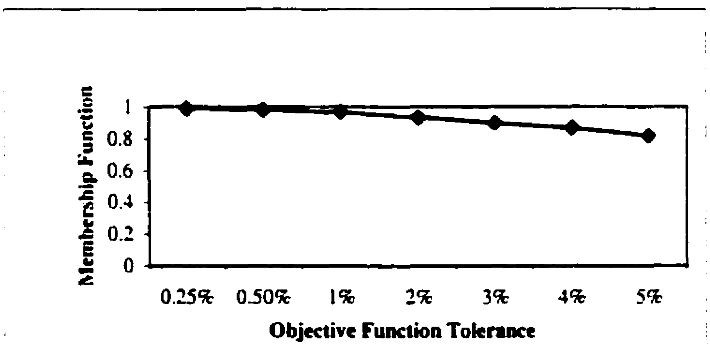
Constraint Tolerance 10%



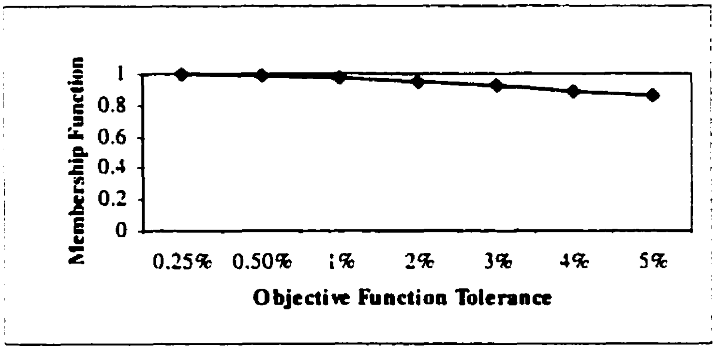
Constraint Tolerance 15%



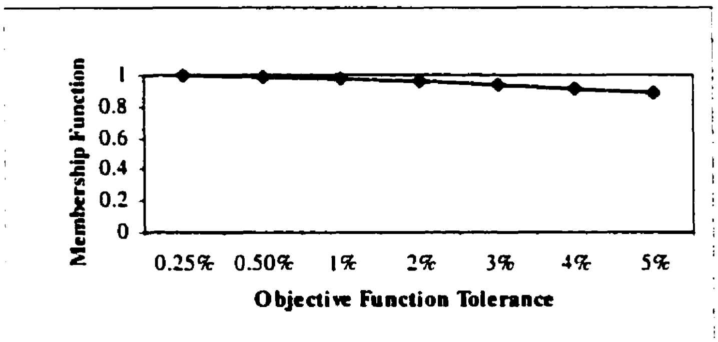
Constraint Tolerance 20%



Constraint Tolerance 25%



Constraint Tolerance 30%

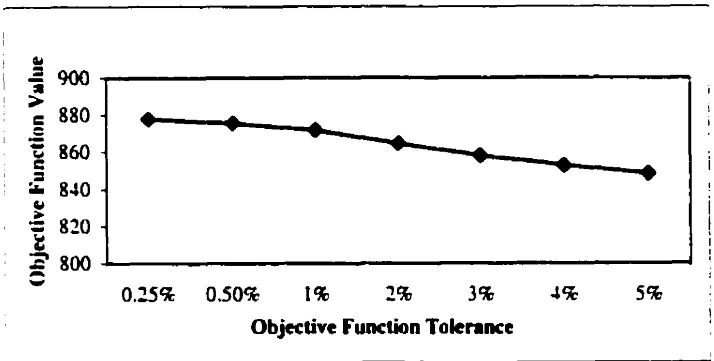


Graphs 5.5

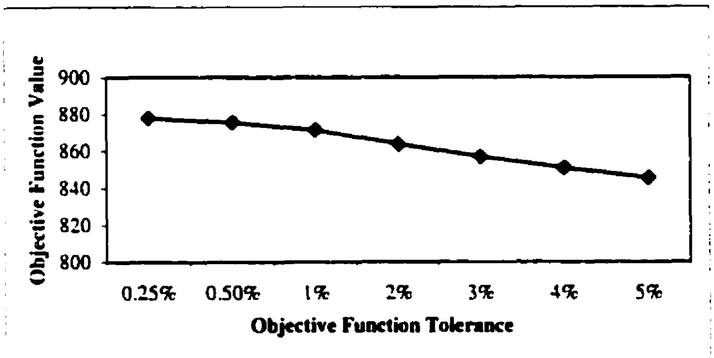
Depicting

Value of Objective Function Corresponding to Constraints and Objective Function Tolerance (as per Table 5.5)

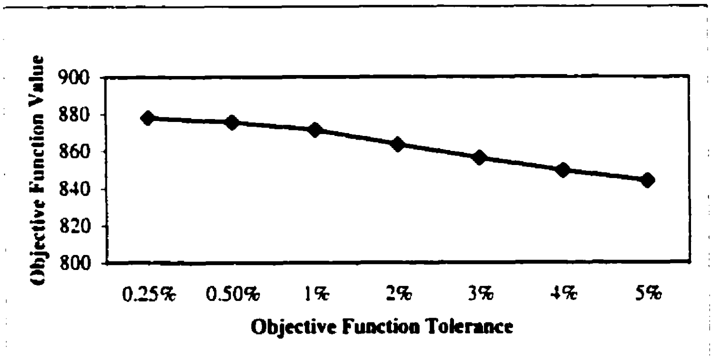
Constraint Tolerance 10%



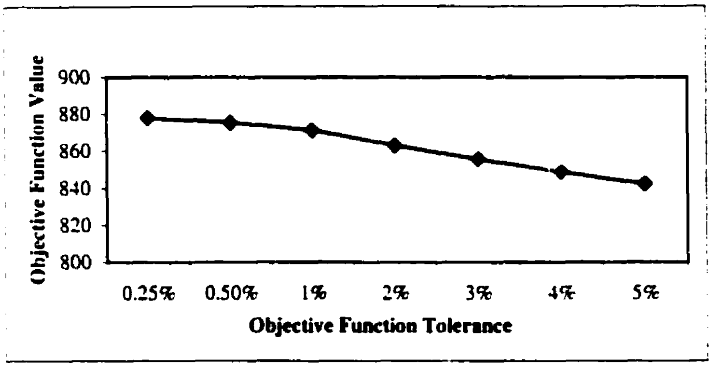
Constraint Tolerance 15%



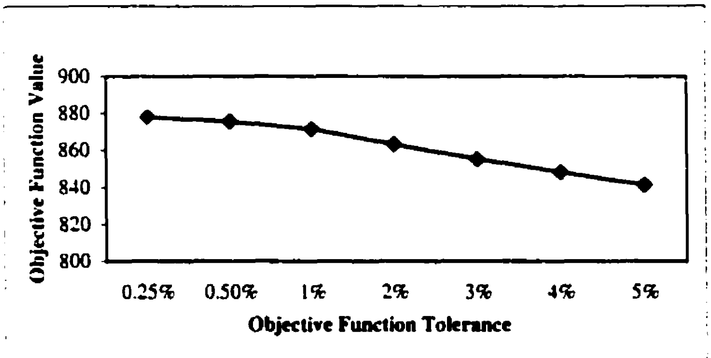
Constraint Tolerance 20%



Constraint Tolerance 25%



Constraint Tolerance 30%



APPENDIX 2

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TABLE 1

LINEAR PROGRAMMING APPROACH TO GRADUATION UNDER CRISP ENVIRONMENT
GRADUATED VALUES, AND MEASURES OF FIT AND SMOOTHNESS FOR $z = 2$

x	u_x	w_x	h = 8.80	h = 10.00	h = 16.60	h = 24.40	h = 32.20	h = 40.00	h = 47.80	h = 55.60	h = 63.40	h = 71.20	h = 79.00
			Graduated Values v_x										
1	34	3	19.67	20.33	22.50	22.50	22.50	22.50	22.50	22.50	23.13	23.13	19.20
2	24	5	25.33	25.67	26.75	26.75	26.75	26.75	26.75	26.75	25.78	25.78	24.00
3	31	8	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	30.22	30.22	28.80
4	40	10	36.67	36.33	35.25	35.25	35.25	35.25	35.25	35.25	34.67	34.67	33.60
5	30	15	42.33	41.67	39.50	39.50	39.50	39.50	39.50	39.50	39.11	39.11	38.40
6	49	20	48.00	47.00	43.75	43.75	43.75	43.75	43.75	43.75	43.56	43.56	43.20
7	48	23	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00
8	48	20	48.00	49.00	52.25	52.25	52.25	52.25	52.25	52.25	52.44	52.44	52.80
9	67	15	54.17	55.00	57.17	56.79	56.79	56.79	56.79	56.79	56.89	56.89	57.60
10	58	13	60.33	61.00	62.08	61.33	61.33	61.33	61.33	61.33	61.33	61.33	62.40
11	67	11	66.50	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.91
12	75	10	72.67	73.00	71.92	72.67	72.67	72.67	72.67	72.67	72.67	72.67	73.43
13	76	9	78.83	79.00	76.83	78.33	78.33	78.33	78.33	78.33	78.33	78.33	78.94
14	76	9	85.00	85.00	84.47	84.00	84.00	84.00	84.00	84.00	84.00	84.00	84.46
15	102	7	92.50	92.50	92.10	89.67	89.67	89.67	89.67	89.67	89.67	89.67	89.97
16	100	5	100.00	100.00	99.73	95.33	95.33	95.33	95.33	95.33	95.33	95.33	95.49
17	101	5	107.50	107.50	107.37	101.00	101.00	101.00	101.00	101.00	101.00	101.00	101.00
18	115	3	115.00	115.00	115.00	106.67	106.67	106.67	106.67	106.67	106.67	106.67	106.51
19	134	1	122.50	124.50	122.63	112.33	112.33	112.33	112.33	112.33	112.33	112.33	112.03
			Measures of Fit and Smoothness (Value of Objective Function)										
			872.53	886.83	913.36	931.94	942.99	954.04	965.09	976.14	986.04	995.58	1001.2

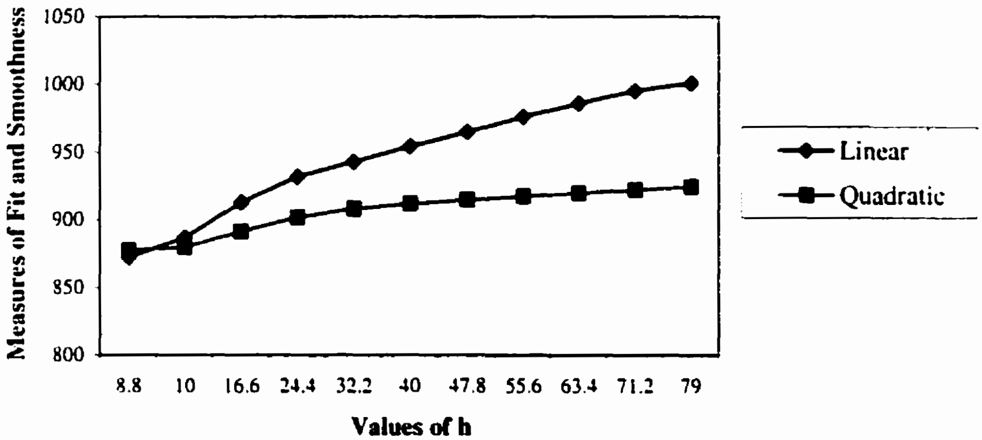
TABLE 2

QUADRATIC PROGRAMMING APPROACH TO GRADUATION UNDER CRISP ENVIRONMENT
GRADUATED VALUES, AND MEASURES OF FIT AND SMOOTHNESS FOR $z = 2$

x	u_x	w_x	h = 8.80	h = 10.00	h = 16.60	h = 24.40	h = 32.20	h = 40.00	h = 47.80	h = 55.60	h = 63.40	h = 71.20	h = 79.00
			Graduated Values v_x										
1	34	3	22.27	22.30	22.41	22.49	22.55	22.58	22.60	22.61	22.62	22.62	22.63
2	24	5	26.55	26.58	26.66	26.71	26.75	26.77	26.79	26.79	26.80	26.80	26.80
3	31	8	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00
4	40	10	35.51	35.47	35.37	35.31	35.27	35.24	35.22	35.22	35.21	35.21	35.20
5	30	15	39.79	39.75	39.63	39.55	39.49	39.46	39.44	39.43	39.42	39.42	39.41
6	49	20	44.15	44.09	43.94	43.84	43.78	43.74	43.72	43.71	43.70	43.69	43.69
7	48	23	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00
8	48	20	51.93	51.98	52.13	52.24	52.33	52.38	52.41	52.42	52.43	52.44	52.44
9	67	15	56.59	56.62	56.73	56.85	56.97	57.03	57.08	57.09	57.09	57.10	57.10
10	58	13	61.49	61.52	61.60	61.71	61.81	61.88	61.92	61.92	61.93	61.93	61.93
11	67	11	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00
12	75	10	72.75	72.76	72.78	72.64	72.45	72.34	72.26	72.27	72.27	72.27	72.27
13	76	9	78.68	78.73	78.88	78.56	78.12	77.85	77.67	77.69	77.70	77.71	77.72
14	76	9	85.28	85.35	85.52	84.92	84.11	83.62	83.30	83.32	83.34	83.35	83.36
15	102	7	92.55	92.58	92.66	91.66	90.40	89.63	89.13	89.14	89.15	89.16	89.17
16	100	5	99.96	99.97	100.00	98.57	96.81	95.74	95.04	95.05	95.06	95.06	95.06
17	101	5	107.39	107.40	107.43	105.56	103.29	101.90	101.00	101.00	101.00	101.00	101.00
18	115	3	115.00	115.00	115.00	112.65	109.84	108.13	107.01	106.99	106.98	106.97	106.97
19	134	1	122.67	122.65	122.60	119.76	116.41	114.36	113.03	113.00	112.97	112.96	112.94
			Measures of Fit and Smoothness (Value of Objective Function)										
			877.38	880.03	891.47	901.76	907.88	911.88	914.81	917.36	919.83	922.25	924.64

Comparison of Measures of Fit and Smoothness

Graph 1



Comparison of Measures of Fit and Smoothness

Graph 2

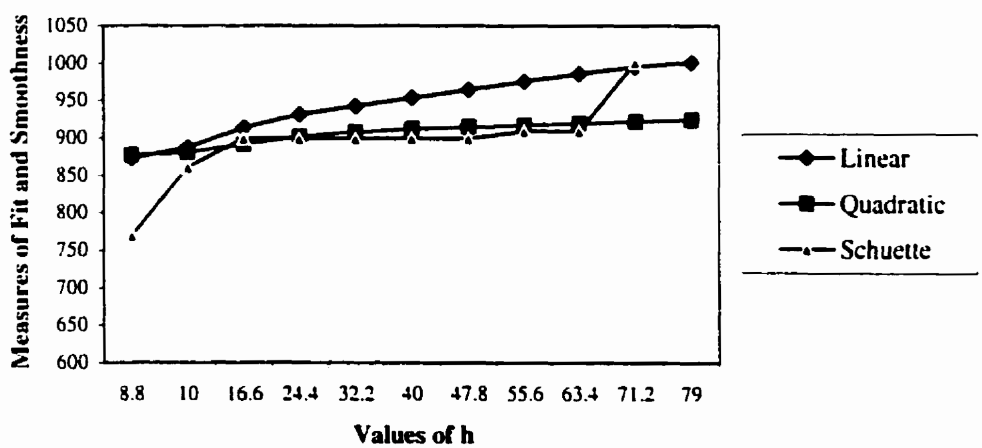


TABLE 3

LINEAR PROGRAMMING APPROACH TO GRADUATION UNDER CRISP ENVIRONMENT

GRADUATED VALUES, AND MEASURES OF FIT AND SMOOTHNESS FOR $z = 3$

x	u_x	w_x	h = 6.91	h = 10.00	h = 13.07	h = 19.23	h = 25.4	h = 31.56	h = 37.72	h = 43.88	h = 50.04	h = 56.20	h = 62.36
			Graduated Values v_x										
1	34	3	20.67	22.32	22.32	22.32	22.32	22.32	22.32	23.13	23.13	23.13	23.13
2	24	5	25.99	26.68	26.68	26.68	26.68	26.68	26.68	27.01	27.01	27.01	27.01
3	31	8	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00
4	40	10	35.71	35.29	35.29	35.29	35.29	35.29	35.29	35.09	35.09	35.09	35.09
5	30	15	40.11	39.56	39.56	39.56	39.56	39.56	39.56	39.29	39.29	39.29	39.29
6	49	20	44.21	43.79	43.79	43.79	43.79	43.79	43.79	43.59	43.59	43.59	43.59
7	48	23	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00
8	48	20	52.18	52.18	52.18	52.18	52.18	52.18	52.18	52.51	52.51	52.51	52.51
9	67	15	56.73	56.73	56.73	56.73	56.73	56.73	56.73	57.13	57.13	57.13	57.13
10	58	13	61.68	61.68	61.68	61.68	61.68	61.68	61.68	61.85	61.85	61.85	61.85
11	67	11	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00
12	75	10	72.71	72.71	72.71	72.71	72.71	72.71	72.71	72.58	72.58	72.58	72.58
13	76	9	78.80	78.80	78.80	78.80	78.80	78.80	78.80	78.58	78.58	78.58	78.58
14	76	9	85.27	85.27	85.27	85.27	85.27	85.27	85.27	85.01	85.01	85.01	85.01
15	102	7	92.13	92.13	92.13	92.13	92.13	92.13	92.13	91.87	91.87	91.87	91.87
16	100	5	99.37	99.37	99.37	99.37	99.37	99.37	99.37	99.15	99.15	99.15	99.15
17	101	5	106.99	106.99	106.99	106.99	106.99	106.99	106.99	106.86	106.86	106.86	106.86
18	115	3	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00
19	134	1	123.39	123.39	123.39	123.39	123.39	123.39	123.39	123.57	123.57	123.57	123.57
			Measures of Fit and Smoothness (Value of Objective Function)										
			874.77	876.76	878.03	880.57	883.12	885.66	888.21	890.21	892.20	894.18	896.17

TABLE 4

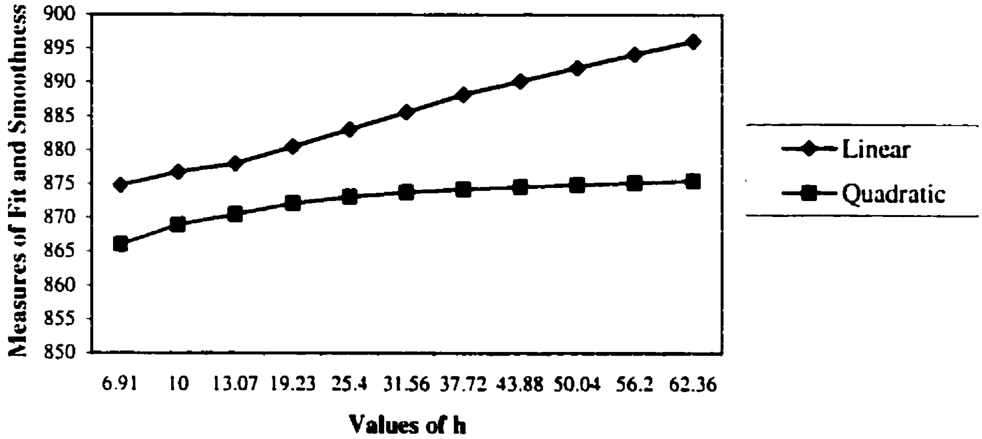
QUADRATIC PROGRAMMING APPROACH TO GRADUATION UNDER CRISP ENVIRONMENT

GRADUATED VALUES, AND MEASURES OF FIT AND SMOOTHNESS FOR $z = 3$

x	u_x	w_x	h = 6.91	h = 10.00	h = 13.07	h = 19.23	h = 25.4	h = 31.56	h = 37.72	h = 43.88	h = 50.04	h = 56.20	h = 62.36
			Graduated Values v_x										
1	34	3	21.83	22.13	22.28	22.45	22.53	22.58	22.62	22.64	22.66	22.68	22.69
2	24	5	26.30	26.48	26.57	26.67	26.72	26.75	26.77	26.78	26.80	26.80	26.81
3	31	8	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00	31.00
4	40	10	35.70	35.54	35.45	35.36	35.32	35.29	35.27	35.25	35.24	35.24	35.23
5	30	15	40.12	39.89	39.77	39.65	39.59	39.55	39.52	39.50	39.49	39.48	39.47
6	49	20	44.24	44.06	43.97	43.87	43.82	43.79	43.77	43.75	43.74	43.73	43.72
7	48	23	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00	48.00
8	48	20	51.95	52.11	52.18	52.27	52.31	52.34	52.35	52.37	52.37	52.38	52.39
9	67	15	56.59	56.73	56.80	56.88	56.92	56.94	56.96	56.97	56.98	56.99	56.99
10	58	13	61.65	61.72	61.76	61.79	61.81	61.82	61.83	61.84	61.84	61.85	61.85
11	67	11	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00
12	75	10	72.27	72.34	72.38	72.41	72.43	72.44	72.45	72.46	72.46	72.47	72.47
13	76	9	77.59	77.84	77.97	78.11	78.18	78.22	78.25	78.27	78.29	78.30	78.31
14	76	9	83.51	83.90	84.10	84.31	84.42	84.48	84.53	84.56	84.58	84.60	84.62
15	102	7	90.29	90.69	90.90	91.12	91.23	91.30	91.34	91.38	91.40	91.42	91.44
16	100	5	97.76	98.11	98.30	98.49	98.59	98.65	98.69	98.72	98.74	98.75	101.23
17	101	5	105.94	106.17	106.30	106.43	106.49	106.53	106.56	106.58	106.59	106.61	106.61
18	115	3	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00	115.00
19	134	1	125.02	124.64	124.45	124.24	124.13	124.06	124.02	123.99	123.97	123.95	123.93
			Measures of Fit and Smoothness (Value of Objective Function)										
			866.06	868.96	870.52	872.21	873.15	873.78	874.25	874.63	874.95	875.23	875.49

Comparison of Measures of Fit and Smoothness

Graph 3



Comparison of Measures of Fit and Smoothness

Graph 4

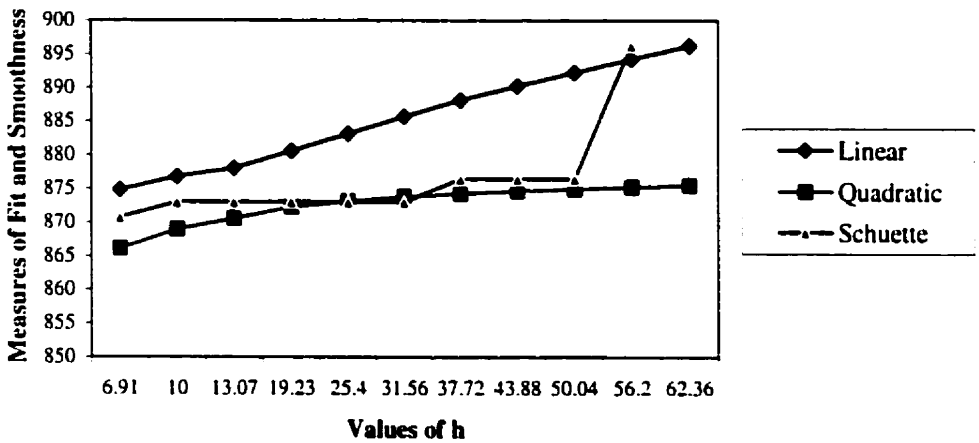


TABLE 5

LINEAR PROGRAMMING APPROACH TO GRADUATION UNDER FUZZY ENVIRONMENT

GRADUATED VALUES, AND MEASURES OF FIT AND SMOOTHNESS FOR $z = 2, h = 10$

$z_0 = 886.8333$, $q_x = 30\%$

x	u_x	w_x	$q_0 = .25\%$	$q_0 = .50\%$	$q_0 = 1\%$	$q_0 = 2\%$	$q_0 = 3\%$	$q_0 = 4\%$	$q_0 = 5\%$
			Graduated Values v_x						
1	34	3	20.37	20.40	20.47	20.59	20.96	21.41	21.84
2	24	5	25.67	25.67	25.66	25.66	25.78	25.94	26.09
3	31	8	31.00	31.00	31.00	31.00	31.00	31.00	31.00
4	40	10	36.34	36.34	36.35	36.37	36.27	36.13	35.99
5	30	15	41.64	41.61	41.55	41.44	41.09	40.65	40.24
6	49	20	47.00	46.99	46.98	46.97	46.60	46.09	45.61
7	48	23	48.00	48.00	48.00	48.00	48.00	48.00	48.00
8	48	20	49.00	49.00	49.01	49.02	49.38	49.88	50.35
9	67	15	54.95	54.90	54.80	54.62	54.67	54.81	54.94
10	58	13	60.96	60.91	60.83	60.67	60.62	60.62	60.62
11	67	11	67.00	67.00	67.00	67.00	67.00	67.00	67.00
12	75	10	73.04	73.08	73.16	73.32	73.35	73.35	73.34
13	76	9	79.07	79.14	79.27	79.53	79.54	79.47	79.41
14	76	9	85.10	85.20	85.38	85.75	85.76	85.63	85.52
15	102	7	92.52	92.54	92.58	92.65	92.58	92.61	92.64
16	100	5	100.00	100.00	100.00	100.00	100.00	100.00	100.00
17	101	5	107.49	107.47	107.45	107.40	107.35	107.30	107.25
18	115	3	115.00	115.00	115.00	115.00	115.00	115.00	115.00
19	134	1	122.52	122.55	122.59	122.68	122.77	122.86	122.95
			Measures of Fit and Smoothness (Value of Objective Function)						
			884.63	882.46	878.21	870.04	862.32	855.08	848.23
			Satisfaction Level (λ)						
			0.9930	0.9862	0.9727	0.9469	0.9212	0.8952	0.8705

TABLE 6

QUADRATIC PROGRAMMING APPROACH TO GRADUATION UNDER CRISP ENVIRONMENT

GRADUATED VALUES, AND MEASURES OF FIT AND SMOOTHNESS FOR $z = 2, h = 10$

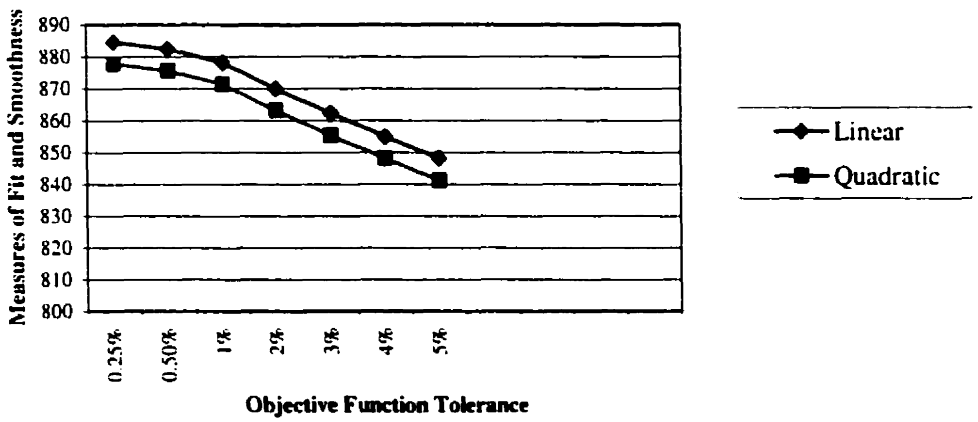
$z_0 = 880.03$, $q_x = 30\%$

x	u_x	w_x	$q_0 = .25\%$	$q_0 = .50\%$	$q_0 = 1\%$	$q_0 = 2\%$	$q_0 = 3\%$	$q_0 = 4\%$	$q_0 = 5\%$
			Graduated Values v_x						
1	34	3	22.31	22.31	22.33	22.36	22.32	22.10	21.80
2	24	5	26.57	26.56	26.53	26.49	26.41	26.24	26.02
3	31	8	31.00	31.00	31.00	31.00	31.00	31.00	31.00
4	40	10	35.49	35.50	35.53	35.58	35.68	35.87	36.10
5	30	15	39.75	39.74	39.73	39.70	39.75	39.98	40.30
6	49	20	44.11	44.13	44.16	44.23	44.34	44.53	44.77
7	48	23	48.00	48.00	48.00	48.00	48.00	48.00	48.00
8	48	20	51.96	51.93	51.88	51.76	51.59	51.39	51.14
9	67	15	56.59	56.55	56.48	56.31	56.06	55.79	55.45
10	58	13	61.50	61.48	61.44	61.34	61.17	61.00	60.76
11	67	11	67.00	67.00	67.00	67.00	67.00	67.00	67.00
12	75	10	72.76	72.75	72.74	72.73	72.81	72.89	73.12
13	76	9	78.71	78.69	78.64	78.56	78.45	78.48	78.89
14	76	9	85.29	85.23	85.10	84.89	84.55	84.49	85.08
15	102	7	92.51	111.56	92.29	92.05	91.66	91.65	92.62
16	100	5	99.92	99.87	99.77	99.64	99.38	99.37	100.00
17	101	5	107.37	107.34	107.27	107.17	106.99	106.94	107.21
18	115	3	115.00	115.00	115.00	115.00	115.00	115.00	115.00
19	134	1	122.69	122.73	122.82	122.95	123.16	123.25	123.02
			Measures of Fit and Smoothness (Value of Objective Function)						
			877.84	875.68	871.42	863.21	855.45	848.15	841.28
			Satisfaction Level (λ)						
			0.9949	0.98969	0.97892	0.9558	0.9311	0.90563	0.8807

Graph 5

Comparison of Measures of Fit and Smoothness

(Constraint Tolerance 30%)



Graph 6

Comparison of Level of Satisfaction

(Constraint Tolerance 30%)