THE REPRESENTATION OF NUMBERS BY SEQUENCES OF ZEROS AND ONES

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PREFACE

It is necessary to extend the rational number field to the real numbers in order to obtain the property of Cauchy completeness. The reals can be constructed from the rationals in several ways, but usually every real is represented as an infinite sequence of rationals. Such a representation may be replaced by an infinite sequence of integers, since each rational may be represented by a pair of integers. Examples of representations of the positive real numbers by sequences of integers are:

(1) The ordinary decimal representation and its extensions.

(2) The simple continued fraction representation.

(3) The representation of numbers by an infinite product. Cantor has shown that every real number $N > 1$ may be represented by the sequence of integers $a_1, a_2, a_3 \cdots a_n, \cdots$ where $a_i \geqq a_{i-1}^2$ and where $N = (1 + 1/a_1)(1 + 1/a_2) \cdots$ He has also shown that a necessary and sufficient condition that N be rational is that, from some fixed integer onward in the sequence, $a_i = a_{i-1}^2$.

In this thesis another representation is obtained and some of its properties studied. A connection is found between it and the continued fraction representation, and a necessary and sufficient condition for rationality obtained. Also a few other isolated properties are obtained.

CHAPTER I

THE STRUCTURE OF THE REPRESENTATION

This thesis was suggested by the following problem[1];

Let $\{f(n)\}$ and $\{g(n)\}$ be two sequences of natural numbers defined by the following three conditions:

(1) $f(1) = 1$,

(2) $g(n) = na - 1 - f(n)$, a being an integer > 4

(3) $f(n+1)$ is the least natural number distinct from the $2n$ numbers $f(1), f(2), f(3) \text{ --- } f(n)$; $g(1), g(2), g(3) \text{ --- } g(n)$.

Prove that there exist uniquely determined constants c and d such that

$$f(n) = [cn] \quad , \quad g(n) = [dn]$$

The brackets denote, as usual, the greatest integer function; i.e. $[x] = n$ if $n \leq x < n+1$, for some integer n.

Upon solving this problem it is found that

$$\frac{1}{c} + \frac{1}{d} = 1$$

so that, setting

$$c = 1 + \frac{1}{\alpha} \quad ,$$

it follows, that

$$d = 1 + \alpha \quad ,$$

and hence the number α may be represented by the set of values of

$f(n) = [(1 + \frac{1}{\alpha})n]$, for all n .

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[1]Ky Fan, Problem 4399, American Mathematical Monthly, Vol.57.

The following system of notation was used for this purpose: Let the number α be represented by the infinite sequence $a_1 a_2 a_3$ -- where $a_n = 1$ if $n < k(1+\frac{1}{\alpha}) < n+1$ for some integer k. If $n = k(1+\frac{1}{\alpha})$, which will occur for rational α, set $a_n = 1, a_{n-1} = 0$ if $\alpha < 1$, and $a_n = 0, a_{n-1} = 1$ if $\alpha > 1$. Otherwise set $a_n = 0$.

Thus with each positive number $\alpha\, (\alpha \neq 1)$ there may be associated an infinite sequence of zeros and ones. This sequence will be called the representation of α.

Some simple properties of this representation are:

(1) If the representation of α is $a_1 a_2 a_3$ ----, and the representation of β is $b_1 b_2 b_3$ ---, and if $a_i = b_i$ for $i < k$, but $a_k = 1$ and $b_k = 0$, then $\alpha > \beta$.

(2) Not all sequences are admissible. For example, consider the sequence 0100110 ---. This does not represent any number, for supposing it represented α, then

$$2 < 1 + \frac{1}{\alpha} < 3\ ,$$

$$5 < 2(1+\frac{1}{\alpha}) < 6\ ,$$

and

$$6 < 3(1+\frac{1}{\alpha}) < 7\ ,$$

which is impossible.

(3) Upon knowing sufficiently many terms of the representations for α and β, the representation of $\alpha + \beta$ may be found to any desired length.

For example, suppose the representation of α is

$$\alpha \equiv 0010010010001 \text{---}$$

and that of β is

$$\beta \equiv 0101010010 1 \text{---}$$

From these $3/7 < \alpha < 4/9$ and $3/4 < \beta < 4/5$, so that $\frac{101}{56} < 1 + \frac{1}{\alpha+\beta} < \frac{61}{33}$, from which the following terms of the representation for $\alpha+\beta$ may be obtained.

$$\alpha+\beta \equiv 10101 0101 \,??\, 1010101 \,??????\, 101 \,???????1?1$$

It is noted that some of the determined terms occur in places after some of the undetermined terms.

(4) If, in the representation of α, all the ones are replaced by zeros except for the K^{th} th one, the $2K$ th one, the $3K$ th one, etc., the resulting sequence represents

$$\frac{1}{K(1+\frac{1}{\alpha})-1}$$

(5) If α is irrational or $\alpha > 1$, and P is a positive integer, the representation for $\alpha + P$ may be obtained from that of α in the following way:

If the n th zero in the representation of α is in the k_n th position, then the n th zero in the representation for $\alpha + P$ is in the $k_n + nP$ th position. This may be easily shown from the first theorem in the next chapter.

CHAPTER II

SOME PROPERTIES OF THE REPRESENTATION

Theorem I[1]

The representation of $\frac{1}{\alpha}$, where α is irrational, may be obtained from that of α by replacing zeros by ones and ones by zeros.

Proof

Consider the two sets of multiples of $(1+\frac{1}{\alpha})$ and $(1+\alpha)$ viz:

$$1+\alpha \quad , \quad 2(1+\alpha) \; , \quad 3(1+\alpha) \; ---$$

and $$1+\frac{1}{\alpha} \quad , \quad 2(1+\frac{1}{\alpha}) \; , \quad 3(1+\frac{1}{\alpha}) \; ---$$

and the intervals

$$(1,2), \; (2,3) \; ---- \; (n, n+1) \, .$$

(1) It will be shown that there are n multiples in the first n intervals. Suppose there are p multiples of $1+\alpha$ and s multiples of $1+\frac{1}{\alpha}$ in the first n intervals. That is;

$$p(1+\alpha) < n+1 < (p+1)(1+\alpha) \, ,$$

and $$s(1+\frac{1}{\alpha}) < n+1 < (s+1)(1+\frac{1}{\alpha}) \, ,$$

from which it follows that

$$p+s < n+1 < p+s+2 \quad ,$$

so that $p+s = n$.

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[1]S. Beatty, Problem 3173, American Mathematical Monthly, Vol.33.

(2) It will be shown that there is only one multiple in each interval. Suppose there were two multiples, $s(1+\alpha)$ and $t(1+\frac{1}{\alpha})$ in the interval $(x, x+1)$.

Then;

$$x < s(1+\alpha) < x+1 ,$$

and

$$x < t(1+\tfrac{1}{\alpha}) < x+1 ,$$

from which it follows that

$$x < s+t < x+1 ,$$

which is impossible for integers x, s and t.

<u>Theorem II</u>

Theorem I is true for rational α, $\alpha \neq 1$.

<u>Proof</u>

If $\alpha = b/a$ and $k(1+\frac{a}{b}) = r$, where a, b, r, k are integers, and $\alpha < 1$, then $a_r = 1$, $a_{r-1} = 0$.

If $r = m(1+b/a)$ then $k/b = m/a$, and therefore $m = \frac{ak}{b}$ is an integer, since $r = k + \frac{ka}{b}$. Thus, in the representation of $1/\alpha$,

$$a_r = 0 , \text{ and } a_{r-1} = 1 .$$

REPRESENTATION OF THE ROOTS OF CERTAIN QUADRATIC EQUATIONS

Let α be a root of

$$x^2 + ax + 1 = 0 ,$$ where a is an integer.

Set $t = 1 + \alpha$, and $s = 1 + \frac{1}{\alpha}$

and obtain

$$s = -t + 2 - a$$

Set $f(n) = [tn]$, and $g(n) = [sn]$ so that the representation of α consists of ones in the $g(1)$, $g(2)$, $g(3)$ ---- places, with zeros elsewhere, and the representation of $\frac{1}{\alpha}$ consists of ones in the $f(1)$, $f(2)$, $f(3)$ ---- places, with zeros elsewhere. Then $g(n) = -f(n) + n(2-a) - 1$, and if $0 < \alpha < 1$, then $f(1) = 1$.

From Theorem I, it follows that every integer is found once among the numbers $f(1)$, $f(2)$, $f(3)$, ----; $g(1)$, $g(2)$, $g(3)$ ----. Thus, $f(n)$ and $g(n)$ may be obtained by the recursion:

(1) $f(1) = 1$,

(2) $g(n) = -f(n) + n(2-a) - 1$,

(3) $f(n+1)$ is the smallest integer distinct from $f(1)$, $f(2)$, $f(3)$ --- $f(n)$; $g(1)$, $g(2)$, $g(3)$, --- $g(n)$.

Thus the representations of α and $\frac{1}{\alpha}$ may be obtained without solving the quadratic equation which has α as a root.

<u>Example</u>

The equation

$$f(x) = x^2 - 3x + 1$$

has a root α, $0 < \alpha < 1$ since $f(1)f(0) < 0$. The representation of α is found to be:

$$\alpha \equiv 0010001001\,0001\,0001001\,0001001---$$

THE REPRESENTATION OF RATIONALS

Theorem III

The representation of a rational consists of a repeated block of ones and zeros.

Proof

Let α be a rational number, $\alpha = a/b < 1$. In the representation of α, the ath one is in the $(a+b)$th position, the $(2a)$th one is in the $2(a+b)$th position, and the nath one is in the $n(a+b)$th position.

Let the rth one lie between the 0th and the $(a+b)$th positions. Then

$k \leq r(1+\frac{b}{a}) < k+1$, where k and r are integers, satisfying the conditions

$$0 < r < a,$$
and
$$0 \leq k \leq a+b-1.$$

Then consider

$(na+r)(1+b/a)$ where n is any integer. Obviously, $(na+r)(1+b/a) = n(a+b)+r(1+b/a)$, so that $n(a+b)+k \leq (na+r)(1+b/a) < n(a+b)+k+1$ and hence the $(na+r)$th one is in the $[n(a+b)+k]$th position.

Hence the representation of any rational number is made up of a block of ones and zeros, repeated indefinitely.

Theorem IV

If the representation of a number is made up of a block of ones and zeros repeated indefinitely, the number is rational.

Proof

Consider a one in the s th space of the first block. Let it be the k th one in the sequence. Let there be t spaces in each block. That is

$$s \leq k(1+\tfrac{1}{\alpha}) \leq s+1 \quad ,$$

and the $(na+k)$ th one in the sequence is in the $(nt+s)$ th position, for all n . Hence

$$nt+s \leq (na+k)(1+\tfrac{1}{\alpha}) \leq nt+s+1$$, and upon dividing by n ,

$$t+\frac{s}{n} \leq (a+\frac{k}{n})(1+\tfrac{1}{\alpha}) \leq t+\frac{s+1}{n} \quad .$$

In order that this be true for all n , we must have

$$t = a(1+\tfrac{1}{\alpha}) \quad ,$$

and so

$$\alpha = \frac{a}{t-a} \quad .$$

Thus α is a rational number.

CONVERSION FROM THE SEQUENCE TO A CONTINUED FRACTION

Suppose the n th one in the representation of α occurs at the r_n th place.

Define the set of numbers a_0 , a_1 , a_2 , a_3 --- by

$$a_0 = r_1 \quad ,$$

$$a_1 = \frac{2}{r_2+1-2r_1} \quad ,$$

$$q_n = \frac{C(n)r_{n+1} - C(n-2)r_{n-1}}{C(n-1)(r_n+1)} \quad \text{, for even } n$$

$$= \frac{C(n)(r_{n+1}+1) - C(n-2)(r_{n-1}+1)}{C(n-1)\, r_n} \quad \text{, for odd } n$$

where the $C(n)$ are defined by

$$C(0) = 1 \quad ,$$

$$C(1) = \frac{1}{r_2+1-2r_1} \quad ,$$

$$C(n)\left[(r_n+1)(n+1) - n r_{n+1}\right] = C(n-2)\left[(r_n+1)(n-1) - n r_{n-1}\right] \text{, for even } n \;;$$

$$C(n)\left[r_n(n+1) - n(r_{n+1}+1)\right] = C(n-2)\left[r_n(n-1) - n(r_{n-1}+1)\right] \text{ for odd } n \;.$$

Define the continued fraction T_n by

$$T_n = q_0 + \frac{1}{q_1 +}\,\frac{1}{q_2 +} \cdots \frac{1}{+\, q_n}$$

It is shown in the theory of continued fractions[2] that

$$T_n = \frac{p_n}{q_n} \quad \text{, where}$$

$$p_0 = q_0,\; p_1 = q_1 q_0 + 1,\; q_0 = 1,\; q_1 = q_1$$

$$p_n = q_n p_{n-1} + p_{n-2},\; q_n = q_n q_{n-1} + q_{n-2} \qquad (n \geqslant 2)$$

From the definitions of the q_n , it follows that

$$p_0 = C(0) r_1 \qquad p_1 = C(1)(r_2+1)$$

$$q_0 = C(0)\cdot 1 \qquad q_1 = C(1)\cdot 2$$

and it may be easily shown by induction that

$$p_i = C(i)(r_{i+1}),\quad q_i = C(i)(i+1) \quad \text{, for even } i$$

and $p_i = C(i)(r_{i+1}+1),\; q_i = C(i)(i+1)$, for odd i .

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[2]Chrystal, Algebra, p.431.

Thus, for even n

$$T_n = \frac{r_{n+1}}{n+1},$$

and, for odd n

$$T_n = \frac{r_{n+1}+1}{n+1}.$$

From the definition of the r_n

$$\frac{r_{n+1}}{n+1} \leq 1+\frac{1}{\alpha} \leq \frac{r_{n+1}+1}{n+1}$$

and hence

$$\lim_{n\to\infty} T_n = 1+\frac{1}{\alpha}$$

CONVERSION OF THE REPRESENTATION TO A SIMPLE CONTINUED FRACTION

Let α, an irrational number, $0<\alpha<1$ be represented by a sequence of zeros and ones, where the nth one appears in the r_nth place in the sequence. Since $r_1 < 1+\frac{1}{\alpha} < r_1+1$, it follows that

$$nr_1 < n(1+\tfrac{1}{\alpha}) < nr_1+n.$$

Thus, the only choices for r_n are

$$r_n = nr_1,\ nr_1+1, \ldots\ nr_1+n-1.$$

If

$$r_n = nr_1 + p$$

and

$$r_{n+1} < (n+1)(1+\tfrac{1}{\alpha}) < r_{n+1}+1$$

then it follows from the definition of r_n that

$$(n+1)r_1+p \le r_{n+1} \le (n+1)r_1+p+1 \quad ,$$

and so there are only two possible choices for r_{n+1} , namely

$$r_{n+1} = (n+1)r_1 + p \quad , \text{ or}$$
$$r_{n+1} = (n+1)r_1 + p+1 \quad .$$

We may picture these choices as

$$\begin{tikzcd}
r_1 & r_1 \arrow[d] \arrow[dr] & & & \\
r_2 & 2r_1 \arrow[d] \arrow[dr] & 2r_1+1 \arrow[d] \arrow[dr] & & \\
r_3 & 3r_1 \arrow[d] \arrow[dr] & 3r_1+1 \arrow[d] \arrow[dr] & 3r_1+2 \arrow[d] \arrow[dr] & \\
r_4 & 4r_1 & 4r_1+1 & 4r_1+2 & 4r_1+3
\end{tikzcd}$$

and define "a jump to the right" by "the $p+1$ st jump to the right occurs at the k th space" if

$$r_k = kr_1 + p \quad ,$$

and

$$r_{k+1} = (k+1)r_1 + p+1 \quad .$$

Suppose the first jump has occurred at the k th space and the p th jump at the ℓ th space.

Then

$$kr_1 < k\left(1+\frac{1}{\alpha}\right) < kr_1+1 \quad ,$$
$$(k+1)r_1+1 < (k+1)\left(1+\frac{1}{\alpha}\right) < (k+1)r_1+2 \quad ,$$
$$\ell r_1+p-1 < \ell\left(1+\frac{1}{\alpha}\right) < \ell r_1+p \quad ,$$
$$(\ell+1)r_1+p < (\ell+1)\left(1+\frac{1}{\alpha}\right) < (\ell+1)r_1+p+1 \quad ,$$

from which it follows that

$$\ell = pk,\ pk+1,\ pk+2 \ \cdots\cdots \ \text{or } pk+p-1 \ .$$

Suppose the $(p-1)$ st jump occurs at $\ell=(p-1)k+s$, $(0\le s\le p-2)$ and the p th jump at $\ell_1=pk+t$ $(0\le t\le p-1)$

Then

$$\ell r_1+p-2<\ell(1+\tfrac{1}{\alpha})<\ell r_1+p-1 \quad ,$$

$$(\ell+1)r_1+p-2<(\ell+1)(1+\tfrac{1}{\alpha})<(\ell+1)r_1+p \quad ,$$

$$\ell_1 r_1+p-1<\ell_1(1+\tfrac{1}{\alpha})<\ell_1 r_1+p \quad ,$$

$$(\ell_1+1)r_1+p<(\ell_1+1)(1+\tfrac{1}{\alpha})<(\ell_1+1)r_1+p+1 \quad ,$$

from which it follows that

$$-1+\left(\frac{p}{p-1}\right)\ell<\ell_1<\left(\frac{p}{p-1}\right)(\ell+1) \quad .$$

Upon substituting

$$\ell=(p-1)k+s$$

and
$$\ell_1=pk+t$$

this becomes

$$s-1<t<s+1+\frac{s+1}{p-1} \quad ,$$

from which it follows that $t=s$ or $t=s+1$.

Summarizing, if the first jump to the right occurs at the k th step, then the second jump occurs at the $2k$ or the $(2k+1)$ st step, the third at $3k$, $3k+1$, or $3k+2$ etc. These choices may be shown as

$$\begin{tikzcd}[row sep=small]
\text{First jump at} & k \arrow[d] \arrow[dr] & & & \\
\text{Second jump at} & 2k \arrow[d] \arrow[dr] & 2k+1 \arrow[d] \arrow[dr] & & \\
\text{Third jump at} & 3k \arrow[d] \arrow[dr] & 3k+1 \arrow[d] \arrow[dr] & 3k+2 \arrow[d] \arrow[dr] & \\
\text{Fourth jump at} & 4k & 4k+1 & 4k+2 & 4k+3
\end{tikzcd}$$

etc.

Let α be expressed as the simple continued fraction

$$\alpha = \frac{1}{q_1+}\,\frac{1}{q_2+}\,\frac{1}{q_3+}\,---\quad .$$

In the next three sections the integers q_1, q_2, q_3 will be determined. Throughout these sections it is assumed that $\alpha < 1$.

THE DETERMINATION OF q_1 FROM THE SET OF Ω_n

We may set

$$\alpha = \frac{1}{q_1 + y} \quad ,$$

where $y = \frac{1}{q_2+}\,\frac{1}{q_3+}\,---$

and hence $0 < y < 1$.

Then obviously

$q_1 = \left[\frac{1}{\alpha}\right]$, where the brackets indicate the greatest integer in $\frac{1}{\alpha}$.

But, from the definition of Ω_1 ,

$$\begin{aligned} \Omega_1 &= \left[1 + \frac{1}{\alpha}\right] \\ &= 1 + \left[\frac{1}{\alpha}\right] \end{aligned}$$

and hence $q_1 = \Omega_1 - 1$.

THE DETERMINATION OF q_2 FROM THE SET OF Ω_n .

We may set

$$\alpha = \frac{1}{q_1 + \frac{1}{q_2 + y}} \quad , \qquad \text{-------} \ (1)$$

where

$$y = \frac{1}{q_3+}\,\frac{1}{q_4+}\,---$$

and hence $0 < y < 1$.

Solving (1) for y we obtain

$$y = \frac{(q_1 q_2 + 1)\alpha - q_2}{1 - q_1 \alpha} ,$$

so that

$$0 < \frac{(q_1 q_2 + 1)\alpha - q_2}{1 - q_1 \alpha} < 1 ,$$

but $0 < 1 - q_1 \alpha < 1$,

and therefore

$$0 < (q_1 q_2 + 1)\alpha - q_2 < 1 - q_1 \alpha \quad , \text{----------}(2)$$

from which it follows that

$$q_2 = \left[\frac{\alpha}{1 - q_1 \alpha} \right] .$$

Thus, we may set

$$q_2 + \epsilon = \frac{\alpha}{1 - q_1 \alpha} \text{ , where } 0 < \epsilon < 1 ,$$

and obtain

$$\frac{1}{\alpha} = q_1 + \frac{1}{q_2 + \epsilon} . \quad \text{--------}(3)$$

Suppose that the first jump to the right occurs at k .

Then

$$k r_1 < k(1 + \tfrac{1}{\alpha}) < k r_1 + 1 ,$$

and $(p+1) r_1 + 1 < (p+1)(1 + \frac{1}{\alpha}) < (p+1) r_1 + 2$,

and thus

$$r_1 + \frac{1}{p+1} < 1 + \frac{1}{\alpha} < r_1 + \frac{1}{k} .$$

Substituting for $\frac{1}{\alpha}$ from (3) and using $r_1 - 1 = q_1$, this becomes

$$\frac{1}{k+1} < \frac{1}{a_2+\epsilon} < \frac{1}{k} \quad ,$$

and therefore

$$k = a_2 \quad .$$

THE DETERMINATION OF a_3 FROM THE SET OF Ω_n .

We may set
$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + y}}} \quad , \qquad \text{-------}(4)$$

where $0 < y < 1$.

Solving this for y we obtain

$$y = \frac{a_2 a_3 + 1 - \alpha(a_1 a_2 a_3 + a_1 + a_3)}{\alpha(1 + a_1 a_2) - a_2} \quad ,$$

so that

$$0 < \frac{a_2 a_3 + 1 - \alpha(a_1 a_2 a_3 + a_1 + a_3)}{\alpha(1 + a_1 a_2) - a_2} < 1 \quad ,$$

From equation (2), $\alpha(1 + a_1 a_2) - a_2$ is positive and therefore

$$0 < a_2 a_3 + 1 - \alpha(a_1 a_2 a_3 + a_1 + a_3) < \alpha(1 + a_1 a_2) - a_2 \, ,$$

from which it follows that

$$\frac{\alpha a_1 - 1}{a_2 - \alpha(a_1 a_2 + 1)} > a_3 > \frac{\alpha a_1 - 1}{a_2 - \alpha(a_1 a_2 + 1)} - 1 \quad .$$

Set

$$a_3 + \epsilon = \frac{\alpha a_1 - 1}{a_2 - \alpha(a_1 a_2 + 1)} \qquad \text{where } 0 < \epsilon < 1 \ .$$

Then we have

$$\frac{1}{\alpha} = q_1 + \frac{1}{q_2} - \frac{1}{q_2(1+q_2(q_3+\epsilon))} \quad . \qquad \text{----------(5)}$$

Suppose in the choice of the r_n the first jump to the right appears at k . Then the second appears at $2k+s$, where $s=0$ or $s=1$.

That is

$$(2k+s)r_1 + 1 < (2k+s)(1+\tfrac{1}{\alpha}) < (2k+s)r_1 + 2 \quad ,$$

and $$(2k+s+1)r_1 + 2 < (2k+s+1)(1+\tfrac{1}{\alpha}) < (2k+s+1)r_1 + 3 \quad ,$$

and thus $$r_1 + \frac{2}{2k+s+1} < 1 + \frac{1}{\alpha} < r_1 + \frac{2}{2k+s} \quad .$$

Substituting for $1/\alpha$ from (5), and using $q_2 = k, q_1 = r_1 - 1$, this becomes

$$\frac{2}{s+1} < q_3 + \epsilon < \frac{2}{s} \quad .$$

If $s=1$, then

$$1 < q_3 + \epsilon < 2 \quad ,$$

and therefore $q_3 = 1$.

If $s=0$, then

$$2 < q_3 + \epsilon \quad ,$$

and therefore $q_3 \geq 2$.

We will now proceed by induction to show: if the first move to the right occurs at k and the second at k_2 and the nth at k_n , then if:

$$k_i = ik \qquad \text{for } i = 1, 2, 3, \cdots p-1$$

and

$$k_p = pk + 1 \qquad , \text{ then } \qquad Q_3 = p - 1 \quad ;$$

but if

$$k_p = pk \qquad , \text{ then } \qquad Q_3 \geq p \quad .$$

That is, referring to the table of choices on page 12, if the first move to the right is from the $(p-1)$ st line to the p th line, then $Q_3 = p-1$. The induction has been shown true for $p = 2$.

Since the p th move is at k_p

$$k_p r_1 + p - 1 < k_p(1 + \tfrac{1}{\alpha}) < k_p r_1 + p \quad ,$$

and $(k_p + 1)r_1 + p < (k_p+1)(1+\tfrac{1}{\alpha}) < (k_p+1)r_1 + p + 1$.

Upon substituting for $1/\alpha$ from (5) and rearranging, these become

$$\frac{-1}{1 + k(Q_3 + \epsilon)} < \frac{pk - k_p}{k_p}$$

and

$$\frac{pk - (k_p + 1)}{k_p + 1} < \frac{-1}{1 + k(Q_3 + \epsilon)} .$$

Since

$$pk - k_p < pk - (pk + 1) = -1$$

we may invert the inequalities, which, upon subtracting one and changing sign, become

$$\frac{p}{k_p + 1 - pk} < Q_3 + \epsilon < \frac{p}{k_p - pk} \quad .$$

If

$$k_p = pk \quad ,$$

then

$$p < a_3 + \epsilon \quad ,$$

and therefore

$$a_3 \geqslant p \quad .$$

If

$$k_p = pk + 1 \quad ,$$

then

$$a_3 + \epsilon < p \quad ,$$

and since by the induction hypothesis

$$a_3 \geqslant p - 1 \quad ,$$

therefore,

$$a_3 = p - 1 \quad .$$

THE REMAINING TERMS OF THE CONTINUED FRACTION

It has been shown above that if the first move to the right in the table on page 12 occurs from the $(p-1)$ st to the p th line, then $a_3 = p - 1$. This same rule applied to the table on page 11 determined a_2. It might be supposed that a_4 could be determined in a similar manner, by making a choice table for the jumps in the table on page 12. This, however, is not the case, as will be shown.

Suppose the moves to the right in the second table occur at the q_3 , $2q_3$, $3q_3$, --------- nq_3, nq_3+1 places. Then according to the rule for q_2 and q_3 , q_4 would be n .

That is, if the nq_3 rd jump is at nq_3k+n-1 but the (nq_3+1) st jump is at $(nq_3+1)k+n$ and the $(n+1)q_3$ rd jump is at $(n+1)q_3k+n$, but the $((n+1)q_3+1)$ st jump is at $((n+1)q_3+1)k+n+1$, then $q_4=n$.

Recalling the definition of jump, this means

$$r_{nq_3k+n-1} = (nq_3k+n-1)r_1 + nq_3-1 ,$$
$$r_{nq_3k+n} = (nq_3k+n)r_1 + nq_3 ,$$
$$r_{(nq_3+1)k+n} = ((nq_3+1)k+n)r_1 + nq_3 ,$$
$$r_{(nq_3+1)k+n+1} = ((nq_3+1)k+n+1)r_1 + nq_3+1 ,$$
$$r_{(n+1)q_3k+n} = ((n+1)q_3k+n)r_1 + (n+1)q_3-1 ,$$
$$r_{(n+1)q_3k+n+1} = ((n+1)q_3k+n+1)r_1 + (n+1)q_3 ,$$
$$r_{((n+1)q_3+1)k+n+1} = (((n+1)q_3+1)k+n+1)r_1 + (n+1)q_3 ,$$
$$r_{((n+1)q_3+1)k+n+2} = (((n+1)q_3+1)k+n+2)r_1 + (n+1)q_3+1 .$$

Consider the continued fraction

$$\frac{-1+\sqrt{5}}{2} = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,---$$

The second last of the above equations gives $r_5 = 12$ whereas actually $r_5=13$, thus the rule is not generally true.

CHAPTER III

THE PERTINENT RECURRENCE FORMULA

The equation to be discussed is

$$g(n) = na + b + cf(n) \qquad \text{-----------(1)}$$

where $f(1)=1$ and $f(n+1)$ is the first integer distinct from $f(1)$, $f(2)$, $f(3)$, ---- $f(n)$; $g(1)$, $g(2)$, $g(3)$, ------ $g(n)$. Here it is assumed that a, b, c are integers of such a nature that there exist unique real numbers α, β for which $f(n)=[n\alpha]$, $g(n)=[n\beta]$. We wish to discuss what conditions concerning a, b, c and α, β are thereby implied.

Theorem V

In order that every integer be of the form $[n\alpha]$ or $[n\beta]$ it is necessary and sufficient that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 , \qquad \text{--------(2)}$$

where α, and β, are irrational.

Proof

The proof of sufficiency has been given in Theorem I, but an alternative proof is given here.

Assuming that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, it will be shown that every integer must be of the form $[n\alpha]$, or $[n\beta]$.

Suppose that $r \neq [n\alpha]$, $r \neq [n\beta]$, for all n. Then, for all n either

$$1 + r < n\alpha \qquad \text{or} \quad r > n\alpha$$

and for all n either

$$1+r < n\beta \qquad \text{or} \quad r > n\beta \; .$$

Let N be the greatest integer for which $n\alpha < r$ and let M be the greatest integer for which $n\beta < r$.

Then we have

$$N\alpha < r \quad , \quad (N+1)\alpha > 1+r \quad ,$$

and $\quad M\beta < r \quad , \quad (M+1)\beta > 1+r \quad ,$

so that

$$N+M < r\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) = r \quad ,$$

and $\quad N+M+2 > (1+r)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) = 1+r \quad ,$

from which it follows that

$$N+M < r < N+M+1 \quad ,$$

which is impossible for N , M , and r , integers.

To show the impossibility of

$$r = [n\alpha] = [m\beta]$$

suppose it to be true. Then

$$r < n\alpha < r+1 \quad , \quad \text{so that} \quad \frac{r}{\alpha} < n < \frac{r+1}{\alpha} \quad ,$$

and $r < m\beta < r+1 \quad , \quad$ so that $\dfrac{r}{\beta} < m < \dfrac{r+1}{\beta}$.

Adding these we obtain

$$r < n+m < r+1 \quad ,$$

which is impossible.

In order to show the necessity, suppose

$$\frac{1}{\alpha}+\frac{1}{\beta} = 1+\epsilon \quad .$$

Assume first that $\epsilon > 0$. Then choose a rational number p/q such that $0 < p/q < \epsilon$. Then

$$\frac{1}{\alpha} + \frac{1}{\beta} > 1 + p/q$$

and hence

$$\frac{q}{\alpha} + \frac{q}{\beta} > q + p .$$

Now, there are $\left[\frac{q}{\alpha}\right]$ multiples of α less than q and $\left[\frac{q}{\beta}\right]$ multiples of β less than q . If every interval (1,2), (2,3), (3,4), ---- $(q-1, q)$ is to have one and only one multiple of α or β in it, then we must have

$$\left[\frac{q}{\alpha}\right] + \left[\frac{q}{\beta}\right] = q-1 .$$

But

$$\left[\frac{q}{\alpha}\right] + \left[\frac{q}{\beta}\right] \geq q + p - 1 > q - 1 ,$$

which is a contradiction.

Assume now that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 - \epsilon ,$$

where $\epsilon > 0$.

Choose a rational number p/q as before. Then

$$\frac{1}{\alpha} + \frac{1}{\beta} < 1 - p/q ,$$

and now

$$\left[\frac{q}{\alpha}\right] + \left[\frac{q}{\beta}\right] \leq q - p - 1 < q - 1 .$$

Thus we have $\frac{1}{\alpha}+\frac{1}{\beta}=1$. This completes the proof of Theorem V.

Another relationship may be found between α and β . Since $g(n)=[n\beta]$ it is seen that $\lim_{n\to\infty}\frac{g(n)}{n}=\beta$, and similarly $\lim_{n\to\infty}\frac{f(n)}{n}=\alpha$.

Thus, using (1) we obtain

$$\beta = a + c\alpha \qquad \text{------(3)}$$

Eliminating β from (2) and (3) we obtain

$$0 = c\alpha^2 + \alpha(a-c-1) - a \;, \qquad \text{------(4)}$$

In order that $f(1)=1$, (4) must have a root between one and two. To avoid ambiguity suppose there is only one.

Then, from (4), $c > 1 - a/2$ ------(5)

The necessary condition $\beta = a + c\alpha$ is not sufficient to guarantee $f(n)=[n\alpha]$ and $g(n)=[n\beta]$. Let us now find the conditions on a , b , c such that $\beta = a + c\alpha$, $f(n)=[n\alpha]$ and yet $g(n)\neq[n\beta]$. If $g(n)\neq[n\beta]$, then either $g(n)>n\beta$ or $\beta n \geq g(n)+1$. If $g(n)>\beta n$, then

$$an + b + cf(n) > n\beta \;,$$

and since $f(n)<\alpha n$ then, for $c>0$

$$an + b + c\alpha n > n\beta \;.$$

But $\beta = a + c\alpha$, hence $b>0$. In the other case if

$\beta n \geq g(n)+1$, then, from (1) ,

$$\beta n \geq c f(n) + a n + b + 1$$

and since $f(n) > \alpha n - 1$, then, for $c > 0$ we have

$$\beta n > c\alpha n - c + a n + b + 1,$$

and thus

$$0 > b - c + 1 \quad .$$

Therefore, if $b \leq 0$ and $0 \leq b - c + 1$; if $f(n) = [\alpha n]$, then $g(n) = [\beta n]$.

Again, if $f(n) = [\alpha n]$ and $g(n) = [\beta n]$, then

$$f(n) < \alpha n < f(n) + 1 \qquad \text{-------(6)}$$

and

$$na + b + c f(n) < \beta n < b + na + c f(n) + 1.$$

If $c > 0$, from (6), we have

$$c f(n) < c\alpha n < c f(n) + c \quad ,$$

so that

$$c\alpha n - c + na + b < \beta n < b + na + c\alpha n + 1 \quad ,$$

which, using (3), becomes

$$b < c \quad \text{and} \quad -1 < b \quad .$$

Thus, in order that whenever $f(n) = [\alpha n]$, then $g(n) = [\beta n]$ the following inequalities must hold: for ($c > 0$)

$$-1 < b \leq 0,$$

and $\quad 0 < c \leq 1 \quad$;

the only integral solutions to which are $b = 0$ and $c = 1$.

In the case $c < 0$, the only solutions are $b = -1$ and

$c = -1$. This is the case dealt with by Ky Fan.

Thus, in order to consider cases other than $c = \pm 1$, the recursion formula (1) must be changed. Consider the case $c > 1$.

From $f(n) = [\alpha n]$,

we have $$c f(n) < c\alpha n < c f(n) + c ,$$

and thus $[c\alpha n] = c f(n) + \delta$, where δ can take on any of the values 0, 1, 2, 3,----- $c-1$. Using (3) this becomes

$$[(\beta - a)n] = c f(n) + \delta ,$$

so that

$$[\beta n] = an + c f(n) + \delta ,$$

and so, if b is made a function of n taking on any one of the values 0, 1, 2, 3,----- $c-1$, then it may be chosen so that if $f(n) = [\alpha n]$, then $g(n) = [\beta n]$. For the case $c < -1$ the δ is found to take on one of the values $c, c+1, c+2, \cdots, -1$. The notation $\chi = (q_1, q_2, \cdots, q_n)$ will mean that χ may take on any of the values $q_1, q_2 \; - - - \; q_n$. Thus $b(n) = (0, 1, 2, \cdots, c-1)$ if $c > 1$ and $b(n) = (c, c+1, \cdots -1)$ if $c < -1$.

It is seen that equation (1) must be changed to $g(n) = an + b(n) + c f(n)$ in order to study it further.

THE DETERMINATION OF THE $b(n)$

First it is noted, upon eliminating α from (2) and (3), that β is a root of

$$\psi(x) = x^2 + x(-a - c - 1) + a ,$$

but, considering the case $a > 0$, $c > 0$, then $\psi(a+c) = -c < 0$ and $\psi(a+c+1) = a > 0$.

Thus a root lies between $a+c$ and $a+c+1$, and further examination shows this to be the only root > 1 . Thus we have $[\beta] = a+c$ and also

$$\begin{aligned} g(n+1) - g(n) &= [\beta(n+1)] - [\beta n] , \\ &= [\beta] + (0,1) , \\ &= a+c+(0,1) . \qquad \text{---------} \ (7) \end{aligned}$$

Suppose $f(n+1) - f(n) = 2$.

Then
$$\begin{aligned} g(n+1) - g(n) &= a + b(n+1) - b(n) + 2c , \\ a+c+(0,1) &= a + b(n+1) - b(n) + 2c , \\ (-c, 1-c) &= (0,1,2,3 \cdots, c-1) - (0,1,2,3 \cdots, c-1) , \end{aligned}$$

and the only possible solutions are $b(n+1) = 0$ and $b(n) = c-1$. It is noted from the above that the jumps $f(n+1) - f(n)$ cannot be greater than 2 .

Suppose $f(n+1) - f(n) = 1$.

Then $g(n+1) - g(n) = a + b(n+1) - b(n) + c$, which using (7), becomes $a+c+(0,1) = b(n+1) - b(n) + a + c$, so that $b(n+1) = b(n) + (0,1)$.

Consider the case $c = 2$, and hence $a > 0$.

<u>Theorem VI</u>

If $c=2$ and $f(n+1) - f(n) = 1$, $f(n+2) - f(n+1) = 2$ then the choice $b(n) = 0$, $b(n+1) = 1$ guarantees that if $f(n) = [\alpha n]$,

$g(n) = [\beta n]$ and $f(n+1) = [\alpha(n+1)]$ then
$g(n+1) = [\beta(n+1)]$.

Proof

Eliminating β from (2) and (3) it is seen that α is a root of $\psi(x) = 2x^2 + (a-3)x - a$.
Since $\psi(1) = -1, \psi(1.5) = a/2$ and the product of the roots is negative, we must have

$$1 < \alpha < 1.5 \quad . \qquad \text{-------(8)}$$

Using (3) and (1) the assumed relationship $g(n) = [n\beta]$ becomes $[2\alpha n] = 2[\alpha n]$.
Substitute $\alpha n = M + \epsilon$, where $M = [\alpha n]$. Then $[2\epsilon] = 0$.

It has been assumed that $[\alpha(n+2)] = [\alpha(n+1)] + 2$. Upon substitution for αn, as before, we obtain $[\epsilon + 2\alpha] = 3$. Combining this with (8) and using $[2\epsilon] = 0$ it follows that $[2(\epsilon+\alpha)] = 3$.
From the assumption $[\alpha(n+1)] = 1 + [\alpha n]$ is obtained $[\epsilon + \alpha] = 1$.
Now, using (1), (3), and the substitution for αn we obtain

$$g(n+1) - [\beta(n+1)] = 1 + 2[\alpha + \epsilon] - [2(\epsilon + \alpha)] \quad ,$$

which, upon substituting the values for the right hand side as found above, is zero.

In the case $f(n+1) - f(n) = 1$, and $f(n+2) - f(n+1) = 1$, it is conjectured that $b(n+1) - b(n) = 0$, but all attempts at a proof have failed.