COVARIANT COMPUTATIONS OF HEAT KERNELS IN PERTURBATION THEORY

BY

YURI VLADIMIROVICH GUSEV

A Thesis
Submitted to the Faculty of Graduate Studies
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COVARIANT COMPUTATIONS OF HEAT KERNELS IN PERTURBATION THEORY

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements for the degree of

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Abstract

We calculate the trace of the heat kernel for the general background of gravitational and gauge fields by means of the covariant perturbation theory. The heat kernel trace is obtained to the third order in the field strengths (curvatures) for noncompact asymptotically flat manifolds of an arbitrary spacetime dimension. The basis of nonlocal tensor invariants is constructed and the form factors are computed in two different integral representations.

The coincidence limit of the heat kernel to second order in the curvatures is derived by the generating function method and by the perturbation theory. The basis set of curvature structures and form factors are derived.

The short time behaviors of both results are compared with the Schwinger-DeWitt expansion. The large time asymptotic for the heat kernel trace is found. The one-loop effective action for the Weyl invariant scalar model is computed in two dimensions. It reveals how third order terms vanish truncating the curvature expansion. The two-dimensional Green's function is obtained similarly. The derivation of the Weyl anomaly in four dimensions is carried out starting from the heat kernel trace.

The presented results can be used for the calculation of the one-loop nonlocal effective action and the Green's functions which contain information about vacuum polarization effects in gauge theories and quantum gravity.

Acknowledgments

I am very grateful to my supervisor Tom Osborn for his constant advice, support and encouragement, as well as for his great help with this thesis.

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Contents

Chapter 1

Introduction

1.1 Heat kernel

The subject of the heat kernel is important as a branch of theoretical physics as well as applied mathematics. Solutions of the heat equation and its modifications such as the diffusion and Schrödinger equations have been widely studied in mathematics [1, 2, 3, 4, 5, 6]. In the present thesis we deal with the heat equation in the form,

$$\frac{\partial}{\partial s}\hat{K}(s|x,y) = \hat{F}(\nabla^x)\hat{K}(s|x,y), \tag{1.1}$$

where s is the proper time along the geodesic connecting two space-time points x and y. We work in curved space-times with metric tensors of constant positive signature, namely Euclidean space-times. The space-time dimension is arbitrary everywhere below except in Chapter 4. $\hat{F}(\nabla)$ in (1.1) is a second order differential operator acting on fields at the point x. The $\hat{}$ notation on $\hat{K}$ and $\hat{F}$ indicates that these objects are matrix valued. The heat kernel then is formally,

$$\hat{K}(s|x,y) = \exp\left[s\hat{F}(\nabla^x)\right]\delta(x,y). \tag{1.2}$$

The solution of the heat equation (1.2) was introduced to theoretical physics by R. Feynman in the form of a propagator [7, 8, 9],

$$\hat{F}(\nabla^x)\hat{G}(x,y) = -\hat{1}\delta(x,y). \tag{1.3}$$

Specifically, the heat kernel defines the propagator or Green's function through the famous Schwinger equation [10],

$$G(x,y) = \int_0^\infty d\,s \hat{K}(s|x,y). \tag{1.4}$$

If $S[\varphi]$ is the action of a field theory model then the induced differential operator is,

$$F_{AB}(\nabla)\delta(x,y) = \frac{\delta}{\delta\varphi^A(x)}\frac{\delta}{\delta\varphi^B(y)}S[\varphi]. \tag{1.5}$$

We restrict our study to models that possess minimal second order operators [11],

$$\hat{F}(\nabla) = \Box\hat{1} + \hat{P} - \frac{1}{6}R\hat{1}, \tag{1.6}$$

where the Laplace-Beltrami operator (or simply Laplacian),

$$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu, \tag{1.7}$$

is constructed in terms of covariant derivatives ∇_μ. The operator $\hat{F}(\nabla)$ acts on small disturbances of an arbitrary set of fields $\varphi^A(x)$. The matrix character of $\hat{F}(\nabla)$ arises from the tensor structure of the target fields $\varphi^A(x)$. The matrix conventions are

$$\hat{1} = \delta^A{}_B, \qquad \hat{P} = P^A{}_B, \text{ etc.} \tag{1.8}$$

The matrix trace over index set A will be denoted by tr :

$$\mathrm{tr}\hat{1} = \delta^A{}_A, \qquad \mathrm{tr}\hat{P} = P^A{}_A, \text{ etc.} \tag{1.9}$$

In (1.7), the metric $g^{\mu\nu}(x)$ is characterized by its Riemann and Ricci curvatures

$$\begin{aligned} R^{\mu}_{\cdot\,\alpha\nu\beta} &= \partial_\nu\Gamma^\mu_{\alpha\beta} - \partial_\beta\Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\lambda\nu}\Gamma^\lambda_{\alpha\beta} - \Gamma^\mu_{\lambda\beta}\Gamma^\lambda_{\alpha\nu}, \\ R_{\alpha\beta} &= R^{\mu}_{\cdot\,\alpha\mu\beta}, \;\; R = g^{\alpha\beta}R_{\alpha\beta}, \end{aligned} \tag{1.10}$$

∇_μ is a covariant derivative with respect to an arbitrary connection characterized by its commutator curvature

$$[\nabla_\mu, \nabla_\nu]\delta\varphi^A \equiv (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\delta\varphi^A = \mathcal{R}^A{}_{B\mu\nu}\delta\varphi^B, \qquad \mathcal{R}^A{}_{B\mu\nu} \equiv \hat{\mathcal{R}}_{\mu\nu}. \tag{1.11}$$

The modification of the potential structure in (1.6) by inclusion of the term in the Ricci scalar R is a matter of convenience and related to conformal invariant model investigations in four dimensions (see sect. 4.3).

Throughout the thesis we employ DeWitt's index notations [12]. In this convention the index A stands for any set of discrete indices of tensor-spinor fields. It allows the interaction term $\hat{P}$ to be an arbitrary matrix potential consistent with a given field model. The nature of this indexing convention is that the same notation $\hat{\mathcal{R}}_{\mu\nu}$ can stand for the field strength of a non-Abelian gauge (Yang-Mills) field with the matrix indices assigned to internal degrees of freedom, or for the electromagnetic tensor when it lacks spinor degrees of freedom at all. It can be even the Riemann tensor in which case there is an additional pair of spacetime indices and $\hat{1} = g_{\mu\nu}$. For the set of the field strengths (generically thought of as curvatures)

$$R^{\alpha\beta\mu\nu}, \qquad \hat{\mathcal{R}}_{\mu\nu}, \qquad \hat{P} \tag{1.12}$$

characterizing the background we use the collective notation $\Re$. Manifolds under consideration are asymptotically flat with the simple topology $R^{2\omega}$ excluding any topologically nontrivial spaces like $R^3 \times S^1$, specifically their gauge and gravitational curvatures and the potential $\hat{P}$ (1.12) vanish at infinity.

Historically a great deal of work on heat kernels has been devoted to the short time expansion also known as the Schwinger-DeWitt series [10, 12]:

$$\hat{K}(s|x,y) = \frac{1}{(4\pi s)^{\omega}} e^{-\frac{\sigma(x,y)}{2s}} D^{1/2}(x,y) \sum_{n=0}^{\infty} s^n \hat{a}_n(x,y), \qquad s \to 0, \tag{1.13}$$

where $\sigma(x,y)$ is the world function [13] or the geodesic interval which is one half the square of the distance along the geodesic between any two space-time points x and y. Here and below 2ω is the space-time dimension. The bi-density $D(x,y)$ is the Van Vleck-Morette determinant [14, 15],

$$D = -\det\Big(- \partial^x_\mu \partial^y_\nu \sigma(x,y)\Big). \tag{1.14}$$

The Schwinger–DeWitt coefficients $\hat{a}_n(x,y)$ may be constructed from recurrence relations [12, 11] or other independent methods [5, 16, 17]. Since their invention these coefficients have played an enormous role in quantum field theory [18, 11, 19] and they are still an important research tool [20, 21].

Although it is important to know the two-point heat kernel $K(s|x,y)$ for multi-loop calculations, a simpler object is its coincidence limit $K(s|x,x)$ which is all that is required for one-loop computations. In this case (1.13) reduces to

$$\hat{K}(s|x,x) = \frac{g^{1/2}(x)}{(4\pi s)^{\omega}} \sum_{n=0}^{\infty} s^n \hat{a}_n(x,x), \quad s \to 0, \tag{1.15}$$

where g is the determinant of the metric tensor $g = \det(g^{\mu\nu})$. The Schwinger-DeWitt coefficients $\hat{a}_n(x,x)$ have been calculated explicitly for $n = 0$ to 3, [12, 11, 4, 22]. Partial results are also available for $n = 4$ [23, 16, 17]. All $\hat{a}_n(x,y)$ are *local* functions of the background fields entering the operator (1.6). The short time expansion will serve as a limiting case and as a consistency check for the results presented in this thesis.

The ultimate goal of the present research is to use the heat kernel for calculation of the one-loop effective action and the Green's functions [8, 12, 11]. The effective action contains all relevant physical information about a field theory and represents the generating function for the one-particle irreducible diagrams [8, 12]. This fact is especially important for quantum gravity where the work with ordinary Feynman diagrams would be formidable. Although it is impossible to derive the effective action in closed form for arbitrary orders of the Planck's constant $\hbar$, it is very often enough to work out the first $\hbar$ order corrections for the classical action, i.e. the one-loop effective action W. Since we work in formalism of the path integral approach [8, 24], the effective action is defined as the Legendre transform of the logarithm of the partition function (the generating function of the Green's functions). Its iterative solution is expressed in one-loop order through the differential operator (1.6) as,

$$W = \frac{1}{2}\mathrm{Tr}\ \ln F - \int d^{2\omega}x \delta^{(2\omega)}(x,x)(\ldots). \tag{1.16}$$

Here the term with ellipses $(\ldots)$ stands for the contribution of the local functional measure [25], proportional to the delta function at coincident points. As shown in [26] this contribution always cancels the volume divergences. For the massless operators (1.6) the result of this cancellation is a subtraction of the zero curvature term.

Knowledge of heat kernel (1.2) allows one to construct the covariant diagrammatic expansion [27, 12, 11, 28] to all loop orders. As can be seen from (1.16) the one-loop effective action is given by the trace of the heat kernel,

$$-W = \frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{Tr}K(s) + \int d^{2\omega}x\delta^{(2\omega)}(x,x)(\ldots), \tag{1.17}$$

where Tr, as distinct from tr in (1.9), denotes the functional trace

$$\mathrm{Tr}K(s) = \int d^{2\omega}x\,\mathrm{tr}\,\hat{K}(s|x,x). \tag{1.18}$$

From now on we will not indicate the dimension of space-time at the integral measure, $dx \equiv d^{2\omega}x$.

The ultraviolet divergences appear in quantum theory as divergences of the loop integrals over s at the lower limits (1.15). They are to be removed by a renormalization procedure [12, 11]. For massive theories there is always the mass factor e^{-sm^2} in (1.13) and (1.15) which makes the series converge also at $s \to \infty$. Unfortunately, for massless theories loop integrals also diverge at the upper limits. These infrared divergences are shortcomings of the mode of computation but not the underlying theory. It is understandable since upon the proper time integration the short time expansion of the heat kernel corresponds to the large mass expansion of the effective action or the Green's functions. Thus, a new method is needed which would allow one to compute heat kernels in the range of the proper time s from 0 to ∞. The general structure of this method, named the covariant perturbation theory, will be explained in the next section.

The covariant perturbation theory [29, 30, 31] corresponds to summation terms of the short time expansion in a given order in curvatures. This summation for the heat kernel as proposed in [32, 33] contains an infinite number of derivatives acting on curvatures, thus it is a *nonlocal* expression. It has the general form

$$\begin{aligned}\mathrm{Tr}\,K(s) &= \frac{1}{(4\pi s)^\omega}\int dx\, g^{1/2}(x)\mathrm{tr}\left\{1 + s\Re + s^2\sum f(s,\Box_1,\Box_2)\Re_1\Re_2\right.\\ &\left. + s^3\sum F(s,\Box_1,\Box_2,\Box_3)\Re_1\Re_2\Re_3 + \mathrm{O}[\Re^4]\right\},\end{aligned} \tag{1.19}$$

where functions f_i and F_i stand for analytic functions of the dimensionless $-s\Box$ arguments - *form factors* - which act on tensor invariants constructed of curvatures $\Re$. Despite the similarity with the Schwinger-DeWitt series (1.13) this expression for the trace of the heat kernel is valid for all values of the proper time s. The structure (1.19) ceases to hold beyond the third order after which it is impossible to express form factors solely in terms of the Laplacians (1.7). At the fourth order new types of mixed derivatives combinations appear [34]. The validity of the covariant perturbation theory is restricted by the condition

$$\nabla\nabla\Re >> \Re^2. \tag{1.20}$$

This means that one can study rapidly oscillating background fields but their magnitude should be small. The opposite case of the slowly fluctuating fields of the large magnitude also presents physical interest [35]. In form of the derivative expansion for an external electromagnetic field it has been studied in [36].

Although, summation (1.19) can be implemented directly [16] only the covariant perturbation theory allows one to do computations beyond second order in curvatures. The calculation of third order of the trace of the heat kernel is the main goal of the present research. The motivation came first from study of the Hawking radiation effect [37], the effect of particle creation by a collapsing gravitational body [38, 33]. As is well known a black hole radiates with the temperature proportional to the inverse of its mass [18]. As was shown in [38, 33] this quantum effect can be reproduced from the nonlocal curvature expansion in two dimensions where the effective action is known exactly (see sect. 4.1). In four dimensions the Hawking radiation effect starts from the third order in the curvature terms in the effective action [27, 39, 40]. Some encouraging results have been obtained recently [41] following this line of investigations. Unfortunately, the calculation of the effective action (1.17) for a generic field model in four dimensions lies beyond the scope of this thesis and is discussed elsewhere [42, 43, 44].

It is worthwhile emphasizing the difference between covariant perturbation theory

[29, 30] and the usual perturbation theory on flat space-time [23]. In the latter, noncovariant vertices of all orders are divergent. The divergences however can cancel in some applications [45]. In fact physical effects are covariant and determined by the covariant effective action W which is finite starting with third order [42]. Covariant renormalization theory for gauge fields is the major area of applications of the present research. The infrared renormalization in quantum electrodynamics was already done this way in [46].

Since the heat kernel trace (1.18) can be used as a generating function for heat kernels [28, 47] it is tempting to derive the heat kernel $K(s|x,x)$ from this starting point. With this trace method one can obtain at most only terms valid to second order in curvatures [48]. After applying the variational method [47] we get the result similar in form to (1.19)

$$K(s|x,x) = \frac{g^{1/2}(x)}{(4\pi s)^{\omega}} \left\{ s \sum g(s,\Box)\Re + s^2 \sum G(s,\Box_1,\Box_2,\Box_3)\Re_1\Re_2 + \mathrm{O}[\Re^3] \right\}. \quad (1.21)$$

We also derive this expression up to the second order directly from the covariant perturbation theory and check consistency of two methods.

We should emphasize that both the Schwinger-DeWitt expansion and the covariant perturbation theory are background field methods, i.e. some fields are treated as classical fields - background, and some as quantum fields. The curvatures (1.12) characterize background manifolds.

An important element of the present study is the use of computer symbolic manipulations. In fact the work could not be completed without it. As will be seen there are two kinds of symbolic manipulations, one is purely algebraic manipulations with form factors which is performed by general purpose software such as *Mathematica* and *MAPLE* [49, 50]. The other is the tensor manipulations needed for work with tensor invariants. These computations have been implemented with help of *MathTensor* [51] and *Ricci* [52] programs, both of which work under *Mathematica*. All TEX forms in appendices were produced directly from *Mathematica* or *MAPLE* outputs.

On the whole the present thesis is organized as follows. In addition to the present

Introduction it consists of three chapters and various appendices. Chapters 2 and 3 contain main results of the study, the trace of the heat kernel up to third order in curvatures and the heat kernel itself up to second order, correspondingly. The structure of these chapters is similar: we start with a perturbation expansion and proceed to a covariant curvature expansion. Having studied form factors we finish with asymptotic behaviors of $\mathrm{Tr}K(s)$ and $K(s)$. The last chapter is different since it deals with a particular class of field models, namely Weyl invariant models. Two sections of chapter 4 are devoted to massless models in two dimensions which admit a closed form of the exact one-loop effective action and Green's function. In third section we derive the four-dimensional Weyl anomaly directly from the heat kernel trace. Several appendices contain relevant form factors and a discussion of the nonlocal tensor invariants in third order.

Results of sects. 2.1-2.3 were completed by the author of the thesis as independent calculations but in parallel to work of A. Barvinsky, while the check for sect. 2.6 was was done by V. Zhytnikov. The rest of Chapter 2 and Chapters 3 and 4 are solely results of the author.

1.2 Covariant perturbation theory

In perturbation theory [10, 23], the heat kernel is expanded in powers of the perturbation:

$$K(s) = \sum_{n=0}^{\infty} K_n(s) \tag{1.22}$$

where $K_n(s)$ is a term of n-th power in the perturbation. For the simple case without gauge fields and flat space-time the perturbation is just a potential term $\hat{P}$. In this case a closed form of $K_n(s)$ for any n can be easily written down [23].

To set up the perturbation theory for covariant calculations we need to introduce splitting of the metric and the covariant derivative into auxiliary parts and perturbations:

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} + h^{\mu\nu}, \tag{1.23}$$

$$\nabla_\mu \varphi = \widetilde{\nabla}_\mu \varphi + \hat{\Gamma}_\mu \varphi. \tag{1.24}$$

The auxiliary metric and derivative are taken to be flat, i.e.

$$R_{\mu\nu\alpha\beta}(\tilde{g}) = 0, \tag{1.25}$$

$$[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu]\delta\varphi = 0. \tag{1.26}$$

Here and below the notation $[,]$ is a commutator introduced in (1.11).

Now there are three independent perturbations, one each for the metric, the connection and the potential

$$h^{\mu\nu}, \qquad \hat{\Gamma}_\mu, \qquad \hat{P} - \frac{1}{6}R\hat{1}. \tag{1.27}$$

In terms of these perturbations the differential operator (1.6) is divided as follows,

$$\hat{F}(\nabla) = \widetilde{\Box} + V(\widetilde{\nabla}), \tag{1.28}$$

with the perturbation

$$V(\widetilde{\nabla}) = h^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu + 2\hat{\Gamma}^\mu\widetilde{\nabla}_\mu - \frac{1}{6}R\hat{1}. \tag{1.29}$$

Generally raising and lowering indices are done with help of the auxiliary metric $\tilde{g}^{\mu\nu}$, with the exception that in the case of (1.29)

$$\hat{\Gamma}^\mu \equiv (\tilde{g}^{\mu\nu} + h^{\mu\nu})\hat{\Gamma}_\nu. \tag{1.30}$$

Perturbative solution to (1.1) can be found iteratively and has the general form,

$$K_n(s) = \int_0^s dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 \mathrm{e}^{(s-t_n)\widetilde{\Box}} V \mathrm{e}^{(t_n - t_{n-1})\widetilde{\Box}} V \ldots \mathrm{e}^{(t_2-t_1)\widetilde{\Box}} V \mathrm{e}^{t_1\widetilde{\Box}}, \tag{1.31}$$

which is in fact merely the Dyson series [53, 8, 24]. An exact solution for the zeroth order of the heat kernel $\mathrm{e}^{s\widetilde{\Box}}$ is known [12, 9],

$$K_0(s|x,y) = \frac{1}{(4\pi s)^\omega}\tilde{g}^{1/4}(x)\tilde{g}^{1/4}(y)\mathrm{e}^{-\frac{\tilde{\sigma}(x,y)}{2s}}\tilde{a}_0(x,y). \tag{1.32}$$

Here $\tilde{a}_0$ is a parallel transport operator along the geodetic connecting y to x [12], $\tilde{g}$ is the determinant of the auxiliary metric $\tilde{g}^{\mu\nu}$, a nd $\tilde{\sigma}$ is an auxiliary world function which in the Cartesian coordinates is simply $(x-y)^2/2$.

With use of (1.32) it is possible to find the result of acting the perturbation (1.29) onto the kernel of $e^{s\widetilde{\Box}}$,

$$V(y_i)K_0(s_i|y_i, y_{i+1}) = \frac{\widetilde{g}^{1/4}(y_i)\widetilde{g}^{1/4}(y_{i+1})}{(4\pi s_i)^{\omega}}\widetilde{a}_0(y_i, x) \times$$
$$\left\{\exp\left[-\frac{\widetilde{\sigma}(y_i, y_{i+1})}{2s_i} - \widetilde{\sigma}_i^{\mu}\widetilde{\nabla}_{\mu}\right]\hat{U}(x|\widetilde{\sigma}_{\alpha}^{i+1} - \widetilde{\sigma}_{\alpha}^{i}, s_i)\right\}\widetilde{a}_0(x, y_{i+1}), \tag{1.33}$$

where $\widetilde{\nabla}_{\mu}$ in the exponent comes from the covariant Taylor series [11] and acts only on the first argument of $\hat{U}$, and $\widetilde{\sigma}_i^{\mu}$ are vectors at x defined as

$$\widetilde{\sigma}_i^{\mu} \equiv \widetilde{g}^{\mu\nu}(x)\widetilde{\sigma}_{\nu}^{i}, \quad \widetilde{\sigma}_{\mu}^{i} \equiv \widetilde{\nabla}_{\nu}\widetilde{\sigma}(x, y_i), \tag{1.34}$$

and

$$\begin{aligned}\hat{U}(x|\xi_{\alpha}, s) \;&=\; \hat{1}\left[-\frac{1}{2s}h^{\mu\nu}(x)\widetilde{g}_{\mu\nu}(x) + \frac{1}{4s^2}h^{\mu\nu}(x)\xi_{\mu}\xi_{\nu}\right] \\ &\quad - \frac{1}{s}\xi_{\mu}\hat{\Gamma}^{\mu}(x) + \hat{P}(x) - \frac{1}{6}R(x)\hat{1}.\end{aligned} \tag{1.35}$$

Now one has to make the change of variables that transform (1.31) into

$$\mathrm{Tr}\,K_n(s) = \int_0^{\infty} ds_1 \ldots \int_0^{\infty} ds_n \delta\Big(1 - \sum_{i=1}^{n} s_i\Big)\int_0^{s_n} dt \times$$
$$\int dx \int dy_1 \ldots dy_n \mathrm{tr}\Big\{K_0(t|x, y_1)V(y_1)K_0(s_1|y_1, y_2)\ldots$$
$$V(y_{n-1})K_0(s_{n-1}|y_{n-1}, y_n)V(y_n)K_0(s_{n-t}|y_n, x)\Big\}. \tag{1.36}$$

After the integrals over x and t are done, and upon another change of variables, we get

$$\mathrm{Tr}\,K_n(s) = \frac{s}{n}\int_{\alpha_i \geq 0} d^n\alpha\,\delta\Big(1 - \sum_{i=1}^{n}\alpha_i\Big)\mathrm{Tr}\Big\{Ve^{s\alpha_1\widetilde{\Box}}\ldots Ve^{s\alpha_n\widetilde{\Box}}\Big\}. \tag{1.37}$$

To do the rest of spacetime integrals one should put $y_1 = x$ and make the change of variables again, this time for spacetime variables,

$$y_i^{\mu} \rightarrow \widetilde{\sigma}_i^{\mu}, \quad \left|\frac{\partial y_i}{\partial \widetilde{\sigma}_i}\right| = \widetilde{g}^{1/2}(x)\widetilde{g}^{-1/2}(y_i). \tag{1.38}$$

Thus, this expression eventually reduces to the Gaussian integrals over $\widetilde{\sigma}^{\mu}$ [30].

When calculated by the algorithm above, the trace of $K_n(s)$ is obtained in the generic form [30],

$$\begin{aligned}\mathrm{Tr}K_n(s) &= \frac{1}{(4\pi s)^\omega}\frac{1}{n}\int dx\,\tilde{g}^{1/2}(x)\int_{\alpha_i\geq 0} d^n\alpha\,\delta(1-\sum_1^n\alpha_i)\times \\ &\quad \mathrm{tr}\Big\{\exp\Big[s\Omega_n(\alpha_1,\ldots\alpha_n|\widetilde{\nabla}^i)\Big]\sum_{l=0}^n s^l\hat{B}_n^l(\alpha_1,\ldots\alpha_n|x_i)\Big\}\Big|_{x_i=x}. \end{aligned} \tag{1.39}$$

With the notation

$$\int_{\alpha_i\geq 0} d^n\alpha\,\delta(1-\sum_1^n\alpha_i)f(\alpha_1,\ldots\alpha_n|x_i)\Big|_{x_i=x} = \langle f\rangle_n, \tag{1.40}$$

this can be rewritten

$$\mathrm{Tr}K_n(s) = \frac{1}{(4\pi s)^\omega}\frac{1}{n}\int dx\,\tilde{g}^{1/2}\sum_{l=0}^n s^l\mathrm{tr}\langle \mathrm{e}^{s\Omega_n}\hat{B}_n^l\rangle_n. \tag{1.41}$$

Here $\Omega_n(\alpha_1,\ldots\alpha_n|\widetilde{\nabla}^i)$ is an operator of second order in $\widetilde{\nabla}^i$, and $\widetilde{\nabla}^i$ acts on the perturbation number i contained in $\hat{B}_n^l$. Each term in $\hat{B}_n^l(\alpha_1,\ldots\alpha_n|x_i)$ where i ranges from 1 to n is a product of n perturbations (1.27) at the points $x_1,\ldots x_n$ respectively, and the label i on a perturbation means that the perturbation is at the point x_i, e.g $\hat{P}_1 = \hat{P}(x_1)$. After the action of $\widetilde{\nabla}^i$, all points x_i are made coincident with the integration point x in (1.39) or (1.41). The simplification following from an integration over x by parts is then expressed by the identity

$$\sum_{i=1}^n\widetilde{\nabla}^i = 0 \tag{1.42}$$

which is used to put third order form factors into the representation given in the next section.

Finally one should make the series (1.22) manifestly covariant. This means we need to replace the perturbations (1.27) and the auxiliary metric and derivative (1.23)–(1.24) by the respective covariant curvatures (1.12), metric and derivative. The calculations in covariant perturbation theory are always carried out with accuracy $\mathrm{O}[\Re^n]$, i.e. up to terms of n-th and higher power in the curvatures (1.12). It is worth noting that, since the calculations are covariant, any term in $g_{\mu\nu}$ is in fact of infinite power in the curvature, and $\mathrm{O}[\Re^n]$ means terms containing n or more curvatures *explicitly.*

The curvature expansions of the perturbations (1.27) can be obtained from eqs. (1.25)–(1.26). Their general solutions are integral equations which are solved by iterations in terms of $R_{\mu\nu\alpha\beta}$ and $\mathcal{R}_{\mu\nu}$ [30]:

$$\begin{aligned} h^{\alpha\beta} &= 2\frac{1}{\widetilde{\Box}}R^{\alpha\beta} - 4\frac{1}{\widetilde{\Box}}\Big\{\Big(\frac{1}{\widetilde{\Box}}R^{\mu\nu}\Big)\Big(\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\frac{1}{\widetilde{\Box}}R^{\alpha\beta}\Big) + 2\Big(\widetilde{\nabla}^{(\alpha}\frac{1}{\widetilde{\Box}}R^{\mu\nu}\Big)\Big(\widetilde{\nabla}_\mu\frac{1}{\widetilde{\Box}}R^{\beta)}_\nu\Big) \\ &- \Big(\widetilde{\nabla}^\mu\frac{1}{\widetilde{\Box}}R^{\nu\alpha}\Big)\Big(\widetilde{\nabla}_\mu\frac{1}{\widetilde{\Box}}R^\beta_\nu\Big) - \Big(\widetilde{\nabla}^\mu\frac{1}{\widetilde{\Box}}R^{\nu\alpha}\Big)\Big(\widetilde{\nabla}_\nu\frac{1}{\widetilde{\Box}}R^\beta_\mu\Big) \\ &- \frac{1}{2}\Big(\widetilde{\nabla}^\alpha\frac{1}{\widetilde{\Box}}R^{\mu\nu}\Big)\Big(\widetilde{\nabla}^\beta\frac{1}{\widetilde{\Box}}R_{\mu\nu}\Big)\Big\} + \mathrm{O}[\Re^3], \end{aligned} \tag{1.43}$$

$$\begin{aligned} \hat{\Gamma}_\mu &= \widetilde{\nabla}^\nu\frac{1}{\widetilde{\Box}}\hat{\mathcal{R}}_{\nu\mu} - \frac{1}{\widetilde{\Box}}\Big\{\Big[\hat{\mathcal{R}}_{\mu\sigma},\Big(\widetilde{\nabla}_\lambda\hat{\mathcal{R}}^{\lambda\sigma}\Big)\Big] + 2\Big(\widetilde{\nabla}_\lambda\frac{1}{\widetilde{\Box}}\hat{\mathcal{R}}^{\lambda\sigma}\Big)\Big(\widetilde{\nabla}_\mu\widetilde{\nabla}^\alpha\frac{1}{\widetilde{\Box}}\hat{\mathcal{R}}_{\alpha\sigma}\Big) \\ &+ 2\widetilde{\nabla}_\mu\Big(\Big(\frac{1}{\widetilde{\Box}}R^{\alpha\beta}\Big)\widetilde{\nabla}_\alpha\widetilde{\nabla}^\lambda\frac{1}{\widetilde{\Box}}\hat{\mathcal{R}}_{\lambda\beta}\Big)\Big\} + \mathrm{O}[\Re^3]. \end{aligned} \tag{1.44}$$

The notation $1/\widetilde{\Box}$ is for the Green's function of the curvature-free Laplacian $\widetilde{\Box}$. Here and below the simplified notations for the Green's functions are used, i.e. tensor properties of them are assumed to be consistent with the functions on which they are acting. This fact is important when we make the transition from $\widetilde{\nabla}$ to ∇.

To lowest order in the curvature the behavior of perturbations at infinity is [30],

$$h^{\alpha\beta} = \mathrm{O}\Big(r^{-2\omega+2}\Big), \quad \hat{\Gamma}_\mu = \mathrm{O}\Big(r^{-2\omega+1}\Big), \tag{1.45}$$

where r is the geodetic distance from an arbitrary fixed point in the metric $\tilde{g}^{\mu\nu}$. This behavior remains unchanged at each order in the curvature. Since the first term in (1.29) is the leading term in the perturbation V at infinity, it behaves like

$$V = \mathrm{O}\Big(r^{-2\omega+2}\Big). \tag{1.46}$$

From this fact it follows that the integral (1.37) converges at spacetime dimensions $n > \omega/(\omega-1)$. Therefore, this expression can be used with caution and when it is not suitable eq. (1.36) should be used instead.

The expression of $\widetilde{\Box}$ through $\Box$ is generally of the form

$$\widetilde{\Box} = \Box + \mathcal{O}(\Re, \nabla) + \mathrm{O}[\Re^2] \tag{1.47}$$

where $\mathcal{O}(\Re, \nabla)$ is an operator containing the curvature linearly. For a scalar (e.g. R) this transition is as follows,

$$\tilde{\Box} X = \Box X - 2\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta X + \mathrm{O}[\Re^2], \tag{1.48}$$

For a matrix like $\hat{P}$,

$$\tilde{\Box}\hat{X} = \Box\hat{X} - 2\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta\hat{X} - 2\Big[\Big(\nabla_\alpha\frac{1}{\Box}\hat{\mathcal{R}}^{\alpha\beta}\Big), \nabla_\beta\hat{X}\Big] + \mathrm{O}[\Re^2], \tag{1.49}$$

And for a vector,

$$\begin{aligned}\tilde{\Box} X^\mu =\; & \Box X^\mu - 2\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta X^\mu + R^\mu_\alpha X^\alpha \\ & + 2\left(\nabla_\alpha\frac{1}{\Box}R^\mu_\beta + \nabla_\beta\frac{1}{\Box}R^\mu_\alpha - \nabla^\mu\frac{1}{\Box}R_{\alpha\beta}\right)\nabla^\alpha X^\beta + \mathrm{O}[\Re^2],\end{aligned} \tag{1.50}$$

The rules (1.48)–(1.50) can produce the transition formulae for the combined cases such as a symmetric tensor,

$$\begin{aligned}\tilde{\Box} X^{\mu\nu} =\; & \Box X^{\mu\nu} - 2\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta X^{\mu\nu} + 2R^{(\mu}_\alpha X^{\alpha\nu)} \\ & + 4\left(\nabla_\alpha\frac{1}{\Box}R^{(\mu}_\beta + \nabla_\beta\frac{1}{\Box}R^{(\mu}_\alpha - \nabla^{(\mu}\frac{1}{\Box}R_{\alpha\beta}\right)\nabla^\alpha X^{\beta\nu)} + \mathrm{O}[\Re^2],\end{aligned} \tag{1.51}$$

and a matrix-valued vector,

$$\begin{aligned}\tilde{\Box}\hat{X}^\mu =\; & \Box\hat{X}^\mu - 2\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta\hat{X}^\mu - 2\Big[\Big(\nabla_\alpha\frac{1}{\Box}\hat{\mathcal{R}}^{\alpha\beta}\Big), \nabla_\beta\hat{X}^\mu\Big] \\ & + 2\left(\nabla_\alpha\frac{1}{\Box}R^\mu_\beta + \nabla_\beta\frac{1}{\Box}R^\mu_\alpha - \nabla^\mu\frac{1}{\Box}R_{\alpha\beta}\right)\nabla^\alpha\hat{X}^\beta + R^\mu_\alpha\hat{X}^\alpha + \mathrm{O}[\Re^2]\end{aligned} \tag{1.52}$$

The symmetrization over indices $\mu\nu$ in (1.51) assumes the factor 1/2,

$$X^{(\mu}Y^{\nu)} = \frac{1}{2}(X^\mu Y^\nu + X^\nu Y^\mu). \tag{1.53}$$

As will be seen shortly we will need additional curvature expansion for both the heat kernel trace and the heat kernel. They are [30],

$$\begin{aligned}\tilde{\nabla}^\mu\tilde{\nabla}^\nu h^{\mu\nu} =\; & R - 2\Big(\nabla^\alpha\frac{1}{\Box}R^{\mu\nu}\Big)\Big(\nabla_\mu\frac{1}{\Box}R_{\alpha\nu}\Big) \\ & + \Big(\nabla^\alpha\frac{1}{\Box}R^{\mu\nu}\Big)\Big(\nabla_\alpha\frac{1}{\Box}R_{\mu\nu}\Big) + \Big(\nabla^\alpha\frac{1}{\Box}R\Big)\Big(\nabla_\alpha\frac{1}{\Box}R\Big) + \mathrm{O}[\Re^3],\end{aligned} \tag{1.54}$$

$$\tilde{\Box} h = 2R - 4\Big(\frac{1}{\Box}R^{\mu\nu}\Big)\Big(\nabla_\mu\nabla_\nu\frac{1}{\Box}R\Big) + 4R_{\mu\nu}\Big(\frac{1}{\Box}R^{\mu\nu}\Big)$$
$$- 4\Big(\nabla^\alpha\frac{1}{\Box}R^{\mu\nu}\Big)\Big(\nabla_\mu\frac{1}{\Box}R_{\alpha\nu}\Big) + 6\Big(\nabla^\alpha\frac{1}{\Box}R^{\mu\nu}\Big)\Big(\nabla_\alpha\frac{1}{\Box}R_{\mu\nu}\Big) + \mathrm{O}[\Re^3], \quad (1.55)$$

$$\widetilde{\nabla}_\mu\hat{\Gamma}^\mu = \Big(\nabla^\mu\frac{1}{\Box}R\Big)\Big(\frac{1}{\Box}\nabla^\nu\hat{\mathcal{R}}_{\nu\mu}\Big) - \Big(\frac{1}{\Box}\nabla^\alpha\hat{\mathcal{R}}_{\alpha\mu}\Big)\Big(\frac{1}{\Box}\nabla_\beta\hat{\mathcal{R}}^{\beta\mu}\Big) + \mathrm{O}[\Re^3]. \quad (1.56)$$

The Riemann tensor does not appear in the covariant perturbation theory [30]. It is expressed through the Ricci tensor in a nonlocal way once the boundary conditions for the gravitational field are specified. Obtaining this expression amounts to solving iteratively a differentiated Bianchi identity:

$$\Box R^{\alpha\beta\mu\nu} \equiv \frac{1}{2}\Big(\nabla^\mu\nabla^\alpha R^{\nu\beta} + \nabla^\alpha\nabla^\mu R^{\nu\beta} - \nabla^\nu\nabla^\alpha R^{\mu\beta} - \nabla^\alpha\nabla^\nu R^{\mu\beta}$$
$$- \nabla^\mu\nabla^\beta R^{\nu\alpha} - \nabla^\beta\nabla^\mu R^{\nu\alpha} + \nabla^\nu\nabla^\beta R^{\mu\alpha} + \nabla^\beta\nabla^\nu R^{\mu\alpha}\Big)$$
$$+ R^{[\mu}_\lambda R^{\nu]\lambda\beta\alpha} - R^{[\alpha}_\lambda R^{\beta]\lambda\mu\nu}$$
$$- 4R^{\alpha\ \ [\mu}_{\ \sigma\ \ \ \lambda}R^{\nu]\lambda\beta\sigma} - R^{\alpha\beta}_{\ \ \ \sigma\lambda}R^{\mu\nu\sigma\lambda}, \quad (1.57)$$

where the same 1/2 factor (1.53) is assigned to the index antisymmetrization. This accuracy of the lowest order in the curvature is sufficient for obtaining nonlocal relations between cubic invariants. However, for the discussion of the Schwinger-DeWitt coefficients in sect. 2.7, the solution for the Riemann tensor is needed with accuracy $\mathrm{O}[R^3_{..}]$. The needed expression is as follows [54, 34, 55]:

$$R^{\alpha\beta\mu\nu} = \frac{1}{\Box}\Big\{\frac{1}{2}\Big(\nabla^\mu\nabla^\alpha R^{\nu\beta} + \nabla^\alpha\nabla^\mu R^{\nu\beta} - \nabla^\nu\nabla^\alpha R^{\mu\beta} - \nabla^\alpha\nabla^\nu R^{\mu\beta}$$
$$- \nabla^\mu\nabla^\beta R^{\nu\alpha} - \nabla^\beta\nabla^\mu R^{\nu\alpha} + \nabla^\nu\nabla^\beta R^{\mu\alpha} + \nabla^\beta\nabla^\nu R^{\mu\alpha}\Big)$$
$$+ 2R^{[\mu}_\lambda\Big(\nabla^\lambda\nabla^{[\alpha}\frac{1}{\Box}R^{\beta]\nu]}\Big) + 2R^{[\alpha}_\lambda\Big(\nabla^\lambda\nabla^{[\mu}\frac{1}{\Box}R^{\beta]\nu]}\Big)$$
$$- 2R^{[\mu}_\lambda\Big(\nabla^{\nu]}\nabla^{[\alpha}\frac{1}{\Box}R^{\beta]\lambda}\Big) - 2R^{[\alpha}_\lambda\Big(\nabla^{\beta]}\nabla^{[\mu}\frac{1}{\Box}R^{\nu]\lambda}\Big)$$
$$- 8\Big(\nabla^\lambda\nabla^{[\alpha}\frac{1}{\Box}R^{\beta]}_\sigma\Big)\Big(\nabla_\lambda\nabla^{[\mu}\frac{1}{\Box}R^{\nu]\sigma}\Big) - 8\Big(\nabla^\lambda\nabla^{[\alpha}\frac{1}{\Box}R^{[\mu}_\sigma\Big)\Big(\nabla_\lambda\nabla^{\beta]}\frac{1}{\Box}R^{\nu]\sigma}\Big)$$
$$- 8\Big(\nabla^\lambda\nabla^{[\mu}\frac{1}{\Box}R^{[\alpha}_\sigma\Big)\Big(\nabla_\lambda\nabla^{\nu]}\frac{1}{\Box}R^{\beta]\sigma}\Big) + 8\Big(\nabla_\lambda\nabla^{[\alpha}\frac{1}{\Box}R^{\beta]}_\sigma\Big)\Big(\nabla^\sigma\nabla^{[\mu}\frac{1}{\Box}R^{\nu]\lambda}\Big)$$
$$- 8\Big(\nabla^\sigma\nabla^{[\mu}\frac{1}{\Box}R^{[\alpha}_\lambda\Big)\Big(\nabla^\lambda\nabla^{\beta]}\frac{1}{\Box}R^{\nu]}_\sigma\Big) + 8\Big(\nabla^\lambda\nabla^\sigma\frac{1}{\Box}R^{[\mu[\alpha}\Big)\Big(\nabla_\lambda\nabla^{\beta]}\frac{1}{\Box}R^{\nu]}_\sigma\Big)$$

$$
\begin{aligned}
&+8\Big(\nabla^\lambda\nabla^\sigma\frac{1}{\Box}R^{[\mu[\alpha}\Big)\Big(\nabla_\lambda\nabla^{\nu]}\frac{1}{\Box}R^{\beta]}_\sigma\Big)+8\Big(\nabla_\lambda\nabla^{[\alpha}\frac{1}{\Box}R^{[\mu}_\sigma\Big)\Big(\nabla^{\nu]}\nabla^{\beta]}\frac{1}{\Box}R^{\lambda\sigma}\Big)\\
&+8\Big(\nabla_\lambda\nabla^{[\mu}\frac{1}{\Box}R^{[\alpha}_\sigma\Big)\Big(\nabla^{\nu]}\nabla^{\beta]}\frac{1}{\Box}R^{\lambda\sigma}\Big)-8\Big(\nabla_\lambda\nabla_\sigma\frac{1}{\Box}R^{[\mu[\alpha}\Big)\Big(\nabla^{\nu]}\nabla^{\beta]}\frac{1}{\Box}R^{\lambda\sigma}\Big)\\
&-8\Big(\nabla_\lambda\nabla_\sigma\frac{1}{\Box}R^{[\mu[\alpha}\Big)\Big(\nabla^\lambda\nabla^\sigma\frac{1}{\Box}R^{\nu]\beta]}\Big)-8\Big(\nabla^{[\mu}\nabla^{[\alpha}\frac{1}{\Box}R_{\lambda\sigma}\Big)\Big(\nabla^{\nu]}\nabla^{\beta]}\frac{1}{\Box}R^{\lambda\sigma}\Big)\Big\}\\
&+\mathrm{O}[R^3_{..}].
\end{aligned}
\tag{1.58}
$$

Here the antisymmetrizations on the right hand side are with respect to $\mu\nu$ and $\alpha\beta$. In this equation the Ricci tensor plays the role of a source which determines the Riemann tensor up to initial or boundary conditions for the operator $\Box$. In the case of positive signature asymptotically flat spaces, the iterational solution is uniquely determined by the Green function $1/\Box$ with zero boundary conditions at infinity.

Chapter 2

The trace of the heat kernel in the third order

2.1 Third order of perturbation theory for the trace of the heat kernel

In this section we directly apply methods of the covariant perturbation theory outlined in the Introduction for computation of the trace of the heat kernel up the third order in perturbations.

We begin with quoting the results of [30] for $n = 1$ and $n = 2$ in (1.39). For $n = 1$ (note that total derivatives can not be discarded here),

$$\Omega_1 = 0, \tag{2.1}$$

$$\hat{B}_1^0 = -\frac{1}{2} h \hat{1}, \tag{2.2}$$

$$\hat{B}_1^1 = \frac{1}{3} \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu h^{\mu\nu} \hat{1} - \frac{1}{12} \tilde{\Box} h \hat{1} - \widetilde{\nabla}_\mu \hat{\Gamma}^\mu + \hat{P} - \frac{1}{6} R \hat{1}. \tag{2.3}$$

Here and below

$$h = h^{\mu\nu} \tilde{g}_{\mu\nu}, \qquad \tilde{\Box} = \tilde{g}^{\mu\nu} \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu, \tag{2.4}$$

and the indices of $\widetilde{\nabla}_\mu$ and the perturbations are raised and lowered with the flat metric $\tilde{g}_{\mu\nu}$ with the exception noted in (1.30).

For $n = 2$,

$$\Omega_2(\alpha_1,\alpha_2|\widetilde{\nabla}^i) = \alpha_1\alpha_2\widetilde{\Box}_2, \tag{2.5}$$

$$\hat{B}_2^0(\alpha_1,\alpha_2|x_i) = \Big(\frac{1}{4}h_1h_2 + \frac{1}{2}h_{1\mu\nu}h_2^{\mu\nu}\Big)\hat{1}, \tag{2.6}$$

$$\begin{aligned}\hat{B}_2^1(\alpha_1,\alpha_2|x_i) &= -\alpha_1^2(\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu h_1^{\mu\nu})h_2\hat{1}\\ &- 2\alpha_1\alpha_2(\widetilde{\nabla}_\nu h_1^{\nu\mu})\widetilde{g}_{\mu\alpha}(\widetilde{\nabla}_\beta h_2^{\beta\alpha})\hat{1} - 2\hat{\Gamma}_1^\mu\widetilde{g}_{\mu\nu}\hat{\Gamma}_2^\nu + 2\alpha_2 h_1(\widetilde{\nabla}_\mu\hat{\Gamma}_2^\mu)\\ &+ 4\alpha_1(\widetilde{\nabla}_\mu h_1^{\mu\nu})\widetilde{g}_{\nu\alpha}\hat{\Gamma}_2^\alpha - h_1\Big(\hat{P}_2 - \frac{1}{6}R_2\hat{1}\Big),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\hat{B}_2^2(\alpha_1,\alpha_2|x_i) &= \hat{1}(\alpha_1\alpha_2)^2(\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu h_1^{\mu\nu})(\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta h_2^{\alpha\beta})\\ &+ 4\alpha_1\alpha_2(\widetilde{\nabla}_\mu\hat{\Gamma}_1^\mu)(\widetilde{\nabla}_\nu\hat{\Gamma}_2^\nu) - 4\alpha_1^2\alpha_2(\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu h_1^{\mu\nu})(\widetilde{\nabla}_\alpha\hat{\Gamma}_2^\alpha)\\ &+ 2\alpha_1^2(\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu h_1^{\mu\nu})\Big(\hat{P}_2 - \frac{1}{6}R_2\hat{1}\Big)\\ &- 4\alpha_2\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)(\widetilde{\nabla}_\mu\hat{\Gamma}_2^\mu) + \Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\Big(\hat{P}_2 - \frac{1}{6}R_2\hat{1}\Big).\end{aligned} \tag{2.8}$$

A calculation by the algorithm described in the Introduction gives for $n = 3$

$$\Omega_3(\alpha_1,\alpha_2,\alpha_3|\widetilde{\nabla}^i) = \alpha_2\alpha_3\widetilde{\Box}_1 + \alpha_1\alpha_3\widetilde{\Box}_2 + \alpha_1\alpha_2\widetilde{\Box}_3, \tag{2.9}$$

$$\hat{B}_3^0(\alpha_1,\alpha_2,\alpha_3|x_i) = -\hat{1}\Big(\frac{1}{8}h_1h_2h_3 + \frac{3}{4}h_1h_2^{\mu\nu}h_{3\mu\nu} + h_{1\mu}^{\alpha}h_{2\nu}^{\mu}h_{3\alpha}^{\nu}\Big), \tag{2.10}$$

$$\begin{aligned}&\hat{B}_3^1(\alpha_1,\alpha_2,\alpha_3|x_i) =\\ &\quad\hat{1}\left[3(\widetilde{g}_{\alpha\beta}\widetilde{g}_{\mu\nu} + 2\widetilde{g}_{\mu\alpha}\widetilde{g}_{\nu\beta})\Big(D_\lambda^1D_\sigma^2 + \frac{1}{4}D_\lambda^2D_\sigma^2\Big)\right]h_1^{\alpha\beta}h_2^{\mu\nu}h_3^{\lambda\sigma}\\ &\quad - 3\Big(\widetilde{g}^{(2)}_{\alpha\nu\lambda\sigma}D_\mu^1 + \widetilde{g}^{(2)}_{\mu\nu\alpha\sigma}D_\lambda^2 + \frac{1}{2}\widetilde{g}^{(2)}_{\mu\nu\lambda\sigma}D_\alpha^3\Big)\hat{\Gamma}_1^\alpha h_2^{\mu\nu}h_3^{\lambda\sigma}\\ &\quad + 3(\widetilde{g}_{\alpha\beta}\hat{\Gamma}_1^\alpha\hat{\Gamma}_2^\beta h_3 + 2\hat{\Gamma}_1^\alpha\hat{\Gamma}_2^\beta h_{3\alpha\beta}) + \frac{3}{4}\widetilde{g}^{(2)}_{\mu\nu\alpha\beta}\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)h_2^{\mu\nu}h_3^{\alpha\beta},\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\hat{B}_3^2(\alpha_1,\alpha_2,\alpha_3|x_i) =\\ &\quad - \hat{1}\Big(\frac{3}{2}\widetilde{g}_{\alpha\beta}D_\mu^1D_\nu^1D_\lambda^2D_\sigma^2 + 6\widetilde{g}_{\mu\alpha}D_\beta^3D_\nu^1D_\lambda^2D_\sigma^2\Big)h_1^{\alpha\beta}h_2^{\mu\nu}h_3^{\lambda\sigma}\end{aligned}$$

$$
\begin{aligned}
&+ 12\widetilde{g}_{\alpha\beta}D^2_\mu\hat{\Gamma}^\alpha_1\hat{\Gamma}^\beta_2\hat{\Gamma}^\mu_3 \\
&+ (3\widetilde{g}_{\mu\nu}D^3_\alpha D^2_\lambda D^2_\sigma + 3\widetilde{g}_{\lambda\sigma}D^3_\alpha D^1_\mu D^1_\nu + 6\widetilde{g}_{\alpha\mu}D^1_\nu D^2_\lambda D^2_\sigma \\
&\quad + 6\widetilde{g}_{\alpha\lambda}D^1_\mu D^1_\nu D^2_\sigma + 12\widetilde{g}_{\mu\lambda}D^3_\alpha D^1_\nu D^2_\sigma)\hat{\Gamma}^\alpha_1 h^{\mu\nu}_2 h^{\lambda\sigma}_3 \\
&- (6\widetilde{g}_{\mu\nu}D^3_\alpha D^1_\beta + 12\widetilde{g}_{\mu\beta}D^3_\alpha D^2_\nu \\
&\quad + 6\widetilde{g}_{\alpha\beta}D^2_\mu D^2_\nu + 12\widetilde{g}_{\alpha\mu}D^1_\beta D^2_\nu)\hat{\Gamma}^\alpha_1\hat{\Gamma}^\beta_2 h^{\mu\nu}_3 \\
&- \frac{3}{2}\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\Big(\hat{P}_2 - \frac{1}{6}R_2\hat{1}\Big)h_3 \\
&+ (3\widetilde{g}_{\mu\nu}D^1_\alpha + 6\widetilde{g}_{\mu\alpha}D^2_\nu - 3\widetilde{g}_{\mu\nu}D^3_\alpha - 6\widetilde{g}_{\mu\alpha}D^1_\nu)\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\hat{\Gamma}^\alpha_2 h^{\mu\nu}_3 \\
&- 6\widetilde{g}_{\alpha\beta}\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\hat{\Gamma}^\alpha_2\hat{\Gamma}^\beta_3 \\
&- \frac{3}{2}(\widetilde{g}_{\mu\nu}D^2_\lambda D^2_\sigma + \widetilde{g}_{\lambda\sigma}D^1_\mu D^1_\nu + 4\widetilde{g}_{\mu\lambda}D^1_\nu D^2_\sigma)\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)h^{\mu\nu}_2 h^{\lambda\sigma}_3,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\hat{B}^3_3(\alpha_1,\alpha_2,\alpha_3|x_i) &= D^3_\alpha D^3_\beta D^1_\mu D^1_\nu D^2_\lambda D^2_\sigma h^{\alpha\beta}_1 h^{\mu\nu}_2 h^{\lambda\sigma}_3\hat{1} \\
&- 8D^3_\alpha D^1_\beta D^2_\mu\hat{\Gamma}^\alpha_1\hat{\Gamma}^\beta_2\hat{\Gamma}^\mu_3 - 6D^3_\alpha D^1_\mu D^1_\nu D^2_\lambda D^2_\sigma\hat{\Gamma}^\alpha_1 h^{\mu\nu}_2 h^{\lambda\sigma}_3 \\
&+ 12D^3_\alpha D^1_\beta D^2_\mu D^2_\nu\hat{\Gamma}^\alpha_1\hat{\Gamma}^\beta_2 h^{\mu\nu}_3 + \Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\Big(\hat{P}_2 - \frac{1}{6}R_2\hat{1}\Big)\Big(\hat{P}_3 - \frac{1}{6}R_3\hat{1}\Big) \\
&+ 3D^2_\alpha D^2_\beta\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\Big(\hat{P}_2 - \frac{1}{6}R_2\hat{1}\Big)h^{\alpha\beta}_3 \\
&- 6D^2_\mu\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\Big(\hat{P}_2 - \frac{1}{6}R_2\hat{1}\Big)\hat{\Gamma}^\mu_3 \\
&+ 6(D^3_\alpha D^1_\mu D^1_\nu - D^1_\alpha D^2_\mu D^2_\nu)\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\hat{\Gamma}^\alpha_2 h^{\mu\nu}_3 \\
&+ 12D^1_\alpha D^2_\beta\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)\hat{\Gamma}^\alpha_2\hat{\Gamma}^\beta_3 \\
&+ 3D^1_\mu D^1_\nu D^2_\lambda D^2_\sigma\Big(\hat{P}_1 - \frac{1}{6}R_1\hat{1}\Big)h^{\mu\nu}_2 h^{\lambda\sigma}_3,
\end{aligned}
\tag{2.13}
$$

where

$$D^1_\mu = \alpha_3\widetilde{\nabla}^2_\mu - \alpha_2\widetilde{\nabla}^3_\mu, \tag{2.14}$$

$$D^2_\mu = \alpha_1\widetilde{\nabla}^3_\mu - \alpha_3\widetilde{\nabla}^1_\mu, \tag{2.15}$$

$$D^3_\mu = \alpha_2\widetilde{\nabla}^1_\mu - \alpha_1\widetilde{\nabla}^2_\mu, \tag{2.16}$$

and

$$\tilde{g}^{(2)}_{\mu\nu\alpha\beta} = \tilde{g}_{\mu\alpha}\tilde{g}_{\nu\beta} + \tilde{g}_{\mu\beta}\tilde{g}_{\nu\alpha} + \tilde{g}_{\mu\nu}\tilde{g}_{\alpha\beta}. \tag{2.17}$$

The expression for $\mathrm{Tr}K(s)$ obtained at this stage is not explicitly covariant since it is formed in terms of the auxiliary metric and derivative and the perturbations. The task is to make it explicitly covariant, and this is done in the next section at the price of introducing the Green's functions of the Laplacian $\Box$.

2.2 The expansion of $\mathrm{Tr}K(s)$ in powers of the curvatures to third order

Our next step is replacing the perturbations $h^{\mu\nu}$ and $\hat{\Gamma}_\mu$ by their expressions through the curvatures, and eliminating the auxiliary quantities $\tilde{g}_{\alpha\beta}$ and $\widetilde{\nabla}_\alpha$. Since each perturbation $h^{\mu\nu}$ or $\hat{\Gamma}_\mu$ is an infinite series in the curvature, the expansion to a given order in the curvature involves all lower orders in perturbations. The iterational solutions for $h^{\mu\nu}$ and $\hat{\Gamma}_\mu$ are needed now up to the third order in the curvature. It is, of course, possible to work out these solutions to third order but we shall avoid this direct calculation by introducing an alternate method. Let us rewrite eq. (1.41) as follows:

$$\begin{aligned}
\mathrm{Tr}K(s) = & \frac{1}{(4\pi s)^\omega}\int dx\, \tilde{g}^{1/2}\,\mathrm{tr}\,\Big[\hat{1} + \hat{B}^0_1 + \frac{1}{2}\langle\hat{B}^0_2\rangle_2 + \frac{1}{3}\langle\hat{B}^0_3\rangle_3\Big]\\
& + \frac{s}{(4\pi s)^\omega}\int dx\, \tilde{g}^{1/2}\,\mathrm{tr}\,\Big[\hat{B}^1_1 + \frac{1}{2}\langle\Omega_2\hat{B}^0_2\rangle_2\\
&\qquad + \frac{1}{2}\langle\hat{B}^1_2\rangle_2 + \frac{1}{3}\langle\Omega_3\hat{B}^0_3\rangle_3 + \frac{1}{3}\langle\hat{B}^1_3\rangle_3\Big]\\
& + \frac{s^2}{2(4\pi s)^\omega}\int dx\, \tilde{g}^{1/2}\,\mathrm{tr}\,\Big[\Big\langle\frac{e^{s\Omega_2} - 1 - s\Omega_2}{s^2}\hat{B}^0_2\Big\rangle_2\\
&\qquad + \Big\langle\frac{e^{s\Omega_2} - 1}{s}\hat{B}^1_2\Big\rangle_2 + \langle e^{s\Omega_2}\hat{B}^2_2\rangle_2\Big]\\
& + \frac{s^3}{3(4\pi s)^\omega}\int dx\, \tilde{g}^{1/2}\,\mathrm{tr}\,\Big[\Big\langle\frac{e^{s\Omega_3} - 1 - s\Omega_3}{s^3}\hat{B}^0_3\Big\rangle_3\\
&\qquad + \Big\langle\frac{e^{s\Omega_3} - 1}{s^2}\hat{B}^1_3\Big\rangle_3 + \Big\langle\frac{e^{s\Omega_3}}{s}\hat{B}^2_3\Big\rangle_3\\
&\qquad + \langle e^{s\Omega_3}\hat{B}^3_3\rangle_3\Big] + \mathrm{O}[\Re^4].
\end{aligned} \tag{2.18}$$

The purpose of this decomposition is to single out the terms of zeroth and first order in the curvature which can only be local and coincident with the coefficients $\hat{a}_0(x,x)$ and $\hat{a}_1(x,x)$ of the Schwinger-DeWitt expansion (1.15). It is easy to observe the connection between covariant perturbation theory and the Schwinger-DeWitt technique. Comparison of the equation,

$$\mathrm{Tr}K(s) = \frac{1}{(4\pi s)^\omega}\int dx\,\tilde{g}^{1/2}\,\mathrm{tr}\left[\hat{1} + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=0}^{n} s^l\left\langle \mathrm{e}^{s\Omega_n}\hat{B}^l_n\right\rangle_n\right], \tag{2.19}$$

with the trace of (1.15) gives an expression for the Schwinger-DeWitt coefficients in terms of perturbation theory,

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_m(x,x)$$
$$= \int dx\,\tilde{g}^{1/2}\,\mathrm{tr}\left[\delta^0_m\hat{1} + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=0}^{\min(m,n)}\frac{1}{(m-l)!}\left\langle(\Omega)^{m-l}\hat{B}^l_n\right\rangle_n\right]. \tag{2.20}$$

In particular,

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_0(x,x) = \int dx\,\tilde{g}^{1/2}\,\mathrm{tr}\left[\hat{1} + \hat{B}^0_1 + \frac{1}{2}\langle\hat{B}^0_2\rangle_2 + \frac{1}{3}\langle\hat{B}^0_3\rangle_3\right] + \mathrm{O}[\Re^4], \tag{2.21}$$

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_1(x,x) = \int dx\,\tilde{g}^{1/2}\,\mathrm{tr}\left[\hat{B}^1_1 + \frac{1}{2}\langle\Omega_2\hat{B}^0_2\rangle_2 + \frac{1}{2}\langle\hat{B}^1_2\rangle_2 + \frac{1}{3}\langle\Omega_3\hat{B}^0_3\rangle_3 + \frac{1}{3}\langle\hat{B}^1_3\rangle_3\right] + \mathrm{O}[\Re^4], \tag{2.22}$$

and, therefore, for the first two integrals in (2.18) we can use the known exact results

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_0(x,x) = \int dx\,g^{1/2}\,\mathrm{tr}\,\hat{1}, \tag{2.23}$$

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_1(x,x) = \int dx\,g^{1/2}\,\mathrm{tr}\,\hat{P}. \tag{2.24}$$

It was shown in [30] that the direct substitution of nonlocal expansions of sect. 1.2 into eqs. (2.21) and (2.22) gives the indentities (2.23) and (2.24) with the accuracy $\mathrm{O}[\Re^3]$. To third order in the curvature such a check would require a knowledge of the iterational solutions for $h^{\mu\nu}$ and $\hat{\Gamma}_\mu$ to third order. Instead of deriving these solutions, we shall take eqs. (2.23) and (2.24) as established. Then, after elimination of the first two integrals in expression (2.18), the remaining terms in this expression are already of second and third order in perturbations, and, therefore, the knowledge of $h^{\mu\nu}$ and $\hat{\Gamma}_\mu$ to second order in the curvature will suffice for their calculation.

Consider now the third integral in (2.18) which involves terms of second order in perturbations: $\hat{B}_2^0, \hat{B}_2^1, \hat{B}_2^2$. It is straightforward to substitute in (2.6)–(2.8) the equations expressing $h^{\mu\nu}$ and $\hat{\Gamma}_\mu$ through $R_{\mu\nu}$ and $\hat{\mathcal{R}}_{\mu\nu}$ (1.43)–(1.44). One has to make use of eqs. (1.48)–(1.52) and (1.54)–(1.56). In the second order form factor itself

$$\mathrm{e}^{s\Omega_2} = \mathrm{e}^{s\alpha_1\alpha_2\widetilde{\Box}}, \tag{2.25}$$

when expressing $\widetilde{\Box}$ through $\Box$, the terms linear in the curvature should also be retained (1.47). With the use of their explicit forms given by eqs. (1.48)–(1.52) the expansion of the form factor $\mathrm{e}^{s\Omega_2}$ is as follows:

$$\begin{aligned}
&\int dx\, g^{1/2}\, \Re_1(\mathrm{e}^{a\widetilde{\Box}} - \mathrm{e}^{a\Box})\Re_2 \\
&\quad= \int dx\, g^{1/2}\, \Re_1 \int_0^a dt\, \mathrm{e}^{(a-t)\Box}\mathcal{O}(\Re, \nabla)\mathrm{e}^{t\Box}\Re_2 + \mathrm{O}[\Re^4] \\
&\quad= \int dx\, g^{1/2} \int_0^a dt\, \mathrm{e}^{(a-t)\Box_1 + t\Box_2}\mathcal{O}(\Re_3, \nabla_2)\Re_1\Re_2 + \mathrm{O}[\Re^4] \\
&\quad= \int dx\, g^{1/2}\, \frac{\mathrm{e}^{a\Box_2} - \mathrm{e}^{a\Box_1}}{\Box_2 - \Box_1}\mathcal{O}(\Re_3, \nabla_2)\Re_1\Re_2 + \mathrm{O}[\Re^4]
\end{aligned} \tag{2.26}$$

with $a = s\alpha_1\alpha_2$. Here the numbers on the arguments of $\mathcal{O}(\Re_3, \nabla_2)$ mean that $\mathcal{O}$ as an operator acts on $\Re_2$, and the curvature that it contains has the index 3. Altogether, expression (2.26) represents a third order contribution with the form factor of a new type.

In this way, for the terms with the second order form factor in (2.18), we obtain the expansions up to third order:

$$\begin{aligned}
&\int dx\, \tilde{g}^{1/2}\, \mathrm{tr}\Big\langle \frac{\mathrm{e}^{s\Omega_2} - 1 - s\Omega_2}{s^2}\hat{B}_2^0 \Big\rangle_2 \\
&\quad= \int dx\, g^{1/2}\, \mathrm{tr}\Big\langle \mathcal{C}_2(R_1R_2 + 2R_1^{\mu\nu}R_{2\mu\nu})\hat{1}\Big\rangle_2 \\
&\quad+ \int dx\, g^{1/2}\, \mathrm{tr}\Big\langle\Big(-\frac{1}{\Box_2}\mathcal{C}_1 + 2\frac{\Box_1}{\Box_2\Box_3}\mathcal{C}_3 - s\mathcal{W}_{12} + \frac{\Box_1}{\Box_3}s\mathcal{W}_{12}\Big)\, R_1R_2R_3\hat{1}
\end{aligned}$$

$$
\begin{aligned}
&+4\frac{\Box_1}{\Box_2\Box_3}\mathcal{C}_1 R^{\mu}_{1\,\alpha} R^{\alpha}_{2\,\beta} R^{\beta}_{3\,\mu}\hat{1}\\
&+\Big(3\frac{\Box_3}{\Box_1\Box_2}\mathcal{C}_3-2\frac{1}{\Box_1}\mathcal{C}_3+2\frac{\Box_1}{\Box_2\Box_3}\mathcal{C}_1-2\frac{1}{\Box_2}\mathcal{C}_1-s\mathcal{W}_{12}\Big)R_1^{\mu\nu}R_{2\,\mu\nu}R_3\hat{1}\\
&+\Big(-4\frac{1}{\Box_2\Box_3}\mathcal{C}_1-4\frac{1}{\Box_3}s\mathcal{W}_{12}+2\frac{1}{\Box_1}s\mathcal{W}_{23}\Big)R_1^{\alpha\beta}\nabla_\alpha R_2\nabla_\beta R_3\hat{1}\\
&+\Big(8\frac{1}{\Box_1\Box_3}\mathcal{C}_2-4\frac{1}{\Box_1\Box_2}\mathcal{C}_3+4\frac{1}{\Box_3}s\mathcal{W}_{12}\Big)\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\,\mu\alpha}R_3\hat{1}\\
&+\Big(8\frac{1}{\Box_1\Box_2}\mathcal{C}_3+4\frac{1}{\Box_2\Box_3}\mathcal{C}_1+4\frac{1}{\Box_1}s\mathcal{W}_{23}\Big)R_1^{\mu\nu}\nabla_\mu R_2^{\alpha\beta}\nabla_\nu R_{3\,\alpha\beta}\hat{1}\\
&+\Big(16\frac{1}{\Box_1\Box_3}\mathcal{C}_2+8\frac{1}{\Box_2\Box_3}\mathcal{C}_1+16\frac{1}{\Box_3}s\mathcal{W}_{12}\Big)R_1^{\mu\nu}\nabla_\alpha R_{2\,\beta\mu}\nabla^\beta R^{\alpha}_{3\,\nu}\hat{1}\Big\rangle_2\\
&+\mathrm{O}[\Re^4],
\end{aligned}
\qquad (2.27)
$$

$$
\begin{aligned}
&\int dx\,\tilde{g}^{1/2}\,\mathrm{tr}\,\Big\langle\frac{\mathrm{e}^{s\Omega_2}-1}{s}\hat{B}_2^1\Big\rangle_2\\
&\quad=\int dx\,g^{1/2}\,\mathrm{tr}\Big\langle\mathcal{B}_2\Big[\hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\mu\nu}\\
&\qquad\quad+\Big(2\alpha_1\alpha_2-2\alpha_1^2+\frac{1}{3}\Big)R_1R_2\hat{1}-2R_1\hat{P}_2\Big]\Big\rangle_2\\
&\quad+\int dx\,g^{1/2}\,\mathrm{tr}\,\Big\langle\Big(-2\frac{1}{\Box_2}\mathcal{B}_1+2\frac{1}{\Box_1}\mathcal{B}_3-\frac{\Box_3}{\Box_1\Box_2}\mathcal{B}_3\\
&\qquad\quad+s\mathcal{V}_{23}+\frac{\Box_2}{\Box_1}s\mathcal{V}_{23}-\frac{\Box_3}{\Box_1}s\mathcal{V}_{23}\Big)R_1R_2\hat{P}_3\\
&\qquad+\Big(2\frac{1}{\Box_1}\mathcal{B}_3-3\frac{\Box_3}{\Box_1\Box_2}\mathcal{B}_3\Big)R_1^{\mu\nu}R_{2\,\mu\nu}\hat{P}_3\\
&\qquad+\Big(\frac{1}{\Box_1}\mathcal{B}_2-\frac{1}{2}s\mathcal{V}_{23}\Big)R_1\hat{\mathcal{R}}_2^{\mu\nu}\hat{\mathcal{R}}_{3\,\mu\nu}\\
&\qquad-4\frac{1}{\Box_1}\mathcal{B}_3R_1^{\alpha\beta}\hat{\mathcal{R}}_{2\,\alpha}{}^{\mu}\hat{\mathcal{R}}_{3\,\beta\mu}\\
&\qquad+\Big[\Big(2\alpha_1\alpha_2-2\alpha_1^2+\frac{1}{6}\Big)\frac{\Box_1}{\Box_2\Box_3}\mathcal{B}_1+2(\alpha_1^2-\alpha_1\alpha_2)\frac{1}{\Box_2}\mathcal{B}_1\\
&\qquad\quad-\Big(\alpha_1\alpha_2-\alpha_1^2+\frac{1}{6}\Big)s\mathcal{V}_{23}\Big]R_1R_2R_3\hat{1}\\
&\qquad+\Big[\Big(2\alpha_1\alpha_2-4\alpha_1^2+\frac{1}{2}\Big)\frac{\Box_3}{\Box_1\Box_2}\mathcal{B}_3\\
&\qquad\quad+\Big(4\alpha_1^2-4\alpha_1\alpha_2-\frac{1}{3}\Big)\frac{1}{\Box_1}\mathcal{B}_3\Big]R_1^{\mu\nu}R_{2\,\mu\nu}R_3\hat{1}\\
&\qquad+\Big(-4\frac{1}{\Box_1\Box_2}\mathcal{B}_3-4\frac{1}{\Box_1}s\mathcal{V}_{23}\Big)R_1^{\mu\nu}\nabla_\mu R_2\nabla_\nu\hat{P}_3\\
&\qquad-4\frac{1}{\Box_1\Box_2}\mathcal{B}_3\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\,\mu\alpha}\hat{P}_3\\
&\qquad-8\frac{1}{\Box_1\Box_3}\mathcal{B}_2R_{1\,\alpha\beta}\nabla_\mu\hat{\mathcal{R}}_2^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_3^{\nu\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\frac{1}{\Box_1} s\mathcal{V}_{23} R_1^{\alpha\beta} \nabla_\alpha \hat{\mathcal{R}}_2^{\mu\nu} \nabla_\beta \hat{\mathcal{R}}_{3\,\mu\nu} \\
&+ \Big[-\Big(4\alpha_1^2 - \frac{2}{3}\Big)\frac{1}{\Box_1\Box_2}\mathcal{B}_3 \\
&\qquad +\Big(4\alpha_1\alpha_2 - 4\alpha_1^2 + \frac{2}{3}\Big)\frac{1}{\Box_1} s\mathcal{V}_{23}\Big] R_1^{\alpha\beta}\nabla_\alpha R_2 \nabla_\beta R_3 \hat{1} \\
&+ \Big(8\alpha_1^2 - 8\alpha_1\alpha_2 - \frac{2}{3}\Big)\frac{1}{\Box_1\Box_2}\mathcal{B}_3 \nabla^\mu R_1^{\nu\alpha} \nabla_\nu R_{2\,\mu\alpha} R_3 \hat{1} \\
&- 4\frac{1}{\Box_1} s\mathcal{V}_{23} \hat{\mathcal{R}}_1^{\alpha\beta} \nabla_\alpha \hat{\mathcal{R}}_2^{\mu\nu} \nabla_\beta \hat{\mathcal{R}}_{3\mu\nu} \\
&+ 4\frac{1}{\Box_1\Box_2\Box_3}(\mathcal{B}_1 + \mathcal{B}_2)\nabla_\mu \nabla_\lambda \hat{\mathcal{R}}_1^{\lambda\nu} \nabla^\alpha \hat{\mathcal{R}}_{2\alpha\nu} \nabla_\sigma \hat{\mathcal{R}}_3^{\sigma\mu} \\
&+ 8\alpha_1 \frac{1}{\Box_2\Box_3}\mathcal{B}_1 \nabla^\alpha \hat{\mathcal{R}}_{1\alpha\beta} R_2^{\beta\mu} \nabla_\mu R_3 \\
&+4\frac{1}{\Box_1} s\mathcal{V}_{23}(\nabla_\alpha R_{1\beta\mu} - \nabla_\mu R_{1\beta\alpha})\nabla^\beta \hat{\mathcal{R}}_2^{\alpha\nu} \hat{\mathcal{R}}_{3\cdot\nu}^{\;\,\mu} \Big\rangle_2 + \mathrm{O}[\Re^4],
\end{aligned} \tag{2.28}
$$

$$
\begin{aligned}
&\int dx\, \tilde{g}^{1/2}\, \mathrm{tr}\, \Big\langle \mathrm{e}^{s\Omega_2} \hat{B}_2^2 \Big\rangle_2 \\
&\quad = \int dx\, g^{1/2}\, \mathrm{tr}\, \Big\langle \mathcal{A}_2 \Big[\hat{P}_1 \hat{P}_2 + \Big(2\alpha_1^2 - \frac{1}{3}\Big) R_1 \hat{P}_2 \\
&\qquad + \Big(\alpha_1^2\alpha_2^2 - \frac{1}{3}\alpha_1^2 + \frac{1}{36}\Big) R_1 R_2 \hat{1}\Big]\Big\rangle_2 \\
&\quad + \int dx\, g^{1/2}\, \mathrm{tr}\, \Big\langle \Big[\alpha_1^2 \frac{\Box_3}{\Box_1\Box_2}\mathcal{A}_3 + 2\alpha_1^2 \frac{1}{\Box_2}\mathcal{A}_1 - 2\alpha_1^2 \frac{1}{\Box_2}\mathcal{A}_3 \\
&\qquad -\frac{1}{3}\frac{1}{\Box_2}\mathcal{A}_1 + \Big(\alpha_1^2 - \frac{1}{6}\Big)\Big(\frac{\Box_3}{\Box_1} - \frac{\Box_2}{\Box_1} - 1\Big) s\,\mathcal{U}_{23}\Big] R_1 R_2 \hat{P}_3 \\
&\quad + \alpha_1^2 \Big(\frac{\Box_3}{\Box_1\Box_2}\mathcal{A}_3 - 2\frac{1}{\Box_1}\mathcal{A}_3\Big) R_1^{\mu\nu} R_{2\,\mu\nu} \hat{P}_3 + \frac{1}{\Box_3}\mathcal{A}_1 \hat{P}_1 \hat{P}_2 R_3 \\
&\quad + \Big[\Big(\alpha_1^2\alpha_2^2 - \frac{1}{6}\alpha_1^2\Big)\frac{\Box_1}{\Box_2\Box_3}\mathcal{A}_1 - \Big(\alpha_1^2\alpha_1^2 - \frac{1}{36}\Big)\frac{1}{\Box_2}\mathcal{A}_1 \\
&\qquad -\frac{1}{2}\Big(\alpha_1^2\alpha_2^2 - \frac{1}{3}\alpha_1^2 + \frac{1}{36}\Big) s\,\mathcal{U}_{23}\Big] R_1 R_2 R_3 \hat{1} \\
&\quad + \Big(\alpha_1^2\alpha_2^2 - \frac{1}{6}\alpha_1^2\Big)\Big(\frac{\Box_3}{\Box_1\Box_2}\mathcal{A}_3 - 2\frac{1}{\Box_1}\mathcal{A}_3\Big) R_1^{\mu\nu} R_{2\,\mu\nu} R_3 \hat{1} \\
&\quad + 4\alpha_2 \frac{1}{\Box_1\Box_2}\mathcal{A}_3 \nabla_\mu \hat{\mathcal{R}}_1^{\mu\alpha} \nabla^\nu \hat{\mathcal{R}}_{2\,\nu\alpha} \hat{P}_3 \\
&\quad + 4\Big(\alpha_1^2 - \frac{1}{6}\Big)\frac{1}{\Box_1} s\,\mathcal{U}_{23} R_1^{\mu\nu} \nabla_\mu R_2 \nabla_\nu \hat{P}_3 \\
&\quad - 4\alpha_1^2 \frac{1}{\Box_1\Box_2}\mathcal{A}_3 \nabla^\mu R_1^{\nu\alpha} \nabla_\nu R_{2\,\mu\alpha} \hat{P}_3 \\
&\quad - 2\frac{1}{\Box_1} s\,\mathcal{U}_{23} R_1^{\mu\nu} \nabla_\mu \nabla_\nu \hat{P}_2 \hat{P}_3 \\
&\quad + 4\alpha_2 \Big(\alpha_1^2 - \frac{1}{6}\Big)\frac{1}{\Box_2\Box_3}\mathcal{A}_1 R_1 \nabla_\alpha \hat{\mathcal{R}}_2^{\alpha\mu} \nabla^\beta \hat{\mathcal{R}}_{3\,\beta\mu}
\end{aligned}
$$

$$
\begin{aligned}
&+2\Big(\alpha_1^2\alpha_2^2-\frac{1}{3}\alpha_1^2+\frac{1}{36}\Big)\frac{1}{\Box_1}s\,\mathcal{U}_{23}R_1^{\alpha\beta}\nabla_\alpha R_2\nabla_\beta R_3\hat{1}\\
&-4\Big(\alpha_1^2\alpha_2^2-\frac{1}{6}\alpha_1^2\Big)\frac{1}{\Box_1\Box_2}\mathcal{A}_3\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\,\mu\alpha}R_3\hat{1}\\
&-4\alpha_2\Big(\alpha_1^2-\frac{1}{6}\Big)\frac{1}{\Box_1\Box_2}\mathcal{A}_3\nabla^\nu\hat{\mathcal{R}}_{1\nu\mu}\nabla^\mu R_2R_3\\
&-4\frac{1}{\Box_1}s\,\mathcal{U}_{23}\hat{\mathcal{R}}_1^{\mu\nu}\nabla_\mu\hat{P}_2\nabla_\nu\hat{P}_3\\
&-4\alpha_2\frac{1}{\Box_1\Box_2}\mathcal{A}_3\nabla^\nu\hat{\mathcal{R}}_{1\nu\mu}\nabla^\mu R_2\hat{P}_3\Big\rangle_2+\mathrm{O}[\Re^4]
\end{aligned}
\tag{2.29}
$$

where $(m,n=1,2,3;\ m\neq n)$

$$\mathcal{A}_m=\mathrm{e}^{s\alpha_1\alpha_2\Box_m},\tag{2.30}$$

$$\mathcal{B}_m=\frac{\mathrm{e}^{s\alpha_1\alpha_2\Box_m}-1}{s\Box_m},\tag{2.31}$$

$$\mathcal{C}_m=\frac{\mathrm{e}^{s\alpha_1\alpha_2\Box_m}-1-s\alpha_1\alpha_2\Box_m}{(s\Box_m)^2},\tag{2.32}$$

$$\mathcal{U}_{mn}=\frac{\mathrm{e}^{s\alpha_1\alpha_2\Box_m}-\mathrm{e}^{s\alpha_1\alpha_2\Box_n}}{s(\Box_m-\Box_n)},\tag{2.33}$$

$$\mathcal{V}_{mn}=\frac{1}{s(\Box_m-\Box_n)}\Big(\frac{\mathrm{e}^{s\alpha_1\alpha_2\Box_m}-1}{s\Box_m}-\frac{\mathrm{e}^{s\alpha_1\alpha_2\Box_n}-1}{s\Box_n}\Big),\tag{2.34}$$

$$
\begin{aligned}
\mathcal{W}_{mn}=\frac{1}{s(\Box_m-\Box_n)}\Big(&\frac{\mathrm{e}^{s\alpha_1\alpha_2\Box_m}-1-s\alpha_1\alpha_2\Box_m}{(s\Box_m)^2}\\
&-\frac{\mathrm{e}^{s\alpha_1\alpha_2\Box_n}-1-s\alpha_1\alpha_2\Box_n}{(s\Box_n)^2}\Big)
\end{aligned}
\tag{2.35}
$$

(the numbers on the form factors refer to the numbers on the boxes appearing in them), and the averaging $\langle\ \rangle_2$ is defined in (1.40).

Finally, the last integral in (2.18), with $\hat{B}_3^l$ is already of third order in perturbations. Therefore, it is sufficient to substitute in eqs. (2.9)–(2.13) the lowest order of the expressions (1.43)–(1.44),

$$h^{\mu\nu}=2\frac{1}{\Box}R^{\mu\nu}+\mathrm{O}[\Re^2],\tag{2.36}$$

$$\hat{\Gamma}_\mu = \nabla^\nu \frac{1}{\Box} \hat{\mathcal{R}}_{\nu\mu} + \mathrm{O}[\Re^2], \tag{2.37}$$

$$\widetilde{\nabla}_\mu = \nabla_\mu + \mathrm{O}[\Re], \qquad \tilde{\Box} = \Box + \mathrm{O}[\Re]. \tag{2.38}$$

We therefore do not reproduce here results of this straightforward substitution.

As distinct from the form factors (2.30)–(2.35) coming from the second order, the form factor appearing in the last terms of (2.18):

$$\mathrm{e}^{s\Omega_3} = \exp[s(\alpha_2\alpha_3\Box_1 + \alpha_1\alpha_3\Box_2 + \alpha_1\alpha_2\Box_3)] + \mathrm{O}[\Re] \tag{2.39}$$

is an irreducible nonlocal function of all the three boxes.

The final result of the curvature expansion is presented in the next section. An important problem of the basis of the nonlocal tensor invariants is discussed in Appendix A.

2.3 The α -polynomial representation of the form factors in the trace of the heat kernel

The final result of the calculations above is as follows [54, 56, 34],

$$\begin{aligned}
\mathrm{Tr}K(s) = & \frac{1}{(4\pi s)^\omega} \int dx\, g^{1/2}\, \mathrm{tr}\Big\{\hat{1} + s\hat{P} + s^2 \sum_{i=1}^{5} f_i(-s\Box_2)\Re_1\Re_2(i) \\
& + \left(s^3 \sum_{i=1}^{11} + s^4 \sum_{i=12}^{25} + s^5 \sum_{i=26}^{28}\right) F_i(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2\Re_3(i) \\
& + s^6 F_{29}(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2\Re_3(29) \\
& + \Big[s^4 \sum_{i=30}^{32} F_i(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2\Re_3(i) \\
& + s^5 F_{33}(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2\Re_3(33)\Big] + \mathrm{O}[\Re^4]\Big\}.
\end{aligned} \tag{2.40}$$

Here terms of zeroth, first and second order in the curvature reproduce the results of [30]. There are five quadratic structures $\Re_1\Re_2(i)$ with $i = 1$ to 5:

$$\Re_1\Re_2(1) = R_{1\,\mu\nu} R_2^{\mu\nu} \hat{1}, \tag{2.41}$$

$$\Re_1\Re_2(2) = R_1 R_2 \hat{1}, \tag{2.42}$$

$$\Re_1\Re_2(3){=}\hat{P}_1 R_2, \tag{2.43}$$

$$\Re_1\Re_2(4){=}\hat{P}_1 \hat{P}_2, \tag{2.44}$$

$$\Re_1\Re_2(5){=}\hat{\mathcal{R}}_{1\,\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu}. \tag{2.45}$$

Their contributions are of the form

$$\int dx\, g^{1/2}\, f(-s\Box_2)\Re_1\Re_2 = \int dx\, g^{1/2}\, \Re f(-s\Box)\Re, \tag{2.46}$$

and the notation on the left-hand side of (2.40) assumes that $\Box_2$ acts on $\Re_2$.

Terms of third order in the curvature $\Re_1\Re_2\Re_3(i)$ with $i = 1$ to 33 in (2.40) are given by a sum of contributions of thirty-three cubic structures. Eleven of them contain no derivatives

$$\Re_1\Re_2\Re_3(1){=}\hat{P}_1\hat{P}_2\hat{P}_3, \tag{2.47}$$

$$\Re_1\Re_2\Re_3(2){=}\hat{\mathcal{R}}_1{}^{\mu}{}_{\alpha}\hat{\mathcal{R}}_2{}^{\alpha}{}_{\beta}\hat{\mathcal{R}}_3{}^{\beta}{}_{\mu}, \tag{2.48}$$

$$\Re_1\Re_2\Re_3(3){=}\hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\,\mu\nu}\hat{P}_3, \tag{2.49}$$

$$\Re_1\Re_2\Re_3(4){=}R_1 R_2\hat{P}_3, \tag{2.50}$$

$$\Re_1\Re_2\Re_3(5){=}R_1^{\mu\nu}R_{2\,\mu\nu}\hat{P}_3, \tag{2.51}$$

$$\Re_1\Re_2\Re_3(6){=}\hat{P}_1\hat{P}_2 R_3, \tag{2.52}$$

$$\Re_1\Re_2\Re_3(7){=}R_1\hat{\mathcal{R}}_2^{\mu\nu}\hat{\mathcal{R}}_{3\,\mu\nu}, \tag{2.53}$$

$$\Re_1\Re_2\Re_3(8){=}R_1^{\alpha\beta}\hat{\mathcal{R}}_{2\,\alpha}{}^{\mu}\hat{\mathcal{R}}_{3\,\beta\mu}, \tag{2.54}$$

$$\Re_1\Re_2\Re_3(9)=R_1R_2R_3\hat{1}, \tag{2.55}$$

$$\Re_1\Re_2\Re_3(10)=R^{\mu}_{1\,\alpha}R^{\alpha}_{2\,\beta}R^{\beta}_{3\,\mu}\hat{1}, \tag{2.56}$$

$$\Re_1\Re_2\Re_3(11)=R^{\mu\nu}_1R_{2\,\mu\nu}R_3\hat{1}, \tag{2.57}$$

fourteen contain second derivatives

$$\Re_1\Re_2\Re_3(12)=\hat{\mathcal{R}}^{\alpha\beta}_1\nabla^{\mu}\hat{\mathcal{R}}_{2\mu\alpha}\nabla^{\nu}\hat{\mathcal{R}}_{3\nu\beta}, \tag{2.58}$$

$$\Re_1\Re_2\Re_3(13)=\hat{\mathcal{R}}^{\mu\nu}_1\nabla_{\mu}\hat{P}_2\nabla_{\nu}\hat{P}_3, \tag{2.59}$$

$$\Re_1\Re_2\Re_3(14)=\nabla_{\mu}\hat{\mathcal{R}}^{\mu\alpha}_1\nabla^{\nu}\hat{\mathcal{R}}_{2\,\nu\alpha}\hat{P}_3, \tag{2.60}$$

$$\Re_1\Re_2\Re_3(15)=R^{\mu\nu}_1\nabla_{\mu}R_2\nabla_{\nu}\hat{P}_3, \tag{2.61}$$

$$\Re_1\Re_2\Re_3(16)=\nabla^{\mu}R^{\nu\alpha}_1\nabla_{\nu}R_{2\,\mu\alpha}\hat{P}_3, \tag{2.62}$$

$$\Re_1\Re_2\Re_3(17)=R^{\mu\nu}_1\nabla_{\mu}\nabla_{\nu}\hat{P}_2\hat{P}_3, \tag{2.63}$$

$$\Re_1\Re_2\Re_3(18)=R_{1\,\alpha\beta}\nabla_{\mu}\hat{\mathcal{R}}^{\mu\alpha}_2\nabla_{\nu}\hat{\mathcal{R}}^{\nu\beta}_3, \tag{2.64}$$

$$\Re_1\Re_2\Re_3(19)=R^{\alpha\beta}_1\nabla_{\alpha}\hat{\mathcal{R}}^{\mu\nu}_2\nabla_{\beta}\hat{\mathcal{R}}_{3\,\mu\nu}, \tag{2.65}$$

$$\Re_1\Re_2\Re_3(20)=R_1\nabla_{\alpha}\hat{\mathcal{R}}^{\alpha\mu}_2\nabla^{\beta}\hat{\mathcal{R}}_{3\,\beta\mu}, \tag{2.66}$$

$$\Re_1\Re_2\Re_3(21)=R^{\mu\nu}_1\nabla_{\mu}\nabla_{\lambda}\hat{\mathcal{R}}^{\lambda\alpha}_2\hat{\mathcal{R}}_{3\,\alpha\nu}, \tag{2.67}$$

$$\Re_1\Re_2\Re_3(22)=R^{\alpha\beta}_1\nabla_{\alpha}R_2\nabla_{\beta}R_3\hat{1}, \tag{2.68}$$

$$\Re_1\Re_2\Re_3(23) = \nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\,\mu\alpha} R_3 \hat{1}, \quad (2.69)$$

$$\Re_1\Re_2\Re_3(24) = R_1^{\mu\nu}\nabla_\mu R_2^{\alpha\beta}\nabla_\nu R_{3\,\alpha\beta}\hat{1}, \quad (2.70)$$

$$\Re_1\Re_2\Re_3(25) = R_1^{\mu\nu}\nabla_\alpha R_{2\,\beta\mu}\nabla^\beta R_{3\,\nu}^{\alpha}\hat{1}, \quad (2.71)$$

three contain fourth derivatives

$$\Re_1\Re_2\Re_3(26) = \nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}\hat{P}_3, \quad (2.72)$$

$$\Re_1\Re_2\Re_3(27) = \nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta} R_3\hat{1}, \quad (2.73)$$

$$\Re_1\Re_2\Re_3(28) = \nabla_\mu R_1^{\alpha\lambda}\nabla_\nu R_{2\,\lambda}^{\beta}\nabla_\alpha\nabla_\beta R_3^{\mu\nu}\hat{1}, \quad (2.74)$$

and one contains sixth derivatives

$$\Re_1\Re_2\Re_3(29) = \nabla_\lambda\nabla_\sigma R_1^{\alpha\beta}\nabla_\alpha\nabla_\beta R_2^{\mu\nu}\nabla_\mu\nabla_\nu R_3^{\lambda\sigma}\hat{1}. \quad (2.75)$$

These twenty nine structures form a complete basis of nonlocal invariants of third order in the curvature. (Ten of them are purely gravitational, and with gravity switched off there are six.)

And there are four additional cubic structures linear in $\hat{\mathcal{R}}_{\mu\nu}$:

$$\Re_1\Re_2\Re_3(30) = \nabla_\beta\hat{\mathcal{R}}_1^{\beta\alpha}\nabla_\alpha R_2 R_3, \quad (2.76)$$

$$\Re_1\Re_2\Re_3(31) = \nabla_\mu\hat{\mathcal{R}}_1^{\mu\alpha} R_{2\,\alpha\beta}\nabla^\beta R_3, \quad (2.77)$$

$$\Re_1\Re_2\Re_3(32) = \hat{P}_1\nabla_\beta\hat{\mathcal{R}}_2^{\beta\alpha}\nabla_\alpha R_3, \quad (2.78)$$

$$\Re_1\Re_2\Re_3(33) = \nabla_\alpha\hat{\mathcal{R}}_1^{\alpha\beta}\nabla_\beta R_2^{\mu\nu}\nabla_\mu\nabla_\nu R_3. \quad (2.79)$$

They are not present in the final result as it will be shown in the next section. However even at this stage not all possible tensor invariants enter the heat kernel trace (see Appendix A).

Form factors

$$f_i(-s\Box_2), \qquad F_i(-s\Box_1, -s\Box_2, -s\Box_3) \tag{2.80}$$

are obtained as integrals over the parameters

$$\langle(\ldots)\rangle_2 = \int_{\alpha\geq 0} d\alpha_1\, d\alpha_2\, \delta(1-\alpha_1-\alpha_2)(\ldots), \tag{2.81}$$

$$\langle(\ldots)\rangle_3 = \int_{\alpha\geq 0} d\alpha_1\, d\alpha_2\, d\alpha_3\, \delta(1-\alpha_1-\alpha_2-\alpha_3)(\ldots), \tag{2.82}$$

and, in this form, are represented by two *nonlocal* kernels:

$$\exp(s\alpha_1\alpha_2\Box) \tag{2.83}$$

and

$$\exp(s\Omega), \qquad \Omega = \alpha_2\alpha_3\Box_1 + \alpha_1\alpha_3\Box_2 + \alpha_1\alpha_2\Box_3. \tag{2.84}$$

The function (2.83) appears in the combinations

$$\mathcal{A}, \quad \mathcal{B}, \quad \mathcal{C}, \quad \mathcal{U}, \quad \mathcal{V}, \quad \mathcal{W} \tag{2.85}$$

introduced in (2.30)–(2.35), and the function (2.84) appears in the combinations, cf.(2.18):

$$\mathrm{e}^{s\Omega}, \qquad \mathrm{e}^{s\Omega}-1, \qquad \mathrm{e}^{s\Omega}-1-s\Omega \tag{2.86}$$

which figure explicitly in the expressions below. The coefficients of these functions are polynomials in α 's, boxes, and inverse boxes. It is assumed that $\Box_1$ acts on the curvature with the label 1, $\Box_2$ acts on the curvature with the label 2, $\Box_3$ acts on the curvature with the label 3. There is no question about a commutativity of $\Box_m$ with ∇'s acting on $\Re_m$ in (2.58)–(2.75) since a contribution of any such commutator is already $\mathrm{O}[\Re^4]$.

In this representation, the second order form factors are of the form [30],

$$f_1 = \langle\mathcal{C}\rangle_2\,, \tag{2.87}$$

$$f_2=\left\langle\frac{1}{2}\left({\alpha_1}^2{\alpha_2}^2-\frac{1}{3}{\alpha_1}^2+\frac{1}{36}\right)\mathcal{A}\right.$$
$$\left.+\left(\alpha_1\alpha_2-{\alpha_1}^2+\frac{1}{6}\right)\mathcal{B}+\frac{1}{2}\mathcal{C}\right\rangle_2, \tag{2.88}$$

$$f_3=\left\langle\left({\alpha_1}^2-\frac{1}{6}\right)\mathcal{A}-\mathcal{B}\right\rangle_2, \tag{2.89}$$

$$f_4=\left\langle\frac{1}{2}\mathcal{A}\right\rangle_2, \tag{2.90}$$

$$f_5=\left\langle\frac{1}{2}\mathcal{B}\right\rangle_2, \tag{2.91}$$

and the results for the third order form factors are as follows [54]:

$$F_1=\left\langle\frac{1}{3}\mathrm{e}^{s\Omega}\right\rangle_3, \tag{2.92}$$

$$F_2=\left\langle\frac{4}{3}\alpha_1\alpha_2\alpha_3\mathrm{e}^{s\Omega}\right\rangle_3+\langle-2\mathcal{V}_{12}\rangle_2, \tag{2.93}$$

$$F_3=\left\langle 2\alpha_1\alpha_2\mathrm{e}^{s\Omega}\right\rangle_3, \tag{2.94}$$

$$F_4=\left\langle\frac{1}{\Box_1\Box_2}\frac{(\mathrm{e}^{s\Omega}-1)}{s^2}+\left[\frac{1}{\Box_1}\left(\frac{1}{3}-{\alpha_1}^2-{\alpha_2}^2+3\alpha_1\alpha_3+\alpha_2\alpha_3-{\alpha_3}^2\right)\right.\right.$$
$$\left.+\frac{\Box_3}{\Box_1\Box_2}(-3\alpha_2\alpha_3)\right]\frac{\mathrm{e}^{s\Omega}}{s}+\left[\left(\frac{1}{36}+\frac{1}{2}{\alpha_1}^4-\frac{1}{6}\alpha_1\alpha_2-\frac{1}{3}{\alpha_2}^2\right.\right.$$
$$\left.-\frac{1}{2}{\alpha_1}^2{\alpha_2}^2-\frac{1}{6}\alpha_2\alpha_3+{\alpha_2}^3\alpha_3-\frac{1}{6}{\alpha_3}^2+\alpha_1\alpha_2{\alpha_3}^2+\frac{3}{2}{\alpha_2}^2{\alpha_3}^2\right)$$
$$+\frac{\Box_2}{\Box_1}\left(-\frac{1}{2}{\alpha_1}^4-\frac{1}{6}\alpha_1\alpha_2-\frac{1}{2}{\alpha_1}^3\alpha_2-\frac{1}{2}{\alpha_1}^2{\alpha_2}^2+\frac{1}{2}\alpha_1{\alpha_2}^3\right.$$
$$\left.+\frac{1}{6}\alpha_1\alpha_3-{\alpha_1}^3\alpha_3-\frac{1}{2}{\alpha_1}^2\alpha_2\alpha_3+\frac{1}{2}\alpha_1{\alpha_2}^2\alpha_3+\frac{1}{2}{\alpha_1}^2{\alpha_3}^2+\alpha_1\alpha_2{\alpha_3}^2\right)$$
$$+\frac{\Box_3}{\Box_1}\left(\frac{1}{2}{\alpha_1}^4+\frac{1}{6}\alpha_1\alpha_2+{\alpha_1}^3\alpha_2+\frac{3}{2}{\alpha_1}^2{\alpha_2}^2-\frac{1}{6}\alpha_1\alpha_3\right.$$
$$\left.+{\alpha_1}^3\alpha_3+2{\alpha_1}^2\alpha_2\alpha_3+\alpha_1{\alpha_2}^2\alpha_3-\frac{1}{2}{\alpha_1}^2{\alpha_3}^2-\alpha_1\alpha_2{\alpha_3}^2\right)$$
$$\left.\left.+\frac{{\Box_3}^2}{\Box_1\Box_2}\left(-\frac{1}{2}{\alpha_1}^3\alpha_2-\frac{1}{2}{\alpha_1}^2{\alpha_2}^2-\frac{3}{2}{\alpha_1}^2\alpha_2\alpha_3\right)\right]\mathrm{e}^{s\Omega}\right\rangle_3$$
$$+\left\langle\frac{1}{\Box_1}\left(-\frac{1}{6}+{\alpha_1}^2\right)\frac{\mathcal{A}_2}{s}+\left[\frac{1}{\Box_1}(-{\alpha_1}^2)+\frac{\Box_3}{\Box_1\Box_2}\left(\frac{1}{2}{\alpha_1}^2\right)\right]\frac{\mathcal{A}_3}{s}\right.$$

$$
\begin{aligned}
&-\frac{1}{\Box_1}\frac{\mathcal{B}_2}{s}+\left(\frac{1}{\Box_1}-\frac{1}{2}\frac{\Box_3}{\Box_1\Box_2}\right)\frac{\mathcal{B}_3}{s}\\
&+\left[\frac{\Box_3}{\Box_1}\left(\frac{1}{2}\alpha_1^2-\frac{1}{12}\right)+\frac{\Box_2}{\Box_1}\left(-\frac{1}{2}\alpha_1^2+\frac{1}{12}\right)\right.\\
&\left.+\left(-\frac{1}{2}\alpha_1^2+\frac{1}{12}\right)\right]\mathcal{U}_{23}+\left(\frac{1}{2}+\frac{1}{2}\frac{\Box_2}{\Box_1}-\frac{1}{2}\frac{\Box_3}{\Box_1}\right)\mathcal{V}_{23}\Big\rangle_2,
\end{aligned}
\tag{2.95}
$$

$$
\begin{aligned}
F_5=&\left\langle 2\frac{1}{\Box_1\Box_2}\frac{(\mathrm{e}^{s\Omega}-1)}{s^2}\right\rangle_3+\left\langle\left[\frac{1}{\Box_1}(-{\alpha_1}^2)+\frac{\Box_3}{\Box_1\Box_2}(\frac{1}{2}{\alpha_1}^2)\right]\frac{\mathcal{A}_3}{s}\right.\\
&\left.+\left(\frac{1}{\Box_1}-\frac{3}{2}\frac{\Box_3}{\Box_1\Box_2}\right)\frac{\mathcal{B}_3}{s}\right\rangle_2,
\end{aligned}
\tag{2.96}
$$

$$
\begin{aligned}
F_6=&\left\langle-\frac{1}{\Box_3}\frac{\mathrm{e}^{s\Omega}}{s}+\left[\left(-\frac{1}{6}+{\alpha_1}^2+\alpha_1\alpha_3\right)+\frac{\Box_1}{\Box_3}(\alpha_1\alpha_3-\alpha_2\alpha_3)\right]\mathrm{e}^{s\Omega}\right\rangle_3\\
&+\left\langle\frac{1}{2}\frac{1}{\Box_3}\frac{\mathcal{A}_1}{s}\right\rangle_2,
\end{aligned}
\tag{2.97}
$$

$$
\begin{aligned}
F_7=&\left\langle\frac{1}{\Box_1}\left(2{\alpha_2}^2-2\alpha_1\alpha_3-4\alpha_2\alpha_3\right)\frac{\mathrm{e}^{s\Omega}}{s}\right.\\
&+\left[\left(-\frac{1}{3}\alpha_2\alpha_3+{\alpha_1}^2\alpha_2\alpha_3+2\alpha_1{\alpha_2}^2\alpha_3+2{\alpha_2}^3\alpha_3\right)\right.\\
&\left.\left.+\frac{\Box_3}{\Box_1}\left(-2\alpha_1{\alpha_2}^2\alpha_3+2\alpha_1\alpha_2{\alpha_3}^2\right)\right]\mathrm{e}^{s\Omega}\right\rangle_3\\
&+\left\langle\frac{1}{2}\frac{1}{\Box_1}\frac{\mathcal{B}_2}{s}+\frac{1}{2}\frac{\Box_2}{\Box_1}\mathcal{V}_{23}\right\rangle_2,
\end{aligned}
\tag{2.98}
$$

$$
\begin{aligned}
F_8=&\left\langle\frac{1}{\Box_1}(-4{\alpha_1}^2+16\alpha_1\alpha_2)\frac{\mathrm{e}^{s\Omega}}{s}+\left[(-4{\alpha_1}^2\alpha_2\alpha_3)+\frac{\Box_3}{\Box_1}(8{\alpha_1}^2\alpha_2\alpha_3)\right]\mathrm{e}^{s\Omega}\right\rangle_3\\
&+\left\langle-2\frac{1}{\Box_1}\frac{\mathcal{B}_3}{s}\right\rangle_2,
\end{aligned}
\tag{2.99}
$$

$$
\begin{aligned}
F_9=&\left\langle-\frac{1}{3}\frac{1}{\Box_1\Box_2\Box_3}\frac{(\mathrm{e}^{s\Omega}-1-s\Omega)}{s^3}+\frac{1}{\Box_1\Box_2}\left(-\frac{1}{6}+{\alpha_1}^2-2\alpha_1\alpha_2\right.\right.\\
&\left.-2\alpha_2\alpha_3+\frac{3}{2}{\alpha_3}^2\right)\frac{(\mathrm{e}^{s\Omega}-1)}{s^2}+\left[\frac{1}{\Box_1}\left(-\frac{1}{36}+\frac{1}{6}{\alpha_1}^2-\frac{1}{2}{\alpha_1}^4\right.\right.\\
&+{\alpha_1}^3\alpha_2+\frac{1}{3}{\alpha_2}^2+2\alpha_1{\alpha_2}^3-\frac{1}{2}{\alpha_2}^4-\frac{1}{2}\alpha_1\alpha_3-\frac{1}{6}\alpha_2\alpha_3\\
&\left.-{\alpha_1}^2\alpha_2\alpha_3+4\alpha_1{\alpha_2}^2\alpha_3+{\alpha_2}^3\alpha_3-\frac{5}{2}{\alpha_1}^2{\alpha_3}^2+\frac{3}{2}{\alpha_2}^2{\alpha_3}^2\right)\\
&+\frac{\Box_1}{\Box_2\Box_3}\left(-2{\alpha_1}^3\alpha_2+\frac{3}{2}{\alpha_1}^2{\alpha_2}^2+\frac{1}{2}{\alpha_2}^4+\frac{1}{2}\alpha_1\alpha_3\right.
\end{aligned}
$$

$$
- \frac{1}{2}{\alpha_1}^2\alpha_2\alpha_3 - \alpha_1{\alpha_2}^2\alpha_3 - \frac{1}{2}{\alpha_2}^2{\alpha_3}^2 - \alpha_1{\alpha_3}^3\Big)\Big] \frac{e^{s\Omega}}{s}
$$

$$
+ \Big[\Big(-\frac{1}{648} + \frac{1}{24}{\alpha_1}^2 - \frac{1}{12}{\alpha_1}^4 - \frac{1}{6}{\alpha_1}^3\alpha_2 + \frac{1}{36}\alpha_1\alpha_3
$$

$$
- \frac{1}{6}{\alpha_1}^2\alpha_2\alpha_3 - {\alpha_1}^3{\alpha_2}^2\alpha_3 - \frac{1}{6}{\alpha_1}^2{\alpha_3}^2 - \frac{1}{3}{\alpha_1}^2{\alpha_2}^2{\alpha_3}^2\Big)
$$

$$
+ \frac{\Box_1}{\Box_2}\Big(\frac{1}{36}\alpha_1\alpha_2 - \frac{1}{12}{\alpha_1}^3\alpha_2 + \frac{1}{6}{\alpha_1}^2{\alpha_2}^2 - \frac{1}{2}{\alpha_1}^4{\alpha_2}^2 - \frac{1}{12}\alpha_1{\alpha_2}^3
$$

$$
- \frac{1}{2}{\alpha_1}^2{\alpha_2}^4 - \frac{1}{36}\alpha_2\alpha_3 + \frac{1}{12}{\alpha_1}^2\alpha_2\alpha_3 - \frac{1}{2}{\alpha_1}^4\alpha_2\alpha_3 - \frac{1}{4}\alpha_1{\alpha_2}^2\alpha_3
$$

$$
+ \frac{1}{2}\alpha_1{\alpha_2}^4\alpha_3 - \frac{1}{3}\alpha_1\alpha_2{\alpha_3}^2 + \frac{1}{2}{\alpha_1}^3\alpha_2{\alpha_3}^2 - \frac{1}{3}{\alpha_2}^2{\alpha_3}^2 + 2{\alpha_1}^2{\alpha_2}^2{\alpha_3}^2
$$

$$
+ \frac{5}{2}\alpha_1{\alpha_2}^3{\alpha_3}^2 + {\alpha_2}^4{\alpha_3}^2 + {\alpha_1}^2\alpha_2{\alpha_3}^3 + 3\alpha_1{\alpha_2}^2{\alpha_3}^3 + {\alpha_2}^3{\alpha_3}^3
$$

$$
+ \alpha_1\alpha_2{\alpha_3}^4 + {\alpha_2}^2{\alpha_3}^4\Big) + \frac{{\Box_1}^2}{\Box_2\Box_3}\Big(\frac{1}{4}\alpha_1\alpha_2{\alpha_3}^2 - {\alpha_1}^3\alpha_2{\alpha_3}^2
$$

$$
+ \frac{1}{12}{\alpha_2}^2{\alpha_3}^2 - {\alpha_1}^2{\alpha_2}^2{\alpha_3}^2 - 2\alpha_1{\alpha_2}^3{\alpha_3}^2 + \frac{1}{12}\alpha_2{\alpha_3}^3
$$

$$
- \frac{1}{2}{\alpha_1}^2\alpha_2{\alpha_3}^3 - \frac{1}{2}{\alpha_2}^3{\alpha_3}^3 - \frac{1}{2}\alpha_1\alpha_2{\alpha_3}^4 - \frac{1}{2}{\alpha_2}^2{\alpha_3}^4\Big)\Big] e^{s\Omega}\Big\rangle_3
$$

$$
+ \Big\langle \frac{\Box_1}{\Box_2\Box_3}\Big(-\frac{1}{12}{\alpha_1}^2 + \frac{1}{2}{\alpha_1}^2{\alpha_2}^2\Big)\frac{\mathcal{A}_1}{s} + \frac{1}{\Box_1}\Big(\frac{1}{72} - \frac{1}{2}{\alpha_1}^2{\alpha_2}^2\Big)\frac{\mathcal{A}_2}{s}
$$

$$
+ \frac{\Box_1}{\Box_2\Box_3}\Big(\frac{1}{12} - {\alpha_1}^2 + \alpha_1\alpha_2\Big)\frac{\mathcal{B}_1}{s} + \frac{1}{\Box_1}({\alpha_1}^2 - \alpha_1\alpha_2)\frac{\mathcal{B}_2}{s}
$$

$$
+ \frac{\Box_1}{\Box_2\Box_3}\frac{\mathcal{C}_1}{s} - \frac{1}{2}\frac{1}{\Box_1}\frac{\mathcal{C}_2}{s} + \Big(\frac{1}{12}\alpha_1^2 - \frac{1}{4}\alpha_1^2\alpha_2^2 - \frac{1}{144}\Big)\mathcal{U}_{12}
$$

$$
+ \Big(\frac{1}{2}\alpha_1^2 - \frac{1}{2}\alpha_1\alpha_2 - \frac{1}{12}\Big)\mathcal{V}_{12} + \Big(\frac{1}{2}\frac{\Box_1}{\Box_3} - \frac{1}{2}\Big)\mathcal{W}_{12}\Big\rangle_2, \tag{2.100}
$$

$$
F_{10} = \Big\langle -\frac{8}{3}\frac{1}{\Box_1\Box_2\Box_3}\frac{(e^{s\Omega} - 1 - s\Omega)}{s^3}\Big\rangle_3 + \Big\langle 2\frac{\Box_1}{\Box_2\Box_3}\frac{\mathcal{C}_1}{s}\Big\rangle_2, \tag{2.101}
$$

$$
F_{11} = \Big\langle -2\frac{1}{\Box_1\Box_2\Box_3}\frac{(e^{s\Omega} - 1 - s\Omega)}{s^3} + \Big[\frac{1}{\Box_1\Box_2}\Big(-\frac{1}{3} + 2{\alpha_1}^2 + 2\alpha_1\alpha_3 + {\alpha_3}^2\Big)
$$

$$
+ \frac{1}{\Box_1\Box_3}(-2\alpha_1\alpha_3 + 2\alpha_2\alpha_3)\Big]\frac{(e^{s\Omega} - 1)}{s^2}\Big\rangle_3
$$

$$
+ \Big\langle \Big[\frac{1}{\Box_1}\Big(\frac{1}{6}{\alpha_1}^2 - {\alpha_1}^2{\alpha_2}^2\Big) + \frac{\Box_3}{\Box_1\Box_2}\Big(-\frac{1}{12}{\alpha_1}^2 + \frac{1}{2}{\alpha_1}^2{\alpha_2}^2\Big)\Big]\frac{\mathcal{A}_3}{s}
$$

$$
+ \Big[\frac{1}{\Box_1}\Big(-\frac{1}{6} + 2{\alpha_1}^2 - 2\alpha_1\alpha_2\Big) + \frac{\Box_3}{\Box_1\Box_2}\Big(\frac{1}{4} - 2{\alpha_1}^2 + \alpha_1\alpha_2\Big)\Big]\frac{\mathcal{B}_3}{s}
$$

$$
+ \Big(-\frac{1}{\Box_2} + \frac{\Box_1}{\Box_2\Box_3}\Big)\frac{\mathcal{C}_1}{s} + \Big(-\frac{1}{\Box_1} + \frac{3}{2}\frac{\Box_3}{\Box_1\Box_2}\Big)\frac{\mathcal{C}_3}{s} - \frac{1}{2}\mathcal{W}_{12}\Big\rangle_2, \tag{2.102}
$$

$$F_{12} = \left\langle \frac{1}{\Box_2\Box_3}(-2\alpha_1 + 2\alpha_2 + 2\alpha_3)\frac{e^{s\Omega}}{s^2} + \left[\frac{1}{\Box_2}(2\alpha_1\alpha_2\alpha_3) + \frac{1}{\Box_3}(2\alpha_1\alpha_2\alpha_3) + \frac{\Box_1}{\Box_2\Box_3}(-2\alpha_1\alpha_2\alpha_3)\right]\frac{e^{s\Omega}}{s}\right\rangle_3 + \left\langle -2\frac{1}{\Box_2\Box_3}\frac{\mathcal{B}_1}{s^2} - 2\frac{1}{\Box_3}\frac{\mathcal{V}_{12}}{s} - 2\frac{1}{\Box_2}\frac{\mathcal{V}_{13}}{s}\right\rangle_2, \tag{2.103}$$

$$F_{13} = \left\langle \frac{1}{\Box_1}(2\alpha_1)\frac{e^{s\Omega}}{s}\right\rangle_3 + \left\langle -2\frac{1}{\Box_1}\frac{\mathcal{U}_{23}}{s}\right\rangle_2, \tag{2.104}$$

$$F_{14} = \left\langle -2\frac{1}{\Box_1\Box_2}\frac{e^{s\Omega}}{s^2} + \left[\frac{1}{\Box_1}(2\alpha_1\alpha_2) + \frac{1}{\Box_2}(2\alpha_1\alpha_2) + \frac{\Box_3}{\Box_1\Box_2}(-2\alpha_1\alpha_2)\right]\frac{e^{s\Omega}}{s}\right\rangle_3 + \left\langle \frac{1}{\Box_1\Box_2}(2\alpha_2)\frac{\mathcal{A}_3}{s^2}\right\rangle_2, \tag{2.105}$$

$$F_{15} = \left\langle \frac{1}{\Box_1\Box_2}(-4\alpha_1 + 12{\alpha_1}^2)\frac{e^{s\Omega}}{s^2} + \left[\frac{1}{\Box_1}\left(\frac{2}{3}{\alpha_1}^2 - 2{\alpha_1}^4 - 2{\alpha_1}^3\alpha_2 - 2{\alpha_1}^2\alpha_2\alpha_3 - 2{\alpha_1}^2{\alpha_3}^2\right) + \frac{1}{\Box_2}\left(-2{\alpha_1}^3\alpha_2 + 2{\alpha_1}^2{\alpha_2}^2 + 2{\alpha_1}^2\alpha_2\alpha_3\right) + \frac{\Box_3}{\Box_1\Box_2}\left(2{\alpha_1}^3\alpha_2 - 2{\alpha_1}^2{\alpha_2}^2 - 2{\alpha_1}^2\alpha_2\alpha_3\right)\right]\frac{e^{s\Omega}}{s}\right\rangle_3 + \left\langle -2\frac{1}{\Box_1\Box_2}\frac{\mathcal{B}_3}{s^2} + \frac{1}{\Box_1}\left(2\alpha_1^2 - \frac{1}{3}\right)\frac{\mathcal{U}_{23}}{s} - 2\frac{1}{\Box_1}\frac{\mathcal{V}_{23}}{s}\right\rangle_2, \tag{2.106}$$

$$F_{16} = \left\langle \frac{1}{\Box_1\Box_2}(8\alpha_1\alpha_2)\frac{e^{s\Omega}}{s^2}\right\rangle_3 + \left\langle \frac{1}{\Box_1\Box_2}(-2{\alpha_1}^2)\frac{\mathcal{A}_3}{s^2} + 2\frac{1}{\Box_1\Box_2}\frac{\mathcal{B}_3}{s^2}\right\rangle_2, \tag{2.107}$$

$$F_{17} = \left\langle \frac{1}{\Box_1}(2{\alpha_1}^2)\frac{e^{s\Omega}}{s}\right\rangle_3 + \left\langle -\frac{1}{\Box_1}\frac{\mathcal{U}_{23}}{s}\right\rangle_2, \tag{2.108}$$

$$F_{18} = \left\langle 4\frac{1}{\Box_1\Box_2\Box_3}\frac{(e^{s\Omega} - 1)}{s^3} + \left[\frac{1}{\Box_2\Box_3}(-8\alpha_1\alpha_2 + 2\alpha_1^2) + \frac{1}{\Box_1\Box_2}(-4\alpha_1^2 + 8\alpha_1\alpha_2 + 8\alpha_1\alpha_3)\right]\frac{e^{s\Omega}}{s^2} + \left[\frac{1}{\Box_1}(4{\alpha_1}^2\alpha_2\alpha_3) + \frac{1}{\Box_2}(-8{\alpha_1}^2\alpha_2\alpha_3) + \frac{\Box_1}{\Box_2\Box_3}(2{\alpha_1}^2\alpha_2\alpha_3) + \frac{\Box_3}{\Box_1\Box_2}(4{\alpha_1}^2\alpha_2\alpha_3)\right]\frac{e^{s\Omega}}{s}\right\rangle_3 + \left\langle -4\frac{1}{\Box_1\Box_3}\frac{\mathcal{B}_2}{s^2}\right\rangle_2, \tag{2.109}$$

$$F_{19} = \left\langle \frac{1}{\Box_1}(-4{\alpha_1}^2\alpha_2\alpha_3)\frac{\mathrm{e}^{s\Omega}}{s} \right\rangle_3 + \left\langle -\frac{1}{\Box_1}\frac{\mathcal{V}_{23}}{s} \right\rangle_2 , \tag{2.110}$$

$$\begin{aligned} F_{20} = & \Big\langle 2\frac{1}{\Box_1\Box_2\Box_3}\frac{(\mathrm{e}^{s\Omega}-1)}{s^3} + \Big[\frac{1}{\Box_1\Box_2}\left(2{\alpha_2}^2 - 4\alpha_1\alpha_3 - 8\alpha_2\alpha_3 + 2{\alpha_3}^2\right) \\ & + \frac{1}{\Box_2\Box_3}\left(\frac{1}{3} - {\alpha_1}^2 - 4{\alpha_2}^2 + 4\alpha_2\alpha_3\right)\Big]\frac{\mathrm{e}^{s\Omega}}{s^2} \\ & + \Big[\frac{1}{\Box_2}\left(-\frac{2}{3}\alpha_2\alpha_3 + 2{\alpha_1}^2\alpha_2\alpha_3 + 4\alpha_1{\alpha_2}^2\alpha_3 + 2{\alpha_2}^3\alpha_3 + 2\alpha_2{\alpha_3}^3\right) \\ & + \frac{\Box_1}{\Box_2\Box_3}\left(\frac{1}{3}\alpha_2\alpha_3 - {\alpha_1}^2\alpha_2\alpha_3 - 2\alpha_1{\alpha_2}^2\alpha_3 - 2{\alpha_2}^3\alpha_3\right) \\ & + \frac{\Box_3}{\Box_1\Box_2}\left(-2\alpha_1{\alpha_2}^2\alpha_3 + 2\alpha_1\alpha_2{\alpha_3}^2\right)\Big]\frac{\mathrm{e}^{s\Omega}}{s}\Big\rangle_3 \\ & + \left\langle \frac{1}{\Box_2\Box_3}\left(-\frac{1}{3}\alpha_2 + 2{\alpha_1}^2\alpha_2\right)\frac{\mathcal{A}_1}{s^2} + \frac{1}{\Box_1}\frac{\mathcal{V}_{23}}{s}\right\rangle_2 , \end{aligned} \tag{2.111}$$

$$\begin{aligned} F_{21} = & \Big\langle \frac{1}{\Box_1\Box_2}(-8{\alpha_1}^2 + 16\alpha_1\alpha_3)\frac{\mathrm{e}^{s\Omega}}{s^2} + \Big[\frac{1}{\Box_1}(8{\alpha_1}^2\alpha_2\alpha_3) \\ & + \frac{1}{\Box_2}(-8{\alpha_1}^2\alpha_2\alpha_3) + \frac{\Box_3}{\Box_1\Box_2}(8{\alpha_1}^2\alpha_2\alpha_3)\Big]\frac{\mathrm{e}^{s\Omega}}{s}\Big\rangle_3 \\ & + \left\langle 4\frac{1}{\Box_1}\frac{\mathcal{V}_{23}}{s}\right\rangle_2 , \end{aligned} \tag{2.112}$$

$$\begin{aligned} F_{22} = & \Big\langle \frac{1}{\Box_1\Box_2\Box_3}\left(-10{\alpha_1}^2 + 24\alpha_1\alpha_3 + 4\alpha_2\alpha_3\right)\frac{(\mathrm{e}^{s\Omega}-1)}{s^3} \\ & + \Big[\frac{1}{\Box_1\Box_2}\Big(2{\alpha_1}^4 - \frac{4}{3}{\alpha_1}^2 + \frac{2}{3}\alpha_1\alpha_2 - 12{\alpha_1}^3\alpha_2 + 18{\alpha_1}^2{\alpha_2}^2 + \frac{2}{3}\alpha_1\alpha_3 \\ & - 4{\alpha_1}^3\alpha_3 + 12{\alpha_1}^2\alpha_2\alpha_3 + 4\alpha_1{\alpha_2}^2\alpha_3 - 6{\alpha_1}^2{\alpha_3}^2 - 4\alpha_1\alpha_2{\alpha_3}^2\Big) \\ & + \frac{1}{\Box_2\Box_3}\left(8{\alpha_1}^3\alpha_2 - 12{\alpha_1}^2{\alpha_2}^2 - 12{\alpha_1}^2\alpha_2\alpha_3 - 4\alpha_1\alpha_2{\alpha_3}^2\right)\Big]\frac{\mathrm{e}^{s\Omega}}{s^2} \\ & + \Big[\frac{1}{\Box_1}\Big(-\frac{1}{18}{\alpha_1}^2 + \frac{1}{3}{\alpha_1}^4 + \frac{1}{3}{\alpha_1}^3\alpha_2 + \frac{1}{3}{\alpha_1}^2\alpha_2\alpha_3 - 2{\alpha_1}^4\alpha_2\alpha_3 \\ & + 2{\alpha_1}^3{\alpha_2}^2\alpha_3 + \frac{1}{3}{\alpha_1}^2{\alpha_3}^2 - 2{\alpha_1}^4{\alpha_3}^2 - 2{\alpha_1}^3{\alpha_3}^3\Big) \\ & + \frac{1}{\Box_2}\Big(\frac{1}{3}{\alpha_1}^3\alpha_2 - \frac{1}{3}{\alpha_1}^2{\alpha_2}^2 + 2{\alpha_1}^4{\alpha_2}^2 - 2{\alpha_1}^3{\alpha_2}^3 - \frac{1}{3}{\alpha_1}^2\alpha_2\alpha_3 \\ & + 2{\alpha_1}^4\alpha_2\alpha_3 - 4{\alpha_1}^3{\alpha_2}^2\alpha_3 - 2{\alpha_1}^3\alpha_2{\alpha_3}^2\Big) + \frac{\Box_1}{\Box_2\Box_3}\left(4{\alpha_1}^3\alpha_2{\alpha_3}^2\right) \\ & + \frac{\Box_2}{\Box_1\Box_3}\Big(-\frac{1}{3}{\alpha_1}^3\alpha_3 + \frac{1}{3}{\alpha_1}^2\alpha_2\alpha_3 - 2{\alpha_1}^4\alpha_2\alpha_3 - 2{\alpha_1}^3{\alpha_2}^2\alpha_3 \\ & + \frac{1}{3}{\alpha_1}^2{\alpha_3}^2 - 2{\alpha_1}^4{\alpha_3}^2 + 2{\alpha_1}^3{\alpha_3}^3\Big)\Big]\frac{\mathrm{e}^{s\Omega}}{s}\Big\rangle_3 + \left\langle \frac{1}{\Box_1\Box_2}\left(\frac{1}{3} - 2{\alpha_1}^2\right)\frac{\mathcal{B}_3}{s^2}\right. \end{aligned}$$

$$- 2\frac{1}{\Box_2\Box_3}\frac{\mathcal{C}_1}{s^2} + \frac{1}{\Box_1}\left(\alpha_1^2\alpha_2^2 - \frac{1}{3}\alpha_1^2 + \frac{1}{36}\right)\frac{\mathcal{U}_{23}}{s}$$

$$+ \frac{1}{\Box_1}\left(2\alpha_1\alpha_2 - 2\alpha_1^2 + \frac{1}{3}\right)\frac{\mathcal{V}_{23}}{s} + \frac{1}{\Box_1}\frac{\mathcal{W}_{23}}{s} - 2\frac{1}{\Box_2}\frac{\mathcal{W}_{13}}{s}\Bigg\rangle_2, \tag{2.113}$$

$$F_{23} = \Bigg\langle \frac{1}{\Box_1\Box_2\Box_3}\left(8{\alpha_1}^2 - 16\alpha_1\alpha_2 - 8\alpha_2\alpha_3\right)\frac{(\mathrm{e}^{s\Omega}-1)}{s^3}$$

$$+ \left[\frac{1}{\Box_1\Box_2}\left(-\frac{4}{3}\alpha_1\alpha_2 + 8{\alpha_1}^3\alpha_2 + 4\alpha_1\alpha_2\alpha_3\right)\right.$$

$$\left. + \frac{1}{\Box_2\Box_3}\left(8{\alpha_1}^2\alpha_2\alpha_3 - 8\alpha_1{\alpha_2}^2\alpha_3\right)\right]\frac{\mathrm{e}^{s\Omega}}{s^2}\Bigg\rangle_3$$

$$+ \Bigg\langle \frac{1}{\Box_1\Box_2}\left(\frac{1}{3}{\alpha_1}^2 - 2{\alpha_1}^2{\alpha_2}^2\right)\frac{\mathcal{A}_3}{s^2} + \frac{1}{\Box_1\Box_2}\left(-\frac{1}{3} + 4{\alpha_1}^2 - 4\alpha_1\alpha_2\right)\frac{\mathcal{B}_3}{s^2}$$

$$+ 4\frac{1}{\Box_1\Box_3}\frac{\mathcal{C}_2}{s^2} - 2\frac{1}{\Box_1\Box_2}\frac{\mathcal{C}_3}{s^2} + 2\frac{1}{\Box_3}\frac{\mathcal{W}_{12}}{s}\Bigg\rangle_2, \tag{2.114}$$

$$F_{24} = \Bigg\langle \frac{1}{\Box_1\Box_2\Box_3}(-4{\alpha_1}^2)\frac{(\mathrm{e}^{s\Omega}-1)}{s^3}\Bigg\rangle_3$$

$$+ \Bigg\langle 2\frac{1}{\Box_2\Box_3}\frac{\mathcal{C}_1}{s^2} + 4\frac{1}{\Box_1\Box_2}\frac{\mathcal{C}_3}{s^2} + 2\frac{1}{\Box_1}\frac{\mathcal{W}_{23}}{s}\Bigg\rangle_2, \tag{2.115}$$

$$F_{25} = \Bigg\langle \frac{1}{\Box_1\Box_2\Box_3}(-16\alpha_2\alpha_3)\frac{(\mathrm{e}^{s\Omega}-1)}{s^3}\Bigg\rangle_3$$

$$+ \Bigg\langle 4\frac{1}{\Box_2\Box_3}\frac{\mathcal{C}_1}{s^2} + 8\frac{1}{\Box_1\Box_3}\frac{\mathcal{C}_2}{s^2} + 8\frac{1}{\Box_3}\frac{\mathcal{W}_{12}}{s}\Bigg\rangle_2, \tag{2.116}$$

$$F_{26} = \Bigg\langle \frac{1}{\Box_1\Box_2}(4{\alpha_1}^2{\alpha_2}^2)\frac{\mathrm{e}^{s\Omega}}{s^2}\Bigg\rangle_3, \tag{2.117}$$

$$F_{27} = \Bigg\langle \frac{1}{\Box_1\Box_2\Box_3}\left(8{\alpha_1}^3\alpha_2 - 12{\alpha_1}^2{\alpha_2}^2 - 8{\alpha_1}^2\alpha_2\alpha_3\right)\frac{\mathrm{e}^{s\Omega}}{s^3}$$

$$+ \left[\frac{1}{\Box_1\Box_2}\left(-\frac{2}{3}{\alpha_1}^2{\alpha_2}^2 + 4{\alpha_1}^4{\alpha_2}^2 + 4{\alpha_1}^3{\alpha_2}^2\alpha_3\right)\right.$$

$$\left. + \frac{1}{\Box_1\Box_3}\left(-4{\alpha_1}^3{\alpha_2}^2\alpha_3 + 4{\alpha_1}^2{\alpha_2}^3\alpha_3\right)\right]\frac{\mathrm{e}^{s\Omega}}{s^2}\Bigg\rangle_3, \tag{2.118}$$

$$F_{28} = \Bigg\langle \frac{1}{\Box_1\Box_2\Box_3}(-16\alpha_1\alpha_2{\alpha_3}^2)\frac{\mathrm{e}^{s\Omega}}{s^3}\Bigg\rangle_3, \tag{2.119}$$

$$F_{29} = \Bigg\langle \frac{1}{\Box_1\Box_2\Box_3}\left(\frac{8}{3}{\alpha_1}^2{\alpha_2}^2{\alpha_3}^2\right)\frac{\mathrm{e}^{s\Omega}}{s^3}\Bigg\rangle_3, \tag{2.120}$$

$$F_{30} = \Big\langle \Big[\frac{1}{\Box_2\Box_3}(2\alpha_3^2 - 4\alpha_1\alpha_3^2 - 4\alpha_1^2\alpha_3)$$
$$+ \frac{1}{\Box_1\Box_2}\Big(-\frac{1}{3} - 2{\alpha_1}^3 - 4{\alpha_1}^2\alpha_2 + 2{\alpha_2}^2$$
$$+ 2{\alpha_2}^3 + 2{\alpha_1}^2\alpha_3 + 4\alpha_1{\alpha_3}^2 - 2\alpha_2{\alpha_3}^2\Big)\Big] \frac{e^{s\Omega}}{s^2} \Big\rangle_3$$
$$+ \Big[\frac{\Box_3}{\Box_1\Box_2}\Big(-\frac{1}{3}\alpha_1\alpha_2 + 2\alpha_1{\alpha_2}^3 + 2{\alpha_1}^2\alpha_2\alpha_3\Big)$$
$$+ \frac{\Box_1}{\Box_2\Box_3}(-2\alpha_1\alpha_2\alpha_3^2) + \frac{1}{\Box_1}\Big(2\alpha_1^2\alpha_3^2 - 2\alpha_1\alpha_2\alpha_3^2 - \frac{1}{3}\alpha_1\alpha_3\Big)$$
$$+ \frac{1}{\Box_2}\Big(\frac{1}{3}\alpha_1\alpha_2 - 2\alpha_1\alpha_2\alpha_3 - 2\alpha_1\alpha_2^3 + 4\alpha_1\alpha_2\alpha_3^2\Big)\Big] \frac{e^{s\Omega}}{s} \Big\rangle_3$$
$$+ \Big\langle \frac{1}{\Box_1\Box_2}\Big(\frac{1}{3}\alpha_2 - 2{\alpha_1}^2\alpha_2\Big)\frac{\mathcal{A}_3}{s^2}\Big\rangle_2, \tag{2.121}$$

$$F_{31} = \Big\langle -4\frac{1}{\Box_1\Box_2\Box_3}\frac{(e^{s\Omega}-1)}{s^3} + \Big[\frac{1}{\Box_1\Box_2}(-4\alpha_1\alpha_2 + 4{\alpha_2}^2)$$
$$+ \frac{1}{\Box_1\Box_3}(4\alpha_2\alpha_3) + \frac{1}{\Box_2\Box_3}(-4\alpha_2\alpha_3)\Big]\frac{e^{s\Omega}}{s^2}\Big\rangle_3$$
$$+ \Big\langle \frac{1}{\Box_2\Box_3}(4\alpha_1)\frac{\mathcal{B}_1}{s^2}\Big\rangle_2, \tag{2.122}$$

$$F_{32} = \Big\langle 2\frac{1}{\Box_2\Box_3}\frac{e^{s\Omega}}{s^2} + \Big[\frac{1}{\Box_2}(2\alpha_1\alpha_2 - 2{\alpha_2}^2) + \frac{1}{\Box_3}(-2\alpha_2\alpha_3)$$
$$+ \frac{\Box_1}{\Box_2\Box_3}(2\alpha_2\alpha_3)\Big]\frac{e^{s\Omega}}{s}\Big\rangle_3 + \Big\langle -\frac{1}{\Box_2\Box_3}\frac{\mathcal{A}_1}{s^2}\Big\rangle_2, \tag{2.123}$$

$$F_{33} = \Big\langle \frac{1}{\Box_1\Box_2\Box_3}(8\alpha_1\alpha_2 - 4{\alpha_2}^2)\frac{e^{s\Omega}}{s^3} + \Big[\frac{1}{\Box_1\Box_3}(-4\alpha_1{\alpha_2}^2\alpha_3)$$
$$+ \frac{1}{\Box_2\Box_3}(4\alpha_1{\alpha_2}^2\alpha_3) + \frac{1}{\Box_1\Box_2}(4{\alpha_1}^2{\alpha_2}^2 - 4\alpha_1{\alpha_2}^3)\Big]\frac{e^{s\Omega}}{s^2}\Big\rangle_3. \tag{2.124}$$

The α -representations is the starting point for all the further derivations. Therefore, we present here several reference formulae concerning the α-integrals. One has

$$\langle \alpha_1^n \alpha_2^m \rangle_2 = \frac{n!m!}{(n+m+1)!}, \tag{2.125}$$

$$\left\langle \alpha_1^n \alpha_2^m \alpha_3^k \right\rangle_3 = \frac{n!m!k!}{(n+m+k+2)!}. \tag{2.126}$$

These equations are, in particular, useful for obtaining the short time asymptotic behaviors of the form factors.

The relevant results for the large time behaviour of form factors are [30]

$$\langle P(\alpha_1,\alpha_2)\exp(s\alpha_1\alpha_2\Box)\rangle_2 = -\frac{1}{s}\frac{P(1,0)+P(0,1)}{\Box} + \mathrm{O}\left(\frac{1}{s^2}\right), \qquad s\to\infty \tag{2.127}$$

$$\langle P(\alpha_1,\alpha_2,\alpha_3)\exp(s\Omega)\rangle_3 = \frac{1}{s^2}\left[\frac{P(1,0,0)}{\Box_2\Box_3} + \frac{P(0,1,0)}{\Box_1\Box_3} + \frac{P(0,0,1)}{\Box_1\Box_2}\right] + \mathrm{O}\left(\frac{1}{s^3}\right), \qquad s\to\infty \tag{2.128}$$

where P 's are any of the α-polynomials above.

2.4 Reduction of the α-polynomial form factors to the basic form factors

The problem with the α-representation is that the $s\Box$-arguments of the form factors are not confined to the kernels (2.83) and (2.84). As seen from the expressions above, they enter also coefficients of the α-polynomials onto which the form factors are mapped. For this reason, the α-representation is not unique because even with the delta-function in (2.82) taken into account, vanishing of the integral like

$$\left\langle P(\alpha,\Box)\mathrm{e}^{s\Omega}\right\rangle_3 \tag{2.129}$$

does not imply vanishing of the polynomial $P(\alpha,\Box)$. This nonuniqueness obscures properties of the form factors and makes difficult various checks like the check of the trace anomaly. In particular, the fact that the contributions of the structures (2.76)–(2.79) vanish (see below) is not seen from (2.121)–(2.124). The origin of the $\Box$'s in the coefficients is the tree formulae which express the perturbations through the curvatures (1.43)–(1.44). The problem of nonuniqueness persists in representations for the form factors in the heat kernel (Chapter 3) and in the effective action too. The technique presented in this section is desiged to remove this defect.

One way of obtaining a unique representation for the form factors in the trace of the heat kernel is to eliminate all polynomials in α. All form factors will then be explicitly expressed through the basic second order form factor,

$$f(\xi) = \left\langle e^{-\alpha_1\alpha_2\xi} \right\rangle_2 = \int_0^1 d\alpha\, e^{-\alpha(1-\alpha)\xi}, \qquad \xi = -s\Box, \tag{2.130}$$

and the basic third order form factor,

$$F(\xi_1, \xi_2, \xi_3) = \left\langle e^{s\Omega} \right\rangle_3 = \tag{2.131}$$
$$\int_{\alpha\geq 0} d^3\alpha\, \delta(1-\alpha_1-\alpha_2-\alpha_3) \exp(-\alpha_1\alpha_2\xi_3 - \alpha_2\alpha_3\xi_1 - \alpha_1\alpha_3\xi_2), \quad \xi_m = -s\Box_m,$$

(which is completely symmetric in ξ_1, ξ_2, ξ_3). The technique of eliminating the polynomials in α is as follows.

After use of the delta-function in (2.81) and (2.82), there remain to be considered the contributions of the monomials:

$$\left\langle \alpha_1^n e^{-\alpha_1\alpha_2\xi} \right\rangle_2 = \int_0^1 d\alpha\, \alpha^n \exp\left[-\alpha(1-\alpha)\xi\right], \tag{2.132}$$

$$\left\langle \alpha_1^n \alpha_2^m e^{s\Omega} \right\rangle_3 = \int_0^1 d\alpha_2 \int_0^{1-\alpha_2} d\alpha_1\, \alpha_1^n \alpha_2^m$$
$$\times \exp\left[-\alpha_2(1-\alpha_1-\alpha_2)\xi_1 - \alpha_1(1-\alpha_1-\alpha_2)\xi_2 - \alpha_1\alpha_2\xi_3\right]. \tag{2.133}$$

For the case (2.132) integration by parts produces the following equation:

$$\int_0^1 d\alpha \frac{d}{d\alpha} \alpha^n \exp\left[-\alpha(1-\alpha)\xi\right] = \begin{cases} 0, & n=0 \\ 1, & n>0 \end{cases} \tag{2.134}$$

which yields the recurrence relations,

$$\left\langle \alpha_1 e^{-\alpha_1\alpha_2\xi} \right\rangle_2 = \frac{1}{2} \left\langle e^{-\alpha_1\alpha_2\xi} \right\rangle_2, \tag{2.135}$$

$$\left\langle \alpha_1^n e^{-\alpha_1\alpha_2\xi} \right\rangle_2 = \frac{1}{2} \left\langle \alpha_1^{n-1} e^{-\alpha_1\alpha_2\xi} \right\rangle_2$$
$$-\frac{1}{2}(n-1) \left\langle \alpha_1^{n-2} \left(\frac{e^{-\alpha_1\alpha_2\xi} - 1}{\xi} \right) \right\rangle_2, \qquad n \geq 2 \tag{2.136}$$

which make it possible to express all integrals (2.132) through the basic form factor (2.130). Note that this procedure automatically leads to the appearance of the form factors with subtractions (2.31)–(2.32). The recurrence relations for them are similar:

$$\left\langle \alpha_1 \left(\frac{e^{-\alpha_1\alpha_2\xi} - 1}{\xi} \right) \right\rangle_2 = \frac{1}{2} \left\langle \left(\frac{e^{-\alpha_1\alpha_2\xi} - 1}{\xi} \right) \right\rangle_2, \tag{2.137}$$

$$\left\langle \alpha_1^n \left(\frac{e^{-\alpha_1\alpha_2\xi} - 1}{\xi} \right) \right\rangle_2 = \frac{1}{2} \left\langle \alpha_1^{n-1} \left(\frac{e^{-\alpha_1\alpha_2\xi} - 1}{\xi} \right) \right\rangle_2$$

$$- \frac{1}{2}(n-1) \left\langle \alpha_1^{n-2} \left(\frac{e^{-\alpha_1\alpha_2\xi} - 1 + \alpha_1\alpha_2\xi}{\xi^2} \right) \right\rangle_2 , \qquad n \geq 2 \qquad (2.138)$$

as one can check with the aid of eq. (2.125). The appearance of the subtractions is explained by analyticity of the integral (2.132) in ξ at $\xi = 0$. Since the recurrence relations imply a division by ξ, the appearing subtractions maintain the analyticity. For the form factors with subtractions one has

$$\left\langle \frac{e^{-\alpha_1\alpha_2\xi} - 1}{\xi} \right\rangle_2 = \frac{f(\xi) - 1}{\xi}, \qquad (2.139)$$

$$\left\langle \frac{e^{-\alpha_1\alpha_2\xi} - 1 + \alpha_1\alpha_2\xi}{\xi^2} \right\rangle_2 = \frac{f(\xi) - 1 + \frac{1}{6}\xi}{\xi^2} \qquad (2.140)$$

in terms of (2.130).

Elimination of the polynomials in α from the third-order form factors is based on integration by parts in (2.133):

$$\int_0^1 d\alpha_2 \int_0^{1-\alpha_2} d\alpha_1 \frac{d}{d\alpha_1} \alpha_1^n \alpha_2^m \exp\left(s\Omega|_{\alpha_3 = 1-\alpha_1-\alpha_2} \right)$$

$$= \begin{cases} \left\langle \alpha_2^m \left(e^{-\alpha_1\alpha_2\xi_3} - e^{-\alpha_1\alpha_2\xi_1} \right) \right\rangle_2 , & n = 0 \\ \left\langle \alpha_1^n \alpha_2^m e^{-\alpha_1\alpha_2\xi_3} \right\rangle_2 , & n > 0, \end{cases} \qquad (2.141)$$

$$\int_0^1 d\alpha_2 \int_0^{1-\alpha_2} d\alpha_1 \frac{d}{d\alpha_2} \alpha_1^n \alpha_2^m \exp\left(s\Omega|_{\alpha_3 = 1-\alpha_1-\alpha_2} \right)$$

$$= \begin{cases} \left\langle \alpha_1^n \left(e^{-\alpha_1\alpha_2\xi_3} - e^{-\alpha_1\alpha_2\xi_2} \right) \right\rangle_2 , & m = 0 \\ \left\langle \alpha_1^n \alpha_2^m e^{-\alpha_1\alpha_2\xi_3} \right\rangle_2 , & m > 0 \end{cases} \qquad (2.142)$$

where the second order form factors appearing on the right-hand sides are subject to the recurrence relations above. By performing the differentiations on the left-hand sides of (2.141) and (2.142), one obtains two linear algebraic equations for the quantities

$$\left\langle \alpha_1^{n+1} \alpha_2^m e^{s\Omega} \right\rangle_3 , \qquad \left\langle \alpha_1^n \alpha_2^{m+1} e^{s\Omega} \right\rangle_3$$

containing the highest-order monomials. The discriminant of this linear system is

$$\Delta = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - 2\xi_2\xi_3, \qquad (2.143)$$

and the recurrence relations obtained in this way are of the form

$$\begin{aligned}\left\langle \alpha_1^{n+1}\alpha_2^m e^{s\Omega}\right\rangle_3 &= -\frac{\xi_1(\xi_3+\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n\alpha_2^m e^{s\Omega}\right\rangle_3 \\ &\quad + 2n\frac{\xi_1}{\Delta}\left\langle \alpha_1^{n-1}\alpha_2^m e^{s\Omega}\right\rangle_3 \\ &\quad + m\frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n\alpha_2^{m-1} e^{s\Omega}\right\rangle_3 \\ &\quad - \frac{(\xi_3+\xi_1-\xi_2)}{\Delta}\left\langle \alpha_1^n\alpha_2^m e^{-\alpha_1\alpha_2\xi_3}\right\rangle_2 + \beta(n,m), \end{aligned} \tag{2.144}$$

$$\beta(n,m) = 0, \qquad n>0, \qquad m>0 \tag{2.145}$$

$$\beta(n,0) = \frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n e^{-\alpha_1\alpha_2\xi_2}\right\rangle_2, \qquad n>0 \tag{2.146}$$

$$\beta(0,m) = 2\frac{\xi_1}{\Delta}\left\langle \alpha_1^m e^{-\alpha_1\alpha_2\xi_1}\right\rangle_2, \qquad m>0 \tag{2.147}$$

$$\beta(0,0) = 2\frac{\xi_1}{\Delta}\left\langle e^{-\alpha_1\alpha_2\xi_1}\right\rangle_2 + \frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle e^{-\alpha_1\alpha_2\xi_2}\right\rangle_2, \tag{2.148}$$

$$\begin{aligned}\left\langle \alpha_1^n\alpha_2^{m+1} e^{s\Omega}\right\rangle_3 &= -\frac{\xi_2(\xi_3+\xi_1-\xi_2)}{\Delta}\left\langle \alpha_1^n\alpha_2^m e^{s\Omega}\right\rangle_3 \\ &\quad + 2m\frac{\xi_2}{\Delta}\left\langle \alpha_1^n\alpha_2^{m-1} e^{s\Omega}\right\rangle_3 \\ &\quad + n\frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^{n-1}\alpha_2^m e^{s\Omega}\right\rangle_3 \\ &\quad - \frac{(\xi_3+\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n\alpha_2^m e^{-\alpha_1\alpha_2\xi_3}\right\rangle_2 + \delta(n,m), \end{aligned} \tag{2.149}$$

$$\delta(n,m) = 0, \qquad n>0, \qquad m>0 \tag{2.150}$$

$$\delta(n,0) = 2\frac{\xi_2}{\Delta}\left\langle \alpha_1^n e^{-\alpha_1\alpha_2\xi_2}\right\rangle_2, \qquad n>0 \tag{2.151}$$

$$\delta(0,m) = \frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^m e^{-\alpha_1\alpha_2\xi_1}\right\rangle_2, \qquad m>0 \tag{2.152}$$

$$\delta(0,0) = 2\frac{\xi_2}{\Delta}\left\langle e^{-\alpha_1\alpha_2\xi_2}\right\rangle_2 + \frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle e^{-\alpha_1\alpha_2\xi_1}\right\rangle_2. \tag{2.153}$$

Together with (2.135)–(2.136) these relations make it possible to express all integrals (2.133) through the basic form factors (2.130) and (2.131). Again one can show that the α-polynomials do not destroy the combinations (2.30)–(2.35) and (2.86) in which the form factors appear. The recurrence relations with subtractions obtained with the aid of eq. (2.126) are of the form

$$\begin{aligned}\left\langle \alpha_1^{n+1}\alpha_2^m\left(\mathrm{e}^{s\Omega}-1\right)\right\rangle_3 &= -\frac{\xi_1(\xi_3+\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n\alpha_2^m\left(\mathrm{e}^{s\Omega}-1\right)\right\rangle_3 \\ &+ 2n\frac{\xi_1}{\Delta}\left\langle \alpha_1^{n-1}\alpha_2^m\left(\mathrm{e}^{s\Omega}-1-s\Omega\right)\right\rangle_3 \\ &+ m\frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n\alpha_2^{m-1}\left(\mathrm{e}^{s\Omega}-1-s\Omega\right)\right\rangle_3 \\ &- \frac{(\xi_3+\xi_1-\xi_2)}{\Delta}\left\langle \alpha_1^n\alpha_2^m\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_3}-1+\alpha_1\alpha_2\xi_3\right)\right\rangle_2 \\ &+ \gamma(n,m), \end{aligned} \tag{2.154}$$

$$\gamma(n,m)=0, \qquad n>0, \qquad m>0 \tag{2.155}$$

$$\gamma(n,0)=\frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_2}-1+\alpha_1\alpha_2\xi_2\right)\right\rangle_2, \qquad n>0 \tag{2.156}$$

$$\gamma(0,m)=2\frac{\xi_1}{\Delta}\left\langle \alpha_1^m\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_1}-1+\alpha_1\alpha_2\xi_1\right)\right\rangle_2, \qquad m>0 \tag{2.157}$$

$$\begin{aligned}\gamma(0,0) &= 2\frac{\xi_1}{\Delta}\left\langle\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_1}-1+\alpha_1\alpha_2\xi_1\right)\right\rangle_2 \\ &+ \frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_2}-1+\alpha_1\alpha_2\xi_2\right)\right\rangle_2, \end{aligned} \tag{2.158}$$

$$\begin{aligned}\left\langle \alpha_1^n\alpha_2^{m+1}\left(\mathrm{e}^{s\Omega}-1\right)\right\rangle_3 &= -\frac{\xi_2(\xi_3+\xi_1-\xi_2)}{\Delta}\left\langle \alpha_1^n\alpha_2^m\left(\mathrm{e}^{s\Omega}-1\right)\right\rangle_3 \\ &+ 2m\frac{\xi_2}{\Delta}\left\langle \alpha_1^n\alpha_2^{m-1}\left(\mathrm{e}^{s\Omega}-1-s\Omega\right)\right\rangle_3 \\ &+ n\frac{(\xi_3-\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^{n-1}\alpha_2^m\left(\mathrm{e}^{s\Omega}-1-s\Omega\right)\right\rangle_3 \\ &- \frac{(\xi_3+\xi_2-\xi_1)}{\Delta}\left\langle \alpha_1^n\alpha_2^m\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_3}-1+\alpha_1\alpha_2\xi_3\right)\right\rangle_2 \\ &+ \sigma(n,m), \end{aligned} \tag{2.159}$$

$$\sigma(n,m)=0, \qquad n>0, \qquad m>0 \tag{2.160}$$

$$\sigma(n,0) = 2\frac{\xi_2}{\Delta}\left\langle \alpha_1^n\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_2} - 1 + \alpha_1\alpha_2\xi_2\right)\right\rangle_2, \qquad n > 0 \tag{2.161}$$

$$\sigma(0,m) = \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta}\left\langle \alpha_1^m\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_1} - 1 + \alpha_1\alpha_2\xi_1\right)\right\rangle_2, \qquad m > 0 \tag{2.162}$$

$$\sigma(0,0) = 2\frac{\xi_2}{\Delta}\left\langle\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_2} - 1 + \alpha_1\alpha_2\xi_2\right)\right\rangle_2 + \frac{(\xi_3 - \xi_2 - \xi_1)}{\Delta}\left\langle\left(\mathrm{e}^{-\alpha_1\alpha_2\xi_1} - 1 + \alpha_1\alpha_2\xi_1\right)\right\rangle_2, \tag{2.163}$$

and, for the combinations (2.86) themselves, one has

$$\left\langle\left(\mathrm{e}^{s\Omega} - 1\right)\right\rangle_3 = F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}, \tag{2.164}$$

$$\left\langle\left(\mathrm{e}^{s\Omega} - 1 - s\Omega\right)\right\rangle_3 = F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{1}{24}(\xi_1 + \xi_2 + \xi_3) \tag{2.165}$$

in terms of (2.131). However, the analyticity in ξ_1, ξ_2, ξ_3 is now maintained by a more general mechanism. The analyticity holds only in the sum of the form factors on the right-hand side of (2.144) (and, similarly, (2.149), (2.154), (2.159)), and it is a nontrivial fact that, when these form factors are expanded in power series in ξ, the denominator Δ gets always cancelled. The mechanism of maintaining analyticity is based on the existence of linear differential equations which the functions (2.130) and (2.131) satisfy.

The differential equations for the basic form factors can be derived with the aid of the recurrence relations above. From (2.130), one has

$$-\frac{d}{d\xi}f(\xi) = \left\langle \alpha_1\alpha_2\mathrm{e}^{-\alpha_1\alpha_2\xi}\right\rangle_2 \tag{2.166}$$

which, by means of (2.135), (2.136), leads to the following equation for the function $f(\xi)$:

$$-\frac{d}{d\xi}f(\xi) = \frac{1}{4}f(\xi) + \frac{1}{2}\frac{f(\xi) - 1}{\xi}. \tag{2.167}$$

The form factor (2.140) with two subtractions expressed through the second derivative of $f(\xi)$ is

$$\frac{d^2}{d\xi^2}f(\xi) = \frac{1}{16}f(\xi) + \frac{1}{4}\frac{f(\xi) - 1}{\xi} + \frac{3}{4}\frac{f(\xi) - 1 + \frac{1}{6}\xi}{\xi^2}. \tag{2.168}$$

Similarly, one obtains the equation for the form factor (2.131):

$$\begin{aligned}
-\frac{\partial}{\partial \xi_1} F(\xi_1, \xi_2, \xi_3) &= \frac{1}{\Delta^2}\Big[(\xi_1 - \xi_2 - \xi_3)\Delta \\
&+ \xi_2\xi_3(2\xi_2\xi_3 - \xi_2^2 - \xi_3^2 + \xi_1^2)\Big] F(\xi_1, \xi_2, \xi_3) \\
&+ \frac{1}{2}\frac{8\xi_1\xi_2\xi_3 + (\xi_2 + \xi_3 - \xi_1)\Delta}{\Delta^2} f(\xi_1) \\
&+ 2\frac{\xi_2\xi_3(\xi_3 - \xi_2 - \xi_1)}{\Delta^2} f(\xi_2) \\
&+ 2\frac{\xi_2\xi_3(\xi_2 - \xi_3 - \xi_1)}{\Delta^2} f(\xi_3).
\end{aligned} \tag{2.169}$$

The function $F(\xi_1, \xi_2, \xi_3)$ is completely symmetric in ξ_1, ξ_2, ξ_3 and, therefore, satisfies two other equations, with $\partial/\partial\xi_2$ and $\partial/\partial\xi_3$, derivable from (2.169) by symmetry. Finally, as a consequence of these equations, one can derive an equation for the form factor (2.131) as a function of s:

$$\begin{aligned}
-s\frac{\partial}{\partial s} F(-s\Box_1, -s\Box_2, -s\Box_3) &= \left(s\frac{\Box_1\Box_2\Box_3}{D} + 1\right) F(-s\Box_1, -s\Box_2, -s\Box_3) \\
&+ \frac{\Box_1(\Box_3 + \Box_2 - \Box_1)}{2D} f(-s\Box_1) \\
&+ \frac{\Box_2(\Box_3 + \Box_1 - \Box_2)}{2D} f(-s\Box_2) \\
&+ \frac{\Box_3(\Box_1 + \Box_2 - \Box_3)}{2D} f(-s\Box_3),
\end{aligned} \tag{2.170}$$

$$D = \Box_1{}^2 + \Box_2{}^2 + \Box_3{}^2 - 2\Box_1\Box_2 - 2\Box_1\Box_3 - 2\Box_2\Box_3. \tag{2.171}$$

This is an important equation since it will be a key tool for study of conformal field models in Chapter 4.

2.5 Final result for the trace of the heat kernel to third order in the curvature

By applying the reduction technique above to expressions (2.87)–(2.124), the form factors f_i with $i = 1$ to 5 and F_i with $i = 1$ to 33 can be brought to their final forms. Quite remarkable is that form factors F_{31}, F_{32}, F_{33} just vanish in this representation:

$$F_{31}(\xi_1, \xi_2, \xi_3) = 0, \tag{2.172}$$

$$F_{32}(\xi_1,\xi_2,\xi_3) = 0, \tag{2.173}$$

$$F_{33}(\xi_1,\xi_2,\xi_3) = 0. \tag{2.174}$$

The result obtained for the form factor F_{30} can be found in Appendix B. This nonvanishing form factor F_{30} is, however, symmetric under a permutation of the labels 2 and 3:

$$F_{30}(\xi_1,\xi_2,\xi_3) = F_{30}(\xi_1,\xi_3,\xi_2) \tag{2.175}$$

as one can check by a direct inspection of expression (B.30). On the other hand, the structure 30 in eq. (2.76) is antisymmetric under this permutation:

$$\begin{aligned}\Re_1\Re_2\Re_3(30) &= \nabla_\beta \hat{\mathcal{R}}_1^{\beta\alpha}\nabla_\alpha R_2 R_3\\ &= -\nabla_\beta \hat{\mathcal{R}}_1^{\beta\alpha}\nabla_\alpha R_3 R_2 + \mathrm{O}[\Re^4] + \text{a total derivative},\end{aligned} \tag{2.176}$$

and, therefore, the contribution of this structure vanishes

$$\int dx\, g^{1/2}\, \mathrm{tr} F_{30}(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2\Re_3(30) = \mathrm{O}[\Re^4]. \tag{2.177}$$

The difference is only that none of the properties (2.172)–(2.174), (2.177) appear before the form factors are brought into a unique representation by eliminating the α-polynomials.

Because the contributions of four extra structures vanish there remain only the contributions of the twenty nine cubic structures. The final result is [34, 54],

$$\begin{aligned}\mathrm{Tr}K(s) \;=\; & \frac{1}{(4\pi s)^\omega}\int dx\, g^{1/2}\,\mathrm{tr}\Big\{\hat{1} + s\hat{P}\\ & + s^2\sum_{i=1}^{5} f_i(-s\Box_2)\Re_1\Re_2(i)\\ & + s^3\sum_{i=1}^{11} F_i(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2\Re_3(i)\\ & + s^4\sum_{i=12}^{25} F_i(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2\Re_3(i)\\ & + s^5\sum_{i=26}^{28} F_i(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2\Re_3(i)\\ & + s^6 F_{29}(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2\Re_3(29)\\ & \quad + \mathrm{O}[\Re^4]\Big\}.\end{aligned} \tag{2.178}$$

The second order form factors $f_i(\xi)$ for $i = 1$ to 5 are [30] expressed through the basic second order form factor (2.130):

$$f_1(\xi) = \frac{(f(\xi) - 1 + \frac{1}{6}\xi)}{\xi^2}, \tag{2.179}$$

$$f_2(\xi) = \frac{1}{8}\left[\frac{1}{36}f(\xi) + \frac{1}{3}\frac{(f(\xi)-1)}{\xi} - \frac{(f(\xi) - 1 + \frac{1}{6}\xi)}{\xi^2}\right], \tag{2.180}$$

$$f_3(\xi) = \frac{1}{12}f(\xi) + \frac{1}{2}\frac{(f(\xi)-1)}{\xi}, \tag{2.181}$$

$$f_4(\xi) = \frac{1}{2}f(\xi), \tag{2.182}$$

$$f_5(\xi) = -\frac{1}{2}\frac{(f(\xi)-1)}{\xi}. \tag{2.183}$$

The form factors $F_i(\xi_1, \xi_2, \xi_3)$, $i = 1$ to 29 are expressed through the basic third order form factor $F_1(\xi_1, \xi_2, \xi_3)$ and the basic second order form factor $f(\xi)$. The coefficients of these expressions are rational functions with a universal denominator (2.143) raised to a certain power. The explicit expressions for the (unsymmetrized) third order form factors can be found in the Appendix C.

In expression (2.178) the form factors F_i automatically acquire symmetries (if any) of their respective curvature structures $\mathrm{tr}\Re_1\Re_2\Re_3(i)$. These symmetries (under permutations of the labels 1,2,3) follow from the table (2.47)–(2.75) and imply the following symmetrization of the form factors:

$$\begin{aligned} F_1^{\mathrm{sym}}(\xi_1, \xi_2, \xi_3) &= \frac{1}{3}\Big(F_1(\xi_1, \xi_2, \xi_3) + F_1(\xi_3, \xi_1, \xi_2) \\ &\quad + F_1(\xi_2, \xi_3, \xi_1)\Big), \end{aligned} \tag{2.184}$$

$$\begin{aligned} F_2^{\mathrm{sym}}(\xi_1, \xi_2, \xi_3) &= \frac{1}{3}\Big(F_2(\xi_1, \xi_2, \xi_3) + F_2(\xi_3, \xi_1, \xi_2) \\ &\quad + F_2(\xi_2, \xi_3, \xi_1)\Big), \end{aligned} \tag{2.185}$$

$$F_3^{\text{sym}}(\xi_1, \xi_2, \xi_3) = F_3(\xi_1, \xi_2, \xi_3), \tag{2.186}$$

$$F_4^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big(F_4(\xi_1, \xi_2, \xi_3) + F_4(\xi_2, \xi_1, \xi_3)\Big), \tag{2.187}$$

$$F_5^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big(F_5(\xi_1, \xi_2, \xi_3) + F_5(\xi_2, \xi_1, \xi_3)\Big), \tag{2.188}$$

$$F_6^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big(F_6(\xi_1, \xi_2, \xi_3) + F_6(\xi_2, \xi_1, \xi_3)\Big), \tag{2.189}$$

$$F_7^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big(F_7(\xi_1, \xi_2, \xi_3) + F_7(\xi_1, \xi_3, \xi_2)\Big), \tag{2.190}$$

$$F_8^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big(F_8(\xi_1, \xi_2, \xi_3) + F_8(\xi_1, \xi_3, \xi_2)\Big), \tag{2.191}$$

$$\begin{aligned} F_9^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{6}\Big(&F_9(\xi_1, \xi_2, \xi_3) + F_9(\xi_3, \xi_1, \xi_2) \\ &+ F_9(\xi_2, \xi_3, \xi_1) + F_9(\xi_2, \xi_1, \xi_3) \\ &+ F_9(\xi_3, \xi_2, \xi_1) + F_9(\xi_1, \xi_3, \xi_2)\Big), \end{aligned} \tag{2.192}$$

$$\begin{aligned} F_{10}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{6}\Big(&F_{10}(\xi_1, \xi_2, \xi_3) + F_{10}(\xi_3, \xi_1, \xi_2) \\ &+ F_{10}(\xi_2, \xi_3, \xi_1) + F_{10}(\xi_2, \xi_1, \xi_3) \\ &+ F_{10}(\xi_3, \xi_2, \xi_1) + F_{10}(\xi_1, \xi_3, \xi_2)\Big), \end{aligned} \tag{2.193}$$

$$F_{11}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big(F_{11}(\xi_1, \xi_2, \xi_3) + F_{11}(\xi_2, \xi_1, \xi_3)\Big), \tag{2.194}$$

$$F_{12}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = F_{12}(\xi_1, \xi_2, \xi_3), \tag{2.195}$$

$$F_{13}^{\text{sym}}(\xi_1, \xi_2, \xi_3) = F_{13}(\xi_1, \xi_2, \xi_3), \tag{2.196}$$

$$F_{14}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = F_{14}(\xi_1,\xi_2,\xi_3), \tag{2.197}$$

$$F_{15}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = F_{15}(\xi_1,\xi_2,\xi_3), \tag{2.198}$$

$$F_{16}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{16}(\xi_1,\xi_2,\xi_3) + F_{16}(\xi_2,\xi_1,\xi_3)\Big), \tag{2.199}$$

$$F_{17}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = F_{17}(\xi_1,\xi_2,\xi_3), \tag{2.200}$$

$$F_{18}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{18}(\xi_1,\xi_2,\xi_3) + F_{18}(\xi_1,\xi_3,\xi_2)\Big), \tag{2.201}$$

$$F_{19}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{19}(\xi_1,\xi_2,\xi_3) + F_{19}(\xi_1,\xi_3,\xi_2)\Big), \tag{2.202}$$

$$F_{20}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{20}(\xi_1,\xi_2,\xi_3) + F_{20}(\xi_1,\xi_3,\xi_2)\Big), \tag{2.203}$$

$$F_{21}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = F_{21}(\xi_1,\xi_2,\xi_3), \tag{2.204}$$

$$F_{22}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{22}(\xi_1,\xi_2,\xi_3) + F_{22}(\xi_1,\xi_3,\xi_2)\Big), \tag{2.205}$$

$$F_{23}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{23}(\xi_1,\xi_2,\xi_3) + F_{23}(\xi_2,\xi_1,\xi_3)\Big), \tag{2.206}$$

$$F_{24}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{24}(\xi_1,\xi_2,\xi_3) + F_{24}(\xi_1,\xi_3,\xi_2)\Big), \tag{2.207}$$

$$F_{25}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{25}(\xi_1,\xi_2,\xi_3) + F_{25}(\xi_1,\xi_3,\xi_2)\Big), \tag{2.208}$$

$$F_{26}^{\mathrm{sym}}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{26}(\xi_1,\xi_2,\xi_3) + F_{26}(\xi_2,\xi_1,\xi_3)\Big), \tag{2.209}$$

$$F_{27}^{\rm sym}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{27}(\xi_1,\xi_2,\xi_3)+F_{27}(\xi_2,\xi_1,\xi_3)\Big), \tag{2.210}$$

$$F_{28}^{\rm sym}(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\Big(F_{28}(\xi_1,\xi_2,\xi_3)+F_{28}(\xi_2,\xi_1,\xi_3)\Big), \tag{2.211}$$

$$F_{29}^{\rm sym}(\xi_1,\xi_2,\xi_3) = \frac{1}{3}(F_{29}(\xi_1,\xi_2,\xi_3)+F_{29}(\xi_3,\xi_1,\xi_2) + F_{29}(\xi_2,\xi_3,\xi_1)). \tag{2.212}$$

When taken separately from their curvature structures, the functions F_i make sense only being explicitly symmetrized as above.

Note that all the structures (2.76)–(2.79) whose contributions vanish are linear in $\hat{\mathcal{R}}_{\mu\nu}$. In the final result for the trace of the heat kernel, there remains only one such a structure, namely, $\Re_1\Re_2\Re_3(13) = \hat{\mathcal{R}}_1^{\mu\nu}\nabla_\mu\hat{P}_2\nabla_\nu\hat{P}_3$ (eq. (2.59)). Its form factor (B.13) is symmetric under a permutation of the labels 2 and 3:

$$F_{13}(\xi_1,\xi_2,\xi_3) = F_{13}(\xi_1,\xi_3,\xi_2) \tag{2.213}$$

but, because all the three curvatures in (2.59) are matrices, this structure possesses no antisymmetry under this permutation. Its contribution can be written down as

$$\int dx\, g^{1/2}\,{\rm tr}F_{13}(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2\Re_3(13) = \frac{1}{2}\int dx\, g^{1/2}\,{\rm tr}F\hat{\mathcal{R}}^{\mu\nu}\Big[\nabla_\mu\hat{P},\nabla_\nu\hat{P}\Big], \tag{2.214}$$

and it does not vanish in the general case, as one can convince oneself by considering simple examples.

2.6 The large time behavior of the trace of the heat kernel

Derivation of the large time behavior of the form factors in the heat kernel was given in [30] to all orders in the curvature (see sect. 2.3). For the basic form factors (2.130) and (2.131) this behavior is

$$f(-s\Box) = -\frac{1}{s}\frac{2}{\Box} + {\rm O}\left(\frac{1}{s^2}\right), \qquad s\to\infty \tag{2.215}$$

$$F(-s\Box_1, -s\Box_2, -s\Box_3) = \frac{1}{s^2}\Big(\frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3}\Big) + \mathrm{O}\left(\frac{1}{s^3}\right), \qquad s \to \infty. \quad (2.216)$$

The large time behavior of all second order and third order form factors follows then from the explicit expressions above. Another way is to use the α-representation of the form factors in sect. 2.3, and eqs. (2.127), (2.128). With the symmetries (2.184)–(2.212) taken into account, one can easily obtain desired asymptotics for all form factors.

Not all third order form factors contribute to the leading asymptotic order. Those absent are form factors F_i with $i = 3, 5, 7, 10, 15 - 17, 25 - 29$. On the whole, the result is that the behavior of the trace of the heat kernel at large s is $s^{-\omega+1}$, and the coefficient of this asymptotic behavior is obtained to third order in the curvature. As shown in [30], this behavior holds at all orders in the curvature except the zeroth. This power asymptotic behavior is characteristic of a noncompact manifold. (For the discussion of compact manifolds see [55]).

The explicit asymptotic form of $\mathrm{Tr}K(s)$ is as follows:

$$\mathrm{Tr}K(s) = \frac{s}{(4\pi s)^\omega} \int dx\, g^{1/2}\, \mathrm{tr}\, \Big\{M_1 + M_2 + M_3 + \mathrm{O}[\Re^4]\Big\} + \mathrm{O}\left(\frac{1}{s^\omega}\right), \qquad s \to \infty, \quad (2.217)$$

where M_i for $i = 1$ to 3 denote the first three orders in curvatures of this expansion. Their explicit form is as follows,

$$M_1 = \hat{P}, \qquad (2.218)$$

$$\begin{aligned} M_2 = & -\hat{P}\frac{1}{\Box}\hat{P} - \frac{1}{2}\hat{\mathcal{R}}_{\mu\nu}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\nu} + \frac{1}{3}\hat{P}\frac{1}{\Box}R \\ & -\frac{1}{6}R_{\mu\nu}\frac{1}{\Box}R^{\mu\nu}\hat{1} + \frac{1}{18}R\frac{1}{\Box}R\hat{1}, \end{aligned} \qquad (2.219)$$

$$\begin{aligned} M_3 = & \hat{P}\frac{1}{\Box}\hat{P}\frac{1}{\Box}\hat{P} - 2\hat{\mathcal{R}}^\mu{}_\alpha\frac{1}{\Box}\hat{\mathcal{R}}^\alpha{}_\beta\frac{1}{\Box}\hat{\mathcal{R}}^\beta{}_\mu \\ & + \frac{1}{36}\frac{1}{\Box}R\frac{1}{\Box}R\hat{P} + \frac{1}{18}R\frac{1}{\Box}R\frac{1}{\Box}\hat{P} \\ & -\frac{1}{6}\frac{1}{\Box}\hat{P}\frac{1}{\Box}\hat{P}R - \frac{1}{3}\hat{P}\frac{1}{\Box}\hat{P}\frac{1}{\Box}R \\ & +2\frac{1}{\Box}R^{\alpha\beta}\frac{1}{\Box}\hat{\mathcal{R}}_\alpha{}^\mu\hat{\mathcal{R}}_{\beta\mu} - \frac{1}{216}R\frac{1}{\Box}R\frac{1}{\Box}R\hat{1} \end{aligned}$$

$$+\frac{1}{12}\frac{1}{\Box}R^{\mu\nu}\frac{1}{\Box}R_{\mu\nu}R\hat{1}-\frac{1}{6}R^{\mu\nu}\frac{1}{\Box}R_{\mu\nu}\frac{1}{\Box}R\hat{1}$$
$$-2\frac{1}{\Box}\hat{\mathcal{R}}^{\alpha\beta}\nabla^{\mu}\frac{1}{\Box}\hat{\mathcal{R}}_{\mu\alpha}\nabla^{\nu}\frac{1}{\Box}\hat{\mathcal{R}}_{\nu\beta}-2\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\nu}\nabla_{\mu}\frac{1}{\Box}\hat{P}\nabla_{\nu}\frac{1}{\Box}\hat{P}$$
$$-2\nabla_{\mu}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\alpha}\nabla^{\nu}\frac{1}{\Box}\hat{\mathcal{R}}_{\nu\alpha}\frac{1}{\Box}\hat{P}+2\frac{1}{\Box}R_{\alpha\beta}\nabla_{\mu}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\alpha}\nabla_{\nu}\frac{1}{\Box}\hat{\mathcal{R}}^{\nu\beta}$$
$$-\frac{1}{\Box}R^{\alpha\beta}\nabla_{\alpha}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\nu}\nabla_{\beta}\frac{1}{\Box}\hat{\mathcal{R}}_{\mu\nu}+\frac{1}{3}\frac{1}{\Box}R\nabla_{\alpha}\frac{1}{\Box}\hat{\mathcal{R}}^{\alpha\mu}\nabla^{\beta}\frac{1}{\Box}\hat{\mathcal{R}}_{\beta\mu}$$
$$+4\frac{1}{\Box}R^{\mu\nu}\nabla_{\mu}\nabla_{\lambda}\frac{1}{\Box}\hat{\mathcal{R}}^{\lambda\alpha}\frac{1}{\Box}\hat{\mathcal{R}}_{\alpha\nu}+\frac{1}{6}\frac{1}{\Box}R^{\alpha\beta}\nabla_{\alpha}\frac{1}{\Box}R\nabla_{\beta}\frac{1}{\Box}R\hat{1}$$
$$-\frac{1}{3}\nabla^{\mu}\frac{1}{\Box}R^{\nu\alpha}\nabla_{\nu}\frac{1}{\Box}R_{\mu\alpha}\frac{1}{\Box}R\hat{1}-\frac{1}{3}\frac{1}{\Box}R^{\mu\nu}\nabla_{\mu}\frac{1}{\Box}R^{\alpha\beta}\nabla_{\nu}\frac{1}{\Box}R_{\alpha\beta}\hat{1}. \qquad (2.220)$$

2.7 The short time behavior of the trace of the heat kernel, and comparison with the Schwinger-DeWitt expansion

Derivation of the short time behavior of the trace the heat kernel presents no problem. One may use either of two form factor representations: the explicit expressions (2.179)–(2.183) of sect. 2.5 and (B.1)–(B.29) of Appendix C, or the α-representation (2.87)–(2.120) of sect. 2.3 combined with eqs. (2.125), (2.126).

We need expansions for the basic second order form factor (2.130) and its modifications with subtractions (2.139)–(2.140) only up to second order in s but in Chapter 2 they are needed up to third order (except for the form factor with two subtractions (2.140)), so a sufficient series is of the form:

$$f(-s\Box)=1+\frac{1}{6}s\Box+\frac{1}{60}s^2\Box^2+\frac{1}{840}s^3\Box^3+\frac{1}{15120}s^4\Box^4+\mathrm{O}\big(s^5\big),\qquad s\to 0. \quad (2.221)$$

Similarly, we need the basic third order form factor (2.131) up to $\mathrm{O}[s^4]$:

$$F(-s\Box_1,-s\Box_2,-s\Box_3)=\frac{1}{2}+s\frac{1}{24}(\Box_1+\Box_2+\Box_3)$$
$$+s^2\frac{1}{2}\Big[\frac{1}{180}(\Box_3\Box_2+\Box_1\Box_2+\Box_1\Box_3+\Box_3{}^2+\Box_1{}^2+\Box_2{}^2)\Big]$$
$$+s^3\frac{1}{6}\Big[\frac{1}{1120}(\Box_3{}^3+\Box_1\Box_2{}^2+\Box_1{}^2\Box_2+\Box_3{}^2\Box_2$$
$$+\Box_3\Box_2{}^2+\Box_1{}^3+\Box_2{}^3+\Box_1{}^2\Box_3+\Box_1\Box_3{}^2)+\frac{1}{840}\Box_1\Box_2\Box_3\Big]$$
$$+s^4\frac{1}{24}\Big[\frac{1}{6300}(\Box_1\Box_3{}^3+\Box_3{}^2\Box_2{}^2+\Box_3\Box_1{}^3+\Box_2{}^4+\Box_2{}^3\Box_3$$

$$
\begin{aligned}
&+ \Box_1{}^2\Box_2{}^2 + \Box_1\Box_2{}^3 + \Box_3{}^2\Box_1{}^2 + \Box_2\Box_1{}^3 + \Box_1{}^4 + \Box_3{}^4 + \Box_2\Box_3{}^3) \\
&+ \frac{1}{4200}(\Box_1\Box_3{}^2\Box_2 + \Box_1\Box_3\Box_2{}^2 + \Box_1{}^2\Box_3\Box_2)\Big] \\
&+ \mathrm{O}\big(s^5\big), \quad s \to 0, \qquad (2.222)
\end{aligned}
$$

It is not worthwhile to reproduce the table of asymptotic behaviors for form factors. Insead we proceed to the short time or the Schwinger-DeWitt expansion (1.15) for the trace of the heat kernel (1.18) which is of the form [12]:

$$
\mathrm{Tr}K(s) = \frac{1}{(4\pi s)^\omega} \sum_{n=0}^{\infty} s^n \int dx\, g^{1/2}\, \mathrm{tr}\ \hat{a}_n(x,x), \qquad (2.223)
$$

where $\hat{a}_n(x,x)$ are the local Schwinger–DeWitt coefficients with coincident arguments. In contrast to (1.15) here the $\hat{a}_n$ coefficients appear under the integral and matrix trace operations.

By inserting short time expansions of the form factors into (2.178), one arrives at eq. (2.223) with the following results for the integrated Schwinger-DeWitt coefficients a_0 to a_4:

$$
\int dx\, g^{1/2}\, \mathrm{tr}\ \hat{a}_0(x,x) = \int dx\, g^{1/2}\, \mathrm{tr}\ \hat{1}, \qquad (2.224)
$$

$$
\int dx\, g^{1/2}\, \mathrm{tr}\ \hat{a}_1(x,x) = \int dx\, g^{1/2}\, \mathrm{tr}\ \hat{P}, \qquad (2.225)
$$

$$
\begin{aligned}
&\int dx\, g^{1/2}\, \mathrm{tr}\ \hat{a}_2(x,x) = \int dx\, g^{1/2}\, \mathrm{tr}\ \Big\{\frac{1}{2}\hat{P}_1\hat{P}_2 + \frac{1}{12}\hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu} \\
&+ \frac{1}{60}R_{1\,\mu\nu}R_2^{\mu\nu}\hat{1} - \frac{1}{180}R_1R_2\hat{1} + \frac{\Box_3}{360\Box_1\Box_2}R_1R_2R_3\hat{1} \\
&+ \Big(-\frac{1}{45\Box_3} + \frac{\Box_3}{90\Box_1\Box_2}\Big)R_1{}^{\mu}{}_{\alpha}R_2{}^{\alpha}{}_{\beta}R_3{}^{\beta}{}_{\mu}\hat{1} + \Big(\frac{1}{90\Box_2} - \frac{\Box_3}{180\Box_1\Box_2}\Big)R_1^{\mu\nu}R_{2\,\mu\nu}R_3\hat{1} \\
&+ \Big(-\frac{1}{90\Box_1\Box_2} - \frac{1}{60\Box_2\Box_3}\Big)R_1^{\alpha\beta}\nabla_\alpha R_2\nabla_\beta R_3\hat{1} + \frac{1}{45\Box_1\Box_2}\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\,\mu\alpha}R_3\hat{1} \\
&+ \frac{1}{45\Box_2\Box_3}R_1^{\mu\nu}\nabla_\mu R_2^{\alpha\beta}\nabla_\nu R_{3\,\alpha\beta}\hat{1} + \Big(\frac{2}{45\Box_1\Box_2} - \frac{1}{45\Box_2\Box_3}\Big)R_1^{\mu\nu}\nabla_\alpha R_{2\,\beta\mu}\nabla^\beta R_3{}^{\alpha}{}_{\nu}\hat{1} \\
&- \frac{1}{45\Box_1\Box_2\Box_3}\nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}R_3\hat{1} \\
&- \frac{2}{45\Box_1\Box_2\Box_3}\nabla_\mu R_1^{\alpha\lambda}\nabla_\nu R_2{}^{\beta}{}_{\lambda}\nabla_\alpha\nabla_\beta R_3^{\mu\nu}\hat{1}\Big\} + \mathrm{O}[\Re^4], \qquad (2.226)
\end{aligned}
$$

$$\int dx\, g^{1/2}\, \mathrm{tr}\, \hat{a}_3(x,x) = \int dx\, g^{1/2}\, \mathrm{tr} \Big\{ \frac{\Box_2}{12} \hat{P}_1 \hat{P}_2 + \frac{\Box_2}{120} \hat{\mathcal{R}}_{1\mu\nu} \hat{\mathcal{R}}_2^{\mu\nu} + \frac{\Box_2}{180} \hat{P}_1 R_2$$
$$+ \frac{\Box_2}{840} R_{1\,\mu\nu} R_2^{\mu\nu} \hat{1} - \frac{\Box_2}{3780} R_1 R_2 \hat{1} + \frac{1}{6} \hat{P}_1 \hat{P}_2 \hat{P}_3 - \frac{1}{45} \hat{\mathcal{R}}_1{}^{\mu}{}_{\alpha} \hat{\mathcal{R}}_2{}^{\alpha}{}_{\beta} \hat{\mathcal{R}}_3{}^{\beta}{}_{\mu}$$
$$+ \frac{1}{12} \hat{\mathcal{R}}_1^{\mu\nu} \hat{\mathcal{R}}_{2\,\mu\nu} \hat{P}_3 + \Big(\frac{1}{180} + \frac{\Box_1}{90\Box_2} - \frac{\Box_3}{45\Box_2} + \frac{\Box_3{}^2}{180\Box_1\Box_2} \Big) R_1^{\mu\nu} R_{2\,\mu\nu} \hat{P}_3$$
$$+ \Big(\frac{1}{45} + \frac{\Box_3}{18\Box_1} \Big) R_1^{\alpha\beta} \hat{\mathcal{R}}_{2\,\alpha}{}^{\mu} \hat{\mathcal{R}}_{3\,\beta\mu} + \Big(-\frac{1}{3240} + \frac{\Box_1}{2520\Box_3} + \frac{\Box_3{}^2}{3360\Box_1\Box_2} \Big) R_1 R_2 R_3 \hat{1}$$
$$+ \Big(-\frac{1}{1890} - \frac{\Box_1}{420\Box_3} + \frac{\Box_3{}^2}{840\Box_1\Box_2} \Big) R^{\mu}_{1\,\alpha} R^{\alpha}_{2\,\beta} R^{\beta}_{3\,\mu} \hat{1}$$
$$+ \Big(\frac{1}{15120} - \frac{\Box_1}{15120\Box_2} + \frac{\Box_3}{1890\Box_2} - \frac{\Box_3{}^2}{4320\Box_1\Box_2} \Big) R_1^{\mu\nu} R_{2\,\mu\nu} R_3 \hat{1}$$
$$+ \Big(\frac{2}{45\Box_2} - \frac{\Box_3}{45\Box_1\Box_2} \Big) \nabla^{\mu} R_1^{\nu\alpha} \nabla_{\nu} R_{2\,\mu\alpha} \hat{P}_3 + \frac{1}{18\Box_1} R_{1\,\alpha\beta} \nabla_{\mu} \hat{\mathcal{R}}_2^{\mu\alpha} \nabla_{\nu} \hat{\mathcal{R}}_3^{\nu\beta}$$
$$- \frac{1}{36\Box_1} R_1^{\alpha\beta} \nabla_{\alpha} \hat{\mathcal{R}}_2^{\mu\nu} \nabla_{\beta} \hat{\mathcal{R}}_{3\,\mu\nu} + \frac{1}{9\Box_1} R_1^{\mu\nu} \nabla_{\mu} \nabla_{\lambda} \hat{\mathcal{R}}_2^{\lambda\alpha} \hat{\mathcal{R}}_{3\,\alpha\nu}$$
$$+ \Big(\frac{1}{2520\Box_1} - \frac{1}{1260\Box_2} - \frac{\Box_1}{630\Box_2\Box_3} - \frac{\Box_3}{1260\Box_1\Box_2} \Big) R_1^{\alpha\beta} \nabla_{\alpha} R_2 \nabla_{\beta} R_3 \hat{1}$$
$$+ \Big(-\frac{1}{3780\Box_2} + \frac{\Box_3}{1080\Box_1\Box_2} \Big) \nabla^{\mu} R_1^{\nu\alpha} \nabla_{\nu} R_{2\,\mu\alpha} R_3 \hat{1}$$
$$+ \Big(-\frac{1}{420\Box_2} + \frac{\Box_1}{504\Box_2\Box_3} \Big) R_1^{\mu\nu} \nabla_{\mu} R_2^{\alpha\beta} \nabla_{\nu} R_{3\,\alpha\beta} \hat{1}$$
$$+ \Big(-\frac{1}{315\Box_1} - \frac{\Box_1}{420\Box_2\Box_3} + \frac{\Box_3}{210\Box_1\Box_2} \Big) R_1^{\mu\nu} \nabla_{\alpha} R_{2\,\beta\mu} \nabla^{\beta} R^{\alpha}_{3\,\nu} \hat{1}$$
$$+ \frac{1}{45\Box_1\Box_2} \nabla_{\alpha} \nabla_{\beta} R_1^{\mu\nu} \nabla_{\mu} \nabla_{\nu} R_2^{\alpha\beta} \hat{P}_3$$
$$+ \Big(-\frac{1}{756\Box_1\Box_2} - \frac{1}{252\Box_2\Box_3} \Big) \nabla_{\alpha} \nabla_{\beta} R_1^{\mu\nu} \nabla_{\mu} \nabla_{\nu} R_2^{\alpha\beta} R_3 \hat{1}$$
$$+ \Big(-\frac{1}{315\Box_1\Box_2} - \frac{1}{105\Box_2\Box_3} \Big) \nabla_{\mu} R_1^{\alpha\lambda} \nabla_{\nu} R^{\beta}_{2\,\lambda} \nabla_{\alpha} \nabla_{\beta} R_3^{\mu\nu} \hat{1}$$
$$+ \frac{1}{1890\Box_1\Box_2\Box_3} \nabla_{\lambda} \nabla_{\sigma} R_1^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} R_2^{\mu\nu} \nabla_{\mu} \nabla_{\nu} R_3^{\lambda\sigma} \hat{1} \Big\} + \mathrm{O}[\Re^4], \qquad (2.227)$$

$$\int dx\, g^{1/2}\, \mathrm{tr}\, \hat{a}_4(x,x) = \int dx\, g^{1/2}\, \mathrm{tr}\, \Big\{ \frac{\Box_2{}^2}{120} \hat{P}_1 \hat{P}_2 + \frac{\Box_2{}^2}{1260} \hat{P}_1 R_2 + \frac{\Box_2{}^2}{1680} \hat{\mathcal{R}}_{1\mu\nu} \hat{\mathcal{R}}_2^{\mu\nu}$$
$$+ \frac{\Box_2{}^2}{15120} R_{1\,\mu\nu} R_2^{\mu\nu} \hat{1} + \frac{\Box_3}{24} \hat{P}_1 \hat{P}_2 \hat{P}_3 - \frac{\Box_3}{630} \hat{\mathcal{R}}_1{}^{\mu}{}_{\alpha} \hat{\mathcal{R}}_2{}^{\alpha}{}_{\beta} \hat{\mathcal{R}}_3{}^{\beta}{}_{\mu}$$
$$+ \Big(\frac{\Box_1}{180} + \frac{\Box_2}{180} + \frac{\Box_3}{90} \Big) \hat{\mathcal{R}}_1^{\mu\nu} \hat{\mathcal{R}}_{2\,\mu\nu} \hat{P}_3 + \Big(\frac{\Box_1}{7560} - \frac{\Box_3}{15120} \Big) R_1 R_2 \hat{P}_3$$
$$+ \Big(\frac{\Box_1}{1680} + \frac{\Box_1{}^2}{1680\Box_2} + \frac{\Box_3}{2520} + \frac{\Box_1\Box_3}{1680\Box_2} - \frac{\Box_3{}^2}{336\Box_2} + \frac{\Box_3{}^3}{1120\Box_1\Box_2} \Big) R_1^{\mu\nu} R_{2\,\mu\nu} \hat{P}_3$$
$$+ \frac{\Box_3}{720} \hat{P}_1 \hat{P}_2 R_3 + \Big(\frac{13\Box_1}{30240} - \frac{\Box_3}{15120} \Big) R_1 \hat{\mathcal{R}}_2^{\mu\nu} \hat{\mathcal{R}}_{3\,\mu\nu}$$

$$
\begin{aligned}
&+\Big(\frac{\Box_1}{840}+\frac{\Box_3}{210}+\frac{\Box_2\Box_3}{210\Box_1}+\frac{{\Box_3}^2}{210\Box_1}\Big)R_1^{\alpha\beta}\hat{\mathcal{R}}_{2\,\alpha}{}^{\mu}\hat{\mathcal{R}}_{3\,\beta\mu}\\
&+\Big(\frac{{\Box_1}^2}{25200\Box_3}+\frac{\Box_1\Box_2}{50400\Box_3}-\frac{\Box_3}{25200}+\frac{{\Box_3}^3}{50400\Box_1\Box_2}\Big)R_1R_2R_3\hat{1}\\
&+\Big(-\frac{{\Box_1}^2}{9450\Box_3}-\frac{\Box_1\Box_2}{18900\Box_3}-\frac{\Box_3}{12600}+\frac{{\Box_3}^3}{12600\Box_1\Box_2}\Big)R^{\mu}_{1\,\alpha}R^{\alpha}_{2\,\beta}R^{\beta}_{3\,\mu}\hat{1}\\
&+\Big(\frac{\Box_1}{151200}-\frac{{\Box_1}^2}{151200\Box_2}+\frac{\Box_3}{25200}+\frac{\Box_1\Box_3}{18900\Box_2}-\frac{13{\Box_3}^2}{151200\Box_2}\\
&+\frac{{\Box_3}^3}{50400\Box_1\Box_2}\Big)R_1^{\mu\nu}R_{2\,\mu\nu}R_3\hat{1}+\frac{1}{252}\hat{\mathcal{R}}_1^{\alpha\beta}\nabla^{\mu}\hat{\mathcal{R}}_{2\mu\alpha}\nabla^{\nu}\hat{\mathcal{R}}_{3\nu\beta}\\
&+\frac{1}{60}\hat{\mathcal{R}}_1^{\mu\nu}\nabla_{\mu}\hat{P}_2\nabla_{\nu}\hat{P}_3+\frac{1}{180}\nabla_{\mu}\hat{\mathcal{R}}_1^{\mu\alpha}\nabla^{\nu}\hat{\mathcal{R}}_{2\,\nu\alpha}\hat{P}_3-\frac{1}{1890}R_1^{\mu\nu}\nabla_{\mu}R_2\nabla_{\nu}\hat{P}_3\\
&+\Big(\frac{1}{630}+\frac{\Box_1}{420\Box_2}+\frac{\Box_3}{210\Box_2}-\frac{{\Box_3}^2}{280\Box_1\Box_2}\Big)\nabla^{\mu}R_1^{\nu\alpha}\nabla_{\nu}R_{2\,\mu\alpha}\hat{P}_3\\
&+\frac{1}{180}R_1^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\hat{P}_2\hat{P}_3+\Big(\frac{1}{1260}+\frac{\Box_3}{105\Box_1}\Big)R_{1\,\alpha\beta}\nabla_{\mu}\hat{\mathcal{R}}_2^{\mu\alpha}\nabla_{\nu}\hat{\mathcal{R}}_3^{\nu\beta}\\
&+\Big(-\frac{1}{1260}-\frac{\Box_3}{210\Box_1}\Big)R_1^{\alpha\beta}\nabla_{\alpha}\hat{\mathcal{R}}_2^{\mu\nu}\nabla_{\beta}\hat{\mathcal{R}}_{3\,\mu\nu}-\frac{1}{7560}R_1\nabla_{\alpha}\hat{\mathcal{R}}_2^{\alpha\mu}\nabla^{\beta}\hat{\mathcal{R}}_{3\,\beta\mu}\\
&+\Big(\frac{1}{630}+\frac{\Box_2}{105\Box_1}+\frac{\Box_3}{105\Box_1}\Big)R_1^{\mu\nu}\nabla_{\mu}\nabla_{\lambda}\hat{\mathcal{R}}_2^{\lambda\alpha}\hat{\mathcal{R}}_{3\,\alpha\nu}+\Big(\frac{1}{226800}-\frac{\Box_1}{8400\Box_2}\\
&-\frac{{\Box_1}^2}{10080\Box_2\Box_3}+\frac{\Box_3}{25200\Box_1}-\frac{\Box_3}{25200\Box_2}-\frac{{\Box_3}^2}{25200\Box_1\Box_2}\Big)R_1^{\alpha\beta}\nabla_{\alpha}R_2\nabla_{\beta}R_3\hat{1}\\
&+\Big(-\frac{\Box_1}{37800\Box_2}+\frac{\Box_3}{5400\Box_2}-\frac{{\Box_3}^2}{12600\Box_1\Box_2}\Big)\nabla^{\mu}R_1^{\nu\alpha}\nabla_{\nu}R_{2\,\mu\alpha}R_3\hat{1}\\
&+\Big(-\frac{1}{9450}-\frac{\Box_1}{12600\Box_2}+\frac{{\Box_1}^2}{8400\Box_2\Box_3}-\frac{\Box_3}{6300\Box_2}\Big)R_1^{\mu\nu}\nabla_{\mu}R_2^{\alpha\beta}\nabla_{\nu}R_{3\,\alpha\beta}\hat{1}\\
&+\Big(-\frac{1}{3150}-\frac{\Box_1}{9450\Box_2}-\frac{{\Box_1}^2}{6300\Box_2\Box_3}-\frac{\Box_3}{3150\Box_1}+\frac{\Box_3}{9450\Box_2}+\frac{{\Box_3}^2}{3150\Box_1\Box_2}\Big)\\
&\times R_1^{\mu\nu}\nabla_{\alpha}R_{2\,\beta\mu}\nabla^{\beta}R^{\alpha}_{3\,\nu}\hat{1}+\Big(\frac{1}{420\Box_2}+\frac{\Box_3}{280\Box_1\Box_2}\Big)\nabla_{\alpha}\nabla_{\beta}R_1^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R_2^{\alpha\beta}\hat{P}_3\\
&+\Big(-\frac{1}{3780\Box_2}-\frac{1}{6300\Box_3}-\frac{\Box_1}{4200\Box_2\Box_3}+\frac{\Box_3}{25200\Box_1\Box_2}\Big)\nabla_{\alpha}\nabla_{\beta}R_1^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R_2^{\alpha\beta}R_3\hat{1}\\
&+\Big(-\frac{1}{1575\Box_2}-\frac{2}{4725\Box_3}-\frac{\Box_1}{1575\Box_2\Box_3}-\frac{\Box_3}{6300\Box_1\Box_2}\Big)\nabla_{\mu}R_1^{\alpha\lambda}\nabla_{\nu}R^{\beta}_{2\,\lambda}\nabla_{\alpha}\nabla_{\beta}R_3^{\mu\nu}\hat{1}\\
&+\frac{1}{6300\Box_1\Box_2}\nabla_{\lambda}\nabla_{\sigma}R_1^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}R_2^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R_3^{\lambda\sigma}\hat{1}\Big\}+\mathrm{O}[\Re^4].
\end{aligned}
\qquad (2.228)
$$

Even if one starts with the exact forms factors of sect. (2.5) there are no Δ-polynomial (2.143) nonlocal factors here, because the form factors are analytical functions of ξ as been explained in sect. 2.4. But the striking feature of this short time expansion is that it is still nonlocal and some of third order structures contribute to

a_2. One can see that such a behavior is characteristic only of gravitational structures and, moreover, the nonlocal operators $1/\Box$ in the asymptotic expressions above act only on the gravitational curvatures. As discussed in the Introduction, these features will persist at all higher orders in $\Re$, and the underlying cause is that the basis set of curvatures for the heat kernel does not contain the Riemann tensor which gets automatically excluded via the Bianchi identities (1.58).

Now we show that restoring of the Riemann tensor restores the local short time expansion. The task is now to bring expressions (2.226)–(2.228) to a local form by restoring the Riemann tensor. The expression for the Riemann tensor solving the Bianchi identities to second order in the Ricci tensor is given in (1.58). The procedure that we use here is as follows. For each a_n, we first consider a linear combination of all possible local invariants of the appropriate dimension with unknown coefficients. Next, in this combination, we exclude the Riemann tensor, and equate the result to the nonlocal expression above. This gives a set of equations for the unknown coefficients, which in each case has a unique solution. In the case of a_2, there is only one local invariant with explicit participation of the Riemann tensor, namely

$$\int dx\, g^{1/2}\, R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \tag{2.229}$$

Its nonlocal expansion up to third order is as follows

$$\begin{aligned}
&\int dx\, g^{1/2}\, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \int dx\, g^{1/2}\, \Big\{4R_{\mu\nu}R^{\mu\nu} - R^2 + \frac{\Box_1}{2\Box_2\Box_3}R_1R_2R_3\\
&+2\Big(\frac{\Box_1}{\Box_2\Box_3} - \frac{2}{\Box_1}\Big)R^{\mu}_{1\,\alpha}R^{\alpha}_{2\,\beta}R^{\beta}_{3\,\mu} + \Big(\frac{2}{\Box_1} - \frac{\Box_3}{\Box_1\Box_2}\Big)R^{\mu\nu}_1R_{2\,\mu\nu}R_3\\
&-\Big(\frac{2}{\Box_1\Box_3} + \frac{3}{\Box_2\Box_3}\Big)R^{\alpha\beta}_1\nabla_\alpha R_2\nabla_\beta R_3 + \frac{4}{\Box_1\Box_2}\nabla^\mu R^{\nu\alpha}_1\nabla_\nu R_{2\,\mu\alpha}R_3\\
&+\frac{4}{\Box_2\Box_3}R^{\mu\nu}_1\nabla_\mu R^{\alpha\beta}_2\nabla_\nu R_{3\,\alpha\beta} + 4\Big(\frac{2}{\Box_1\Box_2} - \frac{1}{\Box_2\Box_3}\Big)R^{\mu\nu}_1\nabla_\alpha R_{2\,\beta\mu}\nabla^\beta R^{\alpha}_{3\,\nu}\\
&-\frac{4}{\Box_1\Box_2\Box_3}\nabla_\alpha\nabla_\beta R^{\mu\nu}_1\nabla_\mu\nabla_\nu R^{\alpha\beta}_2R_3 - \frac{8}{\Box_1\Box_2\Box_3}\nabla_\mu R^{\alpha\lambda}_1\nabla_\nu R^{\beta}_{2\,\lambda}\nabla_\alpha\nabla_\beta R^{\mu\nu}_3\Big\}\\
&+\mathrm{O}[R^4_{..}].
\end{aligned} \tag{2.230}$$

Expressions of this kind were used for other local invariants below.

In the case of a_3, there are seven (the integral over space-time is assumed):

$$\begin{array}{ll} \mathrm{tr}\hat{P}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}, & \mathrm{tr}\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}^{\mu\nu}R_{\alpha\beta\mu\nu}, \\ R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\rho}{}_{\alpha\beta}, & R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\sigma}{}^{\nu}{}_{\rho}R^{\sigma}{}_{\alpha}{}^{\rho}{}_{\beta}, \\ R^{\alpha}_{\beta}R_{\alpha\mu\nu\sigma}R^{\beta\mu\nu\sigma}, & RR^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}, \\ R^{\alpha\mu}R^{\beta\nu}R_{\alpha\beta\mu\nu}, & \end{array} \tag{2.231}$$

and the coefficient of $RR^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ turns out to be zero. In the case of a_4, there are ten (counting only cubic):

$$\begin{array}{ll} \mathrm{tr}\Box\hat{P}R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}, & \mathrm{tr}\hat{P}\nabla_{\mu}\nabla_{\alpha}R_{\nu\beta}R^{\mu\nu\alpha\beta}, \\ \mathrm{tr}\hat{\mathcal{R}}^{\alpha\beta}\Box\hat{\mathcal{R}}^{\mu\nu}R_{\alpha\beta\mu\nu}, & \Box R^{\alpha}_{\beta}R_{\alpha\mu\nu\sigma}R^{\beta\mu\nu\sigma}, \\ \Box RR^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}, & R_{\mu\nu}\nabla^{\mu}R_{\alpha\beta\sigma\rho}\nabla^{\nu}R^{\alpha\beta\sigma\rho}, \\ R\nabla_{\mu}\nabla_{\alpha}R_{\nu\beta}R^{\mu\nu\alpha\beta}, & \nabla_{\mu}R_{\nu\alpha}\nabla^{\alpha}R_{\rho\sigma}R^{\mu\rho\nu\sigma}, \\ \nabla_{\alpha}R_{\beta\lambda}\nabla_{\mu}R^{\lambda}_{\nu}R^{\alpha\beta\mu\nu}, & R^{\alpha\mu}\Box R^{\beta\nu}R_{\alpha\beta\mu\nu}, \end{array} \tag{2.232}$$

and the last one $R^{\alpha\mu}\Box R^{\beta\nu}R_{\alpha\beta\mu\nu}$ has zero coefficient. The number of invariants with the Riemann tensor does not grow fast owing to the Bianchi identities (1.57).

The final results are as follows. The expressions (2.224) and (2.225) are already in the local form. The expression (2.226) is brought to a local form by using eq.(2.230):

$$\begin{aligned} \int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_2(x,x) = \int dx\, g^{1/2}\,\mathrm{tr}\Big\{\frac{1}{2}\hat{P}\hat{P} + \frac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} \\ + \Big[\frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu}\Big]\hat{1}\Big\} + \mathrm{O}[\Re^4], \end{aligned} \tag{2.233}$$

and the expressions (2.227), (2.228) rewritten in terms of invariants (2.231) and (2.232) take the form

$$\begin{aligned} \int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_3(x,x) = \int dx\, g^{1/2}\,\mathrm{tr}\Big\{\frac{1}{12}\hat{P}\Box\hat{P} + \frac{1}{120}\hat{\mathcal{R}}_{\mu\nu}\Box\hat{\mathcal{R}}^{\mu\nu} \\ + \frac{1}{180}\hat{P}\Box R + \Big[\frac{1}{840}R_{\mu\nu}\Box R^{\mu\nu} - \frac{1}{3780}R\Box R\Big]\hat{1} \\ + \frac{1}{6}\hat{P}\hat{P}\hat{P} - \frac{1}{45}\hat{\mathcal{R}}^{\mu}{}_{\alpha}\hat{\mathcal{R}}^{\alpha}{}_{\beta}\hat{\mathcal{R}}^{\beta}{}_{\mu} \\ + \frac{1}{12}\hat{P}\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\alpha\beta} + \frac{1}{72}R^{\mu\nu}{}_{\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu} \\ - \frac{1}{180}R^{\mu\nu}\hat{\mathcal{R}}^{\alpha}{}_{\mu}\hat{\mathcal{R}}_{\alpha\nu} + \frac{1}{180}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\hat{P} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{180}R^{\alpha\beta}R_{\alpha\beta}\hat{P}+\Big[-\frac{1}{1620}R^{\alpha\ \beta}_{\ \mu\ \nu}R^{\mu\ \nu}_{\ \sigma\ \rho}R^{\sigma\ \rho}_{\ \alpha\ \beta}\\
&+\frac{17}{45360}R^{\alpha\beta}_{\ \ \ \mu\nu}R^{\mu\nu}_{\ \ \ \sigma\rho}R^{\sigma\rho}_{\ \ \ \alpha\beta}+\frac{1}{7560}R_{\alpha\beta}R^{\alpha}_{\ \mu\nu\lambda}R^{\beta\mu\nu\lambda}\\
&+\frac{1}{945}R_{\alpha\beta}R^{\mu\nu}R^{\alpha\ \beta}_{\ \mu\ \nu}-\frac{4}{2835}R^{\alpha}_{\beta}R^{\beta}_{\mu}R^{\mu}_{\alpha}\Big]\,\hat{1}\Big\}+\mathrm{O}[\Re^{4}],
\end{aligned}
\qquad (2.234)
$$

$$
\begin{aligned}
\int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_4(x,x)&=\int dx\, g^{1/2}\,\mathrm{tr}\Big\{\frac{1}{120}\hat{P}\Box^2\hat{P}+\frac{1}{1260}\hat{P}\Box^2 R\\
&+\frac{1}{1680}\hat{\mathcal{R}}^{\mu\nu}\Box^2\hat{\mathcal{R}}_{\mu\nu}+\frac{1}{15120}R^{\mu\nu}\Box^2 R_{\mu\nu}\hat{1}\\
&+\frac{1}{24}\Box\hat{P}\hat{P}\hat{P}-\frac{1}{630}\Box\hat{\mathcal{R}}^{\mu}_{\ \alpha}\hat{\mathcal{R}}^{\alpha}_{\ \beta}\hat{\mathcal{R}}^{\beta}_{\ \mu}\\
&+\frac{1}{252}\hat{\mathcal{R}}^{\alpha\beta}\nabla^{\mu}\hat{\mathcal{R}}_{\mu\alpha}\nabla^{\nu}\hat{\mathcal{R}}_{\nu\beta}+\frac{1}{180}\Box\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}\hat{P}\\
&+\frac{1}{180}\hat{\mathcal{R}}^{\mu\nu}\Box\hat{\mathcal{R}}_{\mu\nu}\hat{P}+\frac{1}{90}\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}\Box\hat{P}\\
&+\frac{1}{180}\nabla_{\mu}\hat{\mathcal{R}}^{\mu\alpha}\nabla^{\nu}\hat{\mathcal{R}}_{\nu\alpha}\hat{P}+\frac{1}{720}\hat{P}\hat{P}\Box R\\
&+\frac{1}{180}R^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\hat{P}\hat{P}+\frac{1}{60}\hat{\mathcal{R}}^{\mu\nu}\nabla_{\mu}\hat{P}\nabla_{\nu}\hat{P}\\
&-\frac{1}{1890}R^{\mu\nu}\nabla_{\mu}R\nabla_{\nu}\hat{P}-\frac{1}{15120}\Box\hat{P}RR\\
&+\frac{1}{7560}\hat{P}R\Box R-\frac{1}{1260}\nabla^{\mu}R^{\nu\alpha}\nabla_{\nu}R_{\mu\alpha}\hat{P}\\
&-\frac{1}{840}R^{\mu\nu}\Box R_{\mu\nu}\hat{P}-\frac{1}{5040}R^{\mu\nu}R_{\mu\nu}\Box\hat{P}\\
&+\frac{1}{1120}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\Box\hat{P}+\frac{1}{420}R^{\mu\nu\alpha\beta}\nabla_{\mu}\nabla_{\alpha}R_{\nu\beta}\hat{P}\\
&+\frac{13}{30240}\Box R\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}-\frac{1}{15120}R\hat{\mathcal{R}}^{\mu\nu}\Box\hat{\mathcal{R}}_{\mu\nu}\\
&-\frac{1}{7560}R\nabla_{\alpha}\hat{\mathcal{R}}^{\alpha\mu}\nabla^{\beta}\hat{\mathcal{R}}_{\beta\mu}+\frac{1}{840}\Box R^{\alpha\beta}\hat{\mathcal{R}}_{\alpha}^{\ \mu}\hat{\mathcal{R}}_{\beta\mu}\\
&-\frac{1}{1260}R^{\alpha\beta}\nabla_{\alpha}\hat{\mathcal{R}}^{\mu\nu}\nabla_{\beta}\hat{\mathcal{R}}_{\mu\nu}+\frac{1}{630}R^{\mu\nu}\nabla_{\mu}\nabla_{\lambda}\hat{\mathcal{R}}^{\lambda\alpha}\hat{\mathcal{R}}_{\alpha\nu}\\
&+\frac{1}{1260}R_{\alpha\beta}\nabla_{\mu}\hat{\mathcal{R}}^{\mu\alpha}\nabla_{\nu}\hat{\mathcal{R}}^{\nu\beta}+\frac{1}{420}R_{\mu\nu\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta}\Box\hat{\mathcal{R}}^{\mu\nu}\\
&+\Big[\frac{1}{50400}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\Box R+\frac{1}{6300}\Box R_{\alpha\beta}R^{\alpha}_{\ \mu\nu\lambda}R^{\beta\mu\nu\lambda}\\
&-\frac{1}{25200}R_{\lambda\sigma}\nabla^{\lambda}R^{\mu\nu\alpha\beta}\nabla^{\sigma}R_{\mu\nu\alpha\beta}-\frac{1}{37800}R^{\mu\nu\alpha\beta}\nabla_{\mu}\nabla_{\alpha}R_{\nu\beta}R\\
&-\frac{1}{6300}R^{\mu\alpha\nu\beta}\nabla_{\mu}R_{\nu\lambda}\nabla^{\lambda}R_{\alpha\beta}-\frac{2}{4725}R^{\alpha\beta\mu\nu}\nabla_{\alpha}R_{\beta\lambda}\nabla_{\mu}R^{\lambda}_{\nu}\\
&+\frac{1}{37800}R^{\mu\nu}\nabla_{\alpha}R_{\beta\mu}\nabla^{\beta}R^{\alpha}_{\nu}-\frac{1}{9450}R^{\mu\nu}\nabla_{\mu}R^{\alpha\beta}\nabla_{\nu}R_{\alpha\beta}\\
&-\frac{1}{18900}\nabla^{\mu}R^{\nu\alpha}\nabla_{\nu}R_{\mu\alpha}R+\frac{29}{453600}R^{\alpha\beta}\nabla_{\alpha}R\nabla_{\beta}R
\end{aligned}
$$

$$+\frac{1}{37800}RR^{\mu\nu}\Box R_{\mu\nu}-\frac{1}{75600}\Box RR^{\mu\nu}R_{\mu\nu}$$
$$-\frac{1}{7560}\Box R^{\mu}_{\alpha}R^{\alpha}_{\beta}R^{\beta}_{\mu}-\frac{1}{100800}\Box RRR\Big]\,\hat{1}\Big\}+\mathrm{O}[\Re^4]. \qquad (2.235)$$

As was discussed in the Introduction there exist independent methods for obtaining these coefficients, and for $n = 0, 1, 2, 3$, the $\hat{a}_n(x, x)$ have been calculated explicitly [12, 11, 4, 16, 22]. Some results exist for a_4 as well [16, 6, 23, 17]. Now we carry out a comparison with these known expressions below.

The expressions (2.224), (2.225) and (2.233) for a_0, a_1 and a_2 coincide with the results obtained by other methods [12, 11, 4, 16, 22]. It is easy to compare expression (2.234) for a_3 with other results [4, 16] since they differ only by the definition of $\hat{\mathcal{R}}^2$ term [34].

The information about a_4 available to date is rather incomplete [16, 17, 57]. In all these papers a_4 is derived for the heat kernel expansion (1.15). The paper [57] is devoted to a model with a matrix-valued potential on flat manifolds, therefore it provides only terms constructed of $\hat{P}$ matrices, which are in agreement with (2.235).

The work [17] concerns only a scalar field model in curved space-time without a potential, therefore, it contains only pure gravitatonal terms (terms of (2.235) in square brackets). Our result (2.235) disagrees with this work even in second order in curvature. Futher comparison does not make sense because the transformation from one tensor invariant basis [17] to another (2.235) can only be done in all curvature orders simultaneously.

In Ref. [16] a_4, in fact, is not presented in a final form because a basis of tensor invariants is not chosen and, therefore, the result does not have an explicit representation. After reduction of the result of [16] to the tensor basis of (2.235) the coincidence *does* take place with accuracy $\mathrm{O}[\Re^4]$. Some tensor identities used for this reduction of a_4 can be found in Appendix A.

Finally, in ref. [6] second order curvature terms are obtained for any Schwinger-DeWitt coefficient $\int dx \mathrm{tr} a_n$. The result is consistent with the coefficients reproduced in this section.

We should note, that although all the equations (2.233)–(2.235) are presently obtained with accuracy $O[\Re^4]$, the results for a_2 and a_3 are exact.

Later, in sect. 3.5 we will discuss the comparison of the Schwinger-DeWitt coefficients for the heat kernel (1.15).

Chapter 3

The heat kernel in the second order

3.1 The generating function method for the heat kernel

The present chapter is devoted to computation of the coincidence limit of the heat kernel $\hat{K}(s|x,x)$. The basis of tensor invariants in the second order in curvatures for the heat kernel is constructed and the form factors are obtained in two integral representations. The results are checked by deriving the Schwinger-DeWitt series of the heat kernel and the functional trace operation. This chapter is based significantly on the notations and methods of chapters 1 and 2, therefore references to them are made instead of repetitions.

This task is accomplished by two different methods. One method is a direct application of formulae of the covariant perturbation theory, sect. 1.2; the other, the generating function method [28], is explained in the present section.

The origin of the generating expressions approach is a simple variation principle

$$\frac{1}{s}\delta\Big(\mathrm{Tr}\,\mathrm{e}^{s\hat{F}}\Big) = \mathrm{Tr}\Big(\delta\hat{F}\mathrm{e}^{s\hat{F}}\Big), \tag{3.1}$$

where δ means the variation of the functional $\mathrm{Tr}K$ with respect to background fields it depends upon. The equation (3.1) implies

$$\frac{1}{s}\delta\Big(\mathrm{Tr}K(s)\Big) = \int dx\,\mathrm{tr}\Big[\delta\hat{F}(\nabla^x)\hat{K}(s|x,y)\Big]\Big|_{y=x}. \tag{3.2}$$

Since the differential operator $F(\nabla)$ has three independent background fields, the metric, the gauge field, and the potential, there are three possible variational equations [28, 47, 19]. The one we are interested in is,

$$\hat{K}(s|x,x) = \frac{1}{s}\frac{\delta}{\delta\hat{P}(x)}\mathrm{Tr}K(s). \tag{3.3}$$

Thus, the heat kernel trace, interpreted as a functional of the potential, generates the diagonal heat kernel value.

The method of generating expressions (3.2), or rather its modification for the Green's functions, first has been proposed for analysis of local divergences of the coincidence limits of the Green's functions [28, 19], and reflects the fundamental feature of the effective action as the generating function of the one particle irreducible Green's functions [24, 12]. However, as was shown in [43] the method can be used to treat nonlocal curvature expansions as well. This makes it a good tool for two-loop calculations [28, 20] since to compute even local terms of two-loop graphs one needs to know finite nonlocal terms of one-loop generating functions [28]. The work with heat kernels, instead of the Green's functions, has the advantage that one is not restricted to specific space-time dimensions, and no divergences are present before the proper time integrals are done.

As the generating function we take the trace of the heat kernel $\mathrm{Tr}K(s)$ derived from the covariant perturbation theory in the previous chapter. The only important feature of the form factors in (2.40) required here is that they are functions of the operator $\xi_i = -s\Box_i$ and do not depend explicitly on the curvatures $\Re$. This is a feature of the method which always eliminates one curvature reducing an accuracy $\mathrm{O}[\Re^n]$ by one order. In this circumstance the variation of form factors is not required.

In principle, one needs to derive coincidence limits of the heat kernel of the most general form

$$\nabla^x \ldots \nabla^x \nabla^y \ldots \nabla^y \hat{K}(s|x,y)|_{y=x}. \tag{3.4}$$

The propagators obtained from (3.4) by Schwinger equation (1.4) bear all essential information of quantum field theory [12, 11, 27]. The heat kernels $\nabla_\alpha \hat{K}(s|x,y)|_{y=x}$ and $\nabla_\alpha \nabla_\beta \hat{K}(s|x,y)|_{y=x}$ were derived up to the first order in curvatures in [47] and the computations can be extended to higher orders. But the generating function method is unable to produce heat kernels with more than two derivatives since the operator $F(\nabla)$ is second order in ∇'s. Fortunately, the covariant perturbation theory encompasses all possible cases of (3.4).

To check the consistency, all results of the next three sections are obtained both by the generating function method and by the covariant perturbation theory.

Although, it is possible to state the final result immediately, in this chapter we follow the pattern of Chapter 2, i.e., we start with noncovariant perturbation theory and proceed to the covariant curvature expansion and form factor representations finishing with the short time expansion.

3.2 Second order of the perturbation theory for the heat kernel

In the covariant perturbation theory, sect. 1.2, the heat kernel is first expanded in powers of perturbations (1.22)

$$\hat{K}(s) = \sum_{n=0}^{\infty} \hat{K}_n(s)$$

where $K_n(s)$ is a term of n-th power in the perturbations $h_{\mu\nu}$, $\hat{\Gamma}^{\mu}$, $(\hat{P} - 1/6R\hat{1})$. Then, $K_n(s)$ is obtained in the form (*cf.* (1.39)),

$$\begin{aligned} \hat{K}_n(s)(x,x) &= \frac{1}{(4\pi s)^{\omega}} \tilde{g}^{1/2}(x) \int_{\alpha_i \geq 0} d^n\alpha \, \delta(1 - \sum_1^n \alpha_i) \\ &\times \exp\left[s\Omega_{n+1}(\alpha_1, \ldots \alpha_{n+1}|\tilde{\nabla}^i)\right] \sum_{l=0}^{n} s^l \hat{C}^l_n(\alpha_1, \ldots \alpha_{n+1}|x_i)\Big|_{x_i=x} \end{aligned} \quad (3.5)$$

or, with the notation (1.40):

$$\hat{K}_n(s) = \frac{1}{(4\pi s)^{\omega}} \tilde{g}^{1/2} \sum_{l=0}^{n} s^l \langle \mathrm{e}^{s\Omega_{n+1}} \hat{C}^l_n \rangle_n. \quad (3.6)$$

Here the kernels of nonlocal form factors Ω_{n+1} are the same as in sect. 2.1 but they appear in lower orders than in (1.39).

The variational method (3.3) can be directly applied to the results for the trace of the heat kernel in terms of perturbations derived in sect. 2.1, and result in the same expression (3.5). Let us now display the forms for Ω and $\hat{C}$, for $n = 0$,

$$\Omega_1 = 0, \tag{3.7}$$

$$\hat{C}^0_0 = \hat{1}. \tag{3.8}$$

The results for $n = 1$ are

$$\Omega_2(\alpha_1, \alpha_2 | \tilde{\nabla}^i) = \alpha_1 \alpha_2 \tilde{\Box}, \tag{3.9}$$

$$\hat{C}^0_1(\alpha_1, \alpha_2 | x_i) = \frac{1}{2} h \hat{1}, \tag{3.10}$$

$$\hat{C}^1_1(\alpha_1, \alpha_2 | x_i) = \alpha_1^2 \Big(\tilde{\nabla}_\mu \tilde{\nabla}_\nu h^{\mu\nu} - \frac{1}{6} R\Big)\hat{1} + \hat{P} - 2\alpha_1 \tilde{\nabla}_\mu \hat{\Gamma}^\mu, \tag{3.11}$$

and for $n = 2$

$$\Omega_3(\alpha_1, \alpha_2, \alpha_3 | \tilde{\nabla}^i) = \alpha_2 \alpha_3 \tilde{\Box}_1 + \alpha_1 \alpha_3 \tilde{\Box}_2 + \alpha_1 \alpha_2 \tilde{\Box}_3, \tag{3.12}$$

$$\hat{C}^0_2(\alpha_1, \alpha_2, \alpha_3 | x_i) = \Big(\frac{1}{4} h_1 h_2 + \frac{1}{2} h_1^{\mu\nu} h_2^{\alpha\beta} \tilde{g}_{\alpha\mu} \tilde{g}_{\beta\nu}\Big)\hat{1}, \tag{3.13}$$

$$\begin{aligned} \hat{C}^1_2(\alpha_1, \alpha_2, \alpha_3 | x_i) = & -\hat{P}_1 h_2 - 2\tilde{g}_{\alpha\beta} \hat{\Gamma}^\alpha_1 \hat{\Gamma}^\beta_2 + \Big(\tilde{g}_{\mu\nu}(D^3_\alpha + D^2_\alpha) + 2\tilde{g}_{\mu\alpha}(D^1_\nu - D^3_\nu)\Big)\hat{\Gamma}^\alpha_1 h_2^{\mu\nu} \\ & + \hat{1}\left[\frac{1}{6} R_1 h_2 - \frac{1}{2}(\tilde{g}_{\mu\nu} D^1_\alpha D^1_\beta + \tilde{g}_{\alpha\beta} D^3_\mu D^3_\nu + 4\tilde{g}_{\mu\alpha} D^3_\nu D^1_\beta) h_1^{\mu\nu} h_2^{\alpha\beta}\right], \end{aligned} \tag{3.14}$$

$$\begin{aligned} \hat{C}^2_2(\alpha_1, \alpha_2, \alpha_3 | x_i) = & \Big(\hat{P}_1 - \frac{1}{6} R_1 \hat{1}\Big)\Big(\hat{P}_2 - \frac{1}{6} R_2 \hat{1}\Big) + (D^1_\alpha D^1_\beta + D^3_\alpha D^3_\beta)\Big(\hat{P}_1 - \frac{1}{6} R_1 \hat{1}\Big) h_2^{\alpha\beta} \\ & - 2(D^1_\mu + D^3_\mu)\Big(\hat{P}_1 - \frac{1}{6} R_1 \hat{1}\Big)\hat{\Gamma}^\mu_2 - 2(D^2_\alpha D^3_\mu D^3_\nu + D^3_\alpha D^1_\mu D^1_\nu)\hat{\Gamma}^\alpha_1 h_2^{\mu\nu} \\ & + 4 D^3_\alpha D^1_\beta \hat{\Gamma}^\alpha_1 \hat{\Gamma}^\beta_2 + D^3_\mu D^3_\nu D^1_\alpha D^1_\beta h_1^{\mu\nu} h_2^{\alpha\beta} \hat{1}. \end{aligned} \tag{3.15}$$

3.3 The α-polynomial representation of the form factors in the heat kernel

Now one should apply covariant expansions for the perturbations through the Green's functions to the results of the previous section. Since each perturbation, $h^{\mu\nu}$ and $\hat{\Gamma}_\mu$, is an infinite series in the curvatures, the expansion to the second order in the curvatures involves the first order in the perturbations.

Alternatively, one can use the generating function method (3.3). It is easy to observe that only thirteen of the 29 tensor structures (2.47)–(2.75) contain matrix $\hat{P}$ and thus contribute to the heat kernel itself.

Whether we obtain the heat kernel by the generating function method from tables (2.87)–(2.124), or by a direct substitution of the nonlocal expansions of sect. 1.2 into the perturbations $h^{\mu\nu}$ and $\widehat{\Gamma}_\mu$, the result is

$$\begin{aligned}\hat{K}(s|x,x) \;=\; & \frac{1}{(4\pi s)^\omega} g^{1/2}\Big\{\hat{1} + s\left(g_1(-s\Box)\hat{P} + g_2(-s\Box)R\hat{1}\right) \\ & + s^2 \sum_1^5 G_i(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2[i] \\ & + s^3 \sum_6^{11} G_i(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2[i] \\ & + s^3 G_{12}(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2[12] + \mathrm{O}[\Re^3]\Big\}. \qquad (3.16)\end{aligned}$$

Here $\Re_1\Re_2[i]$ with $i = 1$ to 12 are quadratic structures (note the use of square brackets instead of round ones to distinguish from (2.41)–(2.45)):

$$\Re_1\Re_2[1] = \hat{P}_1\hat{P}_2, \qquad (3.17)$$

$$\Re_1\Re_2[2] = \hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\mu\nu}, \qquad (3.18)$$

$$\Re_1\Re_2[3] = \hat{P}_1 R_2, \qquad (3.19)$$

$$\Re_1\Re_2[4] = R_1 R_2 \hat{1}, \qquad (3.20)$$

$$\Re_1\Re_2[5] = R_1^{\mu\nu} R_{2\mu\nu}\hat{1}, \qquad (3.21)$$

$$\Re_1\Re_2[6] = \nabla_\mu \hat{\mathcal{R}}_1^{\mu\nu}\nabla^\alpha\hat{\mathcal{R}}_{2\alpha\nu}, \qquad (3.22)$$

$$\Re_1\Re_2[7] = [\nabla_\alpha \hat{P}_1, \nabla_\beta \hat{\mathcal{R}}_2^{\beta\alpha}], \tag{3.23}$$

$$\Re_1\Re_2[8] = \nabla_\mu \nabla_\nu \hat{P}_1 R_2^{\mu\nu}, \tag{3.24}$$

$$\Re_1\Re_2[9] = \nabla_\alpha R_{1\mu\nu} \nabla^\mu R_2^{\nu\alpha} \hat{1}, \tag{3.25}$$

$$\Re_1\Re_2[10] = \nabla_\mu \nabla_\nu R_1 R_2^{\mu\nu} \hat{1}, \tag{3.26}$$

$$\Re_1\Re_2[11] = \nabla_\alpha \nabla_\beta R_{1\mu\nu} \nabla^\mu \nabla^\nu R_2^{\alpha\beta} \hat{1}. \tag{3.27}$$

There is an additional quadratic structure linear in $\hat{\mathcal{R}}_{\mu\nu}$

$$\Re_1\Re_2(12) = \nabla_\nu \hat{\mathcal{R}}_1^{\nu\mu} \nabla_\mu R_2, \tag{3.28}$$

which is separated from the others for the same reason the structures $\Re_1\Re_2\Re_3(i)$ for $i = 30$ to 33 were sorted out in the trace of the heat kernel: this structure is absent in the final answer. Without gravity, the basis (3.17)–(3.28) reduces to only four non-vanishing curvature structures. Symmetries of its structures are obvious.

The form factors $g_i(-s\Box)$ and $G_i(-s\Box_1, -s\Box_2, -s\Box_3)$ are represented by two non-local kernels: $\exp(s\alpha_1\alpha_2\Box)$ and $\exp(s\Omega)$ same as (2.83) and (2.84). The function (2.83) appears only in the combinations

$$\mathcal{A},\ \mathcal{B},\ \mathcal{U},\ \mathcal{V} \tag{3.29}$$

of (2.30)–(2.35); the form factors containing two subtractions $\mathcal{C}$, $\mathcal{W}$ do not appear in $\hat{K}(s)$. The same feature is present for form factors formed with the function (2.84), i. e., only two combinations

$$\mathrm{e}^{s\Omega_3},\ \mathrm{e}^{s\Omega_3} - 1 \tag{3.30}$$

of (2.86) appear. Basically, the second order form factors of the heat kernel mimic the third order form factors of the heat kernel trace with their coefficients being polynomials in α 's, the Laplacians, and the Green's functions. But the level of complexity in this order of the heat kernel is less than in the third order of the heat kernel trace.

Here the first order form factors are formed of ones in (2.178) as

$$g_1(\xi) = 2f_4(\xi), \tag{3.31}$$

$$g_2(\xi) = f_3(\xi). \tag{3.32}$$

The second order form factors $G_i(\xi_1, \xi_2, \xi_3)$ for $i = 1$ to 11 are expressed via form factors $F_i(\xi_1, \xi_2, \xi_3)$ of the trace of the heat kernel (2.178) in the following way,

$$G_1(\xi_1, \xi_2, \xi_3) = F_1(\xi_1, \xi_2, \xi_3) + F_1(\xi_2, \xi_3, \xi_1) + F_1(\xi_3, \xi_1, \xi_2), \tag{3.33}$$

$$G_2(\xi_1, \xi_2, \xi_3) = F_3(\xi_1, \xi_2, \xi_3), \tag{3.34}$$

$$G_3(\xi_1, \xi_2, \xi_3) = F_6(\xi_1, \xi_3, \xi_2) + F_6(\xi_3, \xi_1, \xi_2) - \frac{1}{2}(\xi_3 - \xi_1)F_{17}(\xi_2, \xi_3, \xi_1), \tag{3.35}$$

$$G_4(\xi_1, \xi_2, \xi_3) = F_4(\xi_1, \xi_2, \xi_3) + \frac{1}{4}(\xi_3 - \xi_2 - \xi_1)F_{15}(\xi_1, \xi_2, \xi_3), \tag{3.36}$$

$$G_5(\xi_1, \xi_2, \xi_3) = F_5(\xi_1, \xi_2, \xi_3), \tag{3.37}$$

$$G_6(\xi_1, \xi_2, \xi_3) = F_{14}(\xi_1, \xi_2, \xi_3), \tag{3.38}$$

$$G_7(\xi_1, \xi_2, \xi_3) = -F_{13}(\xi_2, \xi_1, \xi_3), \tag{3.39}$$

$$G_8(\xi_1, \xi_2, \xi_3) = F_{17}(\xi_2, \xi_1, \xi_3) + F_{17}(\xi_2, \xi_3, \xi_1), \tag{3.40}$$

$$G_9(\xi_1, \xi_2, \xi_3) = F_{16}(\xi_1, \xi_2, \xi_3), \tag{3.41}$$

$$G_{10}(\xi_1, \xi_2, \xi_3) = -F_{15}(\xi_2, \xi_1, \xi_3), \tag{3.42}$$

$$G_{11}(\xi_1, \xi_2, \xi_3) = F_{26}(\xi_1, \xi_2, \xi_3), \tag{3.43}$$

$$G_{12}(\xi_1,\xi_2,\xi_3) = F_{32}(\xi_3,\xi_1,\xi_2), \tag{3.44}$$

where some interchanges and cyclic substitutions of indices of the arguments ξ_i are made. In the relation (3.39) the symmetry property $F_{13}(\xi_1,\xi_2,\xi_3) = F_{13}(\xi_1,\xi_3,\xi_2)$ is taken into account. It should be emphasized that these rules are not sensitive to a representation of the form factors, they can be in either of them: (2.92)–(2.120) or (B.1)–(B.29).

In the α -polynomial representation the first order form factors are

$$g_1 = \langle \mathcal{A} \rangle_2, \tag{3.45}$$

$$g_2 = \left\langle \left(\alpha_1^2 - \frac{1}{6}\right)\mathcal{A} - \mathcal{B} \right\rangle_2 . \tag{3.46}$$

And the second order form factors admit the form

$$G_1 = \left\langle e^{s\Omega_3} \right\rangle_3, \tag{3.47}$$

$$G_2 = \left\langle 2\alpha_1\alpha_2 e^{s\Omega_3} \right\rangle_3, \tag{3.48}$$

$$\begin{aligned}
G_3 = {} & \left\langle \frac{1}{2}\frac{1}{\Box_2}\frac{\mathcal{A}_1}{s} + \frac{1}{2}\frac{1}{\Box_2}\frac{\mathcal{A}_3}{s} + \frac{1}{2}\frac{\Box_1 - \Box_3}{\Box_2}\mathcal{U}_{13} \right\rangle_2 \\
& - \left\langle 2\frac{1}{\Box_2}\frac{e^{s\Omega_3}}{s} + \left[\left(\frac{2}{3} - 2\alpha_1 + 2{\alpha_1}^2 - \alpha_2 + 2\alpha_1\alpha_2\right) \right.\right. \\
& \left.\left. + \frac{\Box_1}{\Box_2}(2\alpha_1\alpha_2 - \alpha_2) + \frac{\Box_3}{\Box_2}(\alpha_2 - 2\alpha_1\alpha_2)\right] e^{s\Omega_3} \right\rangle_3,
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
G_4 = {} & \left\langle \frac{1}{\Box_1}({\alpha_1}^2 - \frac{1}{6})\frac{\mathcal{A}_2}{s} - \frac{1}{\Box_1}\frac{\mathcal{B}_2}{s} \right. \\
& \left. + \left[\frac{\Box_3}{\Box_1\Box_2}(\frac{1}{2}{\alpha_1}^2) - \frac{1}{\Box_1}({\alpha_1}^2)\right]\frac{\mathcal{A}_3}{s} \right\rangle_2 \\
& + \left\langle \frac{1}{\Box_1\Box_2}\frac{e^{s\Omega_3} - 1}{s^2} + \left[\frac{\Box_3}{\Box_1\Box_2}\left(3\alpha_1\alpha_2 - 2\alpha_2\right) \right.\right. \\
& \left. + \frac{1}{\Box_2}(-\frac{2}{3} + 4\alpha_2 + 2\alpha_1 - 2{\alpha_2}^2 - 6\alpha_1\alpha_2)\right]\frac{e^{s\Omega_3}}{s} \\
& + \left[\left(- \frac{5}{36} + \frac{1}{2}\alpha_1 + \frac{2}{3}{\alpha_2}^2 + \frac{2}{3}\alpha_1\alpha_2 - {\alpha_2}^3 \right.\right. \\
& \left. - 6\alpha_1{\alpha_2}^2 + 2\alpha_1{\alpha_2}^3 + 3{\alpha_1}^2{\alpha_2}^2\right) \\
& + \frac{\Box_1}{\Box_2}\left(\frac{2}{3}\alpha_1\alpha_2 + \frac{1}{6}\alpha_2 - {\alpha_2}^3 - 3\alpha_1{\alpha_2}^2 + 2{\alpha_1}^2{\alpha_2}^2 \right. \\
& \left. - {\alpha_1}^2\alpha_2 + 2\alpha_1{\alpha_2}^3\right) + \frac{\Box_3}{\Box_2}\left(- \frac{1}{6}\alpha_2 + {\alpha_2}^3 - \frac{2}{3}\alpha_1\alpha_2
\end{aligned}$$

$$+ 4\alpha_1\alpha_2{}^2 - 4\alpha_1{}^2\alpha_2{}^2 - 2\alpha_1\alpha_2{}^3 + 2\alpha_1{}^2\alpha_2\Big)$$

$$+\frac{\Box_3{}^2}{\Box_1\Box_2}(\alpha_1{}^2\alpha_2{}^2 - \alpha_1\alpha_2{}^2)\Big] \mathrm{e}^{s\Omega_3}\Big\rangle_3, \tag{3.50}$$

$$G_5 = \left\langle 2\frac{1}{\Box_1\Box_2}\frac{\mathrm{e}^{s\Omega_3}-1}{s^2}\right\rangle_3 + \left\langle\left[\frac{1}{\Box_1}(-\alpha_1{}^2) + \frac{\Box_3}{\Box_1\Box_2}(\frac{1}{2}\alpha_1{}^2)\right]\frac{\mathcal{A}_3}{s}\right.$$

$$\left. + \left(\frac{1}{\Box_1} - \frac{3}{2}\frac{\Box_3}{\Box_1\Box_2}\right)\frac{\mathcal{B}_3}{s}\right\rangle_2, \tag{3.51}$$

$$G_6 = \left\langle -2\frac{1}{\Box_1\Box_2}\frac{\mathrm{e}^{s\Omega_3}}{s^2} + \left[\frac{1}{\Box_1}(2\alpha_1\alpha_2) + \frac{1}{\Box_2}(2\alpha_1\alpha_2)\right.\right.$$

$$\left.\left. +\frac{\Box_3}{\Box_1\Box_2}(-2\alpha_1\alpha_2)\right]\frac{\mathrm{e}^{s\Omega_3}}{s}\right\rangle_3 + \left\langle\frac{1}{\Box_1\Box_2}(2\alpha_2)\frac{\mathcal{A}_3}{s^2}\right\rangle_2, \tag{3.52}$$

$$G_7 = -\left\langle\frac{1}{\Box_2}(2\alpha_2)\mathrm{e}^{s\Omega_3}\right\rangle_3 + \left\langle 2\frac{1}{\Box_2}\frac{\mathcal{U}_{13}}{s}\right\rangle_2, \tag{3.53}$$

$$G_8 = \left\langle\frac{1}{\Box_2}(4\alpha_2{}^2)\mathrm{e}^{s\Omega_3}\right\rangle_3 - \left\langle 2\frac{1}{\Box_2}\frac{\mathcal{U}_{13}}{s}\right\rangle_2, \tag{3.54}$$

$$G_9 = \left\langle\frac{1}{\Box_1\Box_2}(8\alpha_1\alpha_2)\frac{\mathrm{e}^{s\Omega_3}}{s^2}\right\rangle_3 + \left\langle\frac{1}{\Box_1\Box_2}(-2\alpha_1{}^2)\frac{\mathcal{A}_3}{s^2}\right.$$

$$\left. +2\frac{1}{\Box_1\Box_2}\frac{\mathcal{B}_3}{s^2}\right\rangle_2, \tag{3.55}$$

$$G_{10} = \left\langle\frac{1}{\Box_1\Box_2}(4\alpha_2 - 12\alpha_2{}^2)\frac{\mathrm{e}^{s\Omega_3}}{s^2} + \left[\frac{1}{\Box_1}(\alpha_1\alpha_2{}^3 - 2\alpha_1\alpha_2{}^2)\right.\right.$$

$$+ \frac{1}{\Box_2}(\frac{4}{3}\alpha_2{}^2 - 4\alpha_2{}^3 + 4\alpha_2{}^4 + 4\alpha_1\alpha_2{}^3 - 2\alpha_1\alpha_2{}^2)$$

$$\left.\left. + \frac{\Box_3}{\Box_1\Box_2}(2\alpha_1\alpha_2{}^2 - 4\alpha_1\alpha_2{}^3)\right]\frac{\mathrm{e}^{s\Omega_3}}{s}\right\rangle_3$$

$$+ \left\langle 2\frac{1}{\Box_1\Box_2}\frac{\mathcal{B}_3}{s^2} + 2\frac{1}{\Box_2}\frac{\mathcal{V}_{13}}{s} + \frac{1}{\Box_2}\Big(\frac{1}{3} - 2\alpha_2{}^2\Big)\frac{\mathcal{U}_{13}}{s}\right\rangle_2, \tag{3.56}$$

$$G_{11} = \left\langle\frac{1}{\Box_1\Box_2}(4\alpha_1{}^2\alpha_2{}^2)\frac{\mathrm{e}^{s\Omega_3}}{s^2}\right\rangle_3, \tag{3.57}$$

$$G_{12} = \left\langle 2\frac{1}{\Box_1\Box_2}\frac{\mathrm{e}^{s\Omega_3}}{s^2} + \left[\frac{\Box_3}{\Box_1\Box_2}(2\alpha_1\alpha_2) - \frac{1}{\Box_2}(2\alpha_1\alpha_2)\right.\right.$$

$$\left.\left. +\frac{1}{\Box_1}(2\alpha_1 - 4\alpha_1{}^2 - 2\alpha_1\alpha_2)\right]\frac{\mathrm{e}^{s\Omega_3}}{s^2}\right\rangle_3 - \left\langle\frac{1}{\Box_1\Box_2}\frac{\mathcal{A}_3}{s^2}\right\rangle_2. \tag{3.58}$$

3.4 The final result for the heat kernel up to second order in the curvatures

As discussed at the beginning of sect. 2.3 the α-polynomial representation for the form factors (3.47)–(3.58) is not unique. Due to this fact there is an extra quadratic structure (3.28) which will be absent in the final answer for $\hat{K}(s)$. Thus, we need to proceed to the explicit representation of sect. 2.5. This was done again in two ways, first is the treatment of (3.45)–(3.58) with the α -polynomial reduction procedure of sect. 2.4 and the use of a computer program similar to that used to derive form factors of Appendix B. The other is the use of expressions (3.31)–(3.43) and the data of Appendix B.

Both methods result in the following expression for the heat kernel

$$\begin{aligned}\hat{K}(s) &= \frac{1}{(4\pi s)^\omega} g^{1/2}\Big\{\hat{1} + s\Big(g_1(-s\Box)\hat{P} + g_2(-s\Box)R\hat{1}\Big) \\ &\quad + s^2\sum_{i=1}^{5} G_i(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2[i] \\ &\quad + s^3\sum_{i=6}^{10} G_i(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2[i] \\ &\quad + s^4 G_{11}(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2[11] + \mathrm{O}[\Re^3]\Big\},\end{aligned} \tag{3.59}$$

where quadratic tensor structures $\Re_1\Re_2[i]$ are defined by (3.17)–(3.27).

The explicit form of the second order form factors is [47, 48]

$$g_1(\xi) = f(\xi), \tag{3.60}$$

$$g_2(\xi_2) = \frac{1}{12}f(\xi) + \frac{1}{2}\frac{f(\xi)-1}{\xi}. \tag{3.61}$$

The structure and complexity of the second order form factors are similar to those of the third order form factors in the heat kernel trace (2.178), thus they are placed in Appendix C.

The obvious check of the calculations above is the functional trace operation (1.18). We can use the α -representation of the form factors for this consistency check. All total derivative terms should be discarded, and all second order form factors reduced to first order ones by identities like

$$\mathrm{tr}\int d\,x\, g^{1/2}(x) F(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2 = \frac{1}{2}\,\mathrm{tr}\int dx\, g^{1/2}(x) f(-s\Box_2)\Re_1\Re_2 + \mathrm{O}[\Re^3]. \tag{3.62}$$

While first order terms collapse to a single local $\hat{P}$, the tables of second order form factors reduced five form factors of the trace of the heat kernel (2.179)–(2.183). In other words, following the line of the generating function approach we start with the heat kernel trace and end up with the same object but in the lower order in curvatures, This allows one to conclude that there exists a link between two neighboring orders in the curvature of the heat kernel trace, namely, each lower order is completely defined by the higher order, i.e.,

$$\mathcal{K}_{n-1} = \mathrm{tr}\int dx \frac{\delta}{\delta\hat{P}}\,\mathcal{K}_n, \tag{3.63}$$

where

$$\mathcal{K}_n \equiv \mathrm{tr}\int dx g^{1/2} \sum_i F_i(\nabla_1, \ldots, \nabla_n)\Re_1 \ldots \Re_n(i) \tag{3.64}$$

is a given order in curvatures.

3.5 The short time behavior of the heat kernel, and comparison with the Schwinger-DeWitt expansion

This section is devoted to derivation of the short time expansion for the heat kernel $K(s)$ following the way of sect. 2.7. We intend to derive the Schwinger-DeWitt series for the heat kernel (1.13),

$$K(s) = \frac{g^{1/2}}{(4\pi s)^\omega} \sum_{n=0}^{\infty} s^n \hat{a}_n(x,x). \tag{3.65}$$

All we need to know are short time expansions of the form factors in (3.16). Again, the form factors g_i and G_i can be treated in either the explicit representation of sect. 3.4 or in the α -polynomial representation of sect. 3.3. For the latter case, the integrals

(2.125), (2.126) should be used. For the former case, the short time expansions of the basic form factors are already introduced in (2.221) and (2.222).

Generally, all $\hat{a}_n(x,x)$ are local functions of the background fields, but even though the nonlocal denominators Δ cancel in the short time series, nevertheless the tree $1/\Box$ nonlocal terms are still present due to (1.58). By comparing the nonlocal expansion of $K(s)$ with (3.65) we get the following nonlocal form for the Schwinger-DeWitt coefficients:

$$\hat{a}_1(x,x) = \hat{P}, \tag{3.66}$$

$$\begin{aligned}\hat{a}_2(x,x) &= \frac{1}{180}\Box R\hat{1} + \frac{1}{6}\Box\hat{P} + \frac{1}{2}\hat{P}_1\hat{P}_2 + \frac{1}{12}\hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\mu\nu}\\
&+ \Big(\frac{1}{180} - \frac{\Box_3}{45\Box_1} + \frac{\Box_1}{90\Box_2} + \frac{{\Box_3}^2}{180\Box_1\Box_2}\Big)R_1^{\mu\nu}R_{2\mu\nu}\hat{1}\\
&+ \Big(-\frac{\Box_3}{45\Box_1\Box_2} + \frac{2}{45\Box_1}\Big)\nabla_\alpha R_{1\mu\nu}\nabla^\mu R_2^{\nu\alpha}\hat{1}\\
&+ \frac{1}{45\Box_1\Box_2}\nabla_\alpha\nabla_\beta R_{1\mu\nu}\nabla^\mu\nabla^\nu R_2^{\alpha\beta}\hat{1} + \mathrm{O}[\Re^3],\end{aligned} \tag{3.67}$$

$$\begin{aligned}\hat{a}_3(x,x) &= \frac{1}{60}\Box^2\hat{P} + \frac{1}{1260}\Box^2 R\hat{1}\\
&+ \Big(\frac{\Box_3}{24} + \frac{\Box_2}{24} + \frac{\Box_1}{24}\Big)\hat{P}_1\hat{P}_2 + \Big(\frac{\Box_3}{90} + \frac{\Box_2}{180} + \frac{\Box_1}{180}\Big)\hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\mu\nu}\\
&+ \Big(\frac{\Box_3}{360} - \frac{\Box_1}{360} + \frac{\Box_2}{360}\Big)\hat{P}_1 R_2 + \Big(-\frac{\Box_1}{7560} + \frac{\Box_3}{15120}\Big)R_1R_2\hat{1}\\
&+ \Big(\frac{{\Box_1}^2}{1680\Box_2} + \frac{\Box_1}{1680} + \frac{\Box_1\Box_3}{1680\Box_2} - \frac{{\Box_3}^2}{336\Box_2} + \frac{\Box_3}{2520} + \frac{{\Box_3}^3}{1120\Box_1\Box_2}\Big)R_1^{\mu\nu}R_{2\mu\nu}\hat{1}\\
&+ \frac{1}{180}\nabla_\mu\hat{\mathcal{R}}_1^{\mu\nu}\nabla^\alpha\hat{\mathcal{R}}_{2\alpha\nu} - \frac{1}{60}[\nabla_\alpha\hat{P}_1, \nabla_\beta\hat{\mathcal{R}}_2^{\beta\alpha}] + \frac{1}{90}\nabla_\mu\nabla_\nu\hat{P}_1R_2^{\mu\nu}\\
&+ \Big(\frac{1}{630} - \frac{{\Box_3}^2}{280\Box_1\Box_2} + \frac{\Box_1}{420\Box_2} + \frac{\Box_3}{210\Box_1}\Big)\nabla_\alpha R_{1\mu\nu}\nabla^\mu R_2^{\nu\alpha}\hat{1}\\
&+ \frac{1}{1890}\nabla_\mu\nabla_\nu R_1 R_2^{\mu\nu}\hat{1} + \Big(\frac{\Box_3}{280\Box_1\Box_2} + \frac{1}{420\Box_1}\Big)\nabla_\alpha\nabla_\beta R_{1\mu\nu}\nabla^\mu\nabla^\nu R_2^{\alpha\beta}\hat{1}\\
&+ \mathrm{O}[\Re^3],\end{aligned} \tag{3.68}$$

$$\begin{aligned}\hat{a}_4(x,x) &= \frac{1}{840}\Box^3\hat{P} + \frac{1}{15120}\Box^3 R\hat{1}\\
&+ \Big(\frac{\Box_2\Box_3}{360} + \frac{\Box_1\Box_2}{360} + \frac{\Box_1\Box_3}{360} + \frac{{\Box_3}^2}{360} + \frac{{\Box_1}^2}{360} + \frac{{\Box_2}^2}{360}\Big)\hat{P}_1\hat{P}_2\\
&+ \Big(\frac{{\Box_1}^2}{3360} + \frac{{\Box_2}^2}{3360} + \frac{\Box_1\Box_2}{2520} + \frac{\Box_1\Box_3}{1680} + \frac{{\Box_3}^2}{1120} + \frac{\Box_2\Box_3}{1680}\Big)\hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\mu\nu}\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{\Box_1\Box_3}{15120}-\frac{{\Box_1}^2}{3024}+\frac{{\Box_2}^2}{3780}+\frac{{\Box_3}^2}{3780}+\frac{\Box_1\Box_2}{15120}+\frac{\Box_2\Box_3}{3780}\Big)\hat{P}_1R_2\\
&+\Big(-\frac{{\Box_1}^2}{56700}+\frac{\Box_1\Box_3}{453600}-\frac{\Box_1\Box_2}{453600}+\frac{{\Box_3}^2}{129600}\Big)R_1R_2\hat{1}\\
&+\Big(\frac{\Box_3{\Box_1}^2}{37800\Box_2}+\frac{{\Box_3}^2\Box_1}{37800\Box_2}+\frac{\Box_1\Box_3}{25200}+\frac{{\Box_1}^3}{37800\Box_2}+\frac{{\Box_1}^2}{37800}\\
&+\frac{\Box_1\Box_2}{75600}+\frac{{\Box_3}^2}{50400}+\frac{{\Box_3}^4}{12600\Box_1\Box_2}-\frac{{\Box_3}^3}{4200\Box_1}\Big)R_1^{\mu\nu}R_{2\mu\nu}\hat{1}\\
&+\Big(\frac{\Box_3}{2520}+\frac{\Box_2}{2520}+\frac{\Box_1}{2520}\Big)\nabla_\mu\hat{\mathcal{R}}_1^{\mu\nu}\nabla^\alpha\hat{\mathcal{R}}_{2\alpha\nu}\\
&+\Big(-\frac{\Box_1}{630}-\frac{\Box_3}{630}-\frac{\Box_2}{1260}\Big)[\nabla_\alpha\hat{P}_1,\nabla_\beta\hat{\mathcal{R}}_2^{\beta\alpha}]\\
&+\Big(\frac{\Box_1}{840}+\frac{\Box_2}{2520}+\frac{\Box_3}{840}\Big)\nabla_\mu\nabla_\nu\hat{P}_1R_2^{\mu\nu}\\
&+\Big(\frac{{\Box_1}^2}{9450\Box_2}+\frac{\Box_3}{6300}+\frac{{\Box_3}^2}{3150\Box_1}+\frac{\Box_1}{6300}+\frac{\Box_3\Box_1}{4725\Box_2}-\frac{{\Box_3}^3}{3150\Box_1\Box_2}\Big)\nabla_\alpha R_{1\mu\nu}\nabla^\mu R_2^{\nu\alpha}\hat{1}\\
&+\Big(\frac{\Box_2}{75600}+\frac{\Box_1}{15120}+\frac{\Box_3}{15120}\Big)\nabla_\mu\nabla_\nu R_1R_2^{\mu\nu}\hat{1}\\
&+\Big(\frac{1}{12600}+\frac{\Box_3}{3150\Box_1}+\frac{\Box_1}{9450\Box_2}+\frac{{\Box_3}^2}{3150\Box_1\Box_2}\Big)\nabla_\alpha\nabla_\beta R_{1\mu\nu}\nabla^\mu\nabla^\nu R_2^{\alpha\beta}\hat{1}+\mathrm{O}[\Re^3].
\end{aligned}
\tag{3.69}
$$

It is worthwhile to mention that, it is also possible to do the above computations with use of the short time expansions of the trace of the heat kernel (2.225)–(2.228) and the relation (3.3), but if one wants to obtain the same accuracy, i. e., to know the fourth coefficient $a_4(x,x)$, those expansions should be derived up to *fifth* order in the proper time s. This fact follows directly from the Schwinger-DeWitt series (3.65) and the variational principle (3.3):

$$
\hat{a}_{n-1}(x,x)=g^{-1/2}\frac{\delta}{\delta\hat{P}}\int dx\ g^{1/2}\mathrm{tr}\ \hat{a}_n(x,x). \tag{3.70}
$$

As with the heat kernel trace, local coefficients $a_n(x,x)$ are restored by using the Green's function solution of the Bianchi identity (1.58). To do this we have to find an independent basis of quadratic tensor invariants containing the Riemann tensor for each of the coefficients $a_n(x,x)$. Fortunately, it is easy in the second order, such a basis for $a_2(x,x)$ consists of just one structure,

$$
R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \tag{3.71}
$$

Since there are several derivatives eliminating the Riemann tensor with help of (1.57) in possible combinations for the third and fourth coefficients, the situation here is rather simple too. Thus, the basis for $a_3(x,x)$ may contain only

$$\Box(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}), \qquad R_{\mu\nu\alpha\beta}\nabla^\mu\nabla^\alpha R^{\nu\beta}, \tag{3.72}$$

and for $a_4(x,x)$,

$$\Box^2(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}), \qquad \Box(R_{\mu\nu\alpha\beta}\nabla^\mu\nabla^\alpha R^{\nu\beta}), \qquad \Box(R_{\mu\nu\alpha\beta}\Box\nabla^\mu\nabla^\alpha R^{\nu\beta}). \tag{3.73}$$

Then we form the local $a_n(x,x)$ with these invariants and other acceptable tensor invariants which are merely (3.17)–(3.27) mixed with a number of the Laplacians $\Box$. The necessary nonlocal expansions for the invariants (3.71)–(3.73), actually boil down to a single expression

$$\begin{aligned} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} &= 4R_{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha\frac{1}{\Box}R_{\nu\beta} + \mathrm{O}[R^3_{..}] \\ &= \frac{1}{\Box_1\Box_2}\nabla^\nu\nabla^\beta R_1^{\mu\alpha}\nabla_\mu\nabla_\alpha R_{2\nu\beta} + \frac{1}{\Box_1\Box_2}(\Box_1+\Box_2-\Box_3)\nabla^\beta R_1^{\mu\alpha}\nabla_\alpha R_{2\mu\beta} \\ &+ \frac{1}{4}\frac{1}{\Box_1\Box_2}({\Box_1}^2+{\Box_2}^2+{\Box_3}^2+2\Box_1\Box_2-2\Box_1\Box_3-2\Box_2\Box_3)R_1^{\mu\nu}R_2^{\mu\nu} + \mathrm{O}[R^3_{..}], \end{aligned} \tag{3.74}$$

which is substituted into these combinations containing unknown numerical coefficients. Equating the result to (3.67)–(3.69) gives a *unique* solution for unknown coefficients [48]:

$$\begin{aligned} \hat{a}_2(x,x) &= \frac{1}{6}\Box\hat{P} + \frac{1}{180}\Box R\hat{1} + \frac{1}{2}\hat{P}\hat{P} + \frac{1}{12}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \\ &+ \left[\frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu}\right]\hat{1} + \mathrm{O}[\Re^3], \end{aligned} \tag{3.75}$$

$$\begin{aligned} \hat{a}_3(x,x) &= \frac{1}{60}\Box^2\hat{P} + \frac{1}{1260}\Box^2 R\hat{1} \\ &+ \frac{1}{24}\Box(\hat{P}\hat{P}) + \frac{1}{24}\Box\hat{P}\hat{P} + \frac{1}{24}\hat{P}\Box\hat{P} \\ &+ \frac{1}{90}\Box(\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}) + \frac{1}{180}\Box\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu} + \frac{1}{180}\hat{\mathcal{R}}^{\mu\nu}\Box\hat{\mathcal{R}}_{\mu\nu} \\ &+ \frac{1}{360}\Box(\hat{P}R) - \frac{1}{360}\Box\hat{P}R + \frac{1}{360}\hat{P}\Box R \\ &+ \frac{1}{180}\nabla_\mu\hat{\mathcal{R}}^{\mu\nu}\nabla^\alpha\hat{\mathcal{R}}_{\alpha\nu} - \frac{1}{60}\left[\nabla_\alpha\hat{P},\nabla_\beta\hat{\mathcal{R}}^{\beta\alpha}\right] + \frac{1}{90}\nabla_\mu\nabla_\nu\hat{P}R^{\mu\nu} \end{aligned}$$

$$+\left[\frac{1}{1120}\Box(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})+\frac{1}{420}R^{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha R_{\nu\beta}-\frac{1}{840}R^{\mu\nu}\Box R_{\mu\nu}\right.$$
$$-\frac{1}{5040}\Box(R^{\mu\nu}R_{\mu\nu})-\frac{1}{1260}\nabla^\mu R^{\nu\alpha}\nabla_\alpha R_{\mu\nu}+\frac{1}{1890}\nabla_\mu\nabla_\nu RR^{\mu\nu}$$
$$\left.-\frac{1}{7560}R\Box R+\frac{1}{15120}\Box(RR)\right]+\mathrm{O}[\Re^3], \qquad (3.76)$$

$$\hat{a}_4(x,x)=\frac{1}{840}\Box^3\hat{P}+\frac{1}{15120}\Box^3R\hat{1}$$
$$+\frac{1}{360}\Box(\hat{P}\Box\hat{P})+\frac{1}{360}\Box\hat{P}\Box\hat{P}+\frac{1}{360}\Box(\Box\hat{P}\hat{P})$$
$$+\frac{1}{360}\Box^2(\hat{P}\hat{P})+\frac{1}{360}\Box^2\hat{P}\hat{P}+\frac{1}{360}\hat{P}\Box^2\hat{P}$$
$$+\frac{1}{3360}\Box^2\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}+\frac{1}{3360}\hat{\mathcal{R}}^{\mu\nu}\Box^2\hat{\mathcal{R}}_{\mu\nu}+\frac{1}{2520}\Box\hat{\mathcal{R}}^{\mu\nu}\Box\hat{\mathcal{R}}_{\mu\nu}$$
$$+\frac{1}{1680}\Box(\Box\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu})+\frac{1}{1120}\Box^2(\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu})+\frac{1}{1680}\Box(\hat{\mathcal{R}}^{\mu\nu}\Box\hat{\mathcal{R}}_{\mu\nu})$$
$$+\frac{1}{15120}\Box(\Box\hat{P}R)-\frac{1}{3024}\Box^2\hat{P}R+\frac{1}{3780}\hat{P}\Box^2R$$
$$+\frac{1}{3780}\Box^2(\hat{P}R)+\frac{1}{15120}\Box\hat{P}\Box R+\frac{1}{3780}\Box(\hat{P}\Box R)$$
$$+\frac{1}{2520}\Box(\nabla_\mu\hat{\mathcal{R}}^{\mu\nu}\nabla^\alpha\hat{\mathcal{R}}_{\alpha\nu})+\frac{1}{2520}\nabla_\mu\hat{\mathcal{R}}^{\mu\nu}\Box\nabla^\alpha\hat{\mathcal{R}}_{\alpha\nu}+\frac{1}{2520}\Box\nabla_\mu\hat{\mathcal{R}}^{\mu\nu}\nabla^\alpha\hat{\mathcal{R}}_{\alpha\nu}$$
$$-\frac{1}{630}[\Box\nabla_\alpha\hat{P},\nabla_\beta\hat{\mathcal{R}}^{\beta\alpha}]-\frac{1}{630}\Box[\nabla_\alpha\hat{P},\nabla_\beta\hat{\mathcal{R}}^{\beta\alpha}]-\frac{1}{1260}[\nabla_\alpha\hat{P},\Box\nabla_\beta\hat{\mathcal{R}}^{\beta\alpha}]$$
$$+\frac{1}{840}\Box\nabla_\mu\nabla_\nu\hat{P}R^{\mu\nu}+\frac{1}{2520}\nabla_\mu\nabla_\nu\hat{P}\Box R^{\mu\nu}+\frac{1}{840}\Box(\nabla_\mu\nabla_\nu\hat{P}R^{\mu\nu})$$
$$+\left[\frac{1}{12600}\Box^2(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})+\frac{1}{3150}\Box(R^{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha R_{\nu\beta})+\frac{1}{9450}R^{\mu\nu\alpha\beta}\Box\nabla_\mu\nabla_\alpha R_{\nu\beta}\right.$$
$$+\frac{1}{12600}\nabla_\alpha\nabla_\beta R_{\mu\nu}\nabla^\mu\nabla^\nu R^{\alpha\beta}-\frac{1}{6300}\Box(\nabla_\alpha R_{\mu\nu}\nabla^\mu R^{\nu\alpha})+\frac{1}{18900}\nabla_\alpha R_{\mu\nu}\Box\nabla^\mu R^{\nu\alpha}$$
$$+\frac{1}{15120}\Box(\nabla_\mu\nabla_\nu RR^{\mu\nu})+\frac{1}{15120}\Box\nabla_\mu\nabla_\nu RR^{\mu\nu}+\frac{1}{75600}\nabla_\mu\nabla_\nu R\Box R^{\mu\nu}$$
$$+\frac{1}{50400}\Box^2(R^{\mu\nu}R_{\mu\nu})-\frac{11}{75600}\Box(R^{\mu\nu}\Box R_{\mu\nu})-\frac{1}{37800}R^{\mu\nu}\Box^2R_{\mu\nu}$$
$$-\frac{1}{75600}\Box R^{\mu\nu}\Box R_{\mu\nu}-\frac{1}{56700}\Box^2RR+\frac{1}{453600}\Box(\Box RR)$$
$$\left.-\frac{1}{453600}\Box R\Box R+\frac{1}{129600}\Box^2(RR)\right]\hat{1}+\mathrm{O}[\Re^3]. \qquad (3.77)$$

Two terms in (3.77), $R^{\mu\nu}\Box^2R_{\mu\nu}$ and $\Box R^{\mu\nu}\Box R_{\mu\nu}$, are displayed here with corrected numerical coefficients in comparison with [48]. Although, $a_2(x,x)$ was computed with $\mathrm{O}[\Re^3]$ accuracy, its expression above is in fact exact.

We compare the coefficients above with results available in literature. Again, our main concern is the fourth coefficient $a_4(x,x)$, as the expressions (3.75), (3.76) for

$a_2(x,x)$ and $a_3(x,x)$ coincide with the results obtained by other methods [12, 11, 4, 16, 17, 23, 57], taking into account differencies in definitions and curvature conventions.

In ref. [6] the first order curvature terms are presented for any Schwinger-DeWitt coefficient $a_n(x,x)$. This result is fully consistent with similar terms of the coefficients (3.75)–(3.77).

The only generic result for $a_4(x,x)$ is that of Avramidi [16], but this result unfortunately lacks a final representation. The very tedious work of its reduction to a tensor invariant basis, which we choose to be the basis above, was performed, partially with help of the tensor manipulation programs *MathTensor* [51] and *Ricci* [52]. In fact, this reduction is much more difficult in contrast to a similar procedure for the trace $\int dx\, \mathrm{tr} a_4(x,x)$, see sect. 2.7. The use of computers in this context is unavoidable because of the presence of total derivatives. This fact results in computations of objects like a totally index symmetrized combination of the Ricci tensor and six covariant derivatives and its subsequent reduction to the form of $\Box^3 R$ keeping all commutator curvature terms. The result derived this way from [16] is in a full agreement with (3.77).

As mentioned in sect. 2.7, the work [17] reproduces only pure gravitational terms, thus only terms of (3.77) in square brackets can be compared. The expression (3.77) disagrees with that result in several numerical coefficients. Taking into account, that results of the paper [17] fail to produce a correct form for the trace of the fourth Schwinger-DeWitt coefficient, this work must be considered wrong.

In view of these observations, we should remark that the work on the fourth Schwinger-DeWitt coefficient should be continued because there is no reliable complete generic result for $a_4(x,x)$ in the literature. Therefore, such a result for $a_4(x,x)$ must be obtained using all possible sources like [16, 17, 23, 19].

Chapter 4

Weyl invariant models

4.1 The one-loop effective action for the Weyl invariant scalar field model in two dimensions

In this chapter we study field models that are invariant at the classical level under the *local* rescaling of the metric

$$\bar{g}^{\mu\nu}(x) = \mathrm{e}^{\sigma(x)} g^{\mu\nu}(x) \tag{4.1}$$

and the corresponding rescaling of scalar field in N dimensions,

$$\bar{\varphi}(x) = \mathrm{e}^{\frac{(N-2)}{2}\sigma(x)} \varphi(x). \tag{4.2}$$

This transformation was introduced by H. Weyl [58, 59], hence, the term Weyl invariance. Although, two terms, *Weyl* and *conformal*, are used in the literature [9, 18, 60] and this thesis on par, the former is preferable to distinguish from the *global* conformal transformation which forms the conformal group in flat space-times.

As discussed in the Introduction the curvature expansion for the effective action (1.17) in two dimensions does not generally exist due to infrared divergences appearing at every order in the curvature. Nevertheless, it is formally defined by the expression (see Introduction),

$$-W = \frac{1}{2}\int_0^\infty \frac{ds}{s}\left(\mathrm{Tr}K(s) - \mathrm{Tr}K(s)|_{\Re=0}\right). \tag{4.3}$$

There is only one curvature in two dimensions, $\omega = 1$, since

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \tag{4.4}$$

and the term linear in the scalar Ricci $g^{1/2}R$ vanishes because it is a total derivative [18, 30].

There is an exceptional case when the effective action (4.3) proves to be analytic in the curvature. This occures in the conformal invariant scalar quantum field model [18, 30]:

$$\mathrm{tr}\hat{1} = 1, \qquad \hat{\mathcal{R}}_{\mu\nu} = 0, \qquad \hat{P} = \frac{1}{6}R\hat{1}. \tag{4.5}$$

In this case, the effective action is expandable in powers of the curvature, because the integral (4.3) converges at the upper limit at each order of this expansion, owing to specific cancellations in the asymptotic behaviors of the form factors at large s. Furthermore, in the case (4.5), the expansion of W in powers of the curvature should terminate at the second power thereby yielding an exact result [38, 61, 30]; the terms of third and higher powers in the curvature should vanish order by order. In this section we check explicitly the vanishing of the third order terms [42]. We will work out this result directly from the heat kernel trace without computing the generic effective action.

By using the conditions (4.4)–(4.5) in (2.178), we obtain $\mathrm{Tr}K(s)$ as an expansion in powers of the Ricci scalar only:

$$\mathrm{Tr}\,K(s) = \frac{1}{4\pi s}\int dx\, g^{1/2}\left\{1 + s^2\sum_{i=1}^{5} c_i f_i(-s\Box_2)R_1R_2 \right.$$
$$\left. + s^3\sum_{i=1}^{29} C_i F_i(-s\Box_1, -s\Box_2, -s\Box_3)R_1R_2R_3 + \mathrm{O}[R^4]\right\}, \qquad \omega = 1 \tag{4.6}$$

where

$$c_1 = \frac{1}{2}, \qquad c_2 = 1, \qquad c_3 = \frac{1}{6}, \qquad c_4 = \frac{1}{36}, \qquad c_5 = 0, \tag{4.7}$$

and

$$C_1 = \frac{1}{216}, \ C_4 = \frac{1}{6}, \ C_5 = \frac{1}{12}, \ C_6 = \frac{1}{36}, \ C_9 = 1, \ C_{10} = \frac{1}{4}, \ C_{11} = \frac{1}{2},$$
$$C_{15} = \frac{s}{24}(\Box_1 - \Box_2 - \Box_3), \ C_{16} = \frac{s}{48}(\Box_3 - \Box_2 - \Box_1), \ C_{17} = \frac{s}{72}\Box_2,$$
$$C_{22} = \frac{s}{4}(\Box_1 - \Box_2 - \Box_3), \ C_{23} = \frac{s}{8}(\Box_3 - \Box_2 - \Box_1), \ C_{24} = \frac{s}{8}(\Box_1 - \Box_2 - \Box_3),$$

$$C_{25}=\frac{s}{16}(\Box_1-\Box_2-\Box_3),\quad C_{26}=\frac{s^2}{24}\Box_1\Box_2,\quad C_{27}=\frac{s^2}{4}\Box_1\Box_2,$$
$$C_{28}=\frac{s^2}{16}\Box_3(\Box_3-\Box_2-\Box_1),\quad C_{29}=\frac{s^3}{8}\Box_1\Box_2\Box_3,$$
$$C_2=C_3=C_7=C_8=C_{12}=C_{13}=C_{14}=C_{18}=C_{19}=C_{20}=C_{21}=0. \tag{4.8}$$

We insert the coefficients (4.7) and (4.8), and the expressions for the form factors f_i and F_i given in sect. 2.5 and Appendix B into (4.6) using as usual the algebraic manipulation program *MAPLE* [50]. Then, $\mathrm{Tr}K(s)$ divided by s takes the form

$$\begin{aligned}
\frac{1}{s}\mathrm{Tr}K(s) &= \frac{1}{4\pi}\int dx\, g^{1/2}\left\{\frac{1}{s^2}+\left[\frac{1}{32}f(-s\Box_2)-\frac{1}{8}\left(\frac{f(-s\Box_2)-1}{s\Box_2}\right)\right.\right.\\
&\left.+\frac{3}{8}\left(\frac{f(-s\Box_2)-1-\frac{1}{6}s\Box_2}{(s\Box_2)^2}\right)\right]R_1R_2\\
&+\left[-sF(-s\Box_1,-s\Box_2,-s\Box_3)\frac{{\Box_1}^2{\Box_2}^2{\Box_3}^2}{3D^3}\right.\\
&+f(-s\Box_1)\frac{1}{32D^3\Box_2}({\Box_1}^6-4{\Box_1}^5\Box_2-4{\Box_1}^5\Box_3+3{\Box_1}^4\Box_2\Box_3\\
&+24{\Box_1}^3{\Box_2}^2\Box_3+5{\Box_1}^4{\Box_3}^2+24{\Box_1}^3\Box_2{\Box_3}^2-2{\Box_1}^2{\Box_2}^2{\Box_3}^2\\
&+32\Box_1{\Box_2}^3{\Box_3}^2-25{\Box_1}^2\Box_2{\Box_3}^3-36\Box_1{\Box_2}^2{\Box_3}^3+5{\Box_2}^3{\Box_3}^3\\
&-5{\Box_1}^2{\Box_3}^4-9{\Box_2}^2{\Box_3}^4+4\Box_1{\Box_3}^5+5\Box_2{\Box_3}^5-{\Box_3}^6)\\
&-\left(\frac{f(-s\Box_1)-1}{s\Box_1}\right)\frac{1}{8D^2\Box_2}({\Box_1}^4-2{\Box_1}^3\Box_3-12{\Box_1}^2\Box_2\Box_3\\
&-10\Box_1{\Box_2}^2\Box_3+8\Box_1\Box_2{\Box_3}^2-2{\Box_2}^2{\Box_3}^2+2\Box_1{\Box_3}^3\\
&+3\Box_2{\Box_3}^3-{\Box_3}^4)\\
&+\left(\frac{f(-s\Box_1)-1-\frac{1}{6}s\Box_1}{(s\Box_1)^2}\right)\frac{3}{8D\Box_2}({\Box_1}^2+4\Box_1\Box_2+\Box_2\Box_3-{\Box_3}^2)\\
&-\frac{1}{\Box_2-\Box_3}\frac{\Box_2}{32\Box_1}\big(f(-s\Box_2)-f(-s\Box_3)\big)\\
&+\frac{1}{\Box_2-\Box_3}\frac{\Box_2}{8\Box_1}\left(\frac{f(-s\Box_2)-1}{s\Box_2}-\frac{f(-s\Box_3)-1}{s\Box_3}\right)\\
&-\frac{1}{\Box_2-\Box_3}\frac{3\Box_2}{8\Box_1}\left(\frac{f(-s\Box_2)-1-\frac{1}{6}s\Box_2}{(s\Box_2)^2}\right.\\
&\left.\left.\left.-\frac{f(-s\Box_3)-1-\frac{1}{6}s\Box_3}{(s\Box_3)^2}\right)\right]R_1R_2R_3+\mathrm{O}[R^4]\right\},\qquad \omega=1
\end{aligned} \tag{4.9}$$

in terms of the basic form factors (2.130) and (2.131), and the polynomial D (2.171).

By using the large time asymptotic behaviours (2.215) and (2.216), one can now check that, at $s \to \infty$, the leading terms $1/s$ in (4.9) cancel at both second order and third order in the curvature so that

$$\frac{1}{s}\mathrm{Tr}K(s) \;=\; \mathrm{O}\left(\frac{1}{s^2}\right), \qquad s \to \infty. \tag{4.10}$$

As a result, the integral (4.3) converges at the upper limit. The convergence at the lower limit in the curvature-dependent terms holds trivially. Only the term of zeroth order in the curvature is ultraviolet divergent but, in the effective action (4.3), this term gets subtracted out [26, 25] as was showed in eq. (4.3).

For the calculation of the integral (4.3) one can use the differential equations for the basic form factors, eqs. (2.167), (2.168) and (2.170) of sect. 2.4, and make the following substitutions in (4.9):

$$\begin{aligned}
-s\frac{\Box_1\Box_2\Box_3}{D}F(-s\Box_1,-s\Box_2,-s\Box_3) &= \frac{d}{ds}\Big(sF(-s\Box_1,-s\Box_2,-s\Box_3)\Big)\\
&+\frac{\Box_1(\Box_3+\Box_2-\Box_1)}{2D}f(-s\Box_1)+\frac{\Box_2(\Box_1+\Box_3-\Box_2)}{2D}f(-s\Box_2)\\
&+\frac{\Box_3(\Box_1+\Box_2-\Box_3)}{2D}f(-s\Box_3),
\end{aligned} \tag{4.11}$$

$$\frac{f(-s\Box)-1}{s\Box} = \frac{d}{ds}\left(-\frac{2}{\Box}f(-s\Box)\right)+\frac{1}{2}f(-s\Box), \tag{4.12}$$

$$\begin{aligned}
\frac{f(-s\Box)-1-\frac{1}{6}s\Box}{(s\Box)^2} &= \frac{d}{ds}\left(-\frac{2}{3\Box}\frac{f(-s\Box)-1}{s\Box}-\frac{1}{3\Box}f(-s\Box)\right)\\
&+\frac{1}{12}f(-s\Box).
\end{aligned} \tag{4.13}$$

The result of these substitutions is that the expression (4.9) becomes a total derivative in s:

$$\begin{aligned}
\frac{1}{s}\mathrm{Tr}K(s) \;=\; \frac{1}{4\pi}\int dx\, g^{1/2}\,\frac{d}{ds}\Big\{&-\frac{1}{s}+l(s,\Box_2)R_1R_2\\
&+h(s,\Box_1,\Box_2,\Box_3)R_1R_2R_3+\mathrm{O}[R^4]\Big\}, \qquad \omega=1
\end{aligned} \tag{4.14}$$

where

$$l(s,\Box) \;=\; \frac{1}{\Box}\left(\frac{1}{8}f(-s\Box)-\frac{1}{4}\frac{f(-s\Box)-1}{s\Box}\right), \tag{4.15}$$

$$h(s, \Box_1, \Box_2, \Box_3) = h_1^{\rm sym} + h_2^{\rm sym} + h_3^{\rm sym}, \tag{4.16}$$

and $h_1^{\rm sym}, h_2^{\rm sym}, h_3^{\rm sym}$ are the completely symmetrized in $\Box_1, \Box_2, \Box_3$ functions

$$h_1 = sF(-s\Box_1, -s\Box_2, -s\Box_3)\frac{\Box_1\Box_2\Box_3}{3D^2}, \tag{4.17}$$

$$\begin{aligned} h_2 = {}& f(-s\Box_1)\frac{1}{8D^2\Box_1\Box_2}(\Box_1{}^4 - 2\Box_1{}^3\Box_3 + 2\Box_1\Box_3{}^3 - \Box_3{}^4 - 2\Box_1{}^3\Box_2 \\ & + 3\Box_2\Box_3{}^3 - 8\Box_1{}^2\Box_2\Box_3 + 8\Box_1\Box_2\Box_3{}^2 - 10\Box_1\Box_2{}^2\Box_3 - 2\Box_2{}^2\Box_3{}^2) \\ & - \left(\frac{f(-s\Box_1) - 1}{s\Box_1}\right)\frac{1}{4D\Box_1\Box_2}(\Box_1{}^2 + 4\Box_1\Box_2 + \Box_2\Box_3 - \Box_3{}^2), \end{aligned} \tag{4.18}$$

$$\begin{aligned} h_3 = {}& \frac{1}{\Box_2 - \Box_3}\frac{\Box_2}{\Box_1}\left[-\frac{1}{8}\left(\frac{1}{\Box_2}f(-s\Box_2) - \frac{1}{\Box_3}f(-s\Box_3)\right)\right. \\ & \left. + \frac{1}{4}\left(\frac{1}{\Box_2}\frac{f(-s\Box_2) - 1}{s\Box_2} - \frac{1}{\Box_3}\frac{f(-s\Box_3) - 1}{s\Box_3}\right)\right]. \end{aligned} \tag{4.19}$$

The insertion of (4.14) into (4.3) gives for the effective action:

$$\begin{aligned} W = {}& \frac{1}{8\pi}\int dx\, g^{1/2}\Big(l(0, \Box_2)R_1R_2 \\ & + h(0, \Box_1, \Box_2, \Box_3)R_1R_2R_3 + {\rm O}[R^4]\Big), \qquad \omega = 1 \end{aligned} \tag{4.20}$$

where use is made of the fact that the functions l and h vanish at $s \to \infty$. With the short time asymptotic behaviors (2.221) and (2.222) and the explicit expressions above for the functions l and h, we obtain

$$\begin{aligned} h_1^{\rm sym}|_{s=0} &= 0, \\ h_2^{\rm sym}|_{s=0} &= -h_3^{\rm sym}|_{s=0} = -\frac{1}{36}\frac{\Box_1 + \Box_2 + \Box_3}{\Box_1\Box_2\Box_3}, \end{aligned} \tag{4.21}$$

with the final result

$$l(0, \Box) = \frac{1}{12}\frac{1}{\Box}, \qquad h(0, \Box_1, \Box_2, \Box_3) = 0, \tag{4.22}$$

Thus,

$$W = \frac{1}{96\pi}\int dx\, g^{1/2}\, R\frac{1}{\Box}R + {\rm O}[R^4], \qquad \omega = 1. \tag{4.23}$$

Here the second order term in the curvature reproduces the result of [30] and the results of refs. [61, 38] obtained by integrating the anomaly of the stress tensor trace (see sect. 4.3).

Thus the third order contribution to the effective action W actually vanishes, and the mechanism of this vanishing is that, under special conditions like (4.4)–(4.5), the third order contribution in $s^{-1}\mathrm{Tr}K(s)$ becomes a total derivative of a function vanishing at both limits, $s=0$ and $s=\infty$. The same mechanism underlies cancellations of nonlocal terms in the derivation of the Weyl anomaly in four dimensions, sect. 4.3.

4.2 The Green's function for the conformal scalar field model in two dimensions

In this section we consider the same field model as in sect. 4.1 but applied to computation of the Green's function. The Green's function is obtained from the heat kernel derived in Chapter 3 with help of the Schwinger proper time method (1.4).

$$G = \int_0^\infty d\,s\Big(K(s) - K(s)|_{\Re=0}\Big). \tag{4.24}$$

With use of the properties (4.4)–(4.5) to reduce the heat kernel of sect. 3.4 to the form

$$\begin{aligned} K(s) &= \frac{g^{1/2}}{4\pi s}\Bigg\{1 + s\sum_{i=1}^{2}\tilde{c}_i g_i(-s\Box)R \\ &\quad + s^2\sum_{i=1}^{11}\tilde{C}_i\; G_i(-s\Box_1,-s\Box_2,-s\Box_3)R_1R_2 + \mathrm{O}[R^3]\Bigg\}, \end{aligned} \tag{4.25}$$

where for factors $g_i(-s\Box)$ are given by (3.60)–(3.61), and form factors $G_i(-s\Box_1,-s\Box_2,-s\Box_3)$ by (C.1)–(C.11). The coefficients $\tilde{c}_i$ and $\tilde{C}_i$ are:

$$\begin{aligned} &\tilde{c}_1 = \frac{1}{6},\quad \tilde{c}_2 = 1,\\ &\tilde{C}_1 = \frac{1}{36},\quad \tilde{C}_3 = \frac{1}{6},\quad \tilde{C}_4 = 1,\quad \tilde{C}_5 = \frac{1}{2},\quad \tilde{C}_8 = \frac{s}{12}\Box_1,\quad \tilde{C}_9 = \frac{s}{8}(\Box_3 - \Box_2 - \Box_1),\\ &\tilde{C}_{10} = \frac{s}{2}\Box_1,\quad \tilde{C}_{11} = \frac{s^2}{4}\Box_1\Box_2,\quad \tilde{C}_2 = \tilde{C}_6 = \tilde{C}_7 = 0. \end{aligned}$$

After compactification the final expression looks like:

$$
\begin{aligned}
K(s) \;=\; & \frac{g^{1/2}}{4\pi s}\Bigg\{1 + s\left[\frac{1}{4}f(-s\Box) - \frac{1}{2}\frac{f(-s\Box)-1}{s\Box}\right]R \\
& + s\Bigg[sF(-s\Box_1,-s\Box_2,-s\Box_3)\frac{\Box_1\Box_2\Box_3{}^2}{D^2} \\
& + f(-s\Box_1)\frac{\Box_1}{4\Box_2 D^2}(\Box_1{}^3 - 3\Box_2\Box_1{}^2 - 3\Box_3\Box_1{}^2 + 3\Box_1\Box_2{}^2 \\
& - 2\Box_1\Box_2\Box_3 + 3\Box_1\Box_3{}^2 + 5\Box_2\Box_3{}^2 - \Box_2{}^3 + 5\Box_3\Box_2{}^2 - \Box_3{}^3) \\
& - f(-s\Box_3)\frac{1}{4D^2\Box_2}(-\Box_1\Box_3{}^3 + 3\Box_1{}^2\Box_3{}^2 - 3\Box_3\Box_1{}^3 + \Box_1{}^4 - \Box_2\Box_3{}^3 \\
& + 9\Box_1\Box_2\Box_3{}^2 - 4\Box_1{}^2\Box_2\Box_3 - 3\Box_2\Box_1{}^3 + 7\Box_1\Box_3\Box_2{}^2 + 2\Box_1{}^2\Box_2{}^2) \\
& + \left(\frac{f(-s\Box_1)-1}{s\Box_1}\right)\frac{\Box_1}{2\Box_2 D}(\Box_3 + \Box_2 - \Box_1) \\
& + \left(\frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{1}{2\Box_2 D}(-\Box_1\Box_3 + \Box_1{}^2 - 3\Box_2\Box_3 - \Box_1\Box_2) \\
& - \frac{1}{(\Box_1-\Box_3)}\Big(f(-s\Box_1) - f(-s\Box_3)\Big)\frac{\Box_1}{4\Box_2} \\
& + \frac{1}{(\Box_1-\Box_3)}\left(\frac{f(-s\Box_1)-1}{s\Box_1} - \frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{\Box_1}{2\Box_2}\Bigg]R_1R_2 \\
& + \mathrm{O}[R^3]\Bigg\},\quad \omega = 1. \qquad (4.26)
\end{aligned}
$$

One should note that this expression cannot be obtained from (4.9) by the generating function method because there is no potential term $\hat{P}$ with which to make a variation. A straightforward check with help of the large time asymptotics (2.215) and (2.216), shows that the whole expression behaves like $\mathrm{O}[1/s]$ at large s.

To perform the proper time integration we resort to a technique of the previous section, namely, we rewrite $K(s)$ as a total derivative in the proper time s:

$$
\begin{aligned}
& K(s) - K(s)|_{R=0} = \frac{g^{1/2}}{4\pi}\frac{d}{ds}\Big(n(s,\Box) + m^{\mathrm{sym}}(s,\Box_1,\Box_2,\Box_3)\Big) + \mathrm{O}[R^3], \\
& n(s,\Box) = \frac{1}{\Box}f(-s\Box)R, \\
& m(s,\Box_1,\Box_2,\Box_3) = \Bigg[-s\frac{\Box_3}{\Box_2 D}F(-s\Box_1,-s\Box_2,-s\Box_3) \\
& - f(-s\Box_1)\frac{(\Box_3 + \Box_2 - \Box_1)}{\Box_2 D} \\
& - f(-s\Box_3)\frac{(-\Box_1\Box_3 + \Box_1{}^2 - 3\Box_2\Box_3 - \Box_1\Box_2)}{\Box_2\Box_3 D}
\end{aligned}
$$

$$+\frac{1}{(\Box_1-\Box_3)\,\Box_2}\frac{\Box_1}{\Box_2}\left(\frac{f(-s\Box_3)}{\Box_3}-\frac{f(-s\Box_1)}{\Box_1}\right)\Bigg]\,R_1R_2, \qquad (4.27)$$

and $m^{\rm sym}$ is the symmetrized in $\Box_1, \Box_2$ function. Due to (2.215)–(2.216) the expression in the curly brackets vanishes at large s, therefore,

$$G(x,x) = -\frac{1}{4\pi}g^{1/2}\Big(n(0,\Box)+m^{\rm sym}(0,\Box_1,\Box_2,\Box_3)\Big)+{\rm O}[R^3]. \qquad (4.28)$$

As readily seen in this limit only a linear R term survives and the series terminates at the first order:

$$G(x,x) = -\frac{g^{1/2}}{4\pi}\frac{1}{\Box}R+{\rm O}[R^3]. \qquad (4.29)$$

A similar term in the Green function coincidence limit for two-dimensional compact manifolds was found in [62]. Since we showed that the second order is exactly zero we may presume that the answer (4.29) is exact in the same fashion as the one-loop effective action is exact in two dimensions (4.23).

4.3 The Weyl anomaly in four dimensions

A crucial check of the results above is a derivation of the Weyl anomaly (also called the conformal anomaly or the trace anomaly) for the conformal invariant quantum field in four dimensions [63, 64, 9]. As any other quantum anomaly the Weyl anomaly exhibits breaking of classical invariance by the quantization procedure. In this case the Weyl invariance (4.1)–(4.2) of the classical field model results in the vanishing of the trace of the stress tensor [18], $T^{\mu}_{\ \mu}(x) = 0$, but this is violated by quantization, $g^{\mu\nu} < T_{\mu\nu}(x) > \neq 0$. Since the quantum corrected stress tensor is given by the effective action,

$$< T_{\mu\nu} >= 2g^{-1/2}\frac{\delta W}{\delta g^{\mu\nu}} \qquad (4.30)$$

it is important to reproduce this effect from the results obtained in the first chapter of this thesis. Not only should it give an additional check for our derivations but also reveal how the *local* Weyl anomaly appears from the *nonlocal* effective action (the heat kernel trace).

To have as many curvature structures as possible involved in the check, we choose the following model of the quantum scalar field ($\omega = 2$):

$$S[\varphi] = \frac{1}{2}\int dx\, g^{1/2}\left(\nabla_\mu\varphi^T\nabla^\mu\varphi + \frac{R}{6}\varphi^T\varphi + \frac{\lambda^2}{4!}(\varphi^T\varphi)^2\right), \tag{4.31}$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{4.32}$$

In fact, this is the action for the famous φ^4 model for the complex scalar quantum field

$$\varphi = \varphi_1 + i\varphi_2, \tag{4.33}$$

rewritten in terms of the real components.

The gauge abelian (electromagnetic) field A_μ,

$$\nabla_\mu\varphi = \partial_\mu\varphi + A_\mu\hat{J}\varphi, \qquad \nabla_\mu\varphi^T = \partial_\mu\varphi^T + A_\mu\varphi^T\hat{J}^T, \tag{4.34}$$

$$\hat{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.35}$$

and the gravitational field in (4.31) are background classical fields.

The action (4.31) is invariant under the Weyl transformations (4.1) and (4.2):

$$\delta_\sigma g^{\mu\nu}(x) = \sigma(x)g^{\mu\nu}(x), \qquad \delta_\sigma\varphi(x) = \frac{1}{2}\sigma(x)\varphi(x), \qquad \delta_\sigma A_\mu(x) = 0. \tag{4.36}$$

The Hessian (1.5) of the model (4.31) has the form of the second order differential operator (1.6) (times a local matrix) in which the potential term is

$$\hat{P} = -\frac{2\lambda^2}{4!}\begin{pmatrix} 3{\varphi_1}^2 + {\varphi_2}^2 & 2\varphi_1\varphi_2 \\ 2\varphi_1\varphi_2 & 3{\varphi_2}^2 + {\varphi_1}^2 \end{pmatrix}. \tag{4.37}$$

The commutator curvature is defined through (4.34), (4.35):

$$\hat{\mathcal{R}}_{\mu\nu} = \hat{J}(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{4.38}$$

From (4.36)–(4.38) we find the Weyl transformation laws for the curvatures and $\Box$-operators:

$$\delta_\sigma\hat{P} = \sigma\hat{P}, \qquad \delta_\sigma\hat{\mathcal{R}}_{\mu\nu} = 0, \tag{4.39}$$

$$\delta_\sigma R_{\mu\nu} = (\nabla_\mu \nabla_\nu + \frac{1}{2} g_{\mu\nu} \Box)\sigma, \tag{4.40}$$

$$\delta_\sigma R = (3\Box + R)\sigma, \tag{4.41}$$

$$(\delta_\sigma \Box)\hat{P} = \sigma \Box \hat{P} - \nabla_\alpha \sigma \nabla^\alpha \hat{P}, \tag{4.42}$$

$$(\delta_\sigma \Box)\hat{\mathcal{R}}_{\mu\nu} = \sigma \Box \hat{\mathcal{R}}_{\mu\nu} + \hat{\mathcal{R}}_{\mu\nu} \Box \sigma + \nabla_\mu \sigma \nabla^\alpha \hat{\mathcal{R}}_{\alpha\nu} - \nabla_\nu \sigma \nabla^\alpha \hat{\mathcal{R}}_{\alpha\mu}, \tag{4.43}$$

$$\begin{aligned}(\delta_\sigma \Box) R_{\mu\nu} &= \sigma \Box R_{\mu\nu} + R_{\mu\nu} \Box \sigma + \nabla_\alpha \sigma \nabla^\alpha R_{\mu\nu} \\ &\quad + \nabla_{(\mu} \sigma \nabla_{\nu)} R - 2\nabla^\alpha \sigma \nabla_{(\mu} R_{\nu)\alpha},\end{aligned} \tag{4.44}$$

$$(\delta_\sigma \Box) R = \sigma \Box R - \nabla_\alpha \sigma \nabla^\alpha R. \tag{4.45}$$

Having obtained these laws, one can forget the particular content of the model (4.31), (4.37)–(4.38), and merely consider the transformation (4.39)–(4.45) in the effective action. For the dimensionally regularized one-loop effective action (1.17), the result should be exactly [64, 65, 66],

$$-\delta_\sigma W = -\frac{1}{2(4\pi)^2} \int dx\, g^{1/2}\, \sigma(x) \operatorname{tr} \hat{a}_2(x,x) \tag{4.46}$$

where $\hat{a}_2(x,x)$ is the second DeWitt coefficient (3.75) at coincident points. Since the function $\sigma(x)$ is arbitrary in any compact domain, the anomaly (4.46) provides an additional check of $\operatorname{tr} \hat{a}_2(x,x)$.

As mentioned before for the derivation of the four-dimensional effective action the dimensional regularization is used [22, 30, 42]. But in fact, in the present section we will bypass this question since we do not compute the effective action W itself but only its anomaly (4.46).

Expression (4.46) is the general form of the Weyl anomaly in four dimensions

[18, 67, 64, 60, 68]. For the model above,

$$\delta_\sigma = \int dx \left(\sigma(x) g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} + \frac{1}{2}\sigma(x)\varphi \frac{\delta}{\delta \varphi}\right), \tag{4.47}$$

and

$$g^{-1/2}\left(g^{\mu\nu}\frac{\delta W}{\delta g^{\mu\nu}} + \frac{1}{2}\varphi\frac{\delta W}{\delta\varphi}\right) = \frac{1}{2(4\pi)^2}\mathrm{tr}\,\hat{a}_2(x,x). \tag{4.48}$$

In the present technique, eq. (4.46) can be obtained only with a given accuracy $\mathrm{O}[\Re^n]$ and with the Riemann tensor expressed through the Ricci tensor. To lowest order, one can use the expression for $R^2_{\alpha\beta\mu\nu}$ given by (3.74). After elimination of the Riemann tensor in $a_2(x,x)$, eq. (4.46) takes the form

$$\begin{aligned} -\delta_\sigma W \;=\; & \frac{1}{2(4\pi)^2}\int dx\, g^{1/2}\ \mathrm{tr}\,\Big\{-\frac{1}{6}(\Box\hat{P})\sigma \\ & -\frac{1}{180}(\Box R)\sigma\hat{1} - \frac{1}{12}\hat{\mathcal{R}}^2_{\mu\nu}\sigma - \frac{1}{2}\hat{P}^2\sigma \\ & -\frac{1}{180}\left(1 + 2\frac{\Box_1}{\Box_2} - 4\frac{\Box_3}{\Box_1} + \frac{\Box_3{}^2}{\Box_1\Box_2}\right) R_1^{\mu\nu} R_{2\mu\nu}\sigma_3\hat{1} \\ & -\frac{1}{45}\left(\frac{2}{\Box_1} - \frac{\Box_3}{\Box_1\Box_2}\right)\nabla^\mu R_1^{\nu\lambda}\nabla_\nu R_{2\mu\lambda}\sigma_3\hat{1} \\ & -\frac{1}{45}\frac{1}{\Box_1\Box_2}\nabla_\alpha\nabla_\beta R_{1\mu\nu}\nabla^\mu\nabla^\nu R_2^{\alpha\beta}\sigma_3\hat{1}\Big\} + \mathrm{O}[\Re^3] \end{aligned} \tag{4.49}$$

where the notation in the nonlocal terms is the same as before with σ playing the role of the third curvature. It is the latter equation that will be checked below by a direct treatment of $\mathrm{Tr}\,K(s)$.

The way of deriving the Weyl anomaly that we now present is based on carrying out the conformal transformation in the trace of the heat kernel. We use the expansion of $\mathrm{Tr}\,K(s)$ in powers of the curvatures given by eq. (2.178). To enable a comparison with the effective action (1.17), one should subtract from the heat kernel the terms of zeroth and first order in the curvature. These are the first two terms of expression (2.178). For $\mathrm{Tr}K(s)$ with these terms subtracted we introduce the notation

$$\mathrm{Tr}K'(s) = \mathrm{Tr}K(s) - \frac{1}{(4\pi s)^2}\int dx\, g^{1/2}\,\mathrm{tr}\,(\hat{1} + s\hat{P}). \tag{4.50}$$

We begin this check by calculating the result of the transformation (4.39)–(4.45) in the quadratic terms of $\mathrm{Tr}K(s)$:

$$\delta_\sigma \int dx\, g^{1/2}\, \mathrm{tr} \left\{ \sum_{i=1}^{5} f_i(-s\Box_2)\Re_1\Re_2(i) \right\}, \qquad (4.51)$$

where the functions $f_i(-\Box)$ are given by (2.179)–(2.183). In this order one should make the δ_σ-variation not only in the curvature structures but also in the form factors. In the term linear in $R^{\mu\nu}$ which appears after eleminating one of the Ricci tensors by (4.40), in order to employ the Bianchi identity, one should commute $f(-s\Box)$ with the derivatives $\nabla_\mu\nabla_\nu$, and retain the commutator. By the derivation which is similar to that of eq. (2.26) we get for the basic second order form factor:

$$\begin{aligned} &\int dx\, g^{1/2}\, R_{\mu\nu}[f(-s\Box), \nabla^\mu\nabla^\nu]\sigma \\ &= \int dx\, g^{1/2}\, \frac{f(-s\Box_1) - f(-s\Box_3)}{\Box_1 - \Box_3}[\Box_3, \nabla_3^\mu\nabla_3^\nu] R_{1\mu\nu}\sigma_3 + \mathrm{O}[R^3..], \end{aligned} \qquad (4.52)$$

with the expression for the commutator

$$\begin{aligned} [\Box, \nabla_\mu\nabla_\nu]\sigma &= 2\nabla_{(\mu}R_{\nu)\alpha}\nabla^\alpha\sigma + 2R_{\alpha(\mu}\nabla_{\nu)}\nabla^\alpha\sigma \\ &\quad - \nabla_\alpha R_{\mu\nu}\nabla^\alpha\sigma - 2R_{\alpha\nu\beta\mu}\nabla^\alpha\nabla^\beta\sigma \end{aligned} \qquad (4.53)$$

in which the Riemann tensor should eventually be expressed via the Ricci tensor.

In the second order terms (4.51) one has to deal with the second order form factors (2.130) and (2.139)–(2.140). For the basic second form factor the variation is as follows,

$$\begin{aligned} &\int dx\, g^{1/2}\, \mathrm{tr}\, \Re_1 \Big(\delta_\sigma f(-s\Box_2)\Big)\Re_2 \\ &= \int dx\, g^{1/2}\, \mathrm{tr}\, \frac{f(-s\Box_1) - f(-s\Box_2)}{\Box_1 - \Box_2}(\delta_\sigma\Box_2)\Re_1\Re_2 + \mathrm{O}[\Re^3]. \end{aligned} \qquad (4.54)$$

To treat similarly the form factors with subtractions, they need to be rewritten in the integral form, e.g.

$$\frac{f(-s\Box) - 1}{s\Box} = \int_0^1 d\alpha(1-\alpha)\alpha \int_0^1 d\beta e^{\alpha(1-\alpha)\beta s\Box}. \qquad (4.55)$$

The conformal transformation in the cubic terms of the heat kernel trace is easy to carry out because, within the required accuracy, only the curvatures in $\Re_1\Re_2\Re_3$

need be varied, *cf.* sect. 3.1. The result is a sum of contributions of the ten tensor structures:

$$\delta_\sigma \int dx\, g^{1/2}\, \mathrm{tr}\left\{\sum_{i=1}^{29} F_i(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2\Re_3(i)\right\}$$
$$= \int dx\, g^{1/2}\, \mathrm{tr}\left\{\sum_{i=1}^{10} S_i(-s\Box_1, -s\Box_2, -s\Box_3)\Re_1\Re_2\sigma_3(i)\right\} + \mathrm{O}[\Re^3], \quad (4.56)$$

where

$$\Re_1\Re_2\sigma_3(1) = R_1 R_2 \sigma_3 \hat{1}, \quad (4.57)$$

$$\Re_1\Re_2\sigma_3(2) = R_1^{\mu\nu} R_{2\mu\nu}\sigma_3 \hat{1}, \quad (4.58)$$

$$\Re_1\Re_2\sigma_3(3) = R_1^{\mu\nu}\nabla_\nu\nabla_\mu R_2\sigma_3 \hat{1}, \quad (4.59)$$

$$\Re_1\Re_2\sigma_3(4) = \nabla^\mu R_1^{\nu\lambda}\nabla_\nu R_{2\mu\lambda}\sigma_3 \hat{1}, \quad (4.60)$$

$$\Re_1\Re_2\sigma_3(5) = \nabla_\alpha\nabla_\beta R_{1\mu\nu}\nabla^\mu\nabla^\nu R_2^{\alpha\beta}\sigma_3 \hat{1}, \quad (4.61)$$

$$\Re_1\Re_2\sigma_3(6) = \hat{P}_1 R_2 \sigma_3, \quad (4.62)$$

$$\Re_1\Re_2\sigma_3(7) = \nabla_\alpha\nabla_\beta \hat{P}_1 R_2^{\alpha\beta}\sigma_3, \quad (4.63)$$

$$\Re_1\Re_2\sigma_3(8) = \hat{P}_1 \hat{P}_2 \sigma_3, \quad (4.64)$$

$$\Re_1\Re_2\sigma_3(9) = \hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\mu\nu}\sigma_3, \quad (4.65)$$

$$\Re_1\Re_2\sigma_3(10) = \nabla_\mu\hat{\mathcal{R}}_1^{\mu\alpha}\nabla^\nu\hat{\mathcal{R}}_{2\nu\alpha}\sigma_3. \quad (4.66)$$

We do not specify the form factors S_i as they are unimportant here.

The total result for $\mathrm{Tr}K'(s)$ divided by s is of the form

$$\frac{1}{s}\delta_\sigma \mathrm{Tr}K'(s) = \frac{1}{(4\pi)^2}\int dx\, g^{1/2}\, \mathrm{tr}\Big\{\sigma\Box t_1(s,\Box)\hat{P} + \sigma\Box t_2(s,\Box)R\hat{1}$$
$$+ \sum_{i=1}^{10} T_i(s,\Box_1,\Box_2,\Box_3)\Re_1\Re_2\sigma_3(i)\Big\} + \mathrm{O}[\Re^3], \quad (4.67)$$

where $\Re_1\Re_2\sigma_3(i)$ are the tensor structures (4.57)–(4.66). The functions t_1, t_2, T_i are obtained as certain combinations of the form factors in the heat kernel. By making the substitutions (4.11)–(4.13) in the same way as in the previous sections one brings the functions t_1, t_2, T_i into the form of a total derivative:

$$t_1 = \frac{d}{ds}\tilde{t}_1, \qquad t_2 = \frac{d}{ds}\tilde{t}_2, \qquad T_i = \frac{d}{ds}\widetilde{T}_i. \tag{4.68}$$

The final result for $\tilde{t}_1, \tilde{t}_2, \widetilde{T}_i$ is presented in terms of the basic form factors in the heat kernel, $f(-s\Box)$, $F(-s\Box_1, -s\Box_2, -s\Box_3)$, and the determinant D (2.171). First order form factors are,

$$\tilde{t}_1 = \frac{f(-s\Box) - 1}{s\Box}, \tag{4.69}$$

$$\tilde{t}_2 = \frac{1}{12}\frac{f(-s\Box) - 1}{s\Box} - \frac{1}{2}\frac{f(-s\Box) - 1 - \frac{1}{6}s\Box}{(s\Box)^2}, \tag{4.70}$$

and second order form factors can be found in Appendix D.

The conformal variation of the effective action (1.17) can now be obtained as

$$-\delta_\sigma W = \frac{1}{2}\int_0^\infty \frac{ds}{s}\delta_\sigma \mathrm{Tr}K'(s). \tag{4.71}$$

From (4.67) and (4.68) we find

$$\begin{aligned} -\delta_\sigma W \;=\; & -\frac{1}{2(4\pi)^2}\int dx\, g^{1/2}\; \mathrm{tr}\Big\{\sigma\Box\tilde{t}_1(0,\Box)\hat{P} + \sigma\Box\tilde{t}_2(0,\Box)R\hat{1} \\ & + \sum_{i=1}^{10}\widetilde{T}_i(0,\Box_1,\Box_2,\Box_3)\Re_1\Re_2\sigma_3(i)\Big\} + \mathrm{O}[\Re^3] \end{aligned} \tag{4.72}$$

where use is made of the fact that the functions $\tilde{t}_1, \tilde{t}_2, \widetilde{T}_i$ as given in eqs. (4.69)–(D.10) vanish at $s \to \infty$. The behaviors of these functions at $s = 0$ follow from the results of sect. 2.7 :

$$\tilde{t}_1(0,\Box) = \frac{1}{6}, \tag{4.73}$$

$$\tilde{t}_2(0,\Box) = \frac{1}{180}, \tag{4.74}$$

$$\frac{1}{2}\Big(\widetilde{T}_1 + \widetilde{T}_1\big|_{\Box_1 \leftrightarrow \Box_2}\Big) = 0, \qquad s = 0 \tag{4.75}$$

$$\frac{1}{2}\left(\tilde{T}_2+\tilde{T}_2|_{\Box_1\leftrightarrow\Box_2}\right)=\frac{1}{180}+\frac{\Box_1}{180\Box_2}+\frac{\Box_2}{180\Box_1}-\frac{\Box_3}{90\Box_1}-\frac{\Box_3}{90\Box_2}+\frac{\Box_3{}^2}{180\Box_1\Box_2}, \qquad s=0 \tag{4.76}$$

$$\tilde{T}_3=0, \qquad s=0 \tag{4.77}$$

$$\frac{1}{2}(\tilde{T}_4+\tilde{T}_4|_{\Box_1\leftrightarrow\Box_2})=\frac{1}{45\Box_1}+\frac{1}{45\Box_2}-\frac{\Box_3}{45\Box_1\Box_2}, \qquad s=0 \tag{4.78}$$

$$\frac{1}{2}\left(\tilde{T}_5+\tilde{T}_5|_{\Box_1\leftrightarrow\Box_2}\right)=\frac{1}{45\Box_1\Box_2}, \qquad s=0 \tag{4.79}$$

$$\tilde{T}_6=0, \qquad s=0 \tag{4.80}$$

$$\tilde{T}_7=0, \qquad s=0 \tag{4.81}$$

$$\frac{1}{2}\left(\tilde{T}_8+\tilde{T}_8|_{\Box_1\leftrightarrow\Box_2}\right)=\frac{1}{2}, \qquad s=0 \tag{4.82}$$

$$\frac{1}{2}\left(\tilde{T}_9+\tilde{T}_9|_{\Box_1\leftrightarrow\Box_2}\right)=\frac{1}{12}, \qquad s=0 \tag{4.83}$$

$$\frac{1}{2}\left(\tilde{T}_{10}+\tilde{T}_{10}|_{\Box_1\leftrightarrow\Box_2}\right)=0, \qquad s=0. \tag{4.84}$$

With these expressions inserted in (4.72), one arrives at eq. (4.49) which is the correct Weyl anomaly. The derivations of the present section show that to produce the local Weyl anomaly one should take into account both second and third curvature orders, then no nonlocal anomaly terms [69] appear.

Because the Weyl transformation is inhomogeneous in the curvature, the expansion in powers of the curvature does not preserve the exact conformal properties of the effective action. These properties can only be recovered order by order. This shortcoming of covariant perturbation theory has been removed in recent works [70, 71].

To clarify the remark at the end of sect. 4.1 we reproduce here the Weyl anomaly in two dimensions as well. Because eq. (4.46) is now based on $a_1(x,x)$ instead of $a_2(x,x)$ [18], the anomaly is,

$$g^{\mu\nu}\frac{\delta W}{\delta g^{\mu\nu}}=\frac{1}{48\pi^2}g^{1/2}R. \tag{4.85}$$

As mentioned before the two-dimensional effective action (4.23) was originally derived from this anomaly. Now this result is easily obtained from the nonlocal trace of the heat kernel (4.9) following the line of derivations presented in this section.

Chapter 5

Conclusions

Let us summarize and analyze our study. In this thesis we have developed the covariant perturbation theory for the heat kernel trace to the third order in curvatures. Starting with this order there is a qualitative change in derivations – they require use of computers. The third order basis of the curvature invariants has been constructed and nonlocal form factors acting on it computed in two different integral representations. Large time and short time asymptotics are calculated for the trace of the heat kernel. The comparison with the Schwinger-DeWitt series is carried out up to and including the fourth coefficient. It is shown that for the two-dimensional conformal invariant model of scalar field theory that the third order terms cancel producing the Polyakov action. The correct form of the Weyl anomaly in four dimensions is derived from the heat kernel trace showing how the nonlocal effective action creates the local anomaly.

The coincidence limit of the heat kernel is computed up to second order in curvatures by the generating function method from the above results for the heat kernel trace, and by the covariant perturbation theory. The basis set of second order nonlocal tensor invariants and form factors are obtained. The comparison of the short time expansion and the Schwinger-DeWitt series is completed. The diagonal Green's function for the two-dimensional conformal invariant model is derived and is shown to terminate at first curvature order.

In fact, to obtain the one-loop effective action (as well as the Green's function) in four dimensions we need to make just one additional step – to integrate over the proper time (1.17). In the form factor representation of sect. 2.5, this operation is trivial [42]. It is proved that only second order in curvatures is ultraviolet divergent while third and higher ones are finite [44]. To see this fact one needs to know a special feature of the nonlocal invariants basis: hidden identities among gravity invariants which reduce the number of independent invariants in spacetimes with dimension $2\omega < 6$ [34]. In four dimensions there is only one such identity which can reduce the number of nonlocal purely gravitational curvature structures by one. Since these identities depend on spacetime dimensions, but all (except Chapter 4) derivations of the present thesis are valid for manifolds of arbitrary dimensions, we have not utilized these 'hidden' identities. Nevertheless, the analysis of these hidden identities does show how the Gauss-Bonnet invariant becomes topological in four dimensions [34].

The results of sect. 2.5 and Appendix B already has all physical information containing in the one-loop effective action, however it is obviously impossible to work with the effective action in this form. The only nonlocal ingredient that eventually enters the theory is the Green's function, thus we need to transform the nonlocal form factors into the massive Green's functions [29, 31]. Second order form factor transforms, upon proper time integration, into

$$\int_0^\infty dm^2 \left(\frac{1}{m^2 - \Box} - \frac{1}{m^2 + \mu^2} \right),$$

where μ^2 is the parameter of ultraviolet renormalization [30]. For third order form factors the following construction can be obtained [31, 42],

$$\int_0^\infty \frac{dm_1{}^2 dm_2{}^2 dm_3{}^2 \rho(m_1, m_2, m_3)}{(m_1{}^2 - \Box)(m_2{}^2 - \Box)(m_3{}^2 - \Box)}.$$

These spectral forms encode all information about the model into a set of spectral densities $\rho(m_1{}^2, m_2{}^2, m_3{}^2)$ [39].

Even having obtained the above spectral representations one has to do one more procedure before proceeding to physical applications, namely, switch from the Euclidian spacetime to the real world with the Lorentzian signature of the metric. Immediately, one gets instead of a unique Green's function a variety of them [12]. However, first one should obtain the effective equations by varying the Euclidean effective action and only then replace nonlocal Euclidean form factors by the Lorentzian Green's functions. If one is solving a Cauchy problem with data in the remote past (for expectation values of fields in the standard *in*-vacuum) this procedure boils down to using the *retarded* massive Green's functions in the effective equations [38, 29, 40].

The technique explained above would lead one to interesting physical applications in quantum gravity. It was shown that certain asymptotics of the spectral forms govern the energy flux at future infinity [40, 43], which allows one to study some aspects of the Hawking radiation in four dimensions [33, 27, 39]. Although this problem is far from being solved, some interesting physics has been already derived from the third order, e.g. the effect of gravitational wave creation by the vacuum polarization [41]. There are also attempts to derive quantum corrections for the Newton gravity constant from second orders of the nonlocal effective action [72].

In the context of other possible applications of the results of the present study to high-energy physics we would like to make a link to so-called string-inspired methods for gauge theories which are now receiving much attention (for a review see [73]). Although there are several techniques under this name such as supersymmetry relations, color decompositions etc., the core of them is the world-line method [74]. In essence, this is the path integral perturbation theory and the world line parameter is in fact the Schwinger proper time as clear from refs. [75, 76]. As was shown in [77] the form factor structure for some effective actions obtained this way does in fact coincide with that of nonlocal effective action of [30, 42]. On the whole, the string-derived methods have led to significant progress in calculating one-loop corrections to QCD processes [78]. These methods are much more efficient than conventional Feynman rules [73] due to the obvious fact that one works with the effective action instead of the partition function, and one therefore calculates vertices with no external lines [8, 24]. In fact one has to remember that chiral perturbation theory is one of the most significant application areas for the Schwinger-DeWitt series [19]; it is no surprise then that nonlocal extensions of the short time expansion are useful there as well. We conclude by remarking that, of course, it is in combination with other methods like supersymmetry that made the world-line technique so successful in phenomenology. So far an exact relation between the covariant perturbation theory and the string-based methods is yet to be understood.

QCD applications discussed above deal with calculation of 5-point vertices while we studied nonlocal third order terms which are 3-point vertices in field theory language. A natural question arises: do we need to compute higher orders in curvatures? Answering this question we should remember that there was a strong physical motivation to derive the cubic order in curvatures. But if one does want to work out the next orders, one should note that it is even more technically complicated than in third order. For example, the mixed-box arguments appear in form factors beginning with the fourth order [34]. Although the operators $\Box_1, \Box_2, ...$ in form factors commute with each other because they act on different functions, the mixed-box arguments $\Box_{1+2}, \Box_{1+3}, ...$ do not commute with each other and with $\Box_1, \Box_2, ...$. Therefore, beginning with the fourth order one has to order these arguments in a definite way. Also, while working with the accuracy $O[\Re^5]$ and higher it is important to fix the ordering of derivatives in tensor invariants of lower orders.

The fact that the Weyl anomaly can be obtained directly from the heat kernel and the effect lies in the limit $s \to 0$ has been employed for axial anomalies by L. Alvarez-Gaumé and E. Witten [79] and developed further in [80, 81]. The work of S. Ichinose and N. Ikeda [81] deserves special interest since these authors derived the Weyl anomaly for several field models coupled to gravity in terms of noncovariant perturbation theory.

We emphasize however that our derivation of the Weyl anomaly is, first, merely a check of derivations, and second that, it is done starting from the covariant expression for the complete nonlocal heat kernel trace. It should be mentioned that besides the derivation of the Weyl anomaly in the way we did it in sect. 4.3, this anomaly has been also obtained from the one-loop effective action itself starting from two different representations for its form factors found in [42, 44] by A. Barvinsky *et al.*

Further study in this direction lies in the construction of the conformal decomposition for the one-loop effective action [71]. Then the conformally invariant part of the covariant curvature expansion for the generic effective action can be rewritten in a new conformal basis. This also makes a partial summation of the curvature expansion possible. Such partially summed action simplifies drastically [70]. The anomalous part of the effective action exactly reproducing the Weyl anomaly in four dimensions can be found.

Considering further a summation of the curvature expansion, we should mention the fact that such a summation is indeed possible in a closed combinatorial form for the perturbation theory with a scalar potential [23] and even for a model with the Abelian gauge field [36, 82].

The obvious and necessary generalization of the covariant curvature expansion is the derivation of the heat kernel and its associated Green's function with separated points, $\hat{K}(s|x,y)$, which contains much richer physical information and is suitable for two-loop calculations. While the conventional perturbation theory is already formulated for the case of the nondiagonal heat kernel [23], covariant derivations of this kind are not found in the literature. For the case of the Abelian background gauge field, such theory was developed in ref. [82], and for the generic case in (1.6) to the second order [83] (however this result is not explicitly covariant). On the whole, the covariant derivation of $\hat{K}(s|x,y)$ similar to the present study can be done by generalizing ideas of [82].

Covariant derivations of the present study are restricted to asymptotically flat spacetimes and the form of the minimal (having derivatives in the form of a Laplacian) second order differential operator (1.6); this case covers many physically interesting models but of course only a fraction of them. Nevertheless since the main idea of covariant perturbative calculations is sufficiently simple, it might be possible to obtain formulae of this kind for other types of manifolds (e.g. compact) and operators.

Similarly it is well known that the covariant nonlocal technique can be easily modified to include the effect of finite temperature [84].

The heat kernel with separated points is an essential element of two-loop calculations [28]. Einstein quantum gravity is finite at one-loop but divergent at two-loop order [20]. No finite terms of the two-loop effective action in quantum gravity has been computed so far, thus it presents a new challenge and testing ground for covariant perturbation theory to to produce such terms. The form of these two-loop terms is especially important in view of the fact that they are required for the renormalization of divergencies [28].

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Appendix A

The basis of nonlocal curvature invariants in third order

In this appendix a complete basis of nonlocal invariants is constructed for the third order terms in the spacetime curvature and matter field strengths. We assume that the basis for lower orders is known and fixed [30, 34]. The discussion of building a similar basis for an arbitrary n-th order can be found in ref. [34].

We begin with the construction of purely gravitational invariants. In this case, since the Riemann tensor is eliminated, using a dummy indices notation $R_{\bullet\bullet}$ for the Ricci tensor, we write down the general form for nonlocal invariants in the trace of the heat kernel as

$$\int dx\, g^{1/2} \sum_{n=0}^{\infty} c_n (\underbrace{\nabla ... \nabla}_{2n})\, R_{\bullet\bullet} R_{\bullet\bullet} R_{\bullet\bullet}. \tag{A.1}$$

Here the covariant derivatives can be commuted freely because the contribution of their commutator is of order $\mathrm{O}[R^4_{\bullet\bullet}]$. Among $(\nabla ... \nabla)$ only a limited number (≤ 6) of derivatives can have indices contracted with the indices of $R_{\bullet\bullet} R_{\bullet\bullet} R_{\bullet\bullet}$, while the rest of the derivatives contract with one another to form the covariant Laplacians acting on separate Ricci tensors or their pairs. This follows from an identity

$$2\nabla_i \nabla_k = (\nabla_i + \nabla_k)^2 - \nabla_i^2 - \nabla_k^2 \equiv \Box_{i+k} - \Box_i - \Box_k \tag{A.2}$$

where ∇_i denotes the covariant derivative acting on the i-th Ricci tensor in the product $R_{1\bullet\bullet} R_{2\bullet\bullet} R_{3\bullet\bullet}$. The same notation is assumed for covariant boxes $\Box_i$ and $\Box_{i+k}$

acting on the separate i-th Ricci tensor and the separate ik-th pair of those respectively.

The rearrangement of derivatives in (A.1) according to (A.2) leads to the following general structure of the third order nonlocal invariant

$$\int dx\, g^{1/2}\, F(\Box_1, ...\Box_3, \Box_{1+2}, \Box_{1+3}, ...)(\underbrace{\nabla ... \nabla}_{\leq 6})\, R_{1\bullet\bullet} R_{2\bullet\bullet} R_{3\bullet\bullet}, \tag{A.3}$$

where $F(\Box_1, ...\Box_3, \Box_{1+2}, \Box_{1+3}, ...)$ is an operator function of boxes accumulating the result of the infinite summation of derivatives in (A.1). A nonlocal form factor F serves as a coefficient of the invariant $(\underbrace{\nabla ... \nabla}_{\leq 6})\, R_{1\bullet\bullet} R_{2\bullet\bullet} R_{3\bullet\bullet}$ which is a member of the basis of nonlocal invariants.

Under the asymptotically flat boundary conditions which guarantee the absence of surface terms [30], the integration by parts can be written as

$$\nabla_1 + \nabla_2 + \nabla_3 = 0, \tag{A.4}$$

(*cf.* eq. (1.42)). This relation implies that the third order form factor is a function of three boxes acting on three separate Ricci curvatures:

$$F(\Box_1, \Box_2, \Box_3)(\underbrace{\nabla ... \nabla}_{\leq 6})\, R_{1\bullet\bullet} R_{2\bullet\bullet} R_{3\bullet\bullet}. \tag{A.5}$$

The mixed-box arguments $\Box_{1+2}$ etc. appear in form factors beginning only with the fourth order [34].

The full set of invariants $(\underbrace{\nabla ... \nabla}_{\leq 6})\, R_{1\bullet\bullet} R_{2\bullet\bullet} R_{3\bullet\bullet}$ includes the structures with six, four, two and zero number of derivatives whose indices are contracted with the indices of Ricci tensors. We use the following general rule: in any structure with derivatives saturating the contraction of Ricci curvature indices, the covariant derivative acting on one of Ricci curvatures can be contracted only with an index of another curvature, for otherwise the Bianchi identity generates the invariant with two derivatives contracted with one other, which in view of (A.2) should be absorbed into nonlocal form factor and do not enter the tensor invariant itself. This rule together with integration by parts produces the following identification in such invariants

$$R_{1\mu\bullet}\nabla^{\mu}R_{2\bullet\bullet}R_{3\bullet\bullet} = -R_{1\mu\bullet}R_{2\bullet\bullet}\nabla^{\mu}R_{3\bullet\bullet} + (\ldots) \tag{A.6}$$

modulo form factor terms and higher orders of the curvature denoted by (...). Then it immeadiately follows that the only independent invariant with six derivatives is

$$\nabla_{\alpha}\nabla_{\beta}R_1^{\gamma\delta}\nabla_{\gamma}\nabla_{\delta}R_2^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R_3^{\alpha\beta}, \tag{A.7}$$

(*cf.* eq. (2.75)).

In an invariant with four derivatives one pair of indices belonging to the Ricci curvatures should be contracted, the rest of them being contracted with indices of derivatives. There are two such contractions: $R_1^{\mu\nu}R_2^{\alpha\beta}R_3$ and $R_1^{\alpha\lambda}R_{2\lambda}^{\beta}R_3^{\mu\nu}$ which generate in view of (A.6) the following independent invariants with four derivatives (*cf.* eqs. (2.73)–(2.74))

$$\nabla_{\alpha}\nabla_{\beta}R_1^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R_2^{\alpha\beta}R_3, \qquad \nabla_{\mu}R_1^{\alpha\lambda}\nabla_{\nu}R_{2\,\lambda}^{\beta}\nabla_{\alpha}\nabla_{\beta}R_3^{\mu\nu} \tag{A.8}$$

(the independent invariants with ∇_{α} or ∇_{β} acting respectively on $R_1^{\alpha\lambda}$ or $R_{2\,\lambda}^{\beta}$ above are ruled out by integration by parts which reduces them to the first of the structures (A.8)).

There are four tensors with two uncontracted indices $R_1^{\alpha\beta}R_2R_3$, $R_1^{\nu\alpha}R_{2\,\mu\alpha}R_3$, $R_1^{\mu\nu}R_2^{\alpha\beta}R_{3\,\alpha\beta}$ and $R_1^{\mu\nu}R_{2\,\beta\mu}R_{3\,\nu}^{\alpha}$ that give rise to the following invariants with two derivatives

$$\begin{aligned} &R_1^{\alpha\beta}\nabla_{\alpha}R_2\nabla_{\beta}R_3, \qquad \nabla^{\mu}R_1^{\nu\alpha}\nabla_{\nu}R_{2\,\mu\alpha}R_3, \\ &R_1^{\mu\nu}\nabla_{\mu}R_2^{\alpha\beta}\nabla_{\nu}R_{3\,\alpha\beta}, \qquad R_1^{\mu\nu}\nabla_{\alpha}R_{2\,\beta\mu}\nabla^{\beta}R_{3\,\nu}^{\alpha}, \end{aligned} \tag{A.9}$$

(*cf.* eqs. (2.68)–(2.71)).

Finally, there are three obvious invariants that contain no derivatives

$$R_1R_2R_3, \qquad R^{\mu}_{1\,\alpha}R^{\alpha}_{2\,\beta}R^{\beta}_{3\,\mu}, \qquad R_1^{\mu\nu}R_{2\,\mu\nu}R_3, \tag{A.10}$$

(*cf.* eqs. (2.55)–(2.57)).

It should be emphasized that the full set of cubic invariants follows from the above structures (A.7)–(A.10) by all possible permutations of three curvature labels 1, 2 and 3. But according to our notations in (A.3) and (A.5) such permutations in $(\nabla ... \nabla)\, R_{1\bullet\bullet} R_{2\bullet\bullet} R_{3\bullet\bullet}$ can always be replaced by the corresponding permutation of arguments in the form factor $F(\Box_1, \Box_2, \Box_3)$, so that the independent set of invariants can be fixed with the enumeration of curvatures chosen above.

Let us consider now the full set of invariants involving the matter field strengths $\hat{\mathcal{R}}_{\mu\nu}$ and $\hat{P}$ in the cubic order in $\Re$. We first consider $\hat{P}^3$ and $\hat{\mathcal{R}}^3_{\bullet\bullet}$ invariants. The single $\hat{P}^3$ invariant is trivial

$$\mathrm{tr}\, \hat{P}_1 \hat{P}_2 \hat{P}_3, \tag{A.11}$$

(*cf.* eq. (2.47)).

The hirearchy of structures $\underbrace{(\nabla ... \nabla)}_{\leq 6} \hat{\mathcal{R}}_{1\bullet\bullet} \hat{\mathcal{R}}_{2\bullet\bullet} \hat{\mathcal{R}}_{3\bullet\bullet}$ includes *a priori* the invariants with 6, 4, 2 and zero number of derivatives. However, the Jacobi identity for the commutator curvature

$$\nabla_\alpha \hat{\mathcal{R}}_{\mu\nu} + \nabla_\mu \hat{\mathcal{R}}_{\nu\alpha} + \nabla_\nu \hat{\mathcal{R}}_{\alpha\mu} = 0, \tag{A.12}$$

which leads to the relation

$$\nabla_\alpha \hat{\mathcal{R}}_{\mu\nu} \hat{\mathcal{R}}^{\alpha\nu} = \frac{1}{2} \nabla_\mu \hat{\mathcal{R}}_{\nu\alpha} \hat{\mathcal{R}}^{\nu\alpha}, \tag{A.13}$$

essentially reduces their number.

Consider first the invariant with six derivatives. Up to total derivative terms and commutators of ∇'s it can be represented as $\mathrm{tr}\, \nabla_\mu \hat{J}^\alpha_1 \nabla_\alpha \hat{J}^\beta_2 \nabla_\beta \hat{J}^\mu_3$ in terms of the transverse vector

$$\hat{J}^\alpha = \nabla_\lambda \hat{\mathcal{R}}^{\lambda\alpha}, \quad \nabla_\alpha \hat{J}^\alpha = 0. \tag{A.14}$$

In view of (A.12) this vector satisfies the relation

$$\nabla_\mu \hat{J}^\alpha = \nabla^\alpha \hat{J}_\mu - \Box\, \hat{\mathcal{R}}^\alpha_{.\mu} + O\,[\Re^2], \tag{A.15}$$

which means that the above cubic invariant with six derivatives not forming boxes is absent.

The set of invariants with four derivatives again reduces to the only structure $\mathrm{tr}\,\nabla_\mu \hat{J}_{1\nu} \hat{J}_2^\mu \hat{J}_3^\nu$ which does not straightforwardly disappear (up to irrelevant terms) due to eqs.(A.12) and (A.15). However, in view of (A.15) one can rewrite this structure as $\mathrm{tr}\,\nabla_\mu \hat{J}_{1\nu} \hat{J}_2^{(\mu} \hat{J}_3^{\nu)}$, integrate it by parts and again use (A.15) applied to $\hat{J}_2$ and $\hat{J}_3$. Then it takes the form $-\frac{1}{2}\,\hat{J}_{1\nu} \nabla^\nu (\hat{J}_2^\mu \hat{J}_{3\mu})$ which disappears up to the total derivative term in view of the transversality of (A.14). Thus, there is no invariant with four derivatives either.

The invariants with two derivatives originate from all possible differentiations of the following two structures $\mathrm{tr}\,\hat{\mathcal{R}}_1^{\mu\nu} \hat{\mathcal{R}}_2^{\alpha\beta} \hat{\mathcal{R}}_{3\alpha\beta}$ and $\mathrm{tr}\,\hat{\mathcal{R}}_1^{\alpha\beta} \hat{\mathcal{R}}^\mu_{2\,\alpha} \hat{\mathcal{R}}^\nu_{3\,\beta}$ and, by integration by parts and use of the Jacobi identity, boil down to $\mathrm{tr}\,\hat{\mathcal{R}}_1^{\mu\nu} \nabla_\mu \hat{\mathcal{R}}_2^{\alpha\beta} \nabla_\nu \hat{\mathcal{R}}_{3\alpha\beta}$ and $\mathrm{tr}\,\hat{\mathcal{R}}_1^{\alpha\beta} \hat{J}_{2\,\alpha} \hat{J}_{3\,\beta}$. But the first of these invariants reduces to the second one by the following sequence of transformations. First integrate it by parts and use (A.13) to convert it to the form $2\,\mathrm{tr}\,\hat{J}_1^\nu \hat{\mathcal{R}}_2^{\alpha\beta} \nabla_\alpha \hat{\mathcal{R}}_{3\beta\nu}$. Then integrate by parts again and use (A.15) to interchange the indices in $\nabla_\alpha \hat{J}_1^\nu$. The sequence of a new integration by parts and the use of (A.13) eventually leads to the equation

$$\mathrm{tr}\,\hat{\mathcal{R}}_1^{\mu\nu} \nabla_\mu \hat{\mathcal{R}}_2^{\alpha\beta} \nabla_\nu \hat{\mathcal{R}}_{3\alpha\beta} = 2\,\mathrm{tr}\,\hat{\mathcal{R}}_3^{\alpha\beta} \hat{J}_{1\,\alpha} \hat{J}_{2\,\beta} + 2\,\mathrm{tr}\,\hat{\mathcal{R}}_2^{\alpha\beta} \hat{J}_{3\,\alpha} \hat{J}_{1\,\beta} + 2\,\mathrm{tr}\,\Box\,\hat{\mathcal{R}}^{\alpha}_{1\,.\beta} \hat{\mathcal{R}}^{\beta}_{2\,.\nu} \hat{\mathcal{R}}^{\nu}_{3\,.\alpha}$$
$$-\mathrm{tr}\,\hat{\mathcal{R}}_1^{\mu\nu} \nabla_\mu \hat{\mathcal{R}}_2^{\alpha\beta} \nabla_\nu \hat{\mathcal{R}}_{3\alpha\beta} + O\,[\,\Re^4] + \text{a total derivative}, \qquad \text{(A.16)}$$

which can be easily solved for $\mathrm{tr}\,\hat{\mathcal{R}}_1^{\mu\nu} \nabla_\mu \hat{\mathcal{R}}_2^{\alpha\beta} \nabla_\nu \hat{\mathcal{R}}_{3\alpha\beta}$ in terms of the invariant

$$\mathrm{tr}\,\hat{\mathcal{R}}_1^{\alpha\beta} \nabla^\mu \hat{\mathcal{R}}_{2\,\mu\alpha} \nabla^\nu \hat{\mathcal{R}}_{3\,\nu\beta} \qquad \text{(A.17)}$$

and the only possible $\hat{\mathcal{R}}^3$-invariant without derivatives

$$\mathrm{tr}\,\hat{\mathcal{R}}^{\alpha}_{1.\beta} \hat{\mathcal{R}}^{\beta}_{2.\nu} \hat{\mathcal{R}}^{\nu}_{3.\alpha} \qquad \text{(A.18)}$$

(under a proper permutation of curvature labels). As a result there are only two $\hat{\mathcal{R}}^3$-invariants (*cf.* eqs. (2.48), (2.58)).

A similar technique can be applied for a construction of all the rest gravity-matter invariants. Again their number is essentially reduced by using the identities of the above type. Here are some examples of such a reduction which we present without derivation starting with a corollary of (A.16)

$$\begin{aligned}
&\mathrm{tr}\,\hat{\mathcal{R}}_1^{\alpha\beta}\nabla_\alpha\hat{\mathcal{R}}_2^{\mu\nu}\nabla_\beta\hat{\mathcal{R}}_{3\mu\nu} = \mathrm{tr}\Big(\Box_1\hat{\mathcal{R}}^{\mu}_{1\alpha}\hat{\mathcal{R}}^{\alpha}_{2\beta}\hat{\mathcal{R}}^{\beta}_{3\mu} + \hat{\mathcal{R}}_{3\alpha\beta}\nabla_\mu\hat{\mathcal{R}}_1^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_2^{\nu\beta}\\
&\qquad +\hat{\mathcal{R}}_{2\alpha\beta}\nabla_\mu\hat{\mathcal{R}}_3^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_1^{\nu\beta}\Big) + \mathrm{O}[\Re^4] + \text{ a total derivative,} \qquad (A.19)\\
&\mathrm{tr}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_1^{\lambda\nu}\nabla^\alpha\hat{\mathcal{R}}_{2\alpha\nu}\nabla_\sigma\hat{\mathcal{R}}_3^{\sigma\mu} = -\frac{1}{2}\mathrm{tr}\Big(\Box_1\hat{\mathcal{R}}_{1\alpha\beta}\nabla_\mu\hat{\mathcal{R}}_2^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_3^{\nu\beta}\\
&\qquad +\Box_2\hat{\mathcal{R}}_{2\alpha\beta}\nabla_\mu\hat{\mathcal{R}}_3^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_1^{\nu\beta} - \Box_3\hat{\mathcal{R}}_{3\alpha\beta}\nabla_\mu\hat{\mathcal{R}}_1^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_2^{\nu\beta}\Big)\\
&\qquad +\mathrm{O}[\Re^4] + \text{a total derivative,} \qquad (A.20)\\
&\mathrm{tr}\,\nabla_\alpha R_1^{\beta\mu}\nabla_\beta\hat{\mathcal{R}}_2^{\alpha\nu}\hat{\mathcal{R}}_{3\mu\nu} = -\frac{1}{2}\mathrm{tr}\,R_1^{\alpha\beta}\nabla_\alpha\hat{\mathcal{R}}_2^{\mu\nu}\nabla_\beta\hat{\mathcal{R}}_{3\mu\nu}\\
&\qquad +\mathrm{tr}\,R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_2^{\lambda\alpha}\hat{\mathcal{R}}_{3\alpha\nu} + \mathrm{O}[\Re^4] + \text{ a total derivative,} \qquad (A.21)\\
&\mathrm{tr}\nabla^\mu R_{1\beta\alpha}\nabla^\beta\hat{\mathcal{R}}_2^{\alpha\nu}\hat{\mathcal{R}}_{3\mu\nu} = \mathrm{tr}\Big[-\frac{1}{8}(\Box_1+\Box_2+\Box_3)R_1\hat{\mathcal{R}}_2^{\mu\nu}\hat{\mathcal{R}}_{3\mu\nu}\\
&\qquad +\frac{1}{2}R_1^{\alpha\beta}\nabla_\alpha\hat{\mathcal{R}}_2^{\mu\nu}\nabla_\beta\hat{\mathcal{R}}_{3\mu\nu} - \frac{1}{2}R_1\nabla_\alpha\hat{\mathcal{R}}_2^{\alpha\mu}\nabla^\beta\hat{\mathcal{R}}_{3\beta\mu}\\
&\qquad -R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_3^{\lambda\alpha}\hat{\mathcal{R}}_{2\alpha\nu}\Big] + \mathrm{O}[\Re^4] + \text{ a total derivative,} \qquad (A.22)\\
&\mathrm{tr}R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_2^{\lambda\alpha}\nabla_\nu\nabla^\sigma\hat{\mathcal{R}}_{3\sigma\alpha} = \mathrm{tr}\Big[-\Box_2\Box_3R_1^{\mu\nu}\hat{\mathcal{R}}_{2\mu}{}^{\alpha}\hat{\mathcal{R}}_{3\nu\alpha} - \Box_3R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_2^{\lambda\alpha}\hat{\mathcal{R}}_{3\alpha\nu}\\
&\qquad +\frac{1}{2}(\Box_1-\Box_2-\Box_3)R_1^{\mu\nu}\nabla^\alpha\hat{\mathcal{R}}_{2\alpha\mu}\nabla^\beta\hat{\mathcal{R}}_{3\beta\nu} - \Box_2R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_3^{\lambda\alpha}\hat{\mathcal{R}}_{2\alpha\nu}\Big]\\
&\qquad +\mathrm{O}[\Re^4] + \text{a total derivative} \qquad (A.23)
\end{aligned}$$

So, finally, besides twenty nine tensor structures (2.47)–(2.75) present in the final answer for the heat kernel trace and four additional structures (2.76)–(2.79) there are six structures linear in $\hat{\mathcal{R}}_{\mu\nu}$

$$\begin{aligned}
&R^{\mu}_{1\,\alpha}R^{\alpha}_{2\,\beta}\hat{\mathcal{R}}^{\beta}_{3\,\mu}, \qquad \hat{P}_1\nabla_\mu\hat{\mathcal{R}}_{2\,\nu\alpha}\\
&\nabla_\mu\hat{\mathcal{R}}_1^{\mu\alpha}R_{2\,\alpha\beta}\nabla^\beta\hat{P}_3, \quad R_1\nabla_\mu\hat{\mathcal{R}}_{2\,\nu\alpha}\nabla^\nu R_3^{\mu\alpha},\\
&\hat{\mathcal{R}}_1^{\alpha\beta}\nabla_\mu R_{2\,\nu\alpha}\nabla^\nu R^{\mu}_{3\,\beta}, \quad \nabla^\alpha\hat{\mathcal{R}}_1^{\beta\lambda}\nabla_\lambda\nabla_\mu R_{2\,\nu\alpha}\nabla^\nu R^{\mu}_{3\,\beta}. \qquad (A.24)
\end{aligned}$$

These structures form a complete basis of nonlocal invariants in third order in the curvature. Although thirty nine invariants are admissible by the requirements of covariance and asymptotic flatness, in the field theory calculations the last ten structures have a special status. As shown in Chapter 2, the trace of the heat kernel for the operator (1.6) (and the corresponding one-loop effective action) do not contain these structures because their form factors either identically vanish or have such symmetries under the permutation of the box arguments that make their contribution vanishing in view of the symmetries of the structures themselves (see sect. 2.5).

Appendix B

The explicit representation for third order form factors of the trace of the heat kernel

All results of the present appendix were obtained by the symbolic manipulation programs *MAPLE* and *Mathematica*. Although it is really impossible to work with these data without use of computers, we think it is necessary to present them here since they are an essential part of the main results of this thesis.

In terms of basic form factors (2.130),

$$f(\xi) = \left\langle \mathrm{e}^{-\alpha_1\alpha_2\xi}\right\rangle_2 = \int_0^1 d\alpha\, \mathrm{e}^{-\alpha(1-\alpha)\xi}, \qquad \xi = -s\Box,$$

(2.131),

$$F(\xi_1,\xi_2,\xi_3) = \int_{\alpha\geq 0} d^3\alpha\, \delta(1-\alpha_1-\alpha_2-\alpha_3) \times \exp(-\alpha_1\alpha_2\xi_3 - \alpha_2\alpha_3\xi_1 - \alpha_1\alpha_3\xi_2),$$

and the polynomial (2.143),

$$\Delta = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - 2\xi_2\xi_3,$$

the explicit expressions for the (not symmetrized) third order form factors in the trace of the heat kernel are as follows:

$$F_1(\xi_1,\xi_2,\xi_3) = \frac{1}{3}F(\xi_1,\xi_2,\xi_3), \tag{B.1}$$

$$F_2(\xi_1,\xi_2,\xi_3) = F(\xi_1,\xi_2,\xi_3)\Big[\frac{4\xi_1\xi_2\xi_3}{3\Delta^3}(-3{\xi_1}^2\xi_2 - 3{\xi_1}^2\xi_3 + 2\xi_1\xi_2\xi_3 + 3{\xi_3}^3)$$

$$
+\frac{4}{\Delta^2}(-{\xi_1}^2\xi_2-{\xi_1}^2\xi_3+2\xi_1\xi_2\xi_3+{\xi_3}^3)\Big]
$$

$$
+f(\xi_1)\frac{8\xi_1\xi_2\xi_3}{\Delta^3}({\xi_1}^2-{\xi_2}^2+2\xi_2\xi_3-{\xi_3}^2)
$$

$$
+\left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{4\xi_1}{\Delta^2}(3{\xi_1}^2-2\xi_1\xi_2-{\xi_2}^2-2\xi_1\xi_3+2\xi_2\xi_3-{\xi_3}^2)
$$

$$
-2\frac{1}{\xi_1-\xi_2}\left(\frac{f(\xi_1)-1}{\xi_1}-\frac{f(\xi_2)-1}{\xi_2}\right), \tag{B.2}
$$

$$
F_3(\xi_1,\xi_2,\xi_3)=F(\xi_1,\xi_2,\xi_3)\Big[\frac{2\xi_1\xi_2}{\Delta^2}(\xi_1-\xi_2-\xi_3)(-\xi_1+\xi_2-\xi_3)
$$

$$
-\frac{2}{\Delta}(\xi_1+\xi_2-\xi_3)\Big]+f(\xi_1)\frac{4\xi_1\xi_2}{\Delta^2}(-\xi_1+\xi_2-\xi_3)
$$

$$
+f(\xi_2)\frac{4\xi_1\xi_2}{\Delta^2}(\xi_1-\xi_2-\xi_3)
$$

$$
+f(\xi_3)\frac{1}{\Delta^2}({\xi_1}^3-{\xi_1}^2\xi_2-\xi_1{\xi_2}^2+{\xi_2}^3-3{\xi_1}^2\xi_3
$$

$$
+6\xi_1\xi_2\xi_3-3{\xi_2}^2\xi_3+3\xi_1{\xi_3}^2+3\xi_2{\xi_3}^2-{\xi_3}^3), \tag{B.3}
$$

$$
F_4(\xi_1,\xi_2,\xi_3)=F(\xi_1,\xi_2,\xi_3)\Big[\frac{1}{36\Delta^4}(-4{\xi_1}^8-4{\xi_1}^7\xi_2+32{\xi_1}^6{\xi_2}^2-28{\xi_1}^5{\xi_2}^3
$$

$$
+4{\xi_1}^4{\xi_2}^4+2{\xi_1}^7\xi_3+26{\xi_1}^6\xi_2\xi_3-90{\xi_1}^5{\xi_2}^2\xi_3+62{\xi_1}^4{\xi_2}^3\xi_3
$$

$$
+38{\xi_1}^6{\xi_3}^2-60{\xi_1}^5\xi_2{\xi_3}^2+42{\xi_1}^4{\xi_2}^2{\xi_3}^2-20{\xi_1}^3{\xi_2}^3{\xi_3}^2-82{\xi_1}^5{\xi_3}^3
$$

$$
+62{\xi_1}^4\xi_2{\xi_3}^3+20{\xi_1}^3{\xi_2}^2{\xi_3}^3+50{\xi_1}^4{\xi_3}^4-28{\xi_1}^3\xi_2{\xi_3}^4+6{\xi_1}^2{\xi_2}^2{\xi_3}^4
$$

$$
+14{\xi_1}^3{\xi_3}^5+6{\xi_1}^2\xi_2{\xi_3}^5-22{\xi_1}^2{\xi_3}^6-2\xi_1\xi_2{\xi_3}^6+2\xi_1{\xi_3}^7+{\xi_3}^8)
$$

$$
+\frac{4}{3\Delta^3}(-3{\xi_1}^5+3{\xi_1}^3{\xi_2}^2+5{\xi_1}^4\xi_3-2{\xi_1}^3\xi_2\xi_3-3{\xi_1}^2{\xi_2}^2\xi_3
$$

$$
+{\xi_1}^3{\xi_3}^2+2{\xi_1}^2\xi_2{\xi_3}^2-3{\xi_1}^2{\xi_3}^3+\xi_1\xi_2{\xi_3}^3-2\xi_1{\xi_3}^4+{\xi_3}^5)\Big]
$$

$$
-\left(F(\xi_1,\xi_2,\xi_3)-\frac{1}{2}\right)\frac{2}{\Delta^2}(4{\xi_1}^2+2\xi_1\xi_2-2\xi_1\xi_3-{\xi_3}^2)
$$

$$
-f(\xi_1)\frac{1}{24\Delta^4\xi_2}({\xi_1}^8-2{\xi_1}^7\xi_2+34{\xi_1}^6{\xi_2}^2-74{\xi_1}^5{\xi_2}^3+52{\xi_1}^4{\xi_2}^4
$$

$$
-38{\xi_1}^3{\xi_2}^5+46{\xi_1}^2{\xi_2}^6-14\xi_1{\xi_2}^7-5{\xi_2}^8-8{\xi_1}^7\xi_3+38{\xi_1}^6\xi_2\xi_3
$$

$$
-76{\xi_1}^5{\xi_2}^2\xi_3+90{\xi_1}^4{\xi_2}^3\xi_3+16{\xi_1}^3{\xi_2}^4\xi_3-134{\xi_1}^2{\xi_2}^5\xi_3+68\xi_1{\xi_2}^6\xi_3
$$

$$
+6{\xi_2}^7\xi_3+28{\xi_1}^6{\xi_3}^2-106{\xi_1}^5\xi_2{\xi_3}^2+30{\xi_1}^4{\xi_2}^2{\xi_3}^2+28{\xi_1}^3{\xi_2}^3{\xi_3}^2
$$

$$
+88{\xi_1}^2{\xi_2}^4{\xi_3}^2-114\xi_1{\xi_2}^5{\xi_3}^2+46{\xi_2}^6{\xi_3}^2-56{\xi_1}^5{\xi_3}^3+78{\xi_1}^4\xi_2{\xi_3}^3
$$

$$
-8{\xi_1}^3{\xi_2}^2{\xi_3}^3-12{\xi_1}^2{\xi_2}^3{\xi_3}^3+48\xi_1{\xi_2}^4{\xi_3}^3-146{\xi_2}^5{\xi_3}^3+70{\xi_1}^4{\xi_3}^4
$$

$$+ 58\xi_1{}^3\xi_2\xi_3{}^4 + 94\xi_1{}^2\xi_2{}^2\xi_3{}^4 + 78\xi_1\xi_2{}^3\xi_3{}^4 + 180\xi_2{}^4\xi_3{}^4 - 56\xi_1{}^3\xi_3{}^5$$

$$- 110\xi_1{}^2\xi_2\xi_3{}^5 - 108\xi_1\xi_2{}^2\xi_3{}^5 - 110\xi_2{}^3\xi_3{}^5 + 28\xi_1{}^2\xi_3{}^6 + 50\xi_1\xi_2\xi_3{}^6$$

$$+ 34\xi_2{}^2\xi_3{}^6 - 8\xi_1\xi_3{}^7 - 6\xi_2\xi_3{}^7 + \xi_3{}^8)$$

$$- f(\xi_3)\frac{1}{24\Delta^4\xi_2}(-\xi_1{}^8 + \xi_1{}^7\xi_2 - 14\xi_1{}^6\xi_2{}^2 + 46\xi_1{}^5\xi_2{}^3$$

$$- 32\xi_1{}^4\xi_2{}^4 + 8\xi_1{}^7\xi_3 - 30\xi_1{}^6\xi_2\xi_3 + 44\xi_1{}^5\xi_2{}^2\xi_3 - 34\xi_1{}^4\xi_2{}^3\xi_3$$

$$+ 12\xi_1{}^3\xi_2{}^4\xi_3 - 28\xi_1{}^6\xi_3{}^2 + 78\xi_1{}^5\xi_2\xi_3{}^2 - 18\xi_1{}^4\xi_2{}^2\xi_3{}^2 - 32\xi_1{}^3\xi_2{}^3\xi_3{}^2$$

$$+ 56\xi_1{}^5\xi_3{}^3 - 22\xi_1{}^4\xi_2\xi_3{}^3 - 16\xi_1{}^3\xi_2{}^2\xi_3{}^3 - 18\xi_1{}^2\xi_2{}^3\xi_3{}^3 - 70\xi_1{}^4\xi_3{}^4$$

$$- 128\xi_1{}^3\xi_2\xi_3{}^4 - 82\xi_1{}^2\xi_2{}^2\xi_3{}^4 + 56\xi_1{}^3\xi_3{}^5 + 166\xi_1{}^2\xi_2\xi_3{}^5 + 78\xi_1\xi_2{}^2\xi_3{}^5$$

$$- 28\xi_1{}^2\xi_3{}^6 - 78\xi_1\xi_2\xi_3{}^6 + 8\xi_1\xi_3{}^7 + 7\xi_2\xi_3{}^7 - \xi_3{}^8)$$

$$- \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{1}{4\Delta^3\xi_2}(\xi_1{}^6 + 24\xi_1{}^5\xi_2 + 41\xi_1{}^4\xi_2{}^2 - 24\xi_1{}^3\xi_2{}^3 - 13\xi_1{}^2\xi_2{}^4$$

$$- 32\xi_1\xi_2{}^5 + 3\xi_2{}^6 - 6\xi_1{}^5\xi_3 - 8\xi_1{}^4\xi_2\xi_3 + 12\xi_1{}^3\xi_2{}^2\xi_3 + 40\xi_1{}^2\xi_2{}^3\xi_3$$

$$+ 74\xi_1\xi_2{}^4\xi_3 - 16\xi_2{}^5\xi_3 + 15\xi_1{}^4\xi_3{}^2 - 32\xi_1{}^3\xi_2\xi_3{}^2 - 26\xi_1{}^2\xi_2{}^2\xi_3{}^2$$

$$- 24\xi_1\xi_2{}^3\xi_3{}^2 + 35\xi_2{}^4\xi_3{}^2 - 20\xi_1{}^3\xi_3{}^3 - 16\xi_1{}^2\xi_2\xi_3{}^3 - 52\xi_1\xi_2{}^2\xi_3{}^3$$

$$- 40\xi_2{}^3\xi_3{}^3 + 15\xi_1{}^2\xi_3{}^4 + 40\xi_1\xi_2\xi_3{}^4 + 25\xi_2{}^2\xi_3{}^4 - 6\xi_1\xi_3{}^5 - 8\xi_2\xi_3{}^5 + \xi_3{}^6)$$

$$- \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{4\Delta^3\xi_2}(-\xi_1{}^6 + 7\xi_1{}^5\xi_2 - 15\xi_1{}^4\xi_2{}^2 + 9\xi_1{}^3\xi_2{}^3 + 6\xi_1{}^5\xi_3$$

$$- 42\xi_1{}^4\xi_2\xi_3 + 18\xi_1{}^3\xi_2{}^2\xi_3 + 18\xi_1{}^2\xi_2{}^3\xi_3 - 15\xi_1{}^4\xi_3{}^2 + 17\xi_1{}^3\xi_2\xi_3{}^2$$

$$- 2\xi_1{}^2\xi_2{}^2\xi_3{}^2 + 20\xi_1{}^3\xi_3{}^3 + 52\xi_1{}^2\xi_2\xi_3{}^3 + 8\xi_1\xi_2{}^2\xi_3{}^3 - 15\xi_1{}^2\xi_3{}^4$$

$$- 23\xi_1\xi_2\xi_3{}^4 + 6\xi_1\xi_3{}^5 - 5\xi_2\xi_3{}^5 - \xi_3{}^6)$$

$$- \frac{1}{\xi_2 - \xi_3}\big(f(\xi_2) - f(\xi_3)\big)\frac{(\xi_3 - \xi_2 - \xi_1)}{24\xi_1}$$

$$- \frac{1}{\xi_2 - \xi_3}\left(\frac{f(\xi_2) - 1}{\xi_2} - \frac{f(\xi_3) - 1}{\xi_3}\right)\frac{(\xi_3 - \xi_2 - \xi_1)}{4\xi_1}, \tag{B.4}$$

$$F_5(\xi_1, \xi_2, \xi_3) = \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{2}{\xi_1\xi_2} - f(\xi_3)\frac{(-2\xi_2 + \xi_3)}{8\xi_1\xi_2}$$

$$- \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{(-2\xi_2 + 5\xi_3)}{4\xi_1\xi_2}, \tag{B.5}$$

$$F_6(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\Big[- \frac{1}{6\Delta^2}(2\xi_1{}^4 + 4\xi_1{}^3\xi_2 - 6\xi_1{}^2\xi_2{}^2$$

$$- 2\xi_1{}^3\xi_3 + 2\xi_1{}^2\xi_2\xi_3 - 2\xi_1\xi_2\xi_3{}^2 - 2\xi_1\xi_3{}^3 + \xi_3{}^4) - \frac{2\xi_1}{\Delta}\Big]$$

$$- f(\xi_1)\frac{1}{2\Delta^2}(\xi_1{}^3 + 5\xi_1{}^2\xi_2 - 5\xi_1\xi_2{}^2 - \xi_2{}^3$$

$$+ \xi_1{}^2\xi_3 + 6\xi_1\xi_2\xi_3 + \xi_2{}^2\xi_3 - \xi_1\xi_3{}^2 + \xi_2\xi_3{}^2 - \xi_3{}^3)$$

$$- f(\xi_3)\frac{1}{2\Delta^2}(-2\xi_1{}^3 + 2\xi_1{}^2\xi_2 + 2\xi_1{}^2\xi_3 - 2\xi_1\xi_2\xi_3 - 2\xi_1\xi_3{}^2 + \xi_3{}^3), \qquad \text{(B.6)}$$

$$F_7(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\Big[\frac{2\xi_2\xi_3}{3\Delta^4}(\xi_1{}^6 - 5\xi_1{}^5\xi_3 + 6\xi_1{}^4\xi_2\xi_3 + \xi_1{}^4\xi_3{}^2$$

$$- 6\xi_1{}^3\xi_2\xi_3{}^2 + 8\xi_1{}^2\xi_2{}^2\xi_3{}^2 + 6\xi_1{}^3\xi_3{}^3 - 4\xi_1{}^2\xi_2\xi_3{}^3 - 2\xi_1\xi_2{}^2\xi_3{}^3$$

$$+ 8\xi_2{}^3\xi_3{}^3 - 4\xi_1{}^2\xi_3{}^4 + 3\xi_1\xi_2\xi_3{}^4 - 9\xi_2{}^2\xi_3{}^4 - \xi_1\xi_3{}^5 + \xi_3{}^6)$$

$$+ \frac{2}{3\Delta^3}(\xi_1{}^5 - 7\xi_1{}^4\xi_3 + 24\xi_1{}^3\xi_2\xi_3 + 8\xi_1{}^3\xi_3{}^2 - 42\xi_1{}^2\xi_2\xi_3{}^2 + 34\xi_1\xi_2{}^2\xi_3{}^2$$

$$- 2\xi_1{}^2\xi_3{}^3 - 32\xi_1\xi_2\xi_3{}^3 - 34\xi_2{}^2\xi_3{}^3 - 2\xi_1\xi_3{}^4 + 33\xi_2\xi_3{}^4 + \xi_3{}^5)\Big]$$

$$+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{2}{\Delta^2}(\xi_1{}^2 - 4\xi_1\xi_3 + 10\xi_2\xi_3 + 2\xi_3{}^2)$$

$$+ f(\xi_1)\frac{1}{12\Delta^4}(-\xi_1{}^7 + 14\xi_1{}^6\xi_3 - 10\xi_1{}^5\xi_2\xi_3 - 42\xi_1{}^5\xi_3{}^2 + 2\xi_1{}^4\xi_2\xi_3{}^2$$

$$+ 62\xi_1{}^3\xi_2{}^2\xi_3{}^2 + 70\xi_1{}^4\xi_3{}^3 + 8\xi_1{}^3\xi_2\xi_3{}^3 + 4\xi_1{}^2\xi_2{}^2\xi_3{}^3 + 116\xi_1\xi_2{}^3\xi_3{}^3$$

$$- 70\xi_1{}^3\xi_3{}^4 - 46\xi_1{}^2\xi_2\xi_3{}^4 - 178\xi_1\xi_2{}^2\xi_3{}^4 - 58\xi_2{}^3\xi_3{}^4 + 42\xi_1{}^2\xi_3{}^5$$

$$+ 76\xi_1\xi_2\xi_3{}^5 + 90\xi_2{}^2\xi_3{}^5 - 14\xi_1\xi_3{}^6 - 34\xi_2\xi_3{}^6 + 2\xi_3{}^7)$$

$$+ f(\xi_3)\frac{4\xi_2\xi_3}{3\Delta^4}(-2\xi_1{}^5 + 7\xi_1{}^4\xi_2 - 8\xi_1{}^3\xi_2{}^2 + 2\xi_1{}^2\xi_2{}^3 + 2\xi_1\xi_2{}^4 - \xi_2{}^5$$

$$+ 3\xi_1{}^4\xi_3 - 8\xi_1{}^3\xi_2\xi_3 + 6\xi_1{}^2\xi_2{}^2\xi_3 - \xi_2{}^4\xi_3 + 2\xi_1{}^3\xi_3{}^2 - 4\xi_1{}^2\xi_2\xi_3{}^2$$

$$- 6\xi_1\xi_2{}^2\xi_3{}^2 + 8\xi_2{}^3\xi_3{}^2 - 4\xi_1{}^2\xi_3{}^3 + 4\xi_1\xi_2\xi_3{}^3 - 8\xi_2{}^2\xi_3{}^3 + \xi_2\xi_3{}^4 + \xi_3{}^5)$$

$$- \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{1}{2\Delta^3}(-\xi_1{}^5 - 2\xi_1{}^4\xi_3 - 36\xi_1{}^3\xi_2\xi_3 + 20\xi_1{}^3\xi_3{}^2$$

$$+ 28\xi_1{}^2\xi_2\xi_3{}^2 - 54\xi_1\xi_2{}^2\xi_3{}^2 - 28\xi_1{}^2\xi_3{}^3 + 40\xi_1\xi_2\xi_3{}^3$$

$$- 4\xi_2{}^2\xi_3{}^3 + 14\xi_1\xi_3{}^4 + 6\xi_2\xi_3{}^4 - 2\xi_3{}^5)$$

$$+ \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{2\Delta^3\xi_1}(\xi_1{}^6 - 6\xi_1{}^5\xi_2 + 15\xi_1{}^4\xi_2{}^2 - 20\xi_1{}^3\xi_2{}^3 + 15\xi_1{}^2\xi_2{}^4$$

$$- 6\xi_1\xi_2{}^5 + \xi_2{}^6 - 8\xi_1{}^5\xi_3 + 28\xi_1{}^4\xi_2\xi_3 - 32\xi_1{}^3\xi_2{}^2\xi_3 + 8\xi_1{}^2\xi_2{}^3\xi_3$$

$$+ 8\xi_1\xi_2{}^4\xi_3 - 4\xi_2{}^5\xi_3 + 13\xi_1{}^4\xi_3{}^2 - 76\xi_1{}^3\xi_2\xi_3{}^2 + 118\xi_1{}^2\xi_2{}^2\xi_3{}^2$$

$$- 60\xi_1\xi_2{}^3\xi_3{}^2 + 5\xi_2{}^4\xi_3{}^2 - 32\xi_1{}^2\xi_2\xi_3{}^3 - 32\xi_1\xi_2{}^2\xi_3{}^3 - 13\xi_1{}^2\xi_3{}^4$$

$$+ 82\xi_1\xi_2\xi_3{}^4 - 5\xi_2{}^2\xi_3{}^4 + 8\xi_1\xi_3{}^5 + 4\xi_2\xi_3{}^5 - \xi_3{}^6)$$

$$+\frac{1}{\xi_2-\xi_3}\left(\frac{f(\xi_2)-1}{\xi_2}-\frac{f(\xi_3)-1}{\xi_3}\right)\frac{\xi_2}{2\xi_1}, \tag{B.7}$$

$$\begin{aligned}
F_8(\xi_1,\xi_2,\xi_3) = F(\xi_1,\xi_2,\xi_3)\Big[&\frac{4\xi_1\xi_2\xi_3}{\Delta^4}(-{\xi_1}^5+6{\xi_1}^4\xi_3-8{\xi_1}^3\xi_2\xi_3-4{\xi_1}^3{\xi_3}^2 \\
&+12{\xi_1}^2\xi_2{\xi_3}^2-6\xi_1{\xi_2}^2{\xi_3}^2-4{\xi_1}^2{\xi_3}^3+4{\xi_2}^2{\xi_3}^3+6\xi_1{\xi_3}^4-2\xi_2{\xi_3}^4-2{\xi_3}^5) \\
&+\frac{8\xi_2\xi_3}{\Delta^3}(-7{\xi_1}^3+18{\xi_1}^2\xi_3-14\xi_1\xi_2\xi_3+6\xi_1{\xi_3}^2+10\xi_2{\xi_3}^2-10{\xi_3}^3)\Big] \\
&-\left(F(\xi_1,\xi_2,\xi_3)-\frac{1}{2}\right)\frac{8}{\Delta^2\xi_1}(-{\xi_1}^3+6{\xi_1}^2\xi_3+10\xi_1\xi_2\xi_3-6\xi_1{\xi_3}^2 \\
&-2\xi_2{\xi_3}^2+2{\xi_3}^3) \\
&-f(\xi_1)\frac{8\xi_1\xi_2\xi_3}{\Delta^4}({\xi_1}^4-4{\xi_1}^3\xi_3+4{\xi_1}^2\xi_2\xi_3-4\xi_1\xi_2{\xi_3}^2+2{\xi_2}^2{\xi_3}^2 \\
&+4\xi_1{\xi_3}^3-2{\xi_3}^4) \\
&+f(\xi_2)\frac{16\xi_1\xi_2\xi_3}{\Delta^4}({\xi_1}^4-2{\xi_1}^3\xi_2+2\xi_1{\xi_2}^3-{\xi_2}^4-4{\xi_1}^3\xi_3 \\
&+6{\xi_1}^2\xi_2\xi_3-2{\xi_2}^3\xi_3+6{\xi_1}^2{\xi_3}^2-6\xi_1\xi_2{\xi_3}^2-4\xi_1{\xi_3}^3+2\xi_2{\xi_3}^3+{\xi_3}^4) \\
&-\left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{32\xi_1\xi_2\xi_3}{\Delta^3}(3{\xi_1}^2-2\xi_1\xi_3+4\xi_2\xi_3-4{\xi_3}^2) \\
&-\left(\frac{f(\xi_2)-1}{\xi_2}\right)\frac{2}{\Delta^3\xi_1}({\xi_1}^6-2{\xi_1}^5\xi_2-5{\xi_1}^4{\xi_2}^2+20{\xi_1}^3{\xi_2}^3-25{\xi_1}^2{\xi_2}^4 \\
&+14\xi_1{\xi_2}^5-3{\xi_2}^6-6{\xi_1}^5\xi_3+2{\xi_1}^4\xi_2\xi_3-44{\xi_1}^3{\xi_2}^2\xi_3-44{\xi_1}^2{\xi_2}^3\xi_3 \\
&+82\xi_1{\xi_2}^4\xi_3+10{\xi_2}^5\xi_3+15{\xi_1}^4{\xi_3}^2+12{\xi_1}^3\xi_2{\xi_3}^2+114{\xi_1}^2{\xi_2}^2{\xi_3}^2 \\
&-36\xi_1{\xi_2}^3{\xi_3}^2-9{\xi_2}^4{\xi_3}^2-20{\xi_1}^3{\xi_3}^3-28{\xi_1}^2\xi_2{\xi_3}^3-76\xi_1{\xi_2}^2{\xi_3}^3-4{\xi_2}^3{\xi_3}^3 \\
&+15{\xi_1}^2{\xi_3}^4+22\xi_1\xi_2{\xi_3}^4+11{\xi_2}^2{\xi_3}^4-6\xi_1{\xi_3}^5-6\xi_2{\xi_3}^5+{\xi_3}^6),
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
F_9(\xi_1,\xi_2,\xi_3) = F(\xi_1,\xi_2,\xi_3)\Big[&\frac{1}{108\Delta^6}(6{\xi_1}^{11}\xi_2-24{\xi_1}^{10}{\xi_2}^2-26{\xi_1}^9{\xi_2}^3+126{\xi_1}^8{\xi_2}^4 \\
&-108{\xi_1}^7{\xi_2}^5+24{\xi_1}^6{\xi_2}^6-54{\xi_1}^9{\xi_2}^2\xi_3+150{\xi_1}^8{\xi_2}^3\xi_3-156{\xi_1}^7{\xi_2}^4\xi_3 \\
&+60{\xi_1}^6{\xi_2}^5\xi_3-456{\xi_1}^7{\xi_2}^3{\xi_3}^2-60{\xi_1}^6{\xi_2}^4{\xi_3}^2+222{\xi_1}^5{\xi_2}^5{\xi_3}^2-396{\xi_1}^5{\xi_2}^4{\xi_3}^3 \\
&+162{\xi_1}^4{\xi_2}^4{\xi_3}^4+364{\xi_1}^3{\xi_2}^3{\xi_3}^6+186{\xi_1}^2{\xi_2}^2{\xi_3}^8-3\xi_1\xi_2{\xi_3}^{10}+{\xi_3}^{12}) \\
&-\frac{1}{36\Delta^5}(130{\xi_1}^8\xi_2+400{\xi_1}^7{\xi_2}^2-560{\xi_1}^6{\xi_2}^3+172{\xi_1}^5{\xi_2}^4 \\
&-8{\xi_1}^6{\xi_2}^2\xi_3+760{\xi_1}^5{\xi_2}^3\xi_3-570{\xi_1}^4{\xi_2}^4\xi_3+400{\xi_1}^4{\xi_2}^3{\xi_3}^2 \\
&-200{\xi_1}^3{\xi_2}^3{\xi_3}^3-396{\xi_1}^2{\xi_2}^2{\xi_3}^5-156\xi_1\xi_2{\xi_3}^7-71{\xi_3}^9)\Big] \\
&-\left(F(\xi_1,\xi_2,\xi_3)-\frac{1}{2}\right)\frac{4}{\Delta^4\xi_3}(-{\xi_1}^7+5{\xi_1}^6\xi_2-9{\xi_1}^5{\xi_2}^2+5{\xi_1}^4{\xi_2}^3+12{\xi_1}^5\xi_2\xi_3
\end{aligned}$$

$$+ 4\xi_1{}^4\xi_2{}^2\xi_3 - \xi_1{}^3\xi_2{}^3\xi_3 - \xi_1{}^3\xi_2{}^2\xi_3{}^2 + 7\xi_1{}^2\xi_2{}^2\xi_3{}^3 - 8\xi_1\xi_2\xi_3{}^5 - 7\xi_3{}^7)$$

$$+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right) \frac{2}{\Delta^3\xi_1\xi_2\xi_3}(-2\xi_1{}^5\xi_2 - 10\xi_1{}^4\xi_2{}^2$$

$$+ 10\xi_1{}^3\xi_2{}^3 + 18\xi_1{}^3\xi_2{}^2\xi_3 - 20\xi_1{}^2\xi_2{}^2\xi_3{}^2 + 22\xi_1\xi_2\xi_3{}^4 + \xi_3{}^6)$$

$$+ f(\xi_1)\frac{1}{288\Delta^6}(5\xi_1{}^{11} + 50\xi_1{}^{10}\xi_3 - 2\xi_1{}^9\xi_2\xi_3 - 122\xi_1{}^9\xi_3{}^2$$

$$- 134\xi_1{}^8\xi_2\xi_3{}^2 + 508\xi_1{}^7\xi_2{}^2\xi_3{}^2 - 114\xi_1{}^8\xi_3{}^3 - 48\xi_1{}^7\xi_2\xi_3{}^3$$

$$- 664\xi_1{}^6\xi_2{}^2\xi_3{}^3 + 1144\xi_1{}^5\xi_2{}^3\xi_3{}^3 + 228\xi_1{}^7\xi_3{}^4 + 68\xi_1{}^6\xi_2\xi_3{}^4$$

$$- 1196\xi_1{}^5\xi_2{}^2\xi_3{}^4 - 1100\xi_1{}^4\xi_2{}^3\xi_3{}^4 + 1462\xi_1{}^3\xi_2{}^4\xi_3{}^4 + 180\xi_1{}^6\xi_3{}^5$$

$$+ 232\xi_1{}^5\xi_2\xi_3{}^5 + 1292\xi_1{}^4\xi_2{}^2\xi_3{}^5 - 2000\xi_1{}^3\xi_2{}^3\xi_3{}^5 - 1268\xi_1{}^2\xi_2{}^4\xi_3{}^5$$

$$- 140\xi_1\xi_2{}^5\xi_3{}^5 - 180\xi_1{}^5\xi_3{}^6 + 36\xi_1{}^4\xi_2\xi_3{}^6 + 728\xi_1{}^3\xi_2{}^2\xi_3{}^6$$

$$+ 2344\xi_1{}^2\xi_2{}^3\xi_3{}^6 + 476\xi_1\xi_2{}^4\xi_3{}^6 + 180\xi_2{}^5\xi_3{}^6 - 228\xi_1{}^4\xi_3{}^7$$

$$- 304\xi_1{}^3\xi_2\xi_3{}^7 - 1208\xi_1{}^2\xi_2{}^2\xi_3{}^7 - 496\xi_1\xi_2{}^3\xi_3{}^7 - 132\xi_2{}^4\xi_3{}^7$$

$$+ 114\xi_1{}^3\xi_3{}^8 + 10\xi_1{}^2\xi_2\xi_3{}^8 + 86\xi_1\xi_2{}^2\xi_3{}^8 - 210\xi_2{}^3\xi_3{}^8 + 122\xi_1{}^2\xi_3{}^9$$

$$+ 124\xi_1\xi_2\xi_3{}^9 + 202\xi_2{}^2\xi_3{}^9 - 50\xi_1\xi_3{}^{10} - 30\xi_2\xi_3{}^{10} - 10\xi_3{}^{11})$$

$$- \left(\frac{f(\xi_1) - 1}{\xi_1}\right) \frac{1}{12\Delta^5}(-47\xi_1{}^9 - \xi_1{}^8\xi_3 - 103\xi_1{}^7\xi_2\xi_3$$

$$+ 256\xi_1{}^7\xi_3{}^2 + 42\xi_1{}^6\xi_2\xi_3{}^2 - 174\xi_1{}^5\xi_2{}^2\xi_3{}^2 - 124\xi_1{}^6\xi_3{}^3$$

$$+ 418\xi_1{}^5\xi_2\xi_3{}^3 + 294\xi_1{}^4\xi_2{}^2\xi_3{}^3 - 30\xi_1{}^3\xi_2{}^3\xi_3{}^3 - 12\xi_1{}^5\xi_3{}^4$$

$$- 416\xi_1{}^4\xi_2\xi_3{}^4 - 8\xi_1{}^3\xi_2{}^2\xi_3{}^4 + 128\xi_1{}^2\xi_2{}^3\xi_3{}^4 - 190\xi_1\xi_2{}^4\xi_3{}^4$$

$$+ 122\xi_1{}^4\xi_3{}^5 + 278\xi_1{}^3\xi_2\xi_3{}^5 - 186\xi_1{}^2\xi_2{}^2\xi_3{}^5 + 330\xi_1\xi_2{}^3\xi_3{}^5$$

$$+ 16\xi_2{}^4\xi_3{}^5 - 240\xi_1{}^3\xi_3{}^6 + 54\xi_1{}^2\xi_2\xi_3{}^6 - 60\xi_1\xi_2{}^2\xi_3{}^6 - 26\xi_2{}^3\xi_3{}^6$$

$$+ 4\xi_1{}^2\xi_3{}^7 - 170\xi_1\xi_2\xi_3{}^7 + 10\xi_2{}^2\xi_3{}^7 + 90\xi_1\xi_3{}^8 + \xi_2\xi_3{}^8 - \xi_3{}^9)$$

$$+ \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right) \frac{1}{8\Delta^4\xi_2\xi_3}(-4\xi_1{}^9 + 116\xi_1{}^8\xi_3 + 369\xi_1{}^7\xi_2\xi_3$$

$$- 400\xi_1{}^7\xi_3{}^2 - 146\xi_1{}^6\xi_2\xi_3{}^2 + 174\xi_1{}^5\xi_2{}^2\xi_3{}^2 + 536\xi_1{}^6\xi_3{}^3$$

$$+ 362\xi_1{}^5\xi_2\xi_3{}^3 - 46\xi_1{}^4\xi_2{}^2\xi_3{}^3 - 1070\xi_1{}^3\xi_2{}^3\xi_3{}^3 - 176\xi_1{}^5\xi_3{}^4$$

$$- 354\xi_1{}^4\xi_2\xi_3{}^4 + 696\xi_1{}^3\xi_2{}^2\xi_3{}^4 - 740\xi_1{}^2\xi_2{}^3\xi_3{}^4 - 180\xi_1\xi_2{}^4\xi_3{}^4$$

$$- 272\xi_1{}^4\xi_3{}^5 + 70\xi_1{}^3\xi_2\xi_3{}^5 + 1650\xi_1{}^2\xi_2{}^2\xi_3{}^5 + 306\xi_1\xi_2{}^3\xi_3{}^5$$

$$
\begin{aligned}
&- 26\xi_2{}^4\xi_3{}^5 + 304\xi_1{}^3\xi_3{}^6 - 790\xi_1{}^2\xi_2\xi_3{}^6 - 132\xi_1\xi_2{}^2\xi_3{}^6 + 58\xi_2{}^3\xi_3{}^6 \\
&- 120\xi_1{}^2\xi_3{}^7 - 18\xi_1\xi_2\xi_3{}^7 - 50\xi_2{}^2\xi_3{}^7 + 24\xi_1\xi_3{}^8 + 22\xi_2\xi_3{}^8 - 4\xi_3{}^9) \\
&+ \frac{1}{576}\frac{1}{\xi_1 - \xi_2}\big(f(\xi_1) - f(\xi_2)\big) + \frac{1}{48}\frac{1}{\xi_1 - \xi_2}\left(\frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_2) - 1}{\xi_2}\right) \\
&- \frac{1}{\xi_1 - \xi_2}\left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1{}^2} - \frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{\xi_2{}^2}\right)\left(\frac{\xi_1}{2\xi_3} - \frac{3}{16}\right), \qquad \text{(B.9)}
\end{aligned}
$$

$$
\begin{aligned}
F_{10}(\xi_1, \xi_2, \xi_3) &= \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right)\frac{8}{3\xi_1\xi_2\xi_3} \\
&- \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right)\frac{2\xi_1}{\xi_2\xi_3}, \qquad \text{(B.10)}
\end{aligned}
$$

$$
\begin{aligned}
F_{11}(\xi_1, \xi_2, \xi_3) &= -\left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{2}{3\Delta^2\xi_1\xi_2}(\xi_1{}^4 + 2\xi_1{}^3\xi_2 - 3\xi_1{}^2\xi_2{}^2 \\
&- \xi_1{}^3\xi_3 + \xi_1{}^2\xi_2\xi_3 - 3\xi_1{}^2\xi_3{}^2 - 4\xi_1\xi_2\xi_3{}^2 + 5\xi_1\xi_3{}^3 - \xi_3{}^4) \\
&+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right)\frac{2}{\Delta\xi_1\xi_2}(-2\xi_1 + \xi_3) \\
&- f(\xi_3)\frac{1}{96\xi_1\xi_2}(-2\xi_1 + \xi_3) - \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{6\xi_1\xi_2}(-\xi_1 + \xi_3) \\
&- \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right)\frac{1}{\Delta^2\xi_2}(\xi_1{}^4 + 6\xi_1{}^3\xi_2 - 8\xi_1{}^2\xi_2{}^2 + 2\xi_1\xi_2{}^3 \\
&- \xi_2{}^4 + 2\xi_1{}^3\xi_3 + 2\xi_1\xi_2{}^2\xi_3 + 4\xi_2{}^3\xi_3 - 8\xi_1{}^2\xi_3{}^2 \\
&- 10\xi_1\xi_2\xi_3{}^2 - 6\xi_2{}^2\xi_3{}^2 + 6\xi_1\xi_3{}^3 + 4\xi_2\xi_3{}^3 - \xi_3{}^4) \\
&+ \left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right)\frac{1}{8\Delta^2\xi_1\xi_2}(-2\xi_1{}^5 + 6\xi_1{}^4\xi_2 - 4\xi_1{}^3\xi_2{}^2 \\
&+ 2\xi_1{}^4\xi_3 + 24\xi_1{}^3\xi_2\xi_3 - 26\xi_1{}^2\xi_2{}^2\xi_3 + 28\xi_1{}^3\xi_3{}^2 \\
&- 60\xi_1{}^2\xi_2\xi_3{}^2 - 44\xi_1{}^2\xi_3{}^3 + 12\xi_1\xi_2\xi_3{}^3 + 6\xi_1\xi_3{}^4 + 5\xi_3{}^5) \\
&+ \frac{1}{2}\frac{1}{\xi_1 - \xi_2}\left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1{}^2} - \frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{\xi_2{}^2}\right), \qquad \text{(B.11)}
\end{aligned}
$$

$$
\begin{aligned}
F_{12}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3)\Big[-\frac{2\xi_1}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)^2(-\xi_1 - \xi_2 + \xi_3) \times \\
&\times (\xi_1 - \xi_2 + \xi_3) - \frac{8}{\Delta^2}(-2\xi_1{}^2 + \xi_1\xi_2 + \xi_2{}^2 + \xi_1\xi_3 - 2\xi_2\xi_3 + \xi_3{}^2)\Big] \\
&- f(\xi_1)\frac{4\xi_1}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3)(\xi_1 - \xi_2 + \xi_3) \\
&+ f(\xi_2)\frac{4\xi_1}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)^2(-\xi_1 - \xi_2 + \xi_3)
\end{aligned}
$$

$$
\begin{aligned}
&- f(\xi_3)\frac{4\xi_1}{\Delta^3}(\xi_1 - \xi_2 - \xi_3)({\xi_1}^2 - 2\xi_1\xi_2 + {\xi_2}^2 - {\xi_3}^2) \\
&+ \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{2}{\Delta^2\xi_2\xi_3}(-{\xi_1}^3\xi_2 + 3{\xi_1}^2{\xi_2}^2 - 3\xi_1{\xi_2}^3 + {\xi_2}^4 \\
&- {\xi_1}^3\xi_3 + 18{\xi_1}^2\xi_2\xi_3 + 3\xi_1{\xi_2}^2\xi_3 - 4{\xi_2}^3\xi_3 + 3{\xi_1}^2{\xi_3}^2 \\
&+ 3\xi_1\xi_2{\xi_3}^2 + 6{\xi_2}^2{\xi_3}^2 - 3\xi_1{\xi_3}^3 - 4\xi_2{\xi_3}^3 + {\xi_3}^4) \\
&+ \left(\frac{f(\xi_2) - 1}{\xi_2}\right)\frac{2}{\Delta^2\xi_3}({\xi_1}^3 - 3{\xi_1}^2\xi_2 + 3\xi_1{\xi_2}^2 - {\xi_2}^3 \\
&- 3{\xi_1}^2\xi_3 - 6\xi_1\xi_2\xi_3 - 7{\xi_2}^2\xi_3 + 3\xi_1{\xi_3}^2 + 9\xi_2{\xi_3}^2 - {\xi_3}^3) \\
&+ \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{2}{\Delta^2\xi_2}({\xi_1}^3 - 3{\xi_1}^2\xi_2 + 3\xi_1{\xi_2}^2 - {\xi_2}^3 \\
&- 3{\xi_1}^2\xi_3 - 6\xi_1\xi_2\xi_3 + 9{\xi_2}^2\xi_3 + 3\xi_1{\xi_3}^2 - 7\xi_2{\xi_3}^2 - {\xi_3}^3) \\
&+ \frac{1}{\xi_1 - \xi_2}\left(\frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_2) - 1}{\xi_2}\right)\frac{2}{\xi_3} \\
&+ \frac{1}{\xi_1 - \xi_3}\left(\frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_3) - 1}{\xi_3}\right)\frac{2}{\xi_2},
\end{aligned} \tag{B.12}
$$

$$
\begin{aligned}
F_{13}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3)\frac{2}{\Delta}(-\xi_1 + \xi_2 + \xi_3) - f(\xi_1)\frac{4}{\Delta} \\
&- f(\xi_2)\frac{2}{\Delta\xi_1}(-\xi_1 - \xi_2 + \xi_3) - f(\xi_3)\frac{2}{\Delta\xi_1}(-\xi_1 + \xi_2 - \xi_3) \\
&- \frac{1}{\xi_2 - \xi_3}\big(f(\xi_2) - f(\xi_3)\big)\frac{2}{\xi_1},
\end{aligned} \tag{B.13}
$$

$$
\begin{aligned}
F_{14}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3)\Big[-\frac{2}{\Delta^2}(-\xi_1 + \xi_2 - \xi_3) \times \\
&\times (-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3) + \frac{8}{\Delta}\Big] \\
&+ f(\xi_1)\frac{4}{\Delta^2}(-\xi_1 + \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3) \\
&- f(\xi_2)\frac{4}{\Delta^2}(-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3) \\
&- f(\xi_3)\frac{4}{\Delta^2}({\xi_1}^2 - 2\xi_1\xi_2 + {\xi_2}^2 - {\xi_3}^2),
\end{aligned} \tag{B.14}
$$

$$
\begin{aligned}
F_{15}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3)\Big[-\frac{2\xi_1}{3\Delta^4}(-\xi_1 + \xi_2 + \xi_3)^2({\xi_1}^4 + 2{\xi_1}^3\xi_2 \\
&- 6{\xi_1}^2{\xi_2}^2 + 2\xi_1{\xi_2}^3 + {\xi_2}^4 + 2{\xi_1}^3\xi_3 + 4{\xi_1}^2\xi_2\xi_3 - 2\xi_1{\xi_2}^2\xi_3 - 4{\xi_2}^3\xi_3 \\
&- 6{\xi_1}^2{\xi_3}^2 - 2\xi_1\xi_2{\xi_3}^2 + 6{\xi_2}^2{\xi_3}^2 + 2\xi_1{\xi_3}^3 - 4\xi_2{\xi_3}^3 + {\xi_3}^4) \\
&- \frac{4}{3\Delta^3}(19{\xi_1}^4 - 22{\xi_1}^3\xi_2 - 12{\xi_1}^2{\xi_2}^2 + 14\xi_1{\xi_2}^3 + {\xi_2}^4
\end{aligned}
$$

$$- 22\xi_1{}^3\xi_3 + 40\xi_1{}^2\xi_2\xi_3 - 14\xi_1\xi_2{}^2\xi_3 - 4\xi_2{}^3\xi_3 - 12\xi_1{}^2\xi_3{}^2$$

$$- 14\xi_1\xi_2\xi_3{}^2 + 6\xi_2{}^2\xi_3{}^2 + 14\xi_1\xi_3{}^3 - 4\xi_2\xi_3{}^3 + \xi_3{}^4)\Big]$$

$$- \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{48\xi_1}{\Delta^2}$$

$$+ f(\xi_1)\frac{4\xi_1}{3\Delta^4}(-\xi_1 + \xi_2 + \xi_3)(\xi_1{}^4 + 2\xi_1{}^3\xi_2 - 6\xi_1{}^2\xi_2{}^2$$

$$+ 2\xi_1\xi_2{}^3 + \xi_2{}^4 + 2\xi_1{}^3\xi_3 + 4\xi_1{}^2\xi_2\xi_3 - 2\xi_1\xi_2{}^2\xi_3 - 4\xi_2{}^3\xi_3$$

$$- 6\xi_1{}^2\xi_3{}^2 - 2\xi_1\xi_2\xi_3{}^2 + 6\xi_2{}^2\xi_3{}^2 + 2\xi_1\xi_3{}^3 - 4\xi_2\xi_3{}^3 + \xi_3{}^4)$$

$$- f(\xi_2)\frac{1}{6\Delta^4\xi_1}(-\xi_1 - \xi_2 + \xi_3)(9\xi_1{}^6 - 8\xi_1{}^5\xi_2 - 35\xi_1{}^4\xi_2{}^2$$

$$+ 64\xi_1{}^3\xi_2{}^3 - 37\xi_1{}^2\xi_2{}^4 + 8\xi_1\xi_2{}^5 - \xi_2{}^6 - 8\xi_1{}^5\xi_3$$

$$+ 6\xi_1{}^4\xi_2\xi_3 + 20\xi_1{}^2\xi_2{}^3\xi_3 - 24\xi_1\xi_2{}^4\xi_3 + 6\xi_2{}^5\xi_3$$

$$- 35\xi_1{}^4\xi_3{}^2 + 34\xi_1{}^2\xi_2{}^2\xi_3{}^2 + 16\xi_1\xi_2{}^3\xi_3{}^2 - 15\xi_2{}^4\xi_3{}^2$$

$$+ 64\xi_1{}^3\xi_3{}^3 + 20\xi_1{}^2\xi_2\xi_3{}^3 + 16\xi_1\xi_2{}^2\xi_3{}^3 + 20\xi_2{}^3\xi_3{}^3$$

$$- 37\xi_1{}^2\xi_3{}^4 - 24\xi_1\xi_2\xi_3{}^4 - 15\xi_2{}^2\xi_3{}^4 + 8\xi_1\xi_3{}^5 + 6\xi_2\xi_3{}^5 - \xi_3{}^6)$$

$$- f(\xi_3)\frac{1}{6\Delta^4\xi_1}(-9\xi_1{}^7 + 17\xi_1{}^6\xi_2 + 27\xi_1{}^5\xi_2{}^2 - 99\xi_1{}^4\xi_2{}^3$$

$$+ 101\xi_1{}^3\xi_2{}^4 - 45\xi_1{}^2\xi_2{}^5 + 9\xi_1\xi_2{}^6 - \xi_2{}^7 - \xi_1{}^6\xi_3 - 6\xi_1{}^5\xi_2\xi_3$$

$$+ 41\xi_1{}^4\xi_2{}^2\xi_3 - 84\xi_1{}^3\xi_2{}^3\xi_3 + 81\xi_1{}^2\xi_2{}^4\xi_3 - 38\xi_1\xi_2{}^5\xi_3$$

$$+ 7\xi_2{}^6\xi_3 + 43\xi_1{}^5\xi_3{}^2 - 41\xi_1{}^4\xi_2\xi_3{}^2 - 34\xi_1{}^3\xi_2{}^2\xi_3{}^2$$

$$- 2\xi_1{}^2\xi_2{}^3\xi_3{}^2 + 55\xi_1\xi_2{}^4\xi_3{}^2 - 21\xi_2{}^5\xi_3{}^2 - 29\xi_1{}^4\xi_3{}^3$$

$$+ 44\xi_1{}^3\xi_2\xi_3{}^3 - 30\xi_1{}^2\xi_2{}^2\xi_3{}^3 - 20\xi_1\xi_2{}^3\xi_3{}^3 + 35\xi_2{}^4\xi_3{}^3$$

$$- 27\xi_1{}^3\xi_3{}^4 - 33\xi_1{}^2\xi_2\xi_3{}^4 - 25\xi_1\xi_2{}^2\xi_3{}^4 - 35\xi_2{}^3\xi_3{}^4$$

$$+ 29\xi_1{}^2\xi_3{}^5 + 26\xi_1\xi_2\xi_3{}^5 + 21\xi_2{}^2\xi_3{}^5 - 7\xi_1\xi_3{}^6 - 7\xi_2\xi_3{}^6 + \xi_3{}^7)$$

$$- \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{16\xi_1{}^2}{\Delta^3}(3\xi_1{}^2 - \xi_1\xi_2 - 2\xi_2{}^2 - \xi_1\xi_3 + 4\xi_2\xi_3 - 2\xi_3{}^2)$$

$$+ \left(\frac{f(\xi_2) - 1}{\xi_2}\right)\frac{1}{\Delta^3\xi_1}(-\xi_1 - \xi_2 + \xi_3)(3\xi_1{}^4 - 46\xi_1{}^3\xi_2 + 44\xi_1{}^2\xi_2{}^2$$

$$- 2\xi_1\xi_2{}^3 + \xi_2{}^4 - 10\xi_1{}^3\xi_3 + 40\xi_1{}^2\xi_2\xi_3 - 2\xi_1\xi_2{}^2\xi_3 - 4\xi_2{}^3\xi_3$$

$$+ 12\xi_1{}^2\xi_3{}^2 + 10\xi_1\xi_2\xi_3{}^2 + 6\xi_2{}^2\xi_3{}^2 - 6\xi_1\xi_3{}^3 - 4\xi_2\xi_3{}^3 + \xi_3{}^4)$$

$$+ \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{\Delta^3\xi_1}(-3\xi_1{}^5 + 13\xi_1{}^4\xi_2 - 22\xi_1{}^3\xi_2{}^2 + 18\xi_1{}^2\xi_2{}^3$$

$$- 7\xi_1\xi_2{}^4 + \xi_2{}^5 + 43\xi_1{}^4\xi_3 - 76\xi_1{}^3\xi_2\xi_3 + 18\xi_1{}^2\xi_2{}^2\xi_3 + 20\xi_1\xi_2{}^3\xi_3$$

$$- 5\xi_2{}^4\xi_3 + 2\xi_1{}^3\xi_3{}^2 + 6\xi_1{}^2\xi_2\xi_3{}^2 - 18\xi_1\xi_2{}^2\xi_3{}^2 + 10\xi_2{}^3\xi_3{}^2 - 42\xi_1{}^2\xi_3{}^3$$

$$+ 4\xi_1\xi_2\xi_3{}^3 - 10\xi_2{}^2\xi_3{}^3 + \xi_1\xi_3{}^4 + 5\xi_2\xi_3{}^4 - \xi_3{}^5)$$

$$+ \frac{1}{\xi_2 - \xi_3}\big(f(\xi_2) - f(\xi_3)\big)\frac{1}{6\xi_1}$$

$$+ \frac{1}{\xi_2 - \xi_3}\left(\frac{f(\xi_2) - 1}{\xi_2} - \frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{\xi_1}, \tag{B.15}$$

$$F_{16}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\frac{8}{\Delta^2}(-2\xi_1{}^2 + 2\xi_1\xi_2 + \xi_3{}^2)$$

$$- \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{8}{\Delta\xi_1\xi_2}(2\xi_1 - \xi_3)$$

$$- f(\xi_3)\frac{1}{2\xi_1\xi_2} + \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{32\xi_1}{\Delta^2}(-\xi_1 + \xi_2 - \xi_3)$$

$$- \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{\Delta^2\xi_1\xi_2}(2\xi_1{}^4 - 8\xi_1{}^3\xi_2 + 6\xi_1{}^2\xi_2{}^2 - 16\xi_1{}^3\xi_3$$

$$+ 16\xi_1{}^2\xi_2\xi_3 + 36\xi_1{}^2\xi_3{}^2 - 20\xi_1\xi_2\xi_3{}^2 - 32\xi_1\xi_3{}^3 + 5\xi_3{}^4), \tag{B.16}$$

$$F_{17}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\Big[- \frac{2\xi_1}{\Delta^2}(-\xi_1 + \xi_2 + \xi_3)^2 - \frac{4}{\Delta}\Big]$$

$$+ f(\xi_1)\frac{4\xi_1}{\Delta^2}(-\xi_1 + \xi_2 + \xi_3) - f(\xi_2)\frac{1}{\Delta^2\xi_1}(-\xi_1 - \xi_2 + \xi_3) \times$$

$$\times (3\xi_1{}^2 - 4\xi_1\xi_2 + \xi_2{}^2 - 4\xi_1\xi_3 - 2\xi_2\xi_3 + \xi_3{}^2)$$

$$- f(\xi_3)\frac{1}{\Delta^2\xi_1}(-\xi_1 + \xi_2 - \xi_3)(3\xi_1{}^2 - 4\xi_1\xi_2 + \xi_2{}^2 - 4\xi_1\xi_3 - 2\xi_2\xi_3 + \xi_3{}^2)$$

$$- \frac{1}{\xi_2 - \xi_3}\big(f(\xi_2) - f(\xi_3)\big)\frac{1}{\xi_1}, \tag{B.17}$$

$$F_{18}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\Big[- \frac{2\xi_1}{\Delta^4}(\xi_1{}^6 - 8\xi_1{}^5\xi_3 + 14\xi_1{}^4\xi_2\xi_3 + 10\xi_1{}^4\xi_3{}^2$$

$$- 32\xi_1{}^3\xi_2\xi_3{}^2 + 18\xi_1{}^2\xi_2{}^2\xi_3{}^2 + 8\xi_1{}^2\xi_2\xi_3{}^3 - 16\xi_1\xi_2{}^2\xi_3{}^3 + 4\xi_2{}^3\xi_3{}^3$$

$$- 10\xi_1{}^2\xi_3{}^4 + 8\xi_1\xi_2\xi_3{}^4 + 2\xi_2{}^2\xi_3{}^4 + 8\xi_1\xi_3{}^5 - 4\xi_2\xi_3{}^5 - 2\xi_3{}^6)$$

$$- \frac{4}{\Delta^3}(7\xi_1{}^4 - 32\xi_1{}^3\xi_3 + 32\xi_1{}^2\xi_2\xi_3 + 12\xi_1{}^2\xi_3{}^2$$

$$- 32\xi_1\xi_2\xi_3{}^2 + 10\xi_2{}^2\xi_3{}^2 + 16\xi_1\xi_3{}^3 - 10\xi_3{}^4)\Big]$$

$$- \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{32}{\Delta^2\xi_1}(\xi_1{}^2 - \xi_1\xi_3 + \xi_2\xi_3 - \xi_3{}^2)$$

$$+ f(\xi_1)\frac{4\xi_1}{\Delta^4}(-\xi_1{}^5 + 6\xi_1{}^4\xi_3 - 8\xi_1{}^3\xi_2\xi_3 - 4\xi_1{}^3\xi_3{}^2 + 12\xi_1{}^2\xi_2\xi_3{}^2$$

$$
\begin{aligned}
&- 6\xi_1{\xi_2}^2{\xi_3}^2 - 4{\xi_1}^2{\xi_3}^3 + 4{\xi_2}^2{\xi_3}^3 + 6\xi_1{\xi_3}^4 - 2\xi_2{\xi_3}^4 - 2{\xi_3}^5) \\
&- f(\xi_2)\frac{8\xi_1}{\Delta^4}(-\xi_1 + \xi_2 + \xi_3)({\xi_1}^4 - 2{\xi_1}^3\xi_2 + 2\xi_1{\xi_2}^3 - {\xi_2}^4 - 4{\xi_1}^3\xi_3 \\
&+ 6{\xi_1}^2\xi_2\xi_3 - 2{\xi_2}^3\xi_3 + 6{\xi_1}^2{\xi_3}^2 - 6\xi_1\xi_2{\xi_3}^2 - 4\xi_1{\xi_3}^3 + 2\xi_2{\xi_3}^3 + {\xi_3}^4) \\
&+ \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{16\xi_1}{\Delta^3}(-3{\xi_1}^3 + 8{\xi_1}^2\xi_3 - 6\xi_1\xi_2\xi_3 + 2\xi_1{\xi_3}^2 + 4\xi_2{\xi_3}^2 - 4{\xi_3}^3) \\
&+ \left(\frac{f(\xi_2) - 1}{\xi_2}\right)\frac{4}{\Delta^3\xi_1}(-{\xi_1}^5 + 17{\xi_1}^4\xi_2 - 2{\xi_1}^3{\xi_2}^2 - 46{\xi_1}^2{\xi_2}^3 + 35\xi_1{\xi_2}^4 \\
&- 3{\xi_2}^5 + 5{\xi_1}^4\xi_3 - 44{\xi_1}^3\xi_2\xi_3 + 22{\xi_1}^2{\xi_2}^2\xi_3 + 4\xi_1{\xi_2}^3\xi_3 + 13{\xi_2}^4\xi_3 \\
&- 10{\xi_1}^3{\xi_3}^2 + 30{\xi_1}^2\xi_2{\xi_3}^2 - 38\xi_1{\xi_2}^2{\xi_3}^2 - 22{\xi_2}^3{\xi_3}^2 + 10{\xi_1}^2{\xi_3}^3 \\
&+ 4\xi_1\xi_2{\xi_3}^3 + 18{\xi_2}^2{\xi_3}^3 - 5\xi_1{\xi_3}^4 - 7\xi_2{\xi_3}^4 + {\xi_3}^5),
\end{aligned} \tag{B.18}
$$

$$
\begin{aligned}
F_{19}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3)\Big[\frac{4\xi_1\xi_2\xi_3}{\Delta^4}({\xi_1}^4 - 4{\xi_1}^3\xi_3 + 4{\xi_1}^2\xi_2\xi_3 - 4\xi_1\xi_2{\xi_3}^2 \\
&+ 2{\xi_2}^2{\xi_3}^2 + 4\xi_1{\xi_3}^3 - 2{\xi_3}^4) + \frac{4}{\Delta^3}({\xi_1}^4 - 2{\xi_1}^3\xi_3 + 12{\xi_1}^2\xi_2\xi_3 - 6{\xi_1}^2{\xi_3}^2 \\
&- 18\xi_1\xi_2{\xi_3}^2 + 8{\xi_2}^2{\xi_3}^2 + 10\xi_1{\xi_3}^3 - 4\xi_2{\xi_3}^3 - 4{\xi_3}^4)\Big] \\
&+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{16}{\Delta^2\xi_1}({\xi_1}^2 - \xi_1\xi_3 + \xi_2\xi_3 - {\xi_3}^2) \\
&- f(\xi_1)\frac{8\xi_1\xi_2\xi_3}{\Delta^4}(-{\xi_1}^3 + 2{\xi_1}^2\xi_3 - 2\xi_1\xi_2\xi_3 + 2\xi_1{\xi_3}^2 + 2\xi_2{\xi_3}^2 - 2{\xi_3}^3) \\
&+ f(\xi_3)\frac{16\xi_1\xi_2\xi_3}{\Delta^4}(-{\xi_1}^3 + 3{\xi_1}^2\xi_2 - 3\xi_1{\xi_2}^2 + {\xi_2}^3 + {\xi_1}^2\xi_3 \\
&- 2\xi_1\xi_2\xi_3 + {\xi_2}^2\xi_3 + \xi_1{\xi_3}^2 - \xi_2{\xi_3}^2 - {\xi_3}^3) \\
&- \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{8\xi_1}{\Delta^3}(-{\xi_1}^3 - 10\xi_1\xi_2\xi_3 + 6\xi_1{\xi_3}^2 + 4\xi_2{\xi_3}^2 - 4{\xi_3}^3) \\
&- \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{4\xi_3}{\Delta^3\xi_1}(3{\xi_1}^4 - 8{\xi_1}^3\xi_2 + 6{\xi_1}^2{\xi_2}^2 - {\xi_2}^4 \\
&+ 4{\xi_1}^3\xi_3 + 20{\xi_1}^2\xi_2\xi_3 - 28\xi_1{\xi_2}^2\xi_3 + 4{\xi_2}^3\xi_3 - 18{\xi_1}^2{\xi_3}^2 \\
&+ 16\xi_1\xi_2{\xi_3}^2 - 6{\xi_2}^2{\xi_3}^2 + 12\xi_1{\xi_3}^3 + \xi_2{\xi_3}^3 - {\xi_3}^4) \\
&+ \frac{1}{\xi_2 - \xi_3}\left(\frac{f(\xi_2) - 1}{\xi_2} - \frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{\xi_1},
\end{aligned} \tag{B.19}
$$

$$
\begin{aligned}
F_{20}(\xi_1, \xi_2, \xi_3) &= F(\xi_1, \xi_2, \xi_3)\Big[\frac{2}{3\Delta^4}({\xi_1}^7 - 7{\xi_1}^6\xi_3 + 11{\xi_1}^5\xi_2\xi_3 + 6{\xi_1}^5{\xi_3}^2 \\
&- 19{\xi_1}^4\xi_2{\xi_3}^2 + 14{\xi_1}^3{\xi_2}^2{\xi_3}^2 + 5{\xi_1}^4{\xi_3}^3 - 4{\xi_1}^3\xi_2{\xi_3}^3 - 14{\xi_1}^2{\xi_2}^2{\xi_3}^3 \\
&+ 10\xi_1{\xi_2}^3{\xi_3}^3 - 10{\xi_1}^3{\xi_3}^4 + 11{\xi_1}^2\xi_2{\xi_3}^4 - 10\xi_1{\xi_2}^2{\xi_3}^4 - 7{\xi_2}^3{\xi_3}^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 3\xi_1{}^2\xi_3{}^5 - 2\xi_1\xi_2\xi_3{}^5 + 9\xi_2{}^2\xi_3{}^5 + 2\xi_1\xi_3{}^6 - \xi_2\xi_3{}^6 - \xi_3{}^7) \\
&- \frac{2}{3\Delta^3}(-19\xi_1{}^4 + 80\xi_1{}^3\xi_3 - 76\xi_1{}^2\xi_2\xi_3 - 12\xi_1{}^2\xi_3{}^2 \\
&+ 64\xi_1\xi_2\xi_3{}^2 - 42\xi_2{}^2\xi_3{}^2 - 64\xi_1\xi_3{}^3 + 8\xi_2\xi_3{}^3 + 34\xi_3{}^4)\Big] \\
&+ \left(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2}\right)\frac{24}{\Delta^2}(\xi_1 - 2\xi_3) \\
&+ f(\xi_1)\frac{4}{3\Delta^4}(\xi_1{}^6 - 5\xi_1{}^5\xi_3 + 6\xi_1{}^4\xi_2\xi_3 + \xi_1{}^4\xi_3{}^2 \\
&- 6\xi_1{}^3\xi_2\xi_3{}^2 + 8\xi_1{}^2\xi_2{}^2\xi_3{}^2 + 6\xi_1{}^3\xi_3{}^3 - 4\xi_1{}^2\xi_2\xi_3{}^3 - 2\xi_1\xi_2{}^2\xi_3{}^3 \\
&+ 8\xi_2{}^3\xi_3{}^3 - 4\xi_1{}^2\xi_3{}^4 + 3\xi_1\xi_2\xi_3{}^4 - 9\xi_2{}^2\xi_3{}^4 - \xi_1\xi_3{}^5 + \xi_3{}^6) \\
&- f(\xi_3)\frac{4}{3\Delta^4}(2\xi_1{}^6 - 9\xi_1{}^5\xi_2 + 15\xi_1{}^4\xi_2{}^2 - 10\xi_1{}^3\xi_2{}^3 + 3\xi_1\xi_2{}^5 \\
&- \xi_2{}^6 - 5\xi_1{}^5\xi_3 + 18\xi_1{}^4\xi_2\xi_3 - 22\xi_1{}^3\xi_2{}^2\xi_3 + 8\xi_1{}^2\xi_2{}^3\xi_3 \\
&+ 3\xi_1\xi_2{}^4\xi_3 - 2\xi_2{}^5\xi_3 + \xi_1{}^4\xi_3{}^2 - 2\xi_1{}^3\xi_2\xi_3{}^2 + 8\xi_1{}^2\xi_2{}^2\xi_3{}^2 \\
&- 14\xi_1\xi_2{}^3\xi_3{}^2 + 7\xi_2{}^4\xi_3{}^2 + 6\xi_1{}^3\xi_3{}^3 - 12\xi_1{}^2\xi_2\xi_3{}^3 + 6\xi_1\xi_2{}^2\xi_3{}^3 \\
&- 4\xi_1{}^2\xi_3{}^4 + 3\xi_1\xi_2\xi_3{}^4 - 7\xi_2{}^2\xi_3{}^4 - \xi_1\xi_3{}^5 + 2\xi_2\xi_3{}^5 + \xi_3{}^6) \\
&- \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{2}{\Delta^3}(-11\xi_1{}^4 + 24\xi_1{}^3\xi_3 - 20\xi_1{}^2\xi_2\xi_3 + 20\xi_1{}^2\xi_3{}^2 \\
&+ 24\xi_1\xi_2\xi_3{}^2 + 6\xi_2{}^2\xi_3{}^2 - 24\xi_1\xi_3{}^3 - 8\xi_2\xi_3{}^3 + 2\xi_3{}^4) \\
&- \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{2}{\Delta^3\xi_1}(-\xi_1{}^5 + 5\xi_1{}^4\xi_2 - 10\xi_1{}^3\xi_2{}^2 + 10\xi_1{}^2\xi_2{}^3 \\
&- 5\xi_1\xi_2{}^4 + \xi_2{}^5 + 19\xi_1{}^4\xi_3 - 52\xi_1{}^3\xi_2\xi_3 + 42\xi_1{}^2\xi_2{}^2\xi_3 \\
&- 4\xi_1\xi_2{}^3\xi_3 - 5\xi_2{}^4\xi_3 - 10\xi_1{}^3\xi_3{}^2 + 30\xi_1{}^2\xi_2\xi_3{}^2 \\
&- 30\xi_1\xi_2{}^2\xi_3{}^2 + 10\xi_2{}^3\xi_3{}^2 - 34\xi_1{}^2\xi_3{}^3 + 12\xi_1\xi_2\xi_3{}^3 \\
&- 10\xi_2{}^2\xi_3{}^3 + 27\xi_1\xi_3{}^4 + 5\xi_2\xi_3{}^4 - \xi_3{}^5) \\
&- \frac{1}{\xi_2 - \xi_3}\left(\frac{f(\xi_2) - 1}{\xi_2} - \frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{\xi_1},
\end{aligned}
\tag{B.20}
$$

$$
\begin{aligned}
F_{21}(\xi_1,\xi_2,\xi_3) &= F(\xi_1,\xi_2,\xi_3)\Big[-\frac{8\xi_1\xi_3}{\Delta^4}(-\xi_1 + \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3)\times \\
&\times(-\xi_1 + \xi_2 + \xi_3)^3 + \frac{16}{\Delta^3}(-\xi_1 + \xi_2 + \xi_3)\times \\
&\times(-\xi_1{}^3 + 3\xi_1{}^2\xi_2 - 3\xi_1\xi_2{}^2 + \xi_2{}^3 - 6\xi_1{}^2\xi_3 \\
&+ 4\xi_1\xi_2\xi_3 + 2\xi_2{}^2\xi_3 + 3\xi_1\xi_3{}^2 - 7\xi_2\xi_3{}^2 + 4\xi_3{}^3)\Big]
\end{aligned}
$$

$$+\left(F(\xi_1,\xi_2,\xi_3)-\frac{1}{2}\right)\frac{32}{\Delta^2\xi_1}({\xi_1}^2-2\xi_1\xi_2+{\xi_2}^2+4\xi_1\xi_3-2\xi_2\xi_3+{\xi_3}^2)$$
$$+f(\xi_1)\frac{16\xi_3\xi_1}{\Delta^4}(-\xi_1+\xi_2-\xi_3)(-\xi_1-\xi_2+\xi_3)(-\xi_1+\xi_2+\xi_3)^2$$
$$-f(\xi_2)\frac{16\xi_1\xi_3}{\Delta^4}(-\xi_1-\xi_2+\xi_3)(-\xi_1+\xi_2+\xi_3)^3$$
$$-f(\xi_3)\frac{16\xi_1\xi_3}{\Delta^4}(-\xi_1+\xi_2+\xi_3)(-{\xi_1}^3+3{\xi_1}^2\xi_2-3\xi_1{\xi_2}^2$$
$$+{\xi_2}^3+{\xi_1}^2\xi_3-2\xi_1\xi_2\xi_3+{\xi_2}^2\xi_3+\xi_1{\xi_3}^2-\xi_2{\xi_3}^2-{\xi_3}^3)$$
$$-\left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{32\xi_1}{\Delta^3}(-{\xi_1}^3+3{\xi_1}^2\xi_2-3\xi_1{\xi_2}^2+{\xi_2}^3$$
$$-5{\xi_1}^2\xi_3+4\xi_1\xi_2\xi_3+{\xi_2}^2\xi_3+3\xi_1{\xi_3}^2-5\xi_2{\xi_3}^2+3{\xi_3}^3)$$
$$-\left(\frac{f(\xi_2)-1}{\xi_2}\right)\frac{8}{\Delta^3\xi_1}(-{\xi_1}^5+9{\xi_1}^4\xi_2-22{\xi_1}^3{\xi_2}^2+22{\xi_1}^2{\xi_2}^3$$
$$-9\xi_1{\xi_2}^4+{\xi_2}^5+4{\xi_1}^4\xi_3-4{\xi_1}^3\xi_2\xi_3+16{\xi_1}^2{\xi_2}^2\xi_3-12\xi_1{\xi_2}^3\xi_3$$
$$-4{\xi_2}^4\xi_3-6{\xi_1}^3{\xi_3}^2-18{\xi_1}^2\xi_2{\xi_3}^2+10\xi_1{\xi_2}^2{\xi_3}^2+6{\xi_2}^3{\xi_3}^2$$
$$+4{\xi_1}^2{\xi_3}^3+12\xi_1\xi_2{\xi_3}^3-4{\xi_2}^2{\xi_3}^3-\xi_1{\xi_3}^4+\xi_2{\xi_3}^4)$$
$$-\left(\frac{f(\xi_3)-1}{\xi_3}\right)\frac{8\xi_3}{\Delta^3\xi_1}({\xi_1}^4-4{\xi_1}^3\xi_2+6{\xi_1}^2{\xi_2}^2-4\xi_1{\xi_2}^3$$
$$+{\xi_2}^4+20{\xi_1}^3\xi_3-44{\xi_1}^2\xi_2\xi_3+28\xi_1{\xi_2}^2\xi_3-4{\xi_2}^3\xi_3-2{\xi_1}^2{\xi_3}^2$$
$$-4\xi_1\xi_2{\xi_3}^2+6{\xi_2}^2{\xi_3}^2-20\xi_1{\xi_3}^3-4\xi_2{\xi_3}^3+{\xi_3}^4)$$
$$-\frac{1}{\xi_2-\xi_3}\left(\frac{f(\xi_2)-1}{\xi_2}-\frac{f(\xi_3)-1}{\xi_3}\right)\frac{4}{\xi_1}, \tag{B.21}$$

$$F_{22}(\xi_1,\xi_2,\xi_3)=F(\xi_1,\xi_2,\xi_3)\Big[\frac{\xi_1}{18\Delta^6}({\xi_1}^{10}+4{\xi_1}^9\xi_3+2{\xi_1}^8\xi_2\xi_3-30{\xi_1}^8{\xi_3}^2$$
$$-32{\xi_1}^7\xi_2{\xi_3}^2+148{\xi_1}^6{\xi_2}^2{\xi_3}^2+16{\xi_1}^6\xi_2{\xi_3}^3-240{\xi_1}^5{\xi_2}^2{\xi_3}^3+248{\xi_1}^4{\xi_2}^3{\xi_3}^3$$
$$+156{\xi_1}^6{\xi_3}^4-8{\xi_1}^5\xi_2{\xi_3}^4-284{\xi_1}^4{\xi_2}^2{\xi_3}^4-192{\xi_1}^3{\xi_2}^3{\xi_3}^4$$
$$+230{\xi_1}^2{\xi_2}^4{\xi_3}^4-264{\xi_1}^5{\xi_3}^5+136{\xi_1}^4\xi_2{\xi_3}^5+416{\xi_1}^3{\xi_2}^2{\xi_3}^5$$
$$-240{\xi_1}^2{\xi_2}^3{\xi_3}^5-104\xi_1{\xi_2}^4{\xi_3}^5+28{\xi_2}^5{\xi_3}^5+156{\xi_1}^4{\xi_3}^6$$
$$-224{\xi_1}^3\xi_2{\xi_3}^6-72{\xi_1}^2{\xi_2}^2{\xi_3}^6+176\xi_1{\xi_2}^3{\xi_3}^6-28{\xi_2}^4{\xi_3}^6$$
$$+112{\xi_1}^2\xi_2{\xi_3}^7-80\xi_1{\xi_2}^2{\xi_3}^7-16{\xi_2}^3{\xi_3}^7-30{\xi_1}^2{\xi_3}^8$$
$$+4\xi_1\xi_2{\xi_3}^8+26{\xi_2}^2{\xi_3}^8+4\xi_1{\xi_3}^9-12\xi_2{\xi_3}^9+2{\xi_3}^{10})$$
$$+\frac{1}{9\Delta^5}(73{\xi_1}^8-52{\xi_1}^7\xi_3+148{\xi_1}^6\xi_2\xi_3-844{\xi_1}^6{\xi_3}^2$$

$$
\begin{aligned}
&+ 180\xi_1{}^5\xi_2\xi_3{}^2 + 1008\xi_1{}^4\xi_2{}^2\xi_3{}^2 + 1004\xi_1{}^5\xi_3{}^3 - 1648\xi_1{}^4\xi_2\xi_3{}^3 \\
&- 760\xi_1{}^3\xi_2{}^2\xi_3{}^3 + 952\xi_1{}^2\xi_2{}^3\xi_3{}^3 + 320\xi_1{}^4\xi_3{}^4 + 1556\xi_1{}^3\xi_2\xi_3{}^4 \\
&- 1116\xi_1{}^2\xi_2{}^2\xi_3{}^4 - 244\xi_1\xi_2{}^3\xi_3{}^4 + 250\xi_2{}^4\xi_3{}^4 - 796\xi_1{}^3\xi_3{}^5 \\
&+ 72\xi_1{}^2\xi_2\xi_3{}^5 + 612\xi_1\xi_2{}^2\xi_3{}^5 - 256\xi_2{}^3\xi_3{}^5 + 92\xi_1{}^2\xi_3{}^6 \\
&- 532\xi_1\xi_2\xi_3{}^6 - 88\xi_2{}^2\xi_3{}^6 + 164\xi_1\xi_3{}^7 + 128\xi_2\xi_3{}^7 - 34\xi_3{}^8)\Big] \\
&- \left(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2}\right)\frac{2}{\Delta^4\xi_1\xi_2\xi_3}(\xi_1{}^8 - 16\xi_1{}^7\xi_3 - 44\xi_1{}^6\xi_2\xi_3 \\
&+ 44\xi_1{}^6\xi_3{}^2 + 36\xi_1{}^5\xi_2\xi_3{}^2 - 116\xi_1{}^4\xi_2{}^2\xi_3{}^2 - 52\xi_1{}^5\xi_3{}^3 \\
&+ 208\xi_1{}^4\xi_2\xi_3{}^3 + 64\xi_1{}^3\xi_2{}^2\xi_3{}^3 - 64\xi_1{}^2\xi_2{}^3\xi_3{}^3 + 20\xi_1{}^4\xi_3{}^4 \\
&- 72\xi_1{}^3\xi_2\xi_3{}^4 + 164\xi_1{}^2\xi_2{}^2\xi_3{}^4 + 52\xi_1\xi_2{}^3\xi_3{}^4 - 10\xi_2{}^4\xi_3{}^4 \\
&+ 8\xi_1{}^3\xi_3{}^5 - 96\xi_1{}^2\xi_2\xi_3{}^5 - 84\xi_1\xi_2{}^2\xi_3{}^5 + 8\xi_2{}^3\xi_3{}^5 - 4\xi_1{}^2\xi_3{}^6 \\
&+ 36\xi_1\xi_2\xi_3{}^6 + 8\xi_2{}^2\xi_3{}^6 - 4\xi_1\xi_3{}^7 - 8\xi_2\xi_3{}^7 + 2\xi_3{}^8) \\
&+ \left(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2} + \frac{\xi_1+\xi_2+\xi_3}{24}\right)\frac{16}{\Delta^3\xi_1\xi_2}(2\xi_1{}^4 + 10\xi_1{}^3\xi_2 - 5\xi_1{}^3\xi_3 \\
&+ 5\xi_1{}^2\xi_2\xi_3 - \xi_1\xi_2{}^2\xi_3 + 3\xi_1{}^2\xi_3{}^2 - 2\xi_2{}^2\xi_3{}^2 + \xi_1\xi_3{}^3 + 3\xi_2\xi_3{}^3 - \xi_3{}^4) \\
&- f(\xi_1)\frac{\xi_1}{9\Delta^6}(-\xi_1{}^9 - 6\xi_1{}^8\xi_3 - 8\xi_1{}^7\xi_2\xi_3 + 24\xi_1{}^7\xi_3{}^2 \\
&+ 40\xi_1{}^6\xi_2\xi_3{}^2 - 108\xi_1{}^5\xi_2{}^2\xi_3{}^2 + 24\xi_1{}^6\xi_3{}^3 + 48\xi_1{}^5\xi_2\xi_3{}^3 \\
&+ 72\xi_1{}^4\xi_2{}^2\xi_3{}^3 - 176\xi_1{}^3\xi_2{}^3\xi_3{}^3 - 132\xi_1{}^5\xi_3{}^4 - 76\xi_1{}^4\xi_2\xi_3{}^4 \\
&+ 280\xi_1{}^3\xi_2{}^2\xi_3{}^4 + 120\xi_1{}^2\xi_2{}^3\xi_3{}^4 - 110\xi_1\xi_2{}^4\xi_3{}^4 + 132\xi_1{}^4\xi_3{}^5 \\
&- 80\xi_1{}^3\xi_2\xi_3{}^5 - 216\xi_1{}^2\xi_2{}^2\xi_3{}^5 + 144\xi_1\xi_2{}^3\xi_3{}^5 + 28\xi_2{}^4\xi_3{}^5 \\
&- 24\xi_1{}^3\xi_3{}^6 + 120\xi_1{}^2\xi_2\xi_3{}^6 - 24\xi_1\xi_2{}^2\xi_3{}^6 - 56\xi_2{}^3\xi_3{}^6 - 24\xi_1{}^2\xi_3{}^7 \\
&- 16\xi_1\xi_2\xi_3{}^7 + 40\xi_2{}^2\xi_3{}^7 + 6\xi_1\xi_3{}^8 - 14\xi_2\xi_3{}^8 + 2\xi_3{}^9) \\
&+ f(\xi_3)\frac{1}{72\Delta^6\xi_1}(-15\xi_1{}^{11} - 29\xi_1{}^{10}\xi_2 + 335\xi_1{}^9\xi_2{}^2 - 555\xi_1{}^8\xi_2{}^3 \\
&- 198\xi_1{}^7\xi_2{}^4 + 1518\xi_1{}^6\xi_2{}^5 - 1842\xi_1{}^5\xi_2{}^6 + 1050\xi_1{}^4\xi_2{}^7 \\
&- 315\xi_1{}^3\xi_2{}^8 + 63\xi_1{}^2\xi_2{}^9 - 13\xi_1\xi_2{}^{10} + \xi_2{}^{11} \\
&- 59\xi_1{}^{10}\xi_3 + 106\xi_1{}^9\xi_2\xi_3 + 65\xi_1{}^8\xi_2{}^2\xi_3 + 152\xi_1{}^7\xi_2{}^3\xi_3 \\
&- 1366\xi_1{}^6\xi_2{}^4\xi_3 + 2300\xi_1{}^5\xi_2{}^5\xi_3 - 1846\xi_1{}^4\xi_2{}^6\xi_3 + 920\xi_1{}^3\xi_2{}^7\xi_3
\end{aligned}
$$

$$- 367\xi_1^2\xi_2^8\xi_3 + 106\xi_1\xi_2^9\xi_3 - 11\xi_2^{10}\xi_3 + 247\xi_1^9\xi_3^2$$
$$- 305\xi_1^8\xi_2\xi_3^2 - 948\xi_1^7\xi_2^2\xi_3^2 + 1820\xi_1^6\xi_2^3\xi_3^2 - 494\xi_1^5\xi_2^4\xi_3^2$$
$$- 366\xi_1^4\xi_2^5\xi_3^2 - 484\xi_1^3\xi_2^6\xi_3^2 + 844\xi_1^2\xi_2^7\xi_3^2 - 369\xi_1\xi_2^8\xi_3^2$$
$$+ 55\xi_2^9\xi_3^2 + 27\xi_1^8\xi_3^3 + 440\xi_1^7\xi_2\xi_3^3 - 668\xi_1^6\xi_2^2\xi_3^3$$
$$- 1080\xi_1^5\xi_2^3\xi_3^3 + 2242\xi_1^4\xi_2^4\xi_3^3 - 568\xi_1^3\xi_2^5\xi_3^3 - 924\xi_1^2\xi_2^6\xi_3^3$$
$$+ 696\xi_1\xi_2^7\xi_3^3 - 165\xi_2^8\xi_3^3 - 726\xi_1^7\xi_3^4 + 150\xi_1^6\xi_2\xi_3^4$$
$$+ 1874\xi_1^5\xi_2^2\xi_3^4 - 1298\xi_1^4\xi_2^3\xi_3^4 - 50\xi_1^3\xi_2^4\xi_3^4 + 434\xi_1^2\xi_2^5\xi_3^4$$
$$- 714\xi_1\xi_2^6\xi_3^4 + 330\xi_2^7\xi_3^4 + 594\xi_1^6\xi_3^5 - 1028\xi_1^5\xi_2\xi_3^5$$
$$- 258\xi_1^4\xi_2^2\xi_3^5 + 1000\xi_1^3\xi_2^3\xi_3^5 - 98\xi_1^2\xi_2^4\xi_3^5 + 252\xi_1\xi_2^5\xi_3^5$$
$$- 462\xi_2^6\xi_3^5 + 270\xi_1^5\xi_3^6 + 998\xi_1^4\xi_2\xi_3^6 - 260\xi_1^3\xi_2^2\xi_3^6$$
$$+ 252\xi_1^2\xi_2^3\xi_3^6 + 294\xi_1\xi_2^4\xi_3^6 + 462\xi_2^5\xi_3^6 - 522\xi_1^4\xi_3^7$$
$$- 456\xi_1^3\xi_2\xi_3^7 - 364\xi_1^2\xi_2^2\xi_3^7 - 456\xi_1\xi_2^3\xi_3^7 - 330\xi_2^4\xi_3^7$$
$$+ 213\xi_1^3\xi_3^8 + 199\xi_1^2\xi_2\xi_3^8 + 279\xi_1\xi_2^2\xi_3^8 + 165\xi_2^3\xi_3^8$$
$$- 39\xi_1^2\xi_3^9 - 86\xi_1\xi_2\xi_3^9 - 55\xi_2^2\xi_3^9 + 11\xi_1\xi_3^{10} + 11\xi_2\xi_3^{10} - \xi_3^{11})$$
$$- \left(\frac{f(\xi_1) - 1}{\xi_1}\right) \frac{8\xi_1}{3\Delta^5}(-6\xi_1^7 - 7\xi_1^6\xi_3 - 18\xi_1^5\xi_2\xi_3$$
$$+ 62\xi_1^5\xi_3^2 + 9\xi_1^4\xi_2\xi_3^2 - 68\xi_1^3\xi_2^2\xi_3^2 - 25\xi_1^4\xi_3^3$$
$$+ 112\xi_1^3\xi_2\xi_3^3 + 38\xi_1^2\xi_2^2\xi_3^3 - 28\xi_1\xi_2^3\xi_3^3 - 44\xi_1^3\xi_3^4$$
$$- 57\xi_1^2\xi_2\xi_3^4 + 54\xi_1\xi_2^2\xi_3^4 + 15\xi_2^3\xi_3^4 + 19\xi_1^2\xi_3^5$$
$$- 36\xi_1\xi_2\xi_3^5 - 27\xi_2^2\xi_3^5 + 10\xi_1\xi_3^6 + 15\xi_2\xi_3^6 - 3\xi_3^7)$$
$$- \left(\frac{f(\xi_3) - 1}{\xi_3}\right) \frac{1}{6\Delta^5\xi_1}(-3\xi_1^9 + 5\xi_1^8\xi_2 + 38\xi_1^7\xi_2^2 - 166\xi_1^6\xi_2^3$$
$$+ 296\xi_1^5\xi_2^4 - 292\xi_1^4\xi_2^5 + 170\xi_1^3\xi_2^6 - 58\xi_1^2\xi_2^7 + 11\xi_1\xi_2^8$$
$$- \xi_2^9 + 185\xi_1^8\xi_3 - 258\xi_1^7\xi_2\xi_3 - 696\xi_1^6\xi_2^2\xi_3 + 1586\xi_1^5\xi_2^3\xi_3$$
$$- 690\xi_1^4\xi_2^4\xi_3 - 510\xi_1^3\xi_2^5\xi_3 + 488\xi_1^2\xi_2^6\xi_3 - 114\xi_1\xi_2^7\xi_3$$
$$+ 9\xi_2^8\xi_3 + 116\xi_1^7\xi_3^2 + 90\xi_1^6\xi_2\xi_3^2 - 790\xi_1^5\xi_2^2\xi_3^2$$
$$+ 480\xi_1^4\xi_2^3\xi_3^2 + 696\xi_1^3\xi_2^4\xi_3^2 - 854\xi_1^2\xi_2^5\xi_3^2 + 298\xi_1\xi_2^6\xi_3^2$$

$$- 36\xi_2{}^7\xi_3{}^2 - 892\xi_1{}^6\xi_3{}^3 + 1014\xi_1{}^5\xi_2\xi_3{}^3 + 820\xi_1{}^4\xi_2{}^2\xi_3{}^3$$
$$- 1148\xi_1{}^3\xi_2{}^3\xi_3{}^3 + 324\xi_1{}^2\xi_2{}^4\xi_3{}^3 - 202\xi_1\xi_2{}^5\xi_3{}^3 + 84\xi_2{}^6\xi_3{}^3$$
$$+ 198\xi_1{}^5\xi_3{}^4 - 1164\xi_1{}^4\xi_2\xi_3{}^4 + 1050\xi_1{}^3\xi_2{}^2\xi_3{}^4 + 282\xi_1{}^2\xi_2{}^3\xi_3{}^4$$
$$- 240\xi_1\xi_2{}^4\xi_3{}^4 - 126\xi_2{}^5\xi_3{}^4 + 846\xi_1{}^4\xi_3{}^5 + 90\xi_1{}^3\xi_2\xi_3{}^5$$
$$- 352\xi_1{}^2\xi_2{}^2\xi_3{}^5 + 394\xi_1\xi_2{}^3\xi_3{}^5 + 126\xi_2{}^4\xi_3{}^5 - 348\xi_1{}^3\xi_3{}^6$$
$$+ 310\xi_1{}^2\xi_2\xi_3{}^6 - 106\xi_1\xi_2{}^2\xi_3{}^6 - 84\xi_2{}^3\xi_3{}^6 - 140\xi_1{}^2\xi_3{}^7$$
$$- 78\xi_1\xi_2\xi_3{}^7 + 36\xi_2{}^2\xi_3{}^7 + 37\xi_1\xi_3{}^8 - 9\xi_2\xi_3{}^8 + \xi_3{}^9)$$
$$- \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right)\frac{2}{\Delta^4\xi_2\xi_3}(3\xi_1{}^8 - 42\xi_1{}^7\xi_3 - 72\xi_1{}^6\xi_2\xi_3$$
$$+ 104\xi_1{}^6\xi_3{}^2 - 114\xi_1{}^5\xi_2\xi_3{}^2 - 128\xi_1{}^4\xi_2{}^2\xi_3{}^2 - 126\xi_1{}^5\xi_3{}^3$$
$$+ 288\xi_1{}^4\xi_2\xi_3{}^3 - 12\xi_1{}^3\xi_2{}^2\xi_3{}^3 - 48\xi_1{}^2\xi_2{}^3\xi_3{}^3 + 96\xi_1{}^4\xi_3{}^4$$
$$+ 82\xi_1{}^3\xi_2\xi_3{}^4 + 144\xi_1{}^2\xi_2{}^2\xi_3{}^4 + 90\xi_1\xi_2{}^3\xi_3{}^4 + 70\xi_2{}^4\xi_3{}^4$$
$$- 70\xi_1{}^3\xi_3{}^5 - 144\xi_1{}^2\xi_2\xi_3{}^5 - 162\xi_1\xi_2{}^2\xi_3{}^5 - 112\xi_2{}^3\xi_3{}^5$$
$$+ 48\xi_1{}^2\xi_3{}^6 + 90\xi_1\xi_2\xi_3{}^6 + 56\xi_2{}^2\xi_3{}^6 - 18\xi_1\xi_3{}^7 - 16\xi_2\xi_3{}^7 + 2\xi_3{}^8)$$
$$- \left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right)\frac{1}{2\Delta^4\xi_1\xi_2}(8\xi_1{}^8 - 55\xi_1{}^7\xi_2 + 167\xi_1{}^6\xi_2{}^2$$
$$- 295\xi_1{}^5\xi_2{}^3 + 335\xi_1{}^4\xi_2{}^4 - 253\xi_1{}^3\xi_2{}^5 + 125\xi_1{}^2\xi_2{}^6 - 37\xi_1\xi_2{}^7 + 5\xi_2{}^8$$
$$- 72\xi_1{}^7\xi_3 + 285\xi_1{}^6\xi_2\xi_3 - 414\xi_1{}^5\xi_2{}^2\xi_3 + 195\xi_1{}^4\xi_2{}^3\xi_3 + 180\xi_1{}^3\xi_2{}^4\xi_3$$
$$- 333\xi_1{}^2\xi_2{}^5\xi_3 + 210\xi_1\xi_2{}^6\xi_3 - 51\xi_2{}^7\xi_3 + 280\xi_1{}^6\xi_3{}^2 + 189\xi_1{}^5\xi_2\xi_3{}^2$$
$$- 859\xi_1{}^4\xi_2{}^2\xi_3{}^2 - 70\xi_1{}^3\xi_2{}^3\xi_3{}^2 + 714\xi_1{}^2\xi_2{}^4\xi_3{}^2 - 407\xi_1\xi_2{}^5\xi_3{}^2$$
$$+ 153\xi_2{}^6\xi_3{}^2 - 520\xi_1{}^5\xi_3{}^3 + 641\xi_1{}^4\xi_2\xi_3{}^3 + 204\xi_1{}^3\xi_2{}^2\xi_3{}^3$$
$$- 250\xi_1{}^2\xi_2{}^3\xi_3{}^3 + 116\xi_1\xi_2{}^4\xi_3{}^3 - 191\xi_2{}^5\xi_3{}^3 + 456\xi_1{}^4\xi_3{}^4$$
$$- 421\xi_1{}^3\xi_2\xi_3{}^4 + 609\xi_1{}^2\xi_2{}^2\xi_3{}^4 + 573\xi_1\xi_2{}^3\xi_3{}^4 + 95\xi_2{}^4\xi_3{}^4$$
$$- 152\xi_1{}^3\xi_3{}^5 - 873\xi_1{}^2\xi_2\xi_3{}^5 - 718\xi_1\xi_2{}^2\xi_3{}^5 - 25\xi_2{}^3\xi_3{}^5 + 8\xi_1{}^2\xi_3{}^6$$
$$+ 287\xi_1\xi_2\xi_3{}^6 + 51\xi_2{}^2\xi_3{}^6 - 24\xi_1\xi_3{}^7 - 53\xi_2\xi_3{}^7 + 16\xi_3{}^8)$$
$$+ \frac{1}{\xi_2 - \xi_3}\big(f(\xi_2) - f(\xi_3)\big)\frac{1}{144\xi_1} + \frac{1}{\xi_2 - \xi_3}\left(\frac{f(\xi_2) - 1}{\xi_2} - \frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{12\xi_1}$$
$$- \frac{1}{\xi_1 - \xi_3}\left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1{}^2} - \frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3{}^2}\right)\frac{2}{\xi_2}$$

$$
-\frac{1}{\xi_2-\xi_3}\left(\frac{f(\xi_2)-1+\frac{1}{6}\xi_2}{{\xi_2}^2}-\frac{f(\xi_3)-1+\frac{1}{6}\xi_3}{{\xi_3}^2}\right)\frac{1}{4\xi_1}, \tag{B.22}
$$

$$
\begin{aligned}
F_{23}(\xi_1,\xi_2,\xi_3) &= F(\xi_1,\xi_2,\xi_3)\frac{8}{3\Delta^4}({\xi_1}^6-9{\xi_1}^4{\xi_2}^2+8{\xi_1}^3{\xi_2}^3-{\xi_1}^5\xi_3 \\
&+3{\xi_1}^4\xi_2\xi_3-2{\xi_1}^3{\xi_2}^2\xi_3-4{\xi_1}^4{\xi_3}^2-4{\xi_1}^3\xi_2{\xi_3}^2+8{\xi_1}^2{\xi_2}^2{\xi_3}^2 \\
&+6{\xi_1}^3{\xi_3}^3-6{\xi_1}^2\xi_2{\xi_3}^3+{\xi_1}^2{\xi_3}^4+6\xi_1\xi_2{\xi_3}^4-5\xi_1{\xi_3}^5+{\xi_3}^6) \\
&-\left(F(\xi_1,\xi_2,\xi_3)-\frac{1}{2}\right)\frac{8}{3\Delta^3\xi_1\xi_2}(-{\xi_1}^5-33{\xi_1}^4\xi_2+34{\xi_1}^3{\xi_2}^2 \\
&+2{\xi_1}^4\xi_3+32{\xi_1}^3\xi_2\xi_3-34{\xi_1}^2{\xi_2}^2\xi_3+2{\xi_1}^3{\xi_3}^2+42{\xi_1}^2\xi_2{\xi_3}^2 \\
&-8{\xi_1}^2{\xi_3}^3-24\xi_1\xi_2{\xi_3}^3+7\xi_1{\xi_3}^4-{\xi_3}^5) \\
&+\left(F(\xi_1,\xi_2,\xi_3)-\frac{1}{2}+\frac{\xi_1+\xi_2+\xi_3}{24}\right)\frac{8}{\Delta^2\xi_1\xi_2}(2{\xi_1}^2+10\xi_1\xi_2-4\xi_1\xi_3+{\xi_3}^2) \\
&-f(\xi_3)\frac{1}{24\xi_1\xi_2}-\left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{16\xi_1}{3\Delta^4}(-\xi_1+\xi_2-\xi_3)({\xi_1}^4+2{\xi_1}^3\xi_2 \\
&-6{\xi_1}^2{\xi_2}^2+2\xi_1{\xi_2}^3+{\xi_2}^4-{\xi_1}^3\xi_3+{\xi_1}^2\xi_2\xi_3+\xi_1{\xi_2}^2\xi_3-{\xi_2}^3\xi_3 \\
&-3{\xi_1}^2{\xi_3}^2-8\xi_1\xi_2{\xi_3}^2-3{\xi_2}^2{\xi_3}^2+5\xi_1{\xi_3}^3+5\xi_2{\xi_3}^3-2{\xi_3}^4) \\
&-\left(\frac{f(\xi_3)-1}{\xi_3}\right)\frac{2}{3\Delta^4\xi_1\xi_2}({\xi_1}^8-8{\xi_1}^7\xi_2+28{\xi_1}^6{\xi_2}^2-56{\xi_1}^5{\xi_2}^3 \\
&+35{\xi_1}^4{\xi_2}^4-9{\xi_1}^7\xi_3+57{\xi_1}^6\xi_2\xi_3-117{\xi_1}^5{\xi_2}^2\xi_3 \\
&+69{\xi_1}^4{\xi_2}^3\xi_3+35{\xi_1}^6{\xi_3}^2-110{\xi_1}^5\xi_2{\xi_3}^2+125{\xi_1}^4{\xi_2}^2{\xi_3}^2 \\
&-50{\xi_1}^3{\xi_2}^3{\xi_3}^2-77{\xi_1}^5{\xi_3}^3+63{\xi_1}^4\xi_2{\xi_3}^3+14{\xi_1}^3{\xi_2}^2{\xi_3}^3 \\
&+105{\xi_1}^4{\xi_3}^4+36{\xi_1}^3\xi_2{\xi_3}^4-13{\xi_1}^2{\xi_2}^2{\xi_3}^4-91{\xi_1}^3{\xi_3}^5 \\
&-73{\xi_1}^2\xi_2{\xi_3}^5+49{\xi_1}^2{\xi_3}^6+25\xi_1\xi_2{\xi_3}^6-15\xi_1{\xi_3}^7+{\xi_3}^8) \\
&+\left(\frac{f(\xi_1)-1+\frac{1}{6}\xi_1}{\xi_1^2}\right)\frac{4}{\Delta^3\xi_2\xi_3}(-{\xi_1}^5\xi_2+5{\xi_1}^4{\xi_2}^2 \\
&-10{\xi_1}^3{\xi_2}^3+10{\xi_1}^2{\xi_2}^4-5\xi_1{\xi_2}^5+{\xi_2}^6+{\xi_1}^5\xi_3 \\
&+50{\xi_1}^4\xi_2\xi_3-22{\xi_1}^3{\xi_2}^2\xi_3-36{\xi_1}^2{\xi_2}^3\xi_3+13\xi_1{\xi_2}^4\xi_3-6{\xi_2}^5\xi_3 \\
&+{\xi_1}^4{\xi_3}^2-22{\xi_1}^3\xi_2{\xi_3}^2+56{\xi_1}^2{\xi_2}^2{\xi_3}^2-2\xi_1{\xi_2}^3{\xi_3}^2+15{\xi_2}^4{\xi_3}^2 \\
&-10{\xi_1}^3{\xi_3}^3-44{\xi_1}^2\xi_2{\xi_3}^3-22\xi_1{\xi_2}^2{\xi_3}^3-20{\xi_2}^3{\xi_3}^3+14{\xi_1}^2{\xi_3}^4 \\
&+23\xi_1\xi_2{\xi_3}^4+15{\xi_2}^2{\xi_3}^4-7\xi_1{\xi_3}^5-6\xi_2{\xi_3}^5+{\xi_3}^6) \\
&+\left(\frac{f(\xi_3)-1+\frac{1}{6}\xi_3}{\xi_3^2}\right)\frac{1}{2\Delta^3\xi_1\xi_2}(2{\xi_1}^6-12{\xi_1}^5\xi_2+30{\xi_1}^4{\xi_2}^2
\end{aligned}
$$

$$- 20{\xi_1}^3{\xi_2}^3 - 4{\xi_1}^5\xi_3 + 12{\xi_1}^4\xi_2\xi_3 - 8{\xi_1}^3{\xi_2}^2\xi_3 - 26{\xi_1}^4{\xi_3}^2$$

$$- 184{\xi_1}^3\xi_2{\xi_3}^2 + 210{\xi_1}^2{\xi_2}^2{\xi_3}^2 + 72{\xi_1}^3{\xi_3}^3 - 136{\xi_1}^2\xi_2{\xi_3}^3$$

$$- 50{\xi_1}^2{\xi_3}^4 + 162\xi_1\xi_2{\xi_3}^4 - 4\xi_1{\xi_3}^5 + 5{\xi_3}^6)$$

$$+ \frac{1}{\xi_1 - \xi_2}\left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{{\xi_1}^2} - \frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{{\xi_2}^2}\right)\frac{2}{\xi_3}, \tag{B.23}$$

$$F_{24}(\xi_1, \xi_2, \xi_3) = \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{4\xi_1}{\Delta^2\xi_2\xi_3}({\xi_1}^2 - 4\xi_1\xi_3 + 2\xi_2\xi_3 + 2{\xi_3}^2)$$

$$+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right)\frac{8}{\Delta\xi_2\xi_3}$$

$$+ \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right)\frac{2}{\Delta^2\xi_2\xi_3}(5{\xi_1}^4 - 16{\xi_1}^3\xi_3 + 4{\xi_1}^2\xi_2\xi_3$$

$$+ 12{\xi_1}^2{\xi_3}^2 + 8\xi_1\xi_2{\xi_3}^2 + 6{\xi_2}^2{\xi_3}^2 - 8\xi_1{\xi_3}^3 - 8\xi_2{\xi_3}^3 + 2{\xi_3}^4)$$

$$+ \left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right)\frac{4}{\Delta^2\xi_1\xi_2}({\xi_1}^4 - 4{\xi_1}^3\xi_2 + 6{\xi_1}^2{\xi_2}^2$$

$$- 4\xi_1{\xi_2}^3 + {\xi_2}^4 - 7{\xi_1}^3\xi_3 + 11{\xi_1}^2\xi_2\xi_3 - \xi_1{\xi_2}^2\xi_3 - 3{\xi_2}^3\xi_3$$

$$+ 7{\xi_1}^2{\xi_3}^2 + 6\xi_1\xi_2{\xi_3}^2 + 3{\xi_2}^2{\xi_3}^2 - \xi_1{\xi_3}^3 - \xi_2{\xi_3}^3)$$

$$+ \frac{1}{\xi_2 - \xi_3}\left(\frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{{\xi_2}^2} - \frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{{\xi_3}^2}\right)\frac{2}{\xi_1}, \tag{B.24}$$

$$F_{25}(\xi_1, \xi_2, \xi_3) = -\left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{16}{\Delta^2\xi_1}(-{\xi_1}^2 - 2\xi_2\xi_3 + 2{\xi_3}^2)$$

$$+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right)\frac{16}{\Delta\xi_1\xi_2\xi_3}(\xi_1 - 2\xi_3)$$

$$+ \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right)\frac{4}{\Delta^2\xi_2\xi_3}(-{\xi_1}^4 + 4{\xi_1}^3\xi_3 + 16{\xi_1}^2\xi_2\xi_3$$

$$+ 4\xi_1\xi_2{\xi_3}^2 + 6{\xi_2}^2{\xi_3}^2 - 4\xi_1{\xi_3}^3 - 8\xi_2{\xi_3}^3 + 2{\xi_3}^4)$$

$$+ \left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right)\frac{8}{\Delta^2\xi_1\xi_2}({\xi_1}^4 - 4{\xi_1}^3\xi_2 + 6{\xi_1}^2{\xi_2}^2$$

$$- 4\xi_1{\xi_2}^3 + {\xi_2}^4 - 4{\xi_1}^3\xi_3 + 4{\xi_1}^2\xi_2\xi_3 + 4\xi_1{\xi_2}^2\xi_3 - 4{\xi_2}^3\xi_3$$

$$+ 6{\xi_1}^2{\xi_3}^2 - 4\xi_1\xi_2{\xi_3}^2 + 14{\xi_2}^2{\xi_3}^2 - 4\xi_1{\xi_3}^3 - 12\xi_2{\xi_3}^3 + {\xi_3}^4)$$

$$+ \frac{1}{\xi_1 - \xi_2}\left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{{\xi_1}^2} - \frac{f(\xi_2) - 1 + \frac{1}{6}\xi_2}{{\xi_2}^2}\right)\frac{8}{\xi_3}, \tag{B.25}$$

$$F_{26}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\Big[\frac{4\xi_1\xi_2}{\Delta^4}(2{\xi_1}^4 - 8{\xi_1}^3\xi_2 + 6{\xi_1}^2{\xi_2}^2 - 4{\xi_1}^2{\xi_3}^2$$

$$+ 4\xi_1\xi_2{\xi_3}^2 + {\xi_3}^4) - \frac{16}{\Delta^3}(-3{\xi_1}^3 + 3{\xi_1}^2\xi_2 + 4{\xi_1}^2\xi_3$$

$$- 4\xi_1\xi_2\xi_3 + \xi_1{\xi_3}^2 - {\xi_3}^3)\Big]$$

$$+ \left(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2}\right)\frac{8}{\Delta^2\xi_1\xi_2}(2{\xi_1}^2 + 4\xi_1\xi_2 - 4\xi_1\xi_3 + {\xi_3}^2)$$

$$- f(\xi_1)\frac{16\xi_1\xi_2}{\Delta^4}(-\xi_1 + \xi_2 - \xi_3)^2(-\xi_1 + \xi_2 + \xi_3)$$

$$- f(\xi_3)\frac{1}{2\Delta^4\xi_1\xi_2}(-2{\xi_1}^7 + 18{\xi_1}^6\xi_2 - 42{\xi_1}^5{\xi_2}^2 + 26{\xi_1}^4{\xi_2}^3$$

$$+ 14{\xi_1}^6\xi_3 - 76{\xi_1}^5\xi_2\xi_3 + 178{\xi_1}^4{\xi_2}^2\xi_3 - 116{\xi_1}^3{\xi_2}^3\xi_3$$

$$- 42{\xi_1}^5{\xi_3}^2 + 110{\xi_1}^4\xi_2{\xi_3}^2 - 68{\xi_1}^3{\xi_2}^2{\xi_3}^2 + 70{\xi_1}^4{\xi_3}^3$$

$$- 40{\xi_1}^3\xi_2{\xi_3}^3 - 30{\xi_1}^2{\xi_2}^2{\xi_3}^3 - 70{\xi_1}^3{\xi_3}^4 - 50{\xi_1}^2\xi_2{\xi_3}^4$$

$$+ 42{\xi_1}^2{\xi_3}^5 + 26\xi_1\xi_2{\xi_3}^5 - 14\xi_1{\xi_3}^6 + {\xi_3}^7)$$

$$+ \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{32\xi_1}{\Delta^3}(3{\xi_1}^2 - \xi_1\xi_2 - 2{\xi_2}^2 - \xi_1\xi_3 + 4\xi_2\xi_3 - 2{\xi_3}^2)$$

$$- \left(\frac{f(\xi_3) - 1}{\xi_3}\right)\frac{1}{\Delta^3\xi_1\xi_2}(-2{\xi_1}^5 + 6{\xi_1}^4\xi_2 - 4{\xi_1}^3{\xi_2}^2 + 18{\xi_1}^4\xi_3$$

$$+ 24{\xi_1}^3\xi_2\xi_3 - 42{\xi_1}^2{\xi_2}^2\xi_3 - 52{\xi_1}^3{\xi_3}^2 + 52{\xi_1}^2\xi_2{\xi_3}^2$$

$$+ 68{\xi_1}^2{\xi_3}^3 - 20\xi_1\xi_2{\xi_3}^3 - 42\xi_1{\xi_3}^4 + 5{\xi_3}^5), \tag{B.26}$$

$$F_{27}(\xi_1,\xi_2,\xi_3) = F(\xi_1,\xi_2,\xi_3)\Big[-\frac{2\xi_1\xi_2}{3\Delta^6}(2{\xi_1}^8 - 4{\xi_1}^7\xi_2 - 16{\xi_1}^6{\xi_2}^2 + 68{\xi_1}^5{\xi_2}^3$$

$$- 50{\xi_1}^4{\xi_2}^4 - 2{\xi_1}^7\xi_3 + 10{\xi_1}^6\xi_2\xi_3 - 18{\xi_1}^5{\xi_2}^2\xi_3 + 10{\xi_1}^4{\xi_2}^3\xi_3 - 4{\xi_1}^6{\xi_3}^2$$

$$- 4{\xi_1}^5\xi_2{\xi_3}^2 + 52{\xi_1}^4{\xi_2}^2{\xi_3}^2 - 44{\xi_1}^3{\xi_2}^3{\xi_3}^2 + 2{\xi_1}^5{\xi_3}^3 - 6{\xi_1}^4\xi_2{\xi_3}^3$$

$$+ 4{\xi_1}^3{\xi_2}^2{\xi_3}^3 + 4{\xi_1}^4{\xi_3}^4 + 4{\xi_1}^3\xi_2{\xi_3}^4 - 8{\xi_1}^2{\xi_2}^2{\xi_3}^4 + 2{\xi_1}^3{\xi_3}^5$$

$$- 2{\xi_1}^2\xi_2{\xi_3}^5 - 4{\xi_1}^2{\xi_3}^6 + 2\xi_1\xi_2{\xi_3}^6 - 2\xi_1{\xi_3}^7 + {\xi_3}^8)$$

$$+ \frac{8}{3\Delta^5}(-3{\xi_1}^7 - 30{\xi_1}^6\xi_2 + 108{\xi_1}^5{\xi_2}^2 - 75{\xi_1}^4{\xi_2}^3 + {\xi_1}^6\xi_3$$

$$+ 30{\xi_1}^5\xi_2\xi_3 - 129{\xi_1}^4{\xi_2}^2\xi_3 + 98{\xi_1}^3{\xi_2}^3\xi_3 + 15{\xi_1}^5{\xi_3}^2$$

$$+ 21{\xi_1}^4\xi_2{\xi_3}^2 - 36{\xi_1}^3{\xi_2}^2{\xi_3}^2 - 12{\xi_1}^4{\xi_3}^3 - 12{\xi_1}^3\xi_2{\xi_3}^3$$

$$+ 24{\xi_1}^2{\xi_2}^2{\xi_3}^3 - 17{\xi_1}^3{\xi_3}^4 - 12{\xi_1}^2\xi_2{\xi_3}^4 + 21{\xi_1}^2{\xi_3}^5$$

$$+ 3\xi_1\xi_2{\xi_3}^5 - 3\xi_1{\xi_3}^6 - {\xi_3}^7)\Big]$$

$$- \left(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2}\right)\frac{4}{3\Delta^4\xi_1\xi_2\xi_3}(-6{\xi_1}^7 + 30{\xi_1}^6\xi_2 - 54{\xi_1}^5{\xi_2}^2$$

$$+ 30{\xi_1}^4{\xi_2}^3 + 38{\xi_1}^6\xi_3 + 120{\xi_1}^5\xi_2\xi_3 + 258{\xi_1}^4{\xi_2}^2\xi_3$$

$$- 416\xi_1{}^3\xi_2{}^3\xi_3 - 66\xi_1{}^5\xi_3{}^2 - 222\xi_1{}^4\xi_2\xi_3{}^2 + 288\xi_1{}^3\xi_2{}^2\xi_3{}^2$$

$$+ 6\xi_1{}^4\xi_3{}^3 - 12\xi_1{}^3\xi_2\xi_3{}^3 - 174\xi_1{}^2\xi_2{}^2\xi_3{}^3 + 86\xi_1{}^3\xi_3{}^4$$

$$+ 78\xi_1{}^2\xi_2\xi_3{}^4 - 78\xi_1{}^2\xi_3{}^5 - 6\xi_1\xi_2\xi_3{}^5 + 18\xi_1\xi_3{}^6 + \xi_3{}^7)$$

$$+ \left(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2} + \frac{\xi_1+\xi_2+\xi_3}{24}\right)\frac{16}{\Delta^3\xi_1\xi_2\xi_3}(\xi_1{}^4 - 4\xi_1{}^3\xi_2 + 3\xi_1{}^2\xi_2{}^2$$

$$- 7\xi_1{}^3\xi_3 - 23\xi_1{}^2\xi_2\xi_3 + 9\xi_1{}^2\xi_3{}^2 + 11\xi_1\xi_2\xi_3{}^2 - \xi_1\xi_3{}^3 - \xi_3{}^4)$$

$$+ f(\xi_1)\frac{8\xi_1\xi_2}{3\Delta^6}(-\xi_1+\xi_2-\xi_3)^2(-\xi_1+\xi_2+\xi_3)\times$$

$$\times(\xi_1{}^4 + 2\xi_1{}^3\xi_2 - 6\xi_1{}^2\xi_2{}^2 + 2\xi_1\xi_2{}^3 + \xi_2{}^4 - \xi_1{}^3\xi_3 + \xi_1{}^2\xi_2\xi_3$$

$$+ \xi_1\xi_2{}^2\xi_3 - \xi_2{}^3\xi_3 - 2\xi_1\xi_2\xi_3{}^2 - \xi_1\xi_3{}^3 - \xi_2\xi_3{}^3 + \xi_3{}^4)$$

$$- f(\xi_3)\frac{1}{24\Delta^6\xi_1\xi_2}(-2\xi_1{}^{11} + 26\xi_1{}^{10}\xi_2 - 222\xi_1{}^9\xi_2{}^2 + 790\xi_1{}^8\xi_2{}^3$$

$$- 1268\xi_1{}^7\xi_2{}^4 + 676\xi_1{}^6\xi_2{}^5 + 22\xi_1{}^{10}\xi_3 - 212\xi_1{}^9\xi_2\xi_3$$

$$+ 894\xi_1{}^8\xi_2{}^2\xi_3 - 2224\xi_1{}^7\xi_2{}^3\xi_3 + 3692\xi_1{}^6\xi_2{}^4\xi_3 - 2172\xi_1{}^5\xi_2{}^5\xi_3$$

$$- 110\xi_1{}^9\xi_3{}^2 + 738\xi_1{}^8\xi_2\xi_3{}^2 - 1816\xi_1{}^7\xi_2{}^2\xi_3{}^2 + 2120\xi_1{}^6\xi_2{}^3\xi_3{}^2$$

$$- 932\xi_1{}^5\xi_2{}^4\xi_3{}^2 + 330\xi_1{}^8\xi_3{}^3 - 1392\xi_1{}^7\xi_2\xi_3{}^3 + 1912\xi_1{}^6\xi_2{}^2\xi_3{}^3$$

$$- 400\xi_1{}^5\xi_2{}^3\xi_3{}^3 - 450\xi_1{}^4\xi_2{}^4\xi_3{}^3 - 660\xi_1{}^7\xi_3{}^4 + 1428\xi_1{}^6\xi_2\xi_3{}^4$$

$$- 484\xi_1{}^5\xi_2{}^2\xi_3{}^4 - 284\xi_1{}^4\xi_2{}^3\xi_3{}^4 + 924\xi_1{}^6\xi_3{}^5 - 504\xi_1{}^5\xi_2\xi_3{}^5$$

$$- 380\xi_1{}^4\xi_2{}^2\xi_3{}^5 - 40\xi_1{}^3\xi_2{}^3\xi_3{}^5 - 924\xi_1{}^5\xi_3{}^6 - 588\xi_1{}^4\xi_2\xi_3{}^6$$

$$- 504\xi_1{}^3\xi_2{}^2\xi_3{}^6 + 660\xi_1{}^4\xi_3{}^7 + 912\xi_1{}^3\xi_2\xi_3{}^7 + 524\xi_1{}^2\xi_2{}^2\xi_3{}^7$$

$$- 330\xi_1{}^3\xi_3{}^8 - 558\xi_1{}^2\xi_2\xi_3{}^8 + 110\xi_1{}^2\xi_3{}^9$$

$$+ 86\xi_1\xi_2\xi_3{}^9 - 22\xi_1\xi_3{}^{10} + \xi_3{}^{11})$$

$$- \left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{16\xi_1}{3\Delta^5}(3\xi_1{}^6 + 32\xi_1{}^5\xi_2 - 76\xi_1{}^4\xi_2{}^2 + 8\xi_1{}^3\xi_2{}^3 + 67\xi_1{}^2\xi_2{}^4$$

$$- 32\xi_1\xi_2{}^5 - 2\xi_2{}^6 + 2\xi_1{}^5\xi_3 + 3\xi_1{}^4\xi_2\xi_3 + 51\xi_1{}^3\xi_2{}^2\xi_3 - 119\xi_1{}^2\xi_2{}^3\xi_3$$

$$+ 63\xi_1\xi_2{}^4\xi_3 - 13\xi_1{}^4\xi_3{}^2 - 30\xi_1{}^3\xi_2\xi_3{}^2 + 53\xi_1{}^2\xi_2{}^2\xi_3{}^2 - 28\xi_1\xi_2{}^3\xi_3{}^2$$

$$+ 18\xi_2{}^4\xi_3{}^2 - \xi_1{}^3\xi_3{}^3 - 17\xi_1{}^2\xi_2\xi_3{}^3 - 10\xi_1\xi_2{}^2\xi_3{}^3 - 32\xi_2{}^3\xi_3{}^3$$

$$+ 16\xi_1{}^2\xi_3{}^4 + 12\xi_1\xi_2\xi_3{}^4 + 18\xi_2{}^2\xi_3{}^4 - 5\xi_1\xi_3{}^5 - 2\xi_3{}^6)$$

$$- \left(\frac{f(\xi_3)-1}{\xi_3}\right)\frac{1}{3\Delta^5\xi_1\xi_2}(-2\xi_1{}^9 + 20\xi_1{}^8\xi_2 - 70\xi_1{}^7\xi_2{}^2$$

$$+ 110\xi_1{}^6\xi_2{}^3 - 58\xi_1{}^5\xi_2{}^4 + 20\xi_1{}^8\xi_3 - 194\xi_1{}^7\xi_2\xi_3 + 44\xi_1{}^6\xi_2{}^2\xi_3$$

$$+ 1250\xi_1{}^5\xi_2{}^3\xi_3 - 1120\xi_1{}^4\xi_2{}^4\xi_3 - 88\xi_1{}^7\xi_3{}^2 + 390\xi_1{}^6\xi_2\xi_3{}^2$$

$$- 642\xi_1{}^5\xi_2{}^2\xi_3{}^2 + 340\xi_1{}^4\xi_2{}^3\xi_3{}^2 + 224\xi_1{}^6\xi_3{}^3 - 378\xi_1{}^5\xi_2\xi_3{}^3$$

$$+ 552\xi_1{}^4\xi_2{}^2\xi_3{}^3 - 398\xi_1{}^3\xi_2{}^3\xi_3{}^3 - 364\xi_1{}^5\xi_3{}^4 + 326\xi_1{}^4\xi_2\xi_3{}^4$$

$$+ 38\xi_1{}^3\xi_2{}^2\xi_3{}^4 + 392\xi_1{}^4\xi_3{}^5 - 86\xi_1{}^3\xi_2\xi_3{}^5 + 102\xi_1{}^2\xi_2{}^2\xi_3{}^5$$

$$- 280\xi_1{}^3\xi_3{}^6 - 254\xi_1{}^2\xi_2\xi_3{}^6 + 128\xi_1{}^2\xi_3{}^7 + 105\xi_1\xi_2\xi_3{}^7 - 34\xi_1\xi_3{}^8 + 2\xi_3{}^9)$$

$$+ \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2}\right) \frac{4\xi_1}{\Delta^4\xi_2\xi_3}(5\xi_1{}^6 - 22\xi_1{}^5\xi_2 + 35\xi_1{}^4\xi_2{}^2 - 20\xi_1{}^3\xi_2{}^3$$

$$- 5\xi_1{}^2\xi_2{}^4 + 10\xi_1\xi_2{}^5 - 3\xi_2{}^6 - 27\xi_1{}^5\xi_3 - 97\xi_1{}^4\xi_2\xi_3$$

$$- 166\xi_1{}^3\xi_2{}^2\xi_3 + 222\xi_1{}^2\xi_2{}^3\xi_3 + 49\xi_1\xi_2{}^4\xi_3 + 19\xi_2{}^5\xi_3 + 36\xi_1{}^4\xi_3{}^2$$

$$+ 80\xi_1{}^3\xi_2\xi_3{}^2 - 192\xi_1{}^2\xi_2{}^2\xi_3{}^2 - 120\xi_1\xi_2{}^3\xi_3{}^2 - 44\xi_2{}^4\xi_3{}^2$$

$$+ 2\xi_1{}^3\xi_3{}^3 + 50\xi_1{}^2\xi_2\xi_3{}^3 + 62\xi_1\xi_2{}^2\xi_3{}^3 + 46\xi_2{}^3\xi_3{}^3$$

$$- 27\xi_1{}^2\xi_3{}^4 - 10\xi_1\xi_2\xi_3{}^4 - 19\xi_2{}^2\xi_3{}^4 + 9\xi_1\xi_3{}^5 - \xi_2\xi_3{}^5 + 2\xi_3{}^6)$$

$$- \left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2}\right) \frac{1}{2\Delta^4\xi_1\xi_2}(-6\xi_1{}^7 + 30\xi_1{}^6\xi_2 - 54\xi_1{}^5\xi_2{}^2$$

$$+ 30\xi_1{}^4\xi_2{}^3 + 74\xi_1{}^6\xi_3 - 204\xi_1{}^5\xi_2\xi_3 + 150\xi_1{}^4\xi_2{}^2\xi_3 - 20\xi_1{}^3\xi_2{}^3\xi_3$$

$$- 294\xi_1{}^5\xi_3{}^2 - 558\xi_1{}^4\xi_2\xi_3{}^2 + 852\xi_1{}^3\xi_2{}^2\xi_3{}^2 + 378\xi_1{}^4\xi_3{}^3$$

$$+ 728\xi_1{}^3\xi_2\xi_3{}^3 - 1106\xi_1{}^2\xi_2{}^2\xi_3{}^3 - 34\xi_1{}^3\xi_3{}^4 - 382\xi_1{}^2\xi_2\xi_3{}^4$$

$$- 210\xi_1{}^2\xi_3{}^5 + 154\xi_1\xi_2\xi_3{}^5 + 78\xi_1\xi_3{}^6 + 7\xi_3{}^7), \qquad \text{(B.27)}$$

$$F_{28}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\frac{16\xi_3}{\Delta^4}(-2\xi_1{}^4 + 2\xi_1{}^2\xi_2{}^2$$

$$+ 4\xi_1{}^3\xi_3 - 4\xi_1{}^2\xi_2\xi_3 + 4\xi_1\xi_2\xi_3{}^2 - 4\xi_1\xi_3{}^3 + \xi_3{}^4)$$

$$+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right) \frac{16}{\Delta^3\xi_1\xi_2}(-4\xi_1{}^4 - 4\xi_1{}^3\xi_2 + 8\xi_1{}^2\xi_2{}^2 + 10\xi_1{}^3\xi_3$$

$$- 18\xi_1{}^2\xi_2\xi_3 - 6\xi_1{}^2\xi_3{}^2 + 12\xi_1\xi_2\xi_3{}^2 - 2\xi_1\xi_3{}^3 + \xi_3{}^4)$$

$$+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24}\right) \frac{64}{\Delta^2\xi_1\xi_2\xi_3}(-\xi_1{}^2 + \xi_1\xi_2 - \xi_1\xi_3 + \xi_3{}^2)$$

$$+ \left(\frac{f(\xi_1) - 1}{\xi_1}\right) \frac{64\xi_1\xi_3}{\Delta^4}(-\xi_1 + \xi_2 - \xi_3)(\xi_1{}^2 + 2\xi_1\xi_2 + \xi_2{}^2 - 2\xi_1\xi_3$$

$$- 2\xi_2\xi_3 + \xi_3{}^2)$$

$$- \left(\frac{f(\xi_3) - 1}{\xi_3} \right) \frac{32{\xi_3}^2}{\Delta^4} (-2{\xi_1}^3 + 2{\xi_1}^2\xi_2 + 2{\xi_1}^2\xi_3 - 2\xi_1\xi_2\xi_3 + 2\xi_1{\xi_3}^2 - {\xi_3}^3)$$

$$+ \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} \right) \frac{16\xi_1}{\Delta^3\xi_2\xi_3} ({\xi_1}^4 - 4{\xi_1}^3\xi_2 + 6{\xi_1}^2{\xi_2}^2 - 4\xi_1{\xi_2}^3$$

$$+ {\xi_2}^4 - 12{\xi_1}^3\xi_3 - 16{\xi_1}^2\xi_2\xi_3 + 28\xi_1{\xi_2}^2\xi_3 + 18{\xi_1}^2{\xi_3}^2 - 20\xi_1\xi_2{\xi_3}^2$$

$$- 6{\xi_2}^2{\xi_3}^2 - 4\xi_1{\xi_3}^3 + 8\xi_2{\xi_3}^3 - 3{\xi_3}^4)$$

$$- \left(\frac{f(\xi_3) - 1 + \frac{1}{6}\xi_3}{\xi_3^2} \right) \frac{32{\xi_3}^2}{\Delta^3\xi_1\xi_2} (-4{\xi_1}^3 + 4{\xi_1}^2\xi_2 + 6{\xi_1}^2\xi_3$$

$$- 10\xi_1\xi_2\xi_3 - {\xi_3}^3), \tag{B.28}$$

$$F_{29}(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3) \Big[\frac{8\xi_1\xi_2\xi_3}{3\Delta^6} (6{\xi_1}^5\xi_2 + 3{\xi_1}^4{\xi_2}^2 - 12{\xi_1}^3{\xi_2}^3 + 6{\xi_1}^5\xi_3$$

$$+ 12{\xi_1}^3{\xi_2}^2\xi_3 + 3{\xi_1}^4{\xi_3}^2 + 12{\xi_1}^3\xi_2{\xi_3}^2 - 10{\xi_1}^2{\xi_2}^2{\xi_3}^2 - 18\xi_1\xi_2{\xi_3}^4 - 3{\xi_3}^6)$$

$$+ \frac{16}{\Delta^5} (3{\xi_1}^5\xi_2 + 6{\xi_1}^4{\xi_2}^2 - 14{\xi_1}^3{\xi_2}^3 + 3{\xi_1}^5\xi_3 + 18{\xi_1}^3{\xi_2}^2\xi_3$$

$$+ 6{\xi_1}^4{\xi_3}^2 + 18{\xi_1}^3\xi_2{\xi_3}^2 - 16{\xi_1}^2{\xi_2}^2{\xi_3}^2 - 21\xi_1\xi_2{\xi_3}^4 - 2{\xi_3}^6) \Big]$$

$$+ \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \right) \frac{16}{\Delta^4\xi_1\xi_2\xi_3} (9{\xi_1}^4{\xi_2}^2 - 16{\xi_1}^3{\xi_2}^3 + 30{\xi_1}^3{\xi_2}^2\xi_3$$

$$+ 9{\xi_1}^4{\xi_3}^2 + 30{\xi_1}^3\xi_2{\xi_3}^2 - 34{\xi_1}^2{\xi_2}^2{\xi_3}^2 - 30\xi_1\xi_2{\xi_3}^4 - {\xi_3}^6)$$

$$- \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2} + \frac{\xi_1 + \xi_2 + \xi_3}{24} \right) \frac{32}{\Delta^3\xi_1\xi_2\xi_3} (-3{\xi_1}^2\xi_2 - 3{\xi_1}^2\xi_3$$

$$+ 4\xi_1\xi_2\xi_3 + 3{\xi_3}^3)$$

$$+ f(\xi_1) \frac{16\xi_1\xi_2\xi_3}{\Delta^6} (-{\xi_1}^5 + {\xi_1}^4\xi_2 + 2{\xi_1}^3{\xi_2}^2 - 2{\xi_1}^2{\xi_2}^3 - \xi_1{\xi_2}^4$$

$$+ {\xi_2}^5 + {\xi_1}^4\xi_3 - 4{\xi_1}^3\xi_2\xi_3 + 2{\xi_1}^2{\xi_2}^2\xi_3 + 4\xi_1{\xi_2}^3\xi_3 - 3{\xi_2}^4\xi_3$$

$$+ 2{\xi_1}^3{\xi_3}^2 + 2{\xi_1}^2\xi_2{\xi_3}^2 - 6\xi_1{\xi_2}^2{\xi_3}^2 + 2{\xi_2}^3{\xi_3}^2$$

$$- 2{\xi_1}^2{\xi_3}^3 + 4\xi_1\xi_2{\xi_3}^3 + 2{\xi_2}^2{\xi_3}^3 - \xi_1{\xi_3}^4 - 3\xi_2{\xi_3}^4 + {\xi_3}^5)$$

$$+ \left(\frac{f(\xi_1) - 1}{\xi_1} \right) \frac{32\xi_1}{\Delta^5} (-2{\xi_1}^5 + {\xi_1}^4\xi_2 + 7{\xi_1}^3{\xi_2}^2 - 7{\xi_1}^2{\xi_2}^3$$

$$- \xi_1{\xi_2}^4 + 2{\xi_2}^5 + {\xi_1}^4\xi_3 - 18{\xi_1}^3\xi_2\xi_3 + 7{\xi_1}^2{\xi_2}^2\xi_3 + 16\xi_1{\xi_2}^3\xi_3$$

$$- 6{\xi_2}^4\xi_3 + 7{\xi_1}^3{\xi_3}^2 + 7{\xi_1}^2\xi_2{\xi_3}^2 - 30\xi_1{\xi_2}^2{\xi_3}^2 + 4{\xi_2}^3{\xi_3}^2$$

$$- 7{\xi_1}^2{\xi_3}^3 + 16\xi_1\xi_2{\xi_3}^3 + 4{\xi_2}^2{\xi_3}^3 - \xi_1{\xi_3}^4 - 6\xi_2{\xi_3}^4 + 2{\xi_3}^5)$$

$$+ \left(\frac{f(\xi_1) - 1 + \frac{1}{6}\xi_1}{\xi_1^2} \right) \frac{24\xi_1}{\Delta^4\xi_2\xi_3} (-{\xi_1}^5 - 3{\xi_1}^4\xi_2 + 14{\xi_1}^3{\xi_2}^2$$

$$- 14{\xi_1}^2{\xi_2}^3 + 3\xi_1{\xi_2}^4 + {\xi_2}^5 - 3{\xi_1}^4\xi_3 - 36{\xi_1}^3\xi_2\xi_3 + 22{\xi_1}^2{\xi_2}^2\xi_3$$

$$+ 20\xi_1{\xi_2}^3\xi_3 - 3{\xi_2}^4\xi_3 + 14{\xi_1}^3{\xi_3}^2 + 22{\xi_1}^2\xi_2{\xi_3}^2 - 46\xi_1{\xi_2}^2{\xi_3}^2$$

$$+ 2\xi_2{}^3\xi_3{}^2 - 14\xi_1{}^2\xi_3{}^3 + 20\xi_1\xi_2\xi_3{}^3 + 2\xi_2{}^2\xi_3{}^3$$
$$+ 3\xi_1\xi_3{}^4 - 3\xi_2\xi_3{}^4 + \xi_3{}^5), \tag{B.29}$$

$$\begin{aligned}
F_{30} = {} & F(\xi_1, \xi_2, \xi_3)\Big[\frac{1}{3\Delta^4}(-\xi_3 - \xi_2 + \xi_1)(\xi_1{}^6 - 5\xi_1{}^5\xi_2 + 8\xi_1{}^3\xi_2{}^3 \\
& - 13\xi_1{}^2\xi_2{}^4 + 5\xi_3{}^5\xi_1 - 6\xi_3{}^5\xi_2 + 24\xi_3{}^4\xi_2{}^2 - 36\xi_2{}^3\xi_3{}^3 + 24\xi_2{}^4\xi_3{}^2 \\
& + 8\xi_1{}^3\xi_3{}^3 + 4\xi_1{}^4\xi_3{}^2 - 13\xi_1{}^2\xi_3{}^4 - 5\xi_1{}^5\xi_3 + 4\xi_1{}^4\xi_2{}^2 - 12\xi_1{}^3\xi_3{}^2\xi_2 \\
& + 32\xi_1{}^2\xi_3{}^3\xi_2 - 38\xi_1{}^2\xi_3{}^2\xi_2{}^2 - 6\xi_1{}^4\xi_3\xi_2 - 12\xi_1{}^3\xi_3\xi_2{}^2 + 32\xi_1{}^2\xi_2{}^3\xi_3 \\
& - 3\xi_3{}^4\xi_1\xi_2 - 2\xi_1\xi_3{}^3\xi_2{}^2 - 2\xi_1\xi_2{}^3\xi_3{}^2 - 3\xi_1\xi_2{}^4\xi_3 - 6\xi_2{}^5\xi_3 + 5\xi_1\xi_2{}^5) \\
& - \frac{2}{3\xi_1\Delta^3}(84\xi_3{}^2\xi_1\xi_2{}^2 + 17\xi_1{}^2\xi_3\xi_2{}^2 - 68\xi_2{}^3\xi_3\xi_1 - 63\xi_1{}^3\xi_2{}^2 + 7\xi_1{}^2\xi_2{}^3 \\
& + 26\xi_1\xi_2{}^4 - 68\xi_3{}^3\xi_1\xi_2 + 17\xi_3{}^2\xi_1{}^2\xi_2 + 26\xi_1{}^3\xi_2\xi_3 + 31\xi_1{}^4\xi_2 - 12\xi_3{}^3\xi_2{}^2 \\
& - 12\xi_3{}^2\xi_2{}^3 + 18\xi_2{}^4\xi_3 - 6\xi_2{}^5 + 18\xi_3{}^4\xi_2 + 7\xi_3{}^3\xi_1{}^2 - 63\xi_3{}^2\xi_1{}^3 + 31\xi_1{}^4\xi_3 \\
& + 5\xi_1{}^5 + 26\xi_3{}^4\xi_1 - 6\xi_3{}^5)\Big] \\
& + \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right)\frac{4}{\xi_1\xi_2\xi_3\Delta}(\xi_1{}^2 + 2\xi_2\xi_3) \\
& + f(\xi_1)\frac{2}{3\Delta^4}(\xi_1{}^6 - 5\xi_1{}^5\xi_2 + 8\xi_1{}^3\xi_2{}^3 - 13\xi_1{}^2\xi_2{}^4 + 5\xi_3{}^5\xi_1 - 6\xi_3{}^5\xi_2 \\
& + 24\xi_3{}^4\xi_2{}^2 - 36\xi_2{}^3\xi_3{}^3 + 24\xi_2{}^4\xi_3{}^2 + 8\xi_1{}^3\xi_3{}^3 + 4\xi_1{}^4\xi_3{}^2 - 13\xi_1{}^2\xi_3{}^4 \\
& - 5\xi_1{}^5\xi_3 + 4\xi_1{}^4\xi_2{}^2 - 12\xi_1{}^3\xi_3{}^2\xi_2 + 32\xi_1{}^2\xi_3{}^3\xi_2 - 38\xi_1{}^2\xi_3{}^2\xi_2{}^2 \\
& - 6\xi_1{}^4\xi_3\xi_2 - 12\xi_1{}^3\xi_3\xi_2{}^2 + 32\xi_1{}^2\xi_2{}^3\xi_3 - 3\xi_3{}^4\xi_1\xi_2 - 2\xi_1\xi_3{}^3\xi_2{}^2 \\
& - 2\xi_1\xi_2{}^3\xi_3{}^2 - 3\xi_1\xi_2{}^4\xi_3 - 6\xi_2{}^5\xi_3 + 5\xi_1\xi_2{}^5) \\
& - f(\xi_2)\frac{1}{12\xi_1\Delta^4}(20\xi_2{}^3\xi_3{}^3\xi_1 - 66\xi_2{}^3\xi_3{}^2\xi_1{}^2 + 121\xi_2{}^4\xi_3{}^2\xi_1 + 148\xi_2{}^3\xi_1{}^3\xi_3 \\
& + 81\xi_2{}^4\xi_1{}^2\xi_3 - 74\xi_2{}^5\xi_3\xi_1 - 151\xi_2{}^2\xi_3{}^4\xi_1 + 194\xi_2{}^2\xi_3{}^3\xi_1{}^2 + 34\xi_2{}^2\xi_3{}^2\xi_1{}^3 \\
& - 151\xi_2{}^2\xi_1{}^4\xi_3 - 35\xi_2{}^3\xi_3{}^4 + 35\xi_2{}^4\xi_3{}^3 - 21\xi_2{}^5\xi_3{}^2 + 29\xi_2{}^3\xi_1{}^4 - 69\xi_2{}^4\xi_1{}^3 \\
& + 19\xi_2{}^5\xi_1{}^2 + 7\xi_2{}^6\xi_3 + 21\xi_2{}^2\xi_3{}^5 + 53\xi_2{}^2\xi_1{}^5 - 47\xi_1{}^6\xi_2 + 91\xi_1{}^3\xi_3{}^4 \\
& - 189\xi_1{}^4\xi_3{}^3 + 165\xi_1{}^5\xi_3{}^2 - 9\xi_1\xi_3{}^6 - 65\xi_1{}^6\xi_3 - 225\xi_1{}^2\xi_2\xi_3{}^4 \\
& + 180\xi_1{}^3\xi_2\xi_3{}^3 - 41\xi_1{}^4\xi_2\xi_3{}^2 + 54\xi_1{}^5\xi_2\xi_3 + 86\xi_1\xi_2\xi_3{}^5 \\
& - 3\xi_1{}^2\xi_3{}^5 - 7\xi_3{}^6\xi_2 + \xi_3{}^7 + 9\xi_1{}^7 + 7\xi_2{}^6\xi_1 - \xi_2{}^7) \\
& - f(\xi_3)\frac{1}{12\xi_1\Delta^4}(20\xi_2{}^3\xi_3{}^3\xi_1 + 194\xi_2{}^3\xi_3{}^2\xi_1{}^2 - 151\xi_2{}^4\xi_3{}^2\xi_1 + 180\xi_2{}^3\xi_1{}^3\xi_3
\end{aligned}$$

$$
\begin{aligned}
&- 225\xi_2{}^4\xi_1{}^2\xi_3 + 86\xi_2{}^5\xi_3\xi_1 + 121\xi_2{}^2\xi_3{}^4\xi_1 - 66\xi_2{}^2\xi_3{}^3\xi_1{}^2 + 34\xi_2{}^2\xi_3{}^2\xi_1{}^3 \\
&- 41\xi_2{}^2\xi_1{}^4\xi_3 + 35\xi_2{}^3\xi_3{}^4 - 35\xi_2{}^4\xi_3{}^3 + 21\xi_2{}^5\xi_3{}^2 - 189\xi_2{}^3\xi_1{}^4 \\
&+ 91\xi_2{}^4\xi_1{}^3 - 3\xi_2{}^5\xi_1{}^2 - 7\xi_2{}^6\xi_3 - 21\xi_2{}^2\xi_3{}^5 + 165\xi_2{}^2\xi_1{}^5 - 65\xi_1{}^6\xi_2 \\
&- 69\xi_1{}^3\xi_3{}^4 + 29\xi_1{}^4\xi_3{}^3 + 53\xi_1{}^5\xi_3{}^2 + 7\xi_1\xi_3{}^6 - 47\xi_1{}^6\xi_3 + 81\xi_1{}^2\xi_2\xi_3{}^4 \\
&+ 148\xi_1{}^3\xi_2\xi_3{}^3 - 151\xi_1{}^4\xi_2\xi_3{}^2 + 54\xi_1{}^5\xi_2\xi_3 - 74\xi_1\xi_2\xi_3{}^5 \\
&+ 19\xi_1{}^2\xi_3{}^5 + 7\xi_3{}^6\xi_2 - \xi_3{}^7 + 9\xi_1{}^7 - 9\xi_2{}^6\xi_1 + \xi_2{}^7) \\
&+ \left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{1}{\xi_2\xi_3\Delta^3}(\xi_1{}^5\xi_2 + 6\xi_1{}^3\xi_2{}^3 - 4\xi_1{}^2\xi_2{}^4 + \xi_3{}^5\xi_1 - 6\xi_3{}^5\xi_2 \\
&+ 24\xi_3{}^4\xi_2{}^2 - 36\xi_2{}^3\xi_3{}^3 + 24\xi_2{}^4\xi_3{}^2 + 6\xi_1{}^3\xi_3{}^3 - 4\xi_1{}^4\xi_3{}^2 - 4\xi_1{}^2\xi_3{}^4 + \xi_1{}^5\xi_3 \\
&- 4\xi_1{}^4\xi_2{}^2 - 46\xi_1{}^3\xi_3{}^2\xi_2 + 60\xi_1{}^2\xi_3{}^3\xi_2 - 80\xi_1{}^2\xi_3{}^2\xi_2{}^2 - 14\xi_1{}^4\xi_3\xi_2 \\
&- 46\xi_1{}^3\xi_3\xi_2{}^2 + 60\xi_1{}^2\xi_2{}^3\xi_3 + 5\xi_3{}^4\xi_1\xi_2 - 6\xi_1\xi_3{}^3\xi_2{}^2 - 6\xi_1\xi_2{}^3\xi_3{}^2 \\
&+ 5\xi_1\xi_2{}^4\xi_3 - 6\xi_2{}^5\xi_3 + \xi_1\xi_2{}^5) \\
&+ \left(\frac{f(\xi_2)-1}{\xi_2}\right)\frac{1}{2\xi_1\xi_3\Delta^3}(-\xi_3{}^6 + 2\xi_1{}^5\xi_2 + 12\xi_1{}^3\xi_2{}^3 - 8\xi_1{}^2\xi_2{}^4 + 3\xi_3{}^5\xi_1 \\
&- 7\xi_3{}^5\xi_2 + 6\xi_3{}^4\xi_2{}^2 + 34\xi_2{}^3\xi_3{}^3 - 53\xi_2{}^4\xi_3{}^2 - 2\xi_1{}^3\xi_3{}^3 + 3\xi_1{}^4\xi_3{}^2 - 2\xi_1{}^2\xi_3{}^4 \\
&- \xi_1{}^5\xi_3 - 8\xi_1{}^4\xi_2{}^2 + 32\xi_1{}^3\xi_3{}^2\xi_2 - 106\xi_1{}^2\xi_3{}^3\xi_2 + 66\xi_1{}^2\xi_3{}^2\xi_2{}^2 + 9\xi_1{}^4\xi_3\xi_2 \\
&+ 70\xi_1{}^3\xi_3\xi_2{}^2 - 14\xi_1{}^2\xi_2{}^3\xi_3 + 70\xi_3{}^4\xi_1\xi_2 - 102\xi_1\xi_3{}^3\xi_2{}^2 + 112\xi_1\xi_2{}^3\xi_3{}^2 \\
&- 85\xi_1\xi_2{}^4\xi_3 + 21\xi_2{}^5\xi_3 + 2\xi_1\xi_2{}^5) \\
&- \left(\frac{f(\xi_3)-1}{\xi_3}\right)\frac{1}{2\xi_1\xi_2\Delta^3}(\xi_1{}^5\xi_2 + 2\xi_1{}^3\xi_2{}^3 + 2\xi_1{}^2\xi_2{}^4 - 2\xi_3{}^5\xi_1 \\
&- 21\xi_3{}^5\xi_2 + 53\xi_3{}^4\xi_2{}^2 - 34\xi_2{}^3\xi_3{}^3 - 6\xi_2{}^4\xi_3{}^2 - 12\xi_1{}^3\xi_3{}^3 + 8\xi_1{}^4\xi_3{}^2 \\
&+ 8\xi_1{}^2\xi_3{}^4 - 2\xi_1{}^5\xi_3 - 3\xi_1{}^4\xi_2{}^2 - 70\xi_1{}^3\xi_3{}^2\xi_2 + 14\xi_1{}^2\xi_3{}^3\xi_2 - 66\xi_1{}^2\xi_3{}^2\xi_2{}^2 \\
&- 9\xi_1{}^4\xi_3\xi_2 - 32\xi_1{}^3\xi_3\xi_2{}^2 + 106\xi_1{}^2\xi_2{}^3\xi_3 + 85\xi_3{}^4\xi_1\xi_2 - 112\xi_1\xi_3{}^3\xi_2{}^2 \\
&+ 102\xi_1\xi_2{}^3\xi_3{}^2 - 70\xi_1\xi_2{}^4\xi_3 + 7\xi_2{}^5\xi_3 - 3\xi_1\xi_2{}^5 + \xi_2{}^6) \\
&+ \left(\frac{f(\xi_1)-1+\frac{1}{6}\xi_1}{\xi_1^2}\right)\frac{4\xi_1{}^2}{\xi_2\xi_3\Delta^2}(-\xi_3{}^2 + 2\xi_2\xi_3 - 2\xi_1\xi_3 - \xi_2{}^2 - 2\xi_1\xi_2 + 3\xi_1{}^2) \\
&- \left(\frac{f(\xi_2)-1+\frac{1}{6}\xi_2}{\xi_2^2}\right)\frac{4\xi_2}{\xi_3\Delta^2}(\xi_3{}^2 - 2\xi_1\xi_3 - 3\xi_2{}^2 + \xi_1{}^2 + 2\xi_1\xi_2 + 2\xi_2\xi_3) \\
&- \left(\frac{f(\xi_3)-1+\frac{1}{6}\xi_3}{\xi_3^2}\right)\frac{4\xi_3}{\xi_2\Delta^2}(-2\xi_1\xi_2 + 2\xi_1\xi_3 + 2\xi_2\xi_3 + \xi_1{}^2 + \xi_2{}^2 - 3\xi_3{}^2). \qquad \text{(B.30)}
\end{aligned}
$$

Appendix C

The explicit representation for the second order form factors of the heat kernel

Like the third order form factors of the trace of the heat kernel in Appendix B, the second order form factors of the heat kernel are expressed through basic form factors (2.130), (2.131) and Δ (2.143):

$$G_1(\xi_1,\xi_2,\xi_3) = F(\xi_1,\xi_2,\xi_3), \tag{C.1}$$

$$\begin{aligned} G_2(\xi_1,\xi_2,\xi_3) = {} & F(\xi_1,\xi_2,\xi_3)\Big[\frac{2\xi_1\xi_2}{\Delta^2}(\xi_3+\xi_2-\xi_1)(\xi_3+\xi_1-\xi_2) \\ & + \frac{2}{\Delta}(\xi_3-\xi_2-\xi_1)\Big] - f(\xi_1)\frac{4\xi_1\xi_2}{\Delta^2}(\xi_3+\xi_1-\xi_2) \\ & - f(\xi_2)\frac{4\xi_1\xi_2}{\Delta^2}(\xi_3+\xi_2-\xi_1) \\ & - f(\xi_3)\frac{1}{\Delta^2}(-6\xi_1\xi_2\xi_3-3\xi_1{\xi_3}^2 \\ & - 3\xi_2{\xi_3}^2+3\xi_3{\xi_1}^2+3\xi_3{\xi_2}^2+{\xi_3}^3+\xi_1{\xi_2}^2+\xi_2{\xi_1}^2-{\xi_2}^3-{\xi_1}^3), \end{aligned} \tag{C.2}$$

$$\begin{aligned} G_3(\xi_1,\xi_2,\xi_3) = {} & F(\xi_1,\xi_2,\xi_3)\Big[-\frac{1}{3\Delta^2}(-2\xi_1\xi_3{\xi_2}^2-2\xi_1\xi_2{\xi_3}^2 \\ & + 4{\xi_1}^2\xi_2\xi_3+{\xi_1}^4+{\xi_2}^4+{\xi_3}^4-6{\xi_1}^2{\xi_3}^2+2\xi_3{\xi_1}^3+2\xi_1{\xi_3}^3-6{\xi_1}^2{\xi_2}^2 \\ & + 2\xi_2{\xi_1}^3+2\xi_1{\xi_2}^3+6{\xi_2}^2{\xi_3}^2-4{\xi_2}^3\xi_3-4\xi_2{\xi_3}^3)-\frac{4\xi_1}{\Delta}\Big] \\ & + f(\xi_1)\frac{4\xi_1}{\Delta^2}(2\xi_2\xi_3-{\xi_3}^2+\xi_1\xi_3+\xi_1\xi_2-{\xi_2}^2) \\ & + f(\xi_2)\frac{1}{\Delta^2}(\xi_1+\xi_2-\xi_3)({\xi_1}^2+2\xi_2\xi_3-{\xi_2}^2-{\xi_3}^2) \end{aligned}$$

$$+ f(\xi_3)\frac{1}{\Delta^2}(2\xi_1\xi_2\xi_3 - \xi_1{\xi_3}^2 + \xi_3{\xi_1}^2 + {\xi_2}^3 - \xi_2{\xi_1}^2$$
$$+ {\xi_1}^3 - \xi_1{\xi_2}^2 + 3\xi_2{\xi_3}^2 - 3\xi_3{\xi_2}^2 - {\xi_3}^3), \qquad \text{(C.3)}$$

$$G_4(\xi_1,\xi_2,\xi_3) = F(\xi_1,\xi_2,\xi_3)\Big[\frac{1}{36\Delta^4}({\xi_3}^8 - 4\xi_1{\xi_3}^7$$
$$- 16{\xi_1}^2{\xi_3}^6 + 68{\xi_1}^3{\xi_3}^5 - 100{\xi_1}^4{\xi_3}^4 + 68{\xi_1}^5{\xi_3}^3 - 16{\xi_1}^6{\xi_3}^2$$
$$- 4{\xi_1}^7\xi_3 + 2{\xi_1}^8 + 16\xi_2\xi_1{\xi_3}^6 - 60\xi_2{\xi_1}^2{\xi_3}^5 + 8\xi_2{\xi_1}^3{\xi_3}^4$$
$$+ 68\xi_2{\xi_1}^4{\xi_3}^3 - 48\xi_2{\xi_3}^2{\xi_1}^5 - 4\xi_2\xi_3{\xi_1}^6 + 8\xi_2{\xi_1}^7 + 96{\xi_1}^2{\xi_2}^2{\xi_3}^4$$
$$- 136{\xi_1}^3{\xi_2}^2{\xi_3}^3 + 36{\xi_1}^5{\xi_2}^2\xi_3 - 16{\xi_1}^6{\xi_2}^2 + 64{\xi_1}^3{\xi_2}^3{\xi_3}^2 - 28\xi_3{\xi_1}^4{\xi_2}^3$$
$$- 40{\xi_1}^5{\xi_2}^3 + 46{\xi_1}^4{\xi_2}^4)$$
$$+ \frac{1}{3\Delta^3}(3{\xi_3}^5 - 16\xi_1{\xi_3}^4 + 4{\xi_1}^2{\xi_3}^3$$
$$+ 24{\xi_3}^2{\xi_1}^3 - 26\xi_3{\xi_1}^4 + 8{\xi_1}^5 + 28\xi_1{\xi_3}^3\xi_2 - 52{\xi_3}^2{\xi_1}^2\xi_2$$
$$+ 12{\xi_1}^4\xi_2 + 26{\xi_1}^2\xi_3{\xi_2}^2 - 20{\xi_1}^3{\xi_2}^2)\Big]$$
$$+ \Big(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2}\Big)\frac{2}{\Delta^2}({\xi_3}^2 - 4\xi_1\xi_3 + 2{\xi_1}^2 + 4\xi_1\xi_2)$$
$$+ f(\xi_1)\frac{1}{6\Delta^4}(-{\xi_1}^2{\xi_3}^5 + {\xi_3}^2{\xi_1}^5 + {\xi_1}^6\xi_2$$
$$- \xi_1{\xi_2}^6 - {\xi_2}^7 + {\xi_1}^7 + 10\xi_3{\xi_1}^5\xi_2 + 5{\xi_2}^4{\xi_3}^3 - {\xi_3}^7 - 9{\xi_2}^5{\xi_3}^2$$
$$+ 5\xi_3{\xi_2}^6 + 5\xi_2{\xi_3}^6 + 23{\xi_1}^2{\xi_3}^4\xi_2 + 57\xi_1{\xi_3}^4{\xi_2}^2 + 12{\xi_1}^3{\xi_3}^3\xi_2$$
$$- 29{\xi_3}^2{\xi_1}^4\xi_2 - 20{\xi_1}^3\xi_3{\xi_2}^3 - 6\xi_1\xi_3{\xi_2}^5 + 37\xi_1{\xi_3}^2{\xi_2}^4$$
$$+ 11{\xi_1}^2\xi_3{\xi_2}^4 + 3\xi_1{\xi_3}^6 - 42{\xi_1}^2{\xi_3}^3{\xi_2}^2 - 68\xi_1{\xi_3}^3{\xi_2}^3 - 22\xi_1{\xi_3}^5\xi_2$$
$$+ 14{\xi_1}^2{\xi_3}^2{\xi_2}^3 - 14{\xi_3}^2{\xi_1}^3{\xi_2}^2 + 3\xi_3{\xi_1}^4{\xi_2}^2 - 5{\xi_1}^3{\xi_3}^4 + 5{\xi_1}^4{\xi_3}^3$$
$$- 3\xi_3{\xi_1}^6 + 5{\xi_1}^5{\xi_2}^2 - 9{\xi_3}^5{\xi_2}^2 - 27{\xi_1}^4{\xi_2}^3 + 27{\xi_1}^3{\xi_2}^4$$
$$- 5{\xi_1}^2{\xi_2}^5 + 5{\xi_2}^3{\xi_3}^4)$$
$$- f(\xi_3)\frac{1}{24\Delta^4}(-{\xi_3}^7 - 2\xi_1{\xi_3}^6 + 38{\xi_1}^2{\xi_3}^5 - 90{\xi_1}^3{\xi_3}^4$$
$$+ 90{\xi_1}^4{\xi_3}^3 - 38{\xi_3}^2{\xi_1}^5 + 2\xi_3{\xi_1}^6 + 2{\xi_1}^7 - 10\xi_1{\xi_3}^5\xi_2 + 50{\xi_1}^2{\xi_3}^4\xi_2$$
$$- 24{\xi_1}^3{\xi_3}^3\xi_2 + 2{\xi_3}^2{\xi_1}^4\xi_2 - 20\xi_3{\xi_1}^5\xi_2 + 14{\xi_1}^6\xi_2 - 66{\xi_1}^2{\xi_3}^3{\xi_2}^2$$
$$+ 36{\xi_3}^2{\xi_1}^3{\xi_2}^2 + 62\xi_3{\xi_1}^4{\xi_2}^2 - 54{\xi_1}^5{\xi_2}^2 - 44{\xi_1}^3\xi_3{\xi_2}^3 + 38{\xi_1}^4{\xi_2}^3)$$
$$+ \left(\frac{f(\xi_1) - 1}{\xi_1}\right)\frac{2\xi_1}{\Delta^3}(4{\xi_1}^3\xi_2 + 3{\xi_1}^4 + 2{\xi_1}^2{\xi_2}^2 - 8\xi_1{\xi_2}^3$$

$$- \xi_2{}^4 - 8\xi_3\xi_1{}^3 + 4\xi_3\xi_2{}^3 + 16\xi_1\xi_3\xi_2{}^2 - 6\xi_2{}^2\xi_3{}^2 + 6\xi_1{}^2\xi_3{}^2$$
$$- \xi_3{}^4 - 8\xi_1\xi_2\xi_3{}^2 + 4\xi_2\xi_3{}^3)$$
$$+ \left(\frac{f(\xi_3) - 1}{\xi_3}\right) \frac{1}{4\Delta^3}(7\xi_3{}^5 - 22\xi_1\xi_3{}^4 - 20\xi_1{}^2\xi_3{}^3$$
$$+ 52\xi_3{}^2\xi_1{}^3 - 26\xi_3\xi_1{}^4 + 2\xi_1{}^5 + 36\xi_1\xi_3{}^3\xi_2 - 52\xi_3{}^2\xi_1{}^2\xi_2$$
$$+ 8\xi_3\xi_1{}^3\xi_2 - 6\xi_1{}^4\xi_2 + 18\xi_1{}^2\xi_3\xi_2{}^2 + 4\xi_1{}^3\xi_2{}^2), \tag{C.4}$$

$$G_5(\xi_1, \xi_2, \xi_3) = \left(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\right) \frac{2}{\xi_1\xi_2} - f(\xi_3)\frac{(-2\xi_2 + \xi_3)}{8\xi_1\xi_2}$$
$$- \left(\frac{f(\xi_3) - 1}{\xi_3}\right) \frac{(-2\xi_2 + 5\xi_3)}{4\xi_1\xi_2}, \tag{C.5}$$

$$G_6(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\Big[- \frac{2}{\Delta^2}(-\xi_1 + \xi_2 - \xi_3) \times$$
$$\times (-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3) + \frac{8}{\Delta}\Big]$$
$$+ f(\xi_1)\frac{4}{\Delta^2}(-\xi_1 + \xi_2 - \xi_3)(-\xi_1 - \xi_2 + \xi_3)$$
$$- f(\xi_2)\frac{4}{\Delta^2}(-\xi_1 - \xi_2 + \xi_3)(-\xi_1 + \xi_2 + \xi_3)$$
$$- f(\xi_3)\frac{4}{\Delta^2}(\xi_1{}^2 - 2\xi_1\xi_2 + \xi_2{}^2 - \xi_3{}^2), \tag{C.6}$$

$$G_7(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\frac{2}{\Delta}(-\xi_3 - \xi_1 + \xi_2) - f(\xi_1)\frac{2}{\xi_2\Delta}(-\xi_3 + \xi_2 + \xi_1)$$
$$+ f(\xi_2)\frac{4}{\Delta} - f(\xi_3)\frac{2}{\xi_2\Delta}(\xi_3 + \xi_2 - \xi_1)$$
$$+ \frac{1}{(\xi_1 - \xi_3)}\Big(f(\xi_1) - f(\xi_3)\Big)\frac{2}{\xi_2}, \tag{C.7}$$

$$G_8(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\Big[- \frac{4\xi_2}{\Delta^2}(-\xi_3 - \xi_1 + \xi_2)^2 - \frac{8}{\Delta}\Big]$$
$$+ f(\xi_1)\frac{2}{\xi_2\Delta^2}(-\xi_3 + \xi_2 + \xi_1)(-4\xi_2\xi_3 - 4\xi_1\xi_2 + 3\xi_2{}^2 - 2\xi_1\xi_3 + \xi_1{}^2 + \xi_3{}^2)$$
$$+ f(\xi_2)\frac{8\xi_2}{\Delta^2}(\xi_3 + \xi_1 - \xi_2)$$
$$+ f(\xi_3)\frac{2}{\xi_2\Delta^2}(\xi_3 + \xi_2 - \xi_1)(-4\xi_2\xi_3 - 4\xi_1\xi_2 + 3\xi_2{}^2 - 2\xi_1\xi_3 + \xi_1{}^2 + \xi_3{}^2)$$
$$- \frac{1}{(\xi_1 - \xi_3)}\Big(f(\xi_1) - f(\xi_3)\Big)\frac{2}{\xi_2}, \tag{C.8}$$

$$G_9(\xi_1, \xi_2, \xi_3) = F(\xi_1, \xi_2, \xi_3)\frac{8}{\Delta^2}(-2\xi_1{}^2 + 2\xi_1\xi_2 + \xi_3{}^2)$$
$$- \Big(F(\xi_1, \xi_2, \xi_3) - \frac{1}{2}\Big)\frac{8}{\Delta\xi_1\xi_2}(2\xi_1 - \xi_3)$$

$$- f(\xi_3)\frac{1}{2\xi_1\xi_2} + \frac{f(\xi_1)-1}{\xi_1}\frac{32\xi_1}{\Delta^2}(-\xi_1+\xi_2-\xi_3)$$

$$- \frac{f(\xi_3)-1}{\xi_3}\frac{1}{\Delta^2\xi_1\xi_2}(2{\xi_1}^4 - 8{\xi_1}^3\xi_2 + 6{\xi_1}^2{\xi_2}^2 - 16{\xi_1}^3\xi_3$$

$$+ 16{\xi_1}^2\xi_2\xi_3 + 36{\xi_1}^2{\xi_3}^2 - 20\xi_1\xi_2{\xi_3}^2 - 32\xi_1{\xi_3}^3 + 5{\xi_3}^4), \qquad \text{(C.9)}$$

$$G_{10}(\xi_1,\xi_2,\xi_3) = F(\xi_1,\xi_2,\xi_3)\Big[\frac{2}{3\xi_2\Delta^4}(\xi_3+\xi_1-\xi_2)^2(-2\xi_2\xi_3{\xi_1}^2$$

$$- 2\xi_2\xi_1{\xi_3}^2 + 4\xi_1{\xi_2}^2\xi_3 + {\xi_2}^4 + {\xi_1}^4 + {\xi_3}^4 - 6{\xi_2}^2{\xi_3}^2 + 2\xi_3{\xi_2}^3 + 2\xi_2{\xi_3}^3$$

$$- 6{\xi_1}^2{\xi_2}^2 + 2\xi_2{\xi_1}^3 + 2\xi_1{\xi_2}^3 + 6{\xi_1}^2{\xi_3}^2 - 4{\xi_1}^3\xi_3 - 4\xi_1{\xi_3}^3)$$

$$+ \frac{4}{3\Delta^3}(-14\xi_2\xi_3{\xi_1}^2 + 14\xi_2{\xi_3}^3 + 40\xi_1{\xi_2}^2\xi_3 - 14\xi_2\xi_1{\xi_3}^2$$

$$- 22\xi_3{\xi_2}^3 - 12{\xi_2}^2{\xi_3}^2 - 4{\xi_1}^3\xi_3 - 4\xi_1{\xi_3}^3 + 6{\xi_1}^2{\xi_3}^2 + {\xi_3}^4$$

$$+ 19{\xi_2}^4 - 12{\xi_1}^2{\xi_2}^2 + 14\xi_2{\xi_1}^3 - 22\xi_1{\xi_2}^3 + {\xi_1}^4)\Big]$$

$$+ \left(F(\xi_1,\xi_2,\xi_3) - \frac{1}{2}\right)\frac{48\xi_2}{\Delta^2}$$

$$+ f(\xi_1)\frac{1}{6\xi_2\Delta^4}(\xi_2+\xi_1-\xi_3)(-16\xi_2{\xi_1}^2{\xi_3}^3 - 20\xi_3{\xi_2}^2{\xi_1}^3 + 24\xi_2\xi_3{\xi_1}^4$$

$$+ 8\xi_3{\xi_2}^5 - 64{\xi_2}^3{\xi_3}^3 + 35{\xi_3}^2{\xi_2}^4 + 37{\xi_2}^2{\xi_3}^4 + 8\xi_1{\xi_2}^5 - 64{\xi_1}^3{\xi_2}^3$$

$$+ 37{\xi_2}^2{\xi_1}^4 + 35{\xi_1}^2{\xi_2}^4 - 8\xi_2{\xi_3}^5 - 8\xi_2{\xi_1}^5 - 9{\xi_2}^6 + {\xi_1}^6 + {\xi_3}^6$$

$$- 20{\xi_2}^2\xi_1{\xi_3}^3 - 34{\xi_2}^2{\xi_3}^2{\xi_1}^2 - 6\xi_3\xi_1{\xi_2}^4 - 16\xi_2{\xi_1}^3{\xi_3}^2 + 24\xi_2\xi_1{\xi_3}^4$$

$$- 6{\xi_1}^5\xi_3 - 6\xi_1{\xi_3}^5 - 20{\xi_1}^3{\xi_3}^3 + 15{\xi_1}^4{\xi_3}^2 + 15{\xi_1}^2{\xi_3}^4)$$

$$- f(\xi_2)\frac{4\xi_2}{3\Delta^4}(\xi_3+\xi_1-\xi_2)(-2\xi_2\xi_3{\xi_1}^2 - 2\xi_2\xi_1{\xi_3}^2 + 4\xi_1{\xi_2}^2\xi_3$$

$$+ {\xi_2}^4 + {\xi_1}^4 + {\xi_3}^4 - 6{\xi_2}^2{\xi_3}^2 + 2\xi_3{\xi_2}^3 + 2\xi_2{\xi_3}^3 - 6{\xi_1}^2{\xi_2}^2$$

$$+ 2\xi_2{\xi_1}^3 + 2\xi_1{\xi_2}^3 + 6{\xi_1}^2{\xi_3}^2 - 4{\xi_1}^3\xi_3 - 4\xi_1{\xi_3}^3)$$

$$- f(\xi_3)\frac{1}{6\xi_2\Delta^4}(-41\xi_3{\xi_1}^2{\xi_2}^4 + 34{\xi_3}^2{\xi_1}^2{\xi_2}^3 + 30{\xi_2}^2{\xi_3}^3{\xi_1}^2 + 41{\xi_3}^2\xi_1{\xi_2}^4$$

$$- 44\xi_1{\xi_2}^3{\xi_3}^3 + 20\xi_2{\xi_1}^3{\xi_3}^3 - 26\xi_2\xi_1{\xi_3}^5 + 9{\xi_2}^7 + {\xi_1}^7 + 84\xi_3{\xi_1}^3{\xi_2}^3$$

$$- 81\xi_3{\xi_2}^2{\xi_1}^4 + 38\xi_3\xi_2{\xi_1}^5 + 7\xi_2{\xi_3}^6 + 33{\xi_2}^2\xi_1{\xi_3}^4 + 25\xi_2{\xi_1}^2{\xi_3}^4$$

$$+ 2{\xi_3}^2{\xi_2}^2{\xi_1}^3 - 55\xi_2{\xi_3}^2{\xi_1}^4 - 43{\xi_3}^2{\xi_2}^5 + 27{\xi_2}^3{\xi_3}^4 + 29{\xi_3}^3{\xi_2}^4 - 29{\xi_2}^2{\xi_3}^5$$

$$- {\xi_3}^7 + 6\xi_3\xi_1{\xi_2}^5 + \xi_3{\xi_2}^6 - 7\xi_3{\xi_1}^6 + 21{\xi_1}^5{\xi_3}^2 + 7\xi_1{\xi_3}^6$$

$$+ 35{\xi_1}^3{\xi_3}^4 - 35{\xi_1}^4{\xi_3}^3 - 21{\xi_1}^2{\xi_3}^5 - 17\xi_1{\xi_2}^6 + 99{\xi_1}^3{\xi_2}^4 - 101{\xi_2}^3{\xi_1}^4$$

$$- 27{\xi_1}^2{\xi_2}^5 + 45{\xi_2}^2{\xi_1}^5 - 9\xi_2{\xi_1}^6)$$

$$+\left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{1}{\xi_2\Delta^3}(\xi_2+\xi_1-\xi_3)(-2\xi_2\xi_3{\xi_1}^2+10\xi_2\xi_1{\xi_3}^2$$
$$+40\xi_1{\xi_2}^2\xi_3+3{\xi_2}^4+{\xi_1}^4+{\xi_3}^4+12{\xi_2}^2{\xi_3}^2-10\xi_3{\xi_2}^3-6\xi_2{\xi_3}^3$$
$$+44{\xi_1}^2{\xi_2}^2-2\xi_2{\xi_1}^3-46\xi_1{\xi_2}^3+6{\xi_1}^2{\xi_3}^2-4{\xi_1}^3\xi_3-4\xi_1{\xi_3}^3)$$
$$-\left(\frac{f(\xi_2)-1}{\xi_2}\right)\frac{16{\xi_2}^2}{\Delta^3}(-4\xi_1\xi_3+2{\xi_3}^2+\xi_2\xi_3+\xi_1\xi_2+2{\xi_1}^2-3{\xi_2}^2)$$
$$-\left(\frac{f(\xi_3)-1}{\xi_3}\right)\frac{1}{\xi_2\Delta^3}(4\xi_2\xi_1{\xi_3}^3+18\xi_3{\xi_1}^2{\xi_2}^2+20\xi_3\xi_2{\xi_1}^3-3{\xi_2}^5$$
$$-22{\xi_1}^2{\xi_2}^3+18{\xi_1}^3{\xi_2}^2+13\xi_1{\xi_2}^4-7\xi_2{\xi_1}^4+{\xi_1}^5-42{\xi_2}^2{\xi_3}^3$$
$$-76\xi_3\xi_1{\xi_2}^3+6\xi_1{\xi_2}^2{\xi_3}^2+43\xi_3{\xi_2}^4+2{\xi_3}^2{\xi_2}^3-18\xi_2{\xi_3}^2{\xi_1}^2+\xi_2{\xi_3}^4$$
$$-5\xi_3{\xi_1}^4-{\xi_3}^5-10{\xi_1}^2{\xi_3}^3+10{\xi_1}^3{\xi_3}^2+5\xi_1{\xi_3}^4)$$
$$-\frac{1}{(\xi_1-\xi_3)}\Big(f(\xi_1)-f(\xi_3)\Big)\frac{1}{6\xi_2}$$
$$-\frac{1}{(\xi_1-\xi_3)}\left(\frac{f(\xi_1)-1}{\xi_1}-\frac{f(\xi_3)-1}{\xi_3}\right)\frac{1}{\xi_2}, \tag{C.10}$$

$$G_{11}(\xi_1,\xi_2,\xi_3)=F(\xi_1,\xi_2,\xi_3)\Big[\frac{4\xi_1\xi_2}{\Delta^4}(2{\xi_1}^4-8{\xi_1}^3\xi_2+6{\xi_1}^2{\xi_2}^2-4{\xi_1}^2{\xi_3}^2$$
$$+4\xi_1\xi_2{\xi_3}^2+{\xi_3}^4)-\frac{16}{\Delta^3}(-3{\xi_1}^3+3{\xi_1}^2\xi_2+4{\xi_1}^2\xi_3$$
$$-4\xi_1\xi_2\xi_3+\xi_1{\xi_3}^2-{\xi_3}^3)\Big]$$
$$+\Big(F(\xi_1,\xi_2,\xi_3)-\frac{1}{2}\Big)\frac{8}{\Delta^2\xi_1\xi_2}(2{\xi_1}^2+4\xi_1\xi_2-4\xi_1\xi_3+{\xi_3}^2)$$
$$-f(\xi_1)\frac{16\xi_1\xi_2}{\Delta^4}(-\xi_1+\xi_2-\xi_3)^2(-\xi_1+\xi_2+\xi_3)$$
$$-f(\xi_3)\frac{1}{2\Delta^4\xi_1\xi_2}(-2{\xi_1}^7+18{\xi_1}^6\xi_2-42{\xi_1}^5{\xi_2}^2+26{\xi_1}^4{\xi_2}^3$$
$$+14{\xi_1}^6\xi_3-76{\xi_1}^5\xi_2\xi_3+178{\xi_1}^4{\xi_2}^2\xi_3-116{\xi_1}^3{\xi_2}^3\xi_3$$
$$-42{\xi_1}^5{\xi_3}^2+110{\xi_1}^4\xi_2{\xi_3}^2-68{\xi_1}^3{\xi_2}^2{\xi_3}^2+70{\xi_1}^4{\xi_3}^3$$
$$-40{\xi_1}^3\xi_2{\xi_3}^3-30{\xi_1}^2{\xi_2}^2{\xi_3}^3-70{\xi_1}^3{\xi_3}^4-50{\xi_1}^2\xi_2{\xi_3}^4$$
$$+42{\xi_1}^2{\xi_3}^5+26\xi_1\xi_2{\xi_3}^5-14\xi_1{\xi_3}^6+{\xi_3}^7)$$
$$+\left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{32\xi_1}{\Delta^3}(3{\xi_1}^2-\xi_1\xi_2-2{\xi_2}^2-\xi_1\xi_3+4\xi_2\xi_3-2{\xi_3}^2)$$
$$-\left(\frac{f(\xi_3)-1}{\xi_3}\right)\frac{1}{\Delta^3\xi_1\xi_2}(-2{\xi_1}^5+6{\xi_1}^4\xi_2-4{\xi_1}^3{\xi_2}^2+18{\xi_1}^4\xi_3$$
$$+24{\xi_1}^3\xi_2\xi_3-42{\xi_1}^2{\xi_2}^2\xi_3-52{\xi_1}^3{\xi_3}^2+52{\xi_1}^2\xi_2{\xi_3}^2$$

$$+ 68\xi_1{}^2\xi_3{}^3 - 20\xi_1\xi_2\xi_3{}^3 - 42\xi_1\xi_3{}^4 + 5\xi_3{}^5), \tag{C.11}$$

Appendix D

The form factors for the Weyl anomaly calculation

$$
\begin{aligned}
\widetilde{T}_1 = {} & F(-s\Box_1, -s\Box_2, -s\Box_3)\frac{1}{36D^4}(\Box_3{}^8 - 4\Box_1\Box_3{}^7 - 16\Box_1{}^2\Box_3{}^6 \\
& + 68\Box_1{}^3\Box_3{}^5 - 100\Box_1{}^4\Box_3{}^4 + 68\Box_1{}^5\Box_3{}^3 - 16\Box_1{}^6\Box_3{}^2 - 4\Box_1{}^7\Box_3 \\
& + 2\Box_1{}^8 + 16\Box_1\Box_2\Box_3{}^6 - 60\Box_1{}^2\Box_2\Box_3{}^5 + 8\Box_1{}^3\Box_2\Box_3{}^4 + 68\Box_1{}^4\Box_2\Box_3{}^3 \\
& - 48\Box_1{}^5\Box_2\Box_3{}^2 - 4\Box_1{}^6\Box_2\Box_3 + 8\Box_1{}^7\Box_2 + 96\Box_1{}^2\Box_2{}^2\Box_3{}^4 \\
& - 136\Box_1{}^3\Box_2{}^2\Box_3{}^3 + 36\Box_1{}^5\Box_2{}^2\Box_3 - 16\Box_1{}^6\Box_2{}^2 + 64\Box_1{}^3\Box_2{}^3\Box_3{}^2 \\
& - 28\Box_1{}^4\Box_2{}^3\Box_3 - 40\Box_1{}^5\Box_2{}^3 + 46\Box_1{}^4\Box_2{}^4) \\
& - \frac{1}{s}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2}\right)\frac{1}{3D^3}(3\Box_3{}^5 - 16\Box_1\Box_3{}^4 \\
& + 4\Box_3{}^3\Box_1{}^2 + 24\Box_3{}^2\Box_1{}^3 - 26\Box_3\Box_1{}^4 + 8\Box_1{}^5 + 28\Box_3{}^3\Box_1\Box_2 \\
& - 52\Box_3{}^2\Box_2\Box_1{}^2 + 12\Box_2\Box_1{}^4 + 26\Box_3\Box_1{}^2\Box_2{}^2 - 20\Box_2{}^2\Box_1{}^3) \\
& + \frac{1}{s^2}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} - \frac{1}{24}s(\Box_1 + \Box_2 + \Box_3)\right)\frac{2}{D^2} \\
& \times (\Box_3{}^2 - 4\Box_1\Box_3 + 2\Box_1{}^2 + 4\Box_1\Box_2) \\
& + \left(\frac{f(-s\Box_1) - 1}{s\Box_1}\right)\frac{\Box_1}{6D^4}(-\Box_3{}^2\Box_1{}^5 + \Box_1\Box_2{}^6 - \Box_2\Box_1{}^6 + \Box_2{}^7 - \Box_1{}^7 \\
& + \Box_3{}^7 - 37\Box_3{}^2\Box_2{}^4\Box_1 + 9\Box_2{}^5\Box_3{}^2 + 3\Box_3\Box_1{}^6 + 5\Box_3{}^4\Box_1{}^3 - 5\Box_1{}^4\Box_3{}^3 \\
& - 3\Box_3{}^6\Box_1 - 5\Box_1{}^5\Box_2{}^2 - 27\Box_1{}^3\Box_2{}^4 + 27\Box_1{}^4\Box_2{}^3 + 5\Box_1{}^2\Box_2{}^5 - 5\Box_3{}^6\Box_2 \\
& - 3\Box_3\Box_1{}^4\Box_2{}^2 + 42\Box_3{}^3\Box_1{}^2\Box_2{}^2 - 14\Box_1{}^2\Box_2{}^3\Box_3{}^2 - 12\Box_3{}^3\Box_1{}^3\Box_2 \\
& - 23\Box_3{}^4\Box_1{}^2\Box_2 + 20\Box_3\Box_1{}^3\Box_2{}^3 - 57\Box_1\Box_2{}^2\Box_3{}^4 + 22\Box_3{}^5\Box_1\Box_2
\end{aligned}
$$

$$- 11\Box_3\Box_1{}^2\Box_2{}^4 + 6\Box_3\Box_1\Box_2{}^5 + 68\Box_3{}^3\Box_1\Box_2{}^3 + 29\Box_3{}^2\Box_1{}^4\Box_2$$

$$+ 14\Box_1{}^3\Box_2{}^2\Box_3{}^2 - 10\Box_3\Box_1{}^5\Box_2 + \Box_3{}^5\Box_1{}^2 - 5\Box_2{}^4\Box_3{}^3 + 9\Box_2{}^2\Box_3{}^5$$

$$- 5\Box_3{}^4\Box_2{}^3 - 5\Box_2{}^6\Box_3)$$

$$- \left(\frac{f(-s\Box_3) - 1}{s\Box_3}\right)\frac{\Box_3}{24D^4}(\Box_3{}^7 + 2\Box_3{}^6\Box_1 - 38\Box_3{}^5\Box_1{}^2 + 90\Box_3{}^4\Box_1{}^3$$

$$- 90\Box_1{}^4\Box_3{}^3 + 38\Box_3{}^2\Box_1{}^5 - 2\Box_3\Box_1{}^6 - 2\Box_1{}^7 + 10\Box_3{}^5\Box_1\Box_2$$

$$- 50\Box_3{}^4\Box_1{}^2\Box_2 + 24\Box_3{}^3\Box_1{}^3\Box_2 - 2\Box_3{}^2\Box_1{}^4\Box_2 + 20\Box_3\Box_1{}^5\Box_2$$

$$- 14\Box_2\Box_1{}^6 + 66\Box_3{}^3\Box_1{}^2\Box_2{}^2 - 36\Box_1{}^3\Box_2{}^2\Box_3{}^2 - 62\Box_3\Box_1{}^4\Box_2{}^2$$

$$+ 54\Box_1{}^5\Box_2{}^2 + 44\Box_3\Box_1{}^3\Box_2{}^3 - 38\Box_1{}^4\Box_2{}^3)$$

$$- \left(\frac{f(-s\Box_1) - 1 - \frac{1}{6}s\Box_1}{(s\Box_1)^2}\right)\frac{2\Box_1{}^2}{D^3}(\Box_3{}^4 - 4\Box_2\Box_3{}^3 - 6\Box_1{}^2\Box_3{}^2$$

$$+ 6\Box_2{}^2\Box_3{}^2 + 8\Box_1\Box_2\Box_3{}^2 - 16\Box_1\Box_2{}^2\Box_3 - 4\Box_2{}^3\Box_3 + 8\Box_1{}^3\Box_3$$

$$- 2\Box_1{}^2\Box_2{}^2 - 4\Box_1{}^3\Box_2 - 3\Box_1{}^4 + 8\Box_1\Box_2{}^3 + \Box_2{}^4)$$

$$+ \left(\frac{f(-s\Box_3) - 1 - \frac{1}{6}s\Box_3}{(s\Box_3)^2}\right)\frac{\Box_3}{4D^3}(7\Box_3{}^5 - 22\Box_1\Box_3{}^4 - 20\Box_3{}^3\Box_1{}^2$$

$$+ 52\Box_3{}^2\Box_1{}^3 - 26\Box_3\Box_1{}^4 + 2\Box_1{}^5 + 36\Box_3{}^3\Box_1\Box_2 - 52\Box_3{}^2\Box_2\Box_1{}^2$$

$$+ 8\Box_3\Box_1{}^3\Box_2 - 6\Box_2\Box_1{}^4 + 18\Box_3\Box_1{}^2\Box_2{}^2 + 4\Box_2{}^2\Box_1{}^3), \tag{D.1}$$

$$\widetilde{T}_2 = \frac{1}{s^2}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} - \frac{1}{24}s(\Box_1 + \Box_2 + \Box_3)\right)\frac{2}{\Box_1\Box_2}$$

$$- \left(\frac{f(-s\Box_3) - 1}{s\Box_3}\right)\frac{\Box_3(-\Box_3 + 2\Box_1)}{8\Box_2\Box_1}$$

$$+ \left(\frac{f(-s\Box_3) - 1 - \frac{1}{6}s\Box_3}{(s\Box_3)^2}\right)\frac{\Box_3(-5\Box_3 + 2\Box_1)}{4\Box_2\Box_1}, \tag{D.2}$$

$$\widetilde{T}_3 = -F(-s\Box_1, -s\Box_2, -s\Box_3)\frac{2\Box_1}{3D^4}(\Box_1{}^6 - 9\Box_1{}^4\Box_2{}^2 - 9\Box_3{}^2\Box_1{}^4$$

$$- 2\Box_1{}^4\Box_2\Box_3 + 16\Box_1{}^3\Box_2{}^3 + 8\Box_1{}^3\Box_2{}^2\Box_3 + 8\Box_1{}^3\Box_2\Box_3{}^2 + 16\Box_3{}^3\Box_1{}^3$$

$$+ 10\Box_3{}^2\Box_1{}^2\Box_2{}^2 - 9\Box_1{}^2\Box_2{}^4 - 9\Box_3{}^4\Box_1{}^2 - 12\Box_1{}^2\Box_2\Box_3{}^3 - 12\Box_1{}^2\Box_2{}^3\Box_3$$

$$+ 8\Box_1\Box_2{}^4\Box_3 - 8\Box_1\Box_2{}^2\Box_3{}^3 - 8\Box_1\Box_2{}^3\Box_3{}^2 + 8\Box_1\Box_2\Box_3{}^4 - 2\Box_2{}^5\Box_3$$

$$- \Box_2{}^2\Box_3{}^4 - 2\Box_3{}^5\Box_2 + \Box_3{}^6 + 4\Box_2{}^3\Box_3{}^3 - \Box_3{}^2\Box_2{}^4 + \Box_2{}^6)$$

$$+ \frac{1}{s}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2}\right)\frac{4}{3D^3}(40\Box_1{}^2\Box_2\Box_3 + 19\Box_1{}^4$$

$$- 22\Box_1{}^3\Box_2 - 22\Box_1{}^3\Box_3 - 12\Box_1{}^2\Box_3{}^2 - 12\Box_1{}^2\Box_2{}^2 - 14\Box_1\Box_2{}^2\Box_3$$

$$- 14\square_1\square_3^2\square_2 + 14\square_1\square_3^3 + 14\square_1\square_2^3 + \square_2^4 + 6\square_2^2\square_3^2 - 4\square_2^3\square_3$$
$$- 4\square_2\square_3^3 + \square_3^4)$$
$$- \frac{1}{s^2}\left(F(-s\square_1, -s\square_2, -s\square_3) - \frac{1}{2} - \frac{1}{24}s(\square_1 + \square_2 + \square_3)\right)\frac{48\square_1}{D^2}$$
$$+ \left(\frac{f(-s\square_1) - 1}{s\square_1}\right)\frac{4\square_1^2}{3D^4}(\square_1^5 + \square_1^4\square_3 + \square_1^4\square_2 - 8\square_1^3\square_3^2$$
$$- 8\square_1^3\square_2^2 + 8\square_1^2\square_2^3 + 8\square_1^2\square_3^3 - \square_1\square_2^4 - 4\square_1\square_2^3\square_3$$
$$- \square_1\square_3^4 - 4\square_2\square_3^3\square_1 + 10\square_1\square_2^2\square_3^2 + 3\square_2^4\square_3 - \square_2^5 - \square_3^5$$
$$- 2\square_2^2\square_3^3 - 2\square_2^3\square_3^2 + 3\square_2\square_3^4)$$
$$- \left(\frac{f(-s\square_2) - 1}{s\square_2}\right)\frac{1}{6\square_1 D^4}(\square_3^8 + \square_1^8 + 28\square_1^6\square_3^2 - 56\square_1^5\square_3^3$$
$$- 56\square_1^3\square_3^5 + 70\square_1^4\square_3^4 + 28\square_1^2\square_3^6 - 8\square_1^7\square_3 - 8\square_1\square_3^7$$
$$- 61\square_1^3\square_2\square_3^4 - 7\square_2\square_3^7 + 35\square_2^4\square_3^4 - 35\square_2^3\square_3^5 + 21\square_2^2\square_3^6$$
$$+ 99\square_1^4\square_2^4 - 99\square_1^5\square_2^3 - 29\square_1^3\square_2^5 - \square_1^2\square_2^6 + 29\square_1^6\square_2^2$$
$$+ \square_1^7\square_2 - \square_1\square_2^7 - \square_2^7\square_3 - 21\square_2^5\square_3^3 + 7\square_2^6\square_3^2 + 23\square_1^6\square_2\square_3$$
$$- 27\square_1^2\square_2\square_3^5 - 5\square_1^4\square_2^2\square_3^2 + 139\square_1^4\square_2\square_3^3 - 99\square_1^5\square_2\square_3^2$$
$$+ 31\square_1\square_2\square_3^6 + 100\square_1^3\square_2^2\square_3^3 + 50\square_1^3\square_2^3\square_3^2 + 81\square_1^4\square_2^3\square_3$$
$$- 45\square_1^2\square_2^2\square_3^4 + 66\square_1^2\square_2^4\square_3^2 - 39\square_1^2\square_2^5\square_3 - 47\square_1\square_2^5\square_3^2$$
$$+ 14\square_1\square_2^6\square_3 + 60\square_1\square_2^4\square_3^3 - 4\square_1^3\square_2^4\square_3 - 66\square_1^5\square_2^2\square_3$$
$$- 15\square_1\square_2^3\square_3^4 - 34\square_1\square_2^2\square_3^5 + 18\square_1^2\square_2^3\square_3^3)$$
$$- \left(\frac{f(-s\square_3) - 1}{s\square_3}\right)\frac{\square_3}{6\square_1 D^4}(\square_1^6\square_3 + 29\square_1^4\square_3^3 - \square_3^7 + 9\square_1^7 + \square_2^7$$
$$- 7\square_2^6\square_3 - 35\square_2^4\square_3^3 - 21\square_2^2\square_3^5 + 7\square_2\square_3^6 + 35\square_2^3\square_3^4$$
$$+ 7\square_3^6\square_1 - 43\square_1^5\square_3^2 - 27\square_1^5\square_2^2 + 27\square_1^3\square_3^4 - 29\square_1^2\square_3^5$$
$$- 17\square_1^6\square_2 - 9\square_1\square_2^6 + 45\square_1^2\square_2^5 - 101\square_1^3\square_2^4 + 99\square_1^4\square_2^3$$
$$+ 21\square_2^5\square_3^2 + 6\square_1^5\square_2\square_3 - 26\square_2\square_3^5\square_1 + 30\square_1^2\square_2^2\square_3^3$$
$$+ 25\square_1\square_2^2\square_3^4 - 81\square_1^2\square_2^4\square_3 + 84\square_1^3\square_2^3\square_3 + 20\square_1\square_2^3\square_3^3$$
$$+ 34\square_1^3\square_2^2\square_3^2 - 55\square_1\square_2^4\square_3^2 + 2\square_1^2\square_2^3\square_3^2 - 41\square_1^4\square_2^2\square_3$$
$$+ 41\square_1^4\square_2\square_3^2 - 44\square_1^3\square_2\square_3^3 + 33\square_1^2\square_2\square_3^4 + 38\square_1\square_2^5\square_3)$$

$$
\begin{aligned}
&-\left(\frac{f(-s\Box_1)-1-\frac{1}{6}s\Box_1}{(s\Box_1)^2}\right)\frac{16\Box_1{}^3}{D^3}(-2\Box_3{}^2+4\Box_2\Box_3+3\Box_1{}^2\\
&-\Box_1\Box_2-2\Box_2{}^2-\Box_3\Box_1)\\
&+\left(\frac{f(-s\Box_2)-1-\frac{1}{6}s\Box_2}{(s\Box_2)^2}\right)\frac{1}{\Box_1 D^3}(\Box_1{}^6+\Box_3{}^6+58\Box_1{}^4\Box_2{}^2\\
&+15\Box_3{}^2\Box_1{}^4-18\Box_1{}^3\Box_2{}^3-20\Box_3{}^3\Box_1{}^3-27\Box_1{}^2\Box_2{}^4+15\Box_3{}^4\Box_1{}^2\\
&-\Box_2{}^5\Box_3+10\Box_2{}^2\Box_3{}^4-5\Box_3{}^5\Box_2-10\Box_2{}^3\Box_3{}^3+5\Box_3{}^2\Box_2{}^4\\
&+31\Box_1{}^4\Box_2\Box_3-34\Box_1{}^3\Box_2\Box_3{}^2-88\Box_1{}^3\Box_2{}^2\Box_3+6\Box_1{}^2\Box_2\Box_3{}^3\\
&+12\Box_3{}^2\Box_1{}^2\Box_2{}^2+22\Box_1\Box_2{}^4\Box_3+8\Box_1\Box_2{}^2\Box_3{}^3-6\Box_1{}^2\Box_2{}^3\Box_3\\
&-30\Box_1\Box_2{}^3\Box_3{}^2+11\Box_1\Box_2\Box_3{}^4-6\Box_1{}^5\Box_3-9\Box_1{}^5\Box_2\\
&-5\Box_1\Box_2{}^5-6\Box_1\Box_3{}^5)\\
&-\left(\frac{f(-s\Box_3)-1-\frac{1}{6}s\Box_3}{(s\Box_3)^2}\right)\frac{\Box_3}{\Box_1 D^3}(-\Box_2{}^5-10\Box_2{}^3\Box_3{}^2-5\Box_2\Box_3{}^4\\
&+10\Box_2{}^2\Box_3{}^3+\Box_3{}^5-\Box_1\Box_3{}^4+3\Box_1{}^5-6\Box_1{}^2\Box_2\Box_3{}^2-4\Box_2\Box_3{}^3\Box_1\\
&-18\Box_1{}^2\Box_2{}^3+22\Box_1{}^3\Box_2{}^2+18\Box_1\Box_2{}^2\Box_3{}^2+76\Box_1{}^3\Box_2\Box_3\\
&-20\Box_1\Box_2{}^3\Box_3-18\Box_1{}^2\Box_2{}^2\Box_3+5\Box_2{}^4\Box_3-13\Box_1{}^4\Box_2\\
&-43\Box_1{}^4\Box_3-2\Box_1{}^3\Box_3{}^2+42\Box_1{}^2\Box_3{}^3+7\Box_1\Box_2{}^4)\\
&-\frac{1}{\Box_2-\Box_3}\left(\frac{f(-s\Box_2)-1}{s\Box_2}-\frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{\Box_3}{6\Box_1}\\
&+\frac{1}{\Box_2-\Box_3}\Big(\frac{f(-s\Box_2)-1-\frac{1}{6}s\Box_2}{(s\Box_2)^2}-\frac{f(-s\Box_3)-1-\frac{1}{6}s\Box_3}{(s\Box_3)^2}\Big)\frac{\Box_3}{\Box_1},
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
\widetilde{T}_4=&-\frac{1}{s}\left(F(-s\Box_1,-s\Box_2,-s\Box_3)-\frac{1}{2}\right)\frac{8}{D^2}(-\Box_3{}^2+2\Box_1{}^2-2\Box_2\Box_1)\\
&+\frac{1}{s^2}\left(F(-s\Box_1,-s\Box_2,-s\Box_3)-\frac{1}{2}-\frac{1}{24}s(\Box_1+\Box_2+\Box_3)\right)\\
&\quad\times\frac{8(-\Box_3+2\Box_1)}{D\Box_1\Box_2}\\
&-\left(\frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{\Box_3}{2\Box_2\Box_1}\\
&+\left(\frac{f(-s\Box_1)-1-\frac{1}{6}s\Box_1}{(s\Box_1)^2}\right)\frac{32\Box_1{}^2(\Box_3+\Box_1-\Box_2)}{D^2}\\
&+\left(\frac{f(-s\Box_3)-1-\frac{1}{6}s\Box_3}{(s\Box_3)^2}\right)\frac{\Box_3}{D^2\Box_2\Box_1}(5\Box_3{}^4-32\Box_3{}^3\Box_1+36\Box_3{}^2\Box_1{}^2\\
&-16\Box_3\Box_1{}^3+2\Box_1{}^4-20\Box_3{}^2\Box_1\Box_2+16\Box_2\Box_3\Box_1{}^2
\end{aligned}
$$

$$- 8\Box_2\Box_1{}^3 + 6\Box_1{}^2\Box_2{}^2), \tag{D.4}$$

$$\begin{aligned}
\widetilde{T}_5 =& F(-s\Box_1, -s\Box_2, -s\Box_3)\frac{4\Box_1\Box_2}{D^4}(\Box_3{}^4 - 4\Box_1{}^2\Box_3{}^2 + 2\Box_1{}^4 \\
&+ 4\Box_1\Box_2\Box_3{}^2 - 8\Box_2\Box_1{}^3 + 6\Box_1{}^2\Box_2{}^2) \\
&- \frac{1}{s}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2}\right)\frac{16}{D^3}(\Box_3{}^3 - \Box_1\Box_3{}^2 - 4\Box_1{}^2\Box_3 \\
&+ 3\Box_1{}^3 + 4\Box_1\Box_2\Box_3 - 3\Box_1{}^2\Box_2) \\
&+ \frac{1}{s^2}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2} - \frac{1}{24}s(\Box_1 + \Box_2 + \Box_3)\right)\frac{8}{D^2\Box_1\Box_2} \\
&\times (\Box_3{}^2 - 4\Box_1\Box_3 + 2\Box_1{}^2 + 4\Box_1\Box_2) \\
&- \left(\frac{f(-s\Box_1) - 1}{s\Box_1}\right)\frac{16\Box_1{}^2\Box_2}{D^4}(\Box_1{}^3 + \Box_1{}^2\Box_3 - 3\Box_1{}^2\Box_2 + 3\Box_1\Box_2{}^2 \\
&- \Box_1\Box_3{}^2 - 2\Box_1\Box_2\Box_3 - \Box_2{}^3 + \Box_2{}^2\Box_3 + \Box_2\Box_3{}^2 - \Box_3{}^3) \\
&- \left(\frac{f(-s\Box_3) - 1}{s\Box_3}\right)\frac{\Box_3}{2D^4\Box_1\Box_2}(-\Box_3{}^7 + 14\Box_1\Box_3{}^6 - 42\Box_1{}^2\Box_3{}^5 \\
&+ 70\Box_1{}^3\Box_3{}^4 - 70\Box_1{}^4\Box_3{}^3 + 42\Box_1{}^5\Box_3{}^2 - 14\Box_3\Box_1{}^6 + 2\Box_1{}^7 \\
&- 26\Box_1\Box_2\Box_3{}^5 + 50\Box_1{}^2\Box_2\Box_3{}^4 + 40\Box_1{}^3\Box_2\Box_3{}^3 - 110\Box_1{}^4\Box_2\Box_3{}^2 \\
&+ 76\Box_1{}^5\Box_2\Box_3 - 18\Box_1{}^6\Box_2 + 30\Box_1{}^2\Box_2{}^2\Box_3{}^3 + 68\Box_1{}^3\Box_2{}^2\Box_3{}^2 \\
&- 178\Box_1{}^4\Box_2{}^2\Box_3 + 42\Box_1{}^5\Box_2{}^2 + 116\Box_1{}^3\Box_2{}^3\Box_3 - 26\Box_1{}^4\Box_2{}^3) \\
&+ \left(\frac{f(-s\Box_1) - 1 - \frac{1}{6}s\Box_1}{(s\Box_1)^2}\right)\frac{32\Box_1{}^2}{D^3}(3\Box_1{}^2 - \Box_1\Box_3 - \Box_1\Box_2 \\
&+ 4\Box_2\Box_3 - 2\Box_3{}^2 - 2\Box_2{}^2) \\
&+ \left(\frac{f(-s\Box_3) - 1 - \frac{1}{6}s\Box_3}{(s\Box_3)^2}\right)\frac{\Box_3}{\Box_2\Box_1 D^3}(-5\Box_3{}^5 + 42\Box_1\Box_3{}^4 - 68\Box_1{}^2\Box_3{}^3 \\
&+ 52\Box_1{}^3\Box_3{}^2 - 18\Box_1{}^4\Box_3 + 2\Box_1{}^5 + 20\Box_1\Box_2\Box_3{}^3 - 52\Box_1{}^2\Box_2\Box_3{}^2 \\
&- 24\Box_1{}^3\Box_2\Box_3 - 6\Box_1{}^4\Box_2 + 42\Box_1{}^2\Box_2{}^2\Box_3 + 4\Box_1{}^3\Box_2{}^2),
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
\widetilde{T}_6 =& -F(-s\Box_1, -s\Box_2, -s\Box_3)\frac{1}{3D^2}(\Box_2{}^4 - 4\Box_2{}^3\Box_3 + 2\Box_1\Box_2{}^3 + 6\Box_2{}^2\Box_3{}^2 \\
&- 6\Box_1{}^2\Box_2{}^2 - 2\Box_1\Box_2{}^2\Box_3 + 4\Box_1{}^2\Box_2\Box_3 + 2\Box_1{}^3\Box_2 - 4\Box_2\Box_3{}^3 \\
&- 2\Box_1\Box_2\Box_3{}^2 + \Box_3{}^4 - 6\Box_3{}^2\Box_1{}^2 + \Box_1{}^4 + 2\Box_1\Box_3{}^3 + 2\Box_3\Box_1{}^3) \\
&+ \frac{1}{s}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2}\right)\frac{4\Box_1}{D} \\
&+ \left(\frac{f(-s\Box_1) - 1}{s\Box_1}\right)\frac{4\Box_1{}^2}{D^2}(\Box_1\Box_2 + \Box_1\Box_3 - \Box_2{}^2 - \Box_3{}^2 + 2\Box_2\Box_3)
\end{aligned}$$

$$
-\left(\frac{f(-s\Box_2)-1}{s\Box_2}\right)\frac{\Box_2}{D^2}({\Box_1}^3+\Box_2{\Box_1}^2-{\Box_1}^2\Box_3-{\Box_2}^2\Box_1
$$

$$
-\Box_1{\Box_3}^2+2\Box_2\Box_1\Box_3-{\Box_2}^3-3\Box_2{\Box_3}^2+3\Box_3{\Box_2}^2+{\Box_3}^3)
$$

$$
-\left(\frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{\Box_3}{D^2}({\Box_1}^3-\Box_2{\Box_1}^2+{\Box_1}^2\Box_3-{\Box_2}^2\Box_1
$$

$$
-\Box_1{\Box_3}^2+2\Box_2\Box_1\Box_3+{\Box_2}^3+3\Box_2{\Box_3}^2-3\Box_3{\Box_2}^2-{\Box_3}^3), \tag{D.6}
$$

$$
\widetilde{T}_7 = F(-s\Box_1,-s\Box_2,-s\Box_3)\frac{4\Box_2}{D^2}({\Box_3}^2+2\Box_1\Box_3-2\Box_2\Box_3+{\Box_1}^2
$$

$$
-2\Box_1\Box_2+{\Box_2}^2)
$$

$$
-\frac{1}{s}\left(F(-s\Box_1,-s\Box_2,-s\Box_3)-\frac{1}{2}\right)\frac{8}{D}
$$

$$
+\left(\frac{f(-s\Box_1)-1}{s\Box_1}\right)\frac{2}{\Box_2 D^2}({\Box_1}^3\Box_2+{\Box_1}^3\Box_3-7{\Box_1}^2{\Box_2}^2-3{\Box_1}^2{\Box_3}^2
$$

$$
-6{\Box_1}^2\Box_2\Box_3-11\Box_1{\Box_2}^2\Box_3+3\Box_1{\Box_3}^3+7\Box_1{\Box_2}^3+\Box_1\Box_2{\Box_3}^2
$$

$$
+4{\Box_2}^3\Box_3-6{\Box_2}^2{\Box_3}^2+4\Box_2{\Box_3}^3-{\Box_2}^4-{\Box_3}^4)
$$

$$
+\left(\frac{f(-s\Box_2)-1}{s\Box_2}\right)\frac{8{\Box_2}^2(\Box_3+\Box_1-\Box_2)}{D^2}
$$

$$
-\left(\frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{2\Box_3}{\Box_2 D^2}({\Box_1}^3-3\Box_3{\Box_1}^2-5\Box_2{\Box_1}^2+2\Box_3\Box_1\Box_2
$$

$$
+7\Box_1{\Box_2}^2+3\Box_1{\Box_3}^2+3\Box_2{\Box_3}^2+{\Box_2}^2\Box_3-{\Box_3}^3-3{\Box_2}^3)
$$

$$
-\frac{1}{\Box_1-\Box_3}\left(\frac{f(-s\Box_1)-1}{s\Box_1}-\frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{2\Box_3}{\Box_2}, \tag{D.7}
$$

$$
\widetilde{T}_8 = F(-s\Box_1,-s\Box_2,-s\Box_3), \tag{D.8}
$$

$$
\widetilde{T}_9 = -F(-s\Box_1,-s\Box_2,-s\Box_3)\frac{2\Box_1\Box_2}{D^2}(-{\Box_3}^2+2{\Box_1}^2-2\Box_1\Box_2)
$$

$$
+\frac{1}{s}\left(F(-s\Box_1,-s\Box_2,-s\Box_3)-\frac{1}{2}\right)\frac{2(-\Box_3+2\Box_1)}{D}
$$

$$
+\left(\frac{f(-s\Box_1)-1}{s\Box_1}\right)\frac{8\Box_2{\Box_1}^2(\Box_3-\Box_2+\Box_1)}{D^2}
$$

$$
-\left(\frac{f(-s\Box_3)-1}{s\Box_3}\right)\frac{\Box_3}{D^2}(-{\Box_3}^3+6{\Box_3}^2\Box_1-6\Box_3{\Box_1}^2+2{\Box_1}^3
$$

$$
+6\Box_3\Box_2\Box_1-2\Box_2{\Box_1}^2), \tag{D.9}
$$

$$
\widetilde{T}_{10} = -F(-s\Box_1,-s\Box_2,-s\Box_3)\frac{2}{D^2}({\Box_3}^3-2\Box_1{\Box_3}^2-2{\Box_1}^2\Box_3+2{\Box_1}^3
$$

$$
+2\Box_1\Box_2\Box_3-2\Box_2{\Box_1}^2)
$$

$$
\begin{aligned}
&+ \frac{1}{s}\left(F(-s\Box_1, -s\Box_2, -s\Box_3) - \frac{1}{2}\right)\frac{8}{D} \\
&- \left(\frac{f(-s\Box_1) - 1}{s\Box_1}\right)\frac{8\Box_1}{D^2}({\Box_3}^2 - 2\Box_2\Box_3 - {\Box_1}^2 + {\Box_2}^2) \\
&+ \left(\frac{f(-s\Box_3) - 1}{s\Box_3}\right)\frac{4\Box_3({\Box_3}^2 - 2{\Box_1}^2 + 2\Box_1\Box_2)}{D^2}. \qquad \text{(D.10)}
\end{aligned}
$$