ORIENTATION PRESERVING APPROXIMATION

by

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Abstract

In this work we study the following problem on constrained approximation.

Problem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary. What are necessary and sufficient conditions for a continuous mapping $f : \overline{\Omega} \to \mathbb{R}^n$ to be uniformly approximable by C^1-smooth mappings with* ***nonnegative*** *Jacobian?*

When the dimension $n = 1$ this is just approximation by monotone smooth functions. Hence, the necessary and sufficient condition is: f is monotone. On the other hand, for $n \geq 2$ the description is not as clear. We give a simple necessary condition in terms of the topological degree of continuous mapping. We also give some sufficient conditions for $n = 2$. It also turns out that if $n \geq 2$, then there exist real-analytic mappings with nonnegative Jacobian that cannot be approximated by smooth mappings with positive Jacobian.

In our study of the above mentioned question we use topological degree theory, Schoenflies-type extension theorems, and Stoilow's topological characterization of complex analytic functions.

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Chapter 1

Introduction

1.1 Notation

Let us fix the notation that will be used hereafter.

Denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$, that is, if $x = (x_1, x_2, \dots, x_n)$, then $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For a continuous mapping $f : K \to \mathbb{R}^n$, where $K \subset \mathbb{R}^n$ is a compact set, define $\|f\|_K = \max_{x \in K} |f(x)|$. For $x \in \mathbb{R}^n$ and $r > 0$ define $B_r(x)$ to be an open disk of radius r centred at x. For any $A \subset \mathbb{R}^n$ define the open neighbourhood of A by $O_r(A) = \bigcup_{x \in A} B_r(x)$. For sets $F_1, F_2 \subset \mathbb{R}^n$, by definition put $\operatorname{dist}(F_1, F_2) = \inf_{x \in F_1, y \in F_2} |x - y|$. For a set $F \subset \mathbb{R}^n$ define its diameter $\operatorname{diam}(F)$ by $\operatorname{diam}(F) = \sup_{x,y \in F} |x - y|$.

Let $f : \Omega \to \mathbb{R}^n$ be a smooth mapping, and x be a point in Ω. Let $Df(x)$ be the differential of the mapping f at x, i.e, $Df(x)$ is the linear mapping such that

$$|f(x+h) - f(x) - Df(x)[h]| = o(|h|), \quad h \to 0.$$

Denote by $J_f(x)$ the Jacobian of the mapping f at the point x, i.e.

$$J_f(x) = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

We say that $x \in \Omega$ is a critical point of f, if $J_f(x) = 0$. A point $p \in \mathbb{R}^n$ is called a critical value of f, if there exists at least one critical point of f that maps to p. If a point $p \in \mathbb{R}^n$ is not a critical value, we call p a regular value.

We identify $\mathbb{R}^2$ with $\mathbb{C}$ whenever it is convenient. For arbitrary set $A \subset \mathbb{R}^n$, smooth, analytic, and complex analytic (in case of $\mathbb{R}^2$) mappings on A are considered to be defined on some open set $U \supset A$.

In more than one place we are using "bell-shaped" functions with compact support, therefore it is convenient to define them here.

A standard nonnegative C^∞-function supported in $B_1(0)$ is given by:

$$\omega(x) = \begin{cases} c\exp(-\frac{1}{1-|x|^2}), & \text{if } |x| < 1 \\ 0, & \text{else,} \end{cases}$$

where $c\int_{B_1(0)} \exp(-\frac{1}{1-|x|^2})dx = 1$. The scaled version is given by

$$\omega_\varepsilon(x) = \varepsilon^{-n}\omega\left(\frac{x}{\varepsilon}\right). \tag{1.1}$$

Note that each ω_ε is normalized, that is $\int_{\mathbb{R}^n} \omega_\varepsilon(x)dx = 1$.

1.2 Introduction

If $f : [0,1] \to \mathbb{R}$ is a monotone increasing continuous function, then for each $\varepsilon > 0$ there exists a polynomial p such that p is monotone increasing on $[0,1]$, and

$$\|f - p\|_{[0,1]} := \max_{x\in[0,1]} |f(x) - p(x)| \leq \varepsilon.$$

The easiest way to see why this is true is to use Bernstein polynomials

$$B_{n,f}(x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)\binom{n}{j} x^j(1-x)^{n-j}.$$

It is an exercise to show that Bernstein polynomial of a monotone increasing function is also monotone increasing, and it is well known that $\lim_{n\to\infty} \|f - B_{n,f}\|_{[0,1]} = 0$.

A possible generalization of one-dimensional monotone approximation to multi-dimensional case is given in the following problem.

Problem 1. *What are necessary and sufficient conditions for a continuous mapping $f : [0,1]^2 \to \mathbb{R}^2$ to be uniformly approximable by C^1-smooth mappings with nonnegative Jacobian?*

It is natural to call such approximation "orientation-preserving". Here we only want to approximate f by smooth functions, but as will be shown below, in most cases when we know that f can be approximated by smooth mappings with nonnegative Jacobian, f can also be approximated by polynomial mappings with nonnegative Jacobian.

Another possible generalization of monotone approximation is the problem of approximating a given homeomorphism $f : B^d \to \mathbb{R}^d$ of a closed unit ball $B^d \subset \mathbb{R}^d$ by C^1-diffeomorphisms. For $d = 2$ this was done by Franklin and Wiener in [5]. For $d = 3$ the possibility of such approximation follows from a related result about piecewise linear approximation, see, say, [11, pp. 239-246]. Rather surprisingly, for $d = 4$ the answer to this question is negative, i.e., there exists a homeomorphism $f : B^4 \to \mathbb{R}^4$ that cannot be approximated by diffeomorphisms. This was proved by Donaldson and Sullivan in [3]. What is interesting, is that they do not construct such homeomorphism, but deduce its existence from the fact that, in dimension 4, there are topological manifolds that do not admit the so called quasiconformal structure.

Going back to Problem 1, we consider the following slightly more general version.

Problem 1'. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise C^1-smooth boundary. What are necessary and sufficient conditions for a continuous mapping $f : \overline{\Omega} \to \mathbb{R}^2$ to be uniformly approximable by C^1-smooth mappings with nonnegative Jacobian?*

The requirement that Ω has piecewise C^1-smooth boundary means that $\partial\Omega$ is a union of finitely many disjoint piecewise C^1-smooth closed Jordan curves. For a subset $F \subset \mathbb{R}^2$ we denote its closure by $\overline{F}$, and its boundary by ∂F, i.e. $\partial F = \overline{F} \setminus F$. As usual, we say that the mapping $f : A \to \mathbb{R}^2$ is C^n-smooth, if there exists an open set $U \supset A$ and a mapping $g : U \to \mathbb{R}^2$ with continuous partial derivatives of orders up to n such that $f = g|_A$ ($n \in \mathbb{N}$ or $n = \infty$). For a compact set $K \subset \mathbb{R}^2$ denote the uniform norm of a continuous mapping $f : K \to \mathbb{R}^2$ by $\|f\|_K = \max_{x \in K} |f(x)|$.

First we recall that some sufficient condition is known. It follows from the results by Franklin and Wiener [5] that one-to-one mappings can even be approximated by polynomial mappings with nonzero Jacobian. Moreover, we show that the same holds for locally one-to-one mappings $f : \overline{\Omega} \to \mathbb{R}^2$. We say that a mapping $f : \overline{\Omega} \to \mathbb{R}^2$ is locally one-to-one, if for every point $p \in \overline{\Omega}$ there exists an open neighbourhood U of p such that the restriction $f|_U$ is one-to-one. We have

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. If a continuous mapping $f : \overline{\Omega} \to \mathbb{R}^2$ is locally one-to-one, then for each $\varepsilon > 0$ there exists a polynomial mapping $p : \overline{\Omega} \to \mathbb{R}^2$ with nonzero Jacobian such that $\|f - p\|_{\overline{\Omega}} < \varepsilon$.*

Note that the condition given in Theorem 1.1 is not a necessary condition for Problem 1'. An obvious counterexample is $f \equiv const$.

Therefore, we have to look for another necessary and sufficient condition for Problem 1. Say, in the one-dimensional analog of Problem 1 such necessary and sufficient condition is evidently: "f is a nondecreasing function."

We will state a necessary condition in terms of the topological degree. Let F be a closed subset of $\mathbb{R}^d$. The topological degree (sometimes called Brouwer degree) is a certain integer-valued function on the set of triples (f, U, p), where $f : F \to \mathbb{R}^d$ is a continuous mapping, $U \subset \mathbb{R}^d$ is an open subset such that $\overline{U} \subset F$, and p is a point in $\mathbb{R}^2 \setminus f(\partial U)$. The topological degree is denoted by $\deg(f, U, p)$. More detailed discussion is contained in Section 2.2. Here we will just explain how to define the topological degree for $d = 1$, and what this notion means.

Let $f : [0, 1] \to \mathbb{R}$ be a continuous mapping, $U = (a, b) \subset [0, 1]$ be an open interval, and $p \in \mathbb{R}$ be a point different from $f(a)$ and $f(b)$. Then define $\deg(f, U, p)$ as follows

$$\deg(f, U, p) = \begin{cases} 1, \text{ if } f(a) < p < f(b), \\ -1, \text{ if } f(b) < p < f(a), \\ 0, \text{ otherwise.} \end{cases}$$

If $U = \cup_i I_i$ is a union of disjoint open intervals and $p \notin f(\partial U)$, then the degree is defined as

$$\deg(f,U,p)=\sum_i \deg(f,I_i,p).$$

This sum is well-defined, because $\deg(f,I_i,p)\neq 0$ only for a finite number of intervals I_i, as can be easily checked using continuity of f and compactness of $[0,1]$.

We can express the property of f being a nondecreasing function on $[0,1]$ in terms of the topological degree. Namely, f is a nondecreasing function if and only if for each interval $U=(a,b)\subset[0,1]$ and for each $p\in\mathbb{R}$ such that $p\in f(U)\setminus\{f(a),f(b)\}$, the inequality

$$\deg(f,U,p)\geq 0$$

holds.

The usefulness of this reformulation comes from the fact that unlike the notion "f is a nondecreasing function", the notion of topological degree is well-defined for mappings $f:\mathbb{R}^d\to\mathbb{R}^d$.

For the case $d=2$ the topological degree is closely related to the notion of the winding number for closed curves in $\mathbb{R}^2$. In particular, if $U=\{x\in\mathbb{R}^2:|x|<1\}$ is an open disk, the mapping $f:\overline{U}\to\mathbb{R}^2$ is continuous, and $p\in\mathbb{R}^2$ is a point such that $p\notin f(\partial U)$, then $\deg(f,U,p)=w(\gamma,p)$, where $\gamma:[0,1]\to\mathbb{R}^2$ is defined by $\gamma(t)=f(\cos(2\pi t),\sin(2\pi t))$. Recall that the winding number $w(\gamma,p)$ of an oriented closed curve γ around a given point p is the total number of times that the curve γ travels counterclockwise around the point p.

Our necessary condition for Problem 1' is given in the next theorem.

Theorem 1.2. *If the mapping $f:\overline{\Omega}\to\mathbb{R}^2$ can be approximated by C^1-smooth mappings with nonnegative Jacobian, then for every open subset $U\subset\Omega$ and any $p\in f(U)\setminus f(\partial U)$, we have* $\deg(f,U,p)\geq 0$.

Unfortunately, we cannot prove that this necessary condition is also sufficient for Problem 1', but we have Theorem 1.3, where, roughly speaking, we prove that the sufficient condition is

$$\deg(f,U,p)>0.$$

For the formulation of Theorem 1.3 recall that a continuous mapping $f : X \to Y$ is called *light* if $f^{-1}(y)$ is totally disconnected (connected components in $f^{-1}(y)$ are one-point sets) for all $y \in Y$.

Theorem 1.3. *Let Δ be a bounded domain and $f : \Delta \to \mathbb{R}^2$ be a light continuous mapping. Suppose that for every open set U with $\overline{U} \subset \Delta$ and any $p \in f(U) \setminus f(\partial U)$ we have* $\deg(f, U, p) > 0$.

Then for any domain Ω such that $\overline{\Omega} \subset \Delta$ and each $\varepsilon > 0$, there exists a C^∞-smooth mapping $g : \overline{\Omega} \to \mathbb{R}^2$ with nonnegative Jacobian such that $\|f - g\|_{\overline{\Omega}} < \varepsilon$.

If additionally Δ is simply connected, then there exists a polynomial mapping g with the above properties.

We believe that the additional conditions of Theorem 1.3 are not necessary, therefore we formulate a conjecture.

Conjecture 1.1. *Let $f : \overline{\Omega} \to \mathbb{R}^2$ be a continuous mapping such that for any open set $U \subset \Omega$ and for any point $p \in \mathbb{R}^2 \setminus f(\partial U)$ the inequality* $\deg(f, U, p) \geq 0$ *holds. Then for each $\varepsilon > 0$ there exists a C^∞-smooth mapping $g : \overline{\Omega} \to \mathbb{R}^2$ with nonnegative Jacobian such that $\|f - g\|_{\overline{\Omega}} < \varepsilon$.*

We give counterexamples that show that there is a difference between approximation by smooth mappings with nonnegative Jacobian, and approximation by smooth mappings with strictly positive Jacobian. This is in contrast to one-dimensional case, where a function can be approximated by smooth nondecreasing functions if and only if it can be approximated by smooth strictly increasing functions.

Theorem 1.4. *Define the mapping $f : \overline{B_1(0)} \to \mathbb{R}^2$ by the formula*

$$f(x,y) = \begin{cases} \left(8(\frac{1}{2} - \sqrt{x^2+y^2})^2, 0\right), & \text{if } x^2+y^2 \leq \frac{1}{4} \\ \left(x\frac{(\sqrt{x^2+y^2}-1/2)^2}{x^2+y^2}, y\frac{(\sqrt{x^2+y^2}-1/2)^2}{x^2+y^2}\right), & \text{if } \frac{1}{4} < x^2+y^2 \leq 1. \end{cases}$$

Then f is C^1-smooth, has nonnegative Jacobian, and cannot be approximated by smooth mappings with positive Jacobian.

Theorem 1.5. *Let the mapping* $f : \overline{B_1(0)} \to \mathbb{C}$ *be defined by the formula* $f(x, y) = (x^2 - y^2, 2xy)$*. Then for any* C^1*-smooth mapping* $g : \overline{B_1(0)} \to \mathbb{R}^2$ *with strictly positive Jacobian we have* $\|f - g\|_{\overline{B_1(0)}} \geq 1/4$.

The conclusion of both theorems is the same, but the counterexamples illustrate two different properties that the mappings with positive Jacobian have. The relevant property in Theorem 1.4 is that these mappings satisfy the maximum modulus principle, and the relevant property in Theorem 1.5 is that they have the local topological degree ± 1 (we do not give the definition here). Mappings with nonnegative Jacobian do not necessarily possess these properties.

This work is organized as follows. In Chapter 2 we will provide all the necessary tools and definitions, including the topological degree theory and theorems of Stoilow and Schoenflies. In Chapter 3 we will prove all the main results. The proof of Theorem 1.2 is given in Section 3.2. The proofs of Theorem 1.1 and Theorem 1.3 are given in Section 3.3. Finally, the proofs of Theorem 1.4 and Theorem 1.5 are given in Section 3.4.

Theorems 1.1 - 1.5 are new results. Theorems 1.1 - 1.3 and 1.5 will appear in a paper in the journal “Mathematical Notes”, where the corresponding manuscript has already been accepted for publication.

Chapter 2

Preliminaries

2.1 Topological degree theory

In this section we give a very brief introduction to topological degree theory. We take the analytic approach to the definition of degree because it directly involves smooth mappings and their Jacobians. We mainly follow the exposition of [4] and [13]. We will not touch upon the homological approach to degree theory, see, say, [6].

The degree is defined in three steps. First step is to define the degree of a smooth mapping at a regular value.

Definition 2.1. *Let $U \subset \mathbb{R}^n$ be open and bounded, a mapping $f : \overline{U} \to \mathbb{R}^n$ be C^1-smooth, and $p \notin f(\partial U)$ be a regular value of f. Define*

$$\deg(f, U, p) = \sum_{x \in f^{-1}(p)} \operatorname{sgn} J_f(x),$$

where $\deg(f, U, p)$ is considered to be 0 if $f^{-1}(p)$ is empty, and sgn is the signum function.

Note that the sum on the right hand side is finite, since $p \notin f(\partial U)$ is regular. Indeed, if $f^{-1}(p)$ has a limit point in U it would contradict regularity, and if it has a limit point in ∂U, it would contradict $p \notin f(\partial U)$.

The next step is to define the degree of smooth mapping at any point $p \notin f(\partial U)$.

Definition 2.2. *Let $U \subset \mathbb{R}^n$ be open and bounded, a mapping $f : \overline{U} \to \mathbb{R}^n$ be C^2-smooth, and $p \notin f(\partial U)$. Define*

$$\deg(f, U, p) = \deg(f, U, p'),$$

where p' is a regular value of f, such that $|p - p'| < \operatorname{dist}(p, f(\partial U))$.

Note, that for this definition to make sense one must know that there exists at least one regular value close to p. This follows from the following special case of Sard's theorem (see [10, p.69, Th.1.3]).

Proposition 2.1. *Let $U \subset \mathbb{R}^n$ be open, and a mapping $f : U \to \mathbb{R}^n$ be C^1-smooth. Then the set of critical values of the mapping f has Lebesgue measure 0.*

To prove that the choice of regular value p' does not change the value of degree we use the following integral representation that was first observed by Heinz [7].

Proposition 2.2. *Let a mapping $f : \overline{U} \to \mathbb{R}^n$ be C^1-smooth, $p \notin f(\partial U)$ be a regular value of f, and ω_ε be defined as in (1.1). Then there exists $\varepsilon_0 = \varepsilon_0(f, p) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\deg(f, U, p) = \int_U \omega_\varepsilon(f(x) - p) J_f(x) dx.$$

The proof, which is an application of the Inverse Function Theorem and change of variables in the integral, can be found in [4, Prop. 1.7] or [13, Prop. 1.2.2]. The following proposition may give the reader a taste of ideas that are used in this theory.

Proposition 2.3 ([4], Prop. 1.8)**.** *Let a mapping $f : \overline{U} \to \mathbb{R}^n$ be C^2-smooth, Ω be a connected component of $\mathbb{R}^n \setminus f(\partial U)$, and $p_1, p_2 \in \Omega$ be regular values of f. Then*

$$\deg(f, U, p_1) = \deg(f, U, p_2).$$

Proof. By previous proposition there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$

$$\deg(f, U, p_i) = \int_U \omega_\varepsilon(f(x) - p) J_f(x) dx.$$

Fix $\varepsilon_1 \in (0, \varepsilon_0)$ and suppose that there exists a smooth function $v : \Omega \to \mathbb{R}^n$ with compact support such that

$$\operatorname{div}(v(x)) = \omega_{\varepsilon_1}(x - p_1) - \omega_{\varepsilon_1}(x - p_2),$$

where div denotes the divergence of a vector field. Let $A(x)$ be the gradient matrix of $f(x)$, and $B(x)$ be the adjoint matrix of $A(x)$, so that $B(x)A(x) = A(x)B(x) = \det(A(x)) \cdot I_n = J_f(x)I_n$. Put $u(x) = B(x)v(f(x))$, then

$$\operatorname{div}(u(x)) = \operatorname{div}v(f(x))J_f(x) = \Big(\omega_{\varepsilon_1}(f(x) - p_1) - \omega_{\varepsilon_1}(f(x) - p_2)\Big)J_f(x).$$

Indeed,

$$\begin{aligned}\operatorname{div}(u(x)) &= \sum_{i,j} \frac{\partial b_{ij}}{\partial x_i} v_j(f(x)) + \sum_{i,j,k} b_{ij} \frac{\partial v_j}{\partial x_k}(f(x)) \frac{\partial f_k}{\partial x_i} \\ &= \sum_j v_j(f(x)) \left(\sum_i \frac{\partial b_{ij}}{\partial x_i} \right) + \sum_{j,k} \frac{\partial v_j}{\partial x_k}(f(x)) \delta_{j,k} J_f(x) \\ &= \sum_j v_j(f(x)) \left(\sum_i \frac{\partial b_{ij}}{\partial x_i} \right) + \operatorname{div}v(f(x))J_f(x),\end{aligned}$$

therefore we only need to show that

$$\sum_i \frac{\partial b_{ij}}{\partial x_i} = 0.$$

If we denote $\psi = (f_1, \ldots, \hat{f}_j, \ldots, f_n)^T$, where ˆ means omission, we get

$$\begin{aligned}(-1)^j \frac{\partial b_{ij}}{\partial x_i} = (-1)^i \Bigg(&\sum_{k<i} (-1)^{k+1} \det\Big(\frac{\partial^2 \psi}{\partial x_i \partial x_k}, \frac{\partial \psi}{\partial x_1}, \ldots, \widehat{\frac{\partial \psi}{\partial x_k}}, \ldots, \widehat{\frac{\partial \psi}{\partial x_i}}, \ldots, \frac{\partial \psi}{\partial x_n}\Big) \\ &+ \sum_{k>i} (-1)^k \det\Big(\frac{\partial^2 \psi}{\partial x_i \partial x_k}, \frac{\partial \psi}{\partial x_1}, \ldots, \widehat{\frac{\partial \psi}{\partial x_i}}, \ldots, \widehat{\frac{\partial \psi}{\partial x_k}}, \ldots, \frac{\partial \psi}{\partial x_n}\Big) \Bigg),\end{aligned}$$

and when we take sum over all i, notice that every term appears twice with opposite signs, hence they cancel out, and therefore $\sum_i \frac{\partial b_{ij}}{\partial x_i} = 0$.

Since $v(x)$ has support in interior of Ω, $u(x)$ has support in interior of U (otherwise there exists a point of $f(\partial U)$ in support of v, which contradicts the definition of Ω).

Then, using Green's formula we conclude that

$$\begin{aligned}\deg(f,U,p_1)-\deg(f,U,p_2) &= \int_U \Big(\omega_{\varepsilon_1}(f(x)-p_1)-\omega_{\varepsilon_1}(f(x)-p_2)\Big)J_f(x)dx \\ &= \int_U \mathrm{div} v(f(x))J_f(x)dx \\ &= \int_U \mathrm{div}(u(x))dx \\ &= 0.\end{aligned}$$

Let us outline the proof of existence of the vector field v. As Ω is open and connected, it is path connected, so let $\gamma : [0,1] \to \Omega$ be a continuous path such that $\gamma(0) = p_1$, $\gamma(1) = p_2$. We choose ε_1 small enough, so that $B_{\varepsilon_1}(\gamma(t)) \subset \Omega$ for all $t \in [0,1]$. Now, $O_{\varepsilon_1}(\gamma([0,1]))$ is a connected open set, so there exists a piecewise linear curve $\tilde{\gamma}$ that connects $\gamma(0)$ to $\gamma(1)$ and lies in $O_{\varepsilon_1}(\gamma([0,1]))$. Since $\mathrm{div}(v_1+v_2) = \mathrm{div}(v_1)+\mathrm{div}(v_2)$, this reduces the problem to the case when $\tilde{\gamma}(t) = x_0 + t(x_1 - x_0)$. By shifting, we may assume that $x_0 = 0$. Then take $F(x) = \int_0^1 \omega_{\varepsilon_1}(x - \theta x_1)d\theta$, and $v(x) = F(x) \cdot x_1$. It is then easy to check that $v(x)$ satisfies

$$\mathrm{div}(v(x)) = \omega_{\varepsilon_1}(x - x_0) - \omega_{\varepsilon_1}(x - x_1).$$

□

For general continuous functions f the topological degree is defined as below.

Definition 2.3. *Let $U \subset \mathbb{R}^n$ be open and bounded, the mapping $f : \overline{U} \to \mathbb{R}^n$ be continuous, and $p \notin f(\partial U)$. Define*

$$\deg(f,U,p) = \deg(g,U,p),$$

where g is C^2-smooth, and $\|f-g\|_{\overline{U}} < \mathrm{dist}(p, f(\partial U))$.

Once again, this notion is well-defined, see, e.g., [4, Ch. 1.2]. For the sake of convenience, we collect the main properties of topological degree in the following proposition.

Proposition 2.4. *Topological degree* $\deg(f,U,p)$ *has the following properties.*

(P1) **Homotopy invariance.** *If* $h : [0,1] \times \overline{U} \to \mathbb{R}^n$ *is a continuous mapping, and if* $p \notin h([0,1] \times \partial U)$*, then if we denote* $f_t(x) = h(t,x)$*, we have*

$$\deg(f_0, U, p) = \deg(f_1, U, p).$$

(P2) **Dependence on boundary values only.** *If* $f|_{\partial U} = g|_{\partial U}$*, and* $p \notin f(\partial U) = g(\partial U)$*, then*

$$\deg(f, U, p) = \deg(g, U, p).$$

(P3) **Decomposition of domain.** *If* $U = \bigcup_{i=1}^m U_i$*, with* U_i *open, pairwise disjoint, and* $\partial U_i \subset \partial U$*, then for* $p \notin f(\partial U)$ *we have*

$$\deg(f, U, p) = \sum_{i=1}^m \deg(f, U_i, p).$$

(P4) **Continuity in** f**.** *Given* f *and* $p \notin f(\partial U)$ *there exists* $\varepsilon > 0$ *such that for any continuous* $g : \overline{U} \to \mathbb{R}^n$ *with* $\|f - g\|_{\overline{U}} < \varepsilon$ *we have* $p \notin g(\partial U)$ *and*

$$\deg(f, U, p) = \deg(g, U, p).$$

(P5) **Continuity in** p**.** *Given* f*, the degree* $\deg(f, U, p)$ *as function of* p *is constant on any connected component of* $\mathbb{R}^n \setminus f(\partial U)$*.*

(P6) **Solvability.** *If* $\deg(f, U, p) \neq 0$*, then* $f(x) = p$ *has a solution in* U*.*

(P7) **Excision.** *If* $K \subset \overline{U}$ *is a compact set such that* $p \notin f(K)$*, then*

$$\deg(f, U, p) = \deg(f, U \setminus K, p).$$

(P8) **Normality.** *If we denote by* id_U *the identity mapping of* U*, then* $\deg(id_U, U, p) = 1$ *if* $p \in U$*, and* $\deg(id_U, U, p) = 0$ *otherwise.*

Proofs of these statements can be found in [4, Th 2.1, 2.3, 2.4, 2.6 and 2.7] and [13, Prop. 1.2.6].

The abundance of nice properties makes topological degree a very convenient tool to work with. Moreover, in modern treatments the degree is usually introduced axiomatically, similarly to the way homology theory is treated with MacLane-Eilenberg axioms. Namely, there exists a unique integer valued function $\deg(f, U, p)$ that satisfies the above properties.

Degree theory can be used to prove a number of important results in topological analysis, which are usually proved using techniques of algebraic topology: invariance of domain, Brouwer fixed point theorem, Borsuk theorem, Jordan curve theorem, etc.

2.2 Piecewise linear topology

Recall some definitions from piecewise linear topology (see [11, pp. 2-5]).

Definition 2.4. *A **simplicial complex** $\mathcal{K}$ is a set of simplices (in $\mathbb{R}^n$) that satisfies the following conditions:*

(i) Any face of a simplex in $\mathcal{K}$ is also in $\mathcal{K}$.

(ii) The intersection of any two simplices $\sigma_1, \sigma_2 \in \mathcal{K}$ is a face of both σ_1 and σ_2.

Unless stated otherwise, we always assume that simplicial complex is finite.

The dimension of a simplex given as a convex hull of $n+1$ points in $\mathbb{R}^m$ is defined to be equal to n. We say that a simplicial complex $\mathcal{K}$ is a **simplicial k-complex**, if k is the maximal dimension of simplices in $\mathcal{K}$. We say that simplicial k-complex is **homogenous**, if every simplex in $\mathcal{K}$ of dimension less than k is a face of some simplex in $\mathcal{K}$. The **n-skeleton** of a simplicial complex $\mathcal{K}$ is the subset of all simplices of $\mathcal{K}$ of dimension at most n, it is denoted by $\mathcal{K}^n$.

For a simplicial complex $\mathcal{K}$ denote by $|\mathcal{K}|$ the union of its simplices, we refer to this set as the **underlying space** of $\mathcal{K}$. A simplicial complex $\mathcal{L}$ is called a **subdivision** of a simplicial complex $\mathcal{K}$ if every simplex of $\mathcal{L}$ is contained in some simplex of $\mathcal{K}$ and $|\mathcal{K}| = |\mathcal{L}|$.

Definition 2.5. *A mapping $f : |\mathcal{K}| \to \mathbb{R}^n$ is called **linear** (with respect to $\mathcal{K}$) if for any simplex $\sigma \in \mathcal{K}$ the restriction $f|_\sigma$ is a linear function. A mapping $f : |\mathcal{K}| \to \mathbb{R}^n$ is called **piecewise linear** if there exists a subdivision $\mathcal{L}$ of $\mathcal{K}$ such that f is linear with respect to $\mathcal{L}$.*

Piecewise linear (abbr. PL) mappings have several properties similar to those of smooth mappings, and we use them as an intermediate class in our approximation problem.

2.3 Stoilow theorem

Recall the following definitions.

Definition 2.6. *A mapping $f : X \to Y$ between topological spaces is called **open**, if for every open set U in X the image $f(U)$ is open in Y.*

Definition 2.7. *A topological space C is called totally disconnected, if every connected component of C is a one-point set.*

Some typical examples of totally disconnected spaces are: discrete spaces, the Cantor set, the set of rational numbers $\mathbb{Q}$ with topology induced from $\mathbb{R}$.

Definition 2.8. *A mapping $f : X \to Y$ between topological spaces is called **light**, if for every point $y \in f(U)$ the set $f^{-1}(p)$ with topology induced from X is a totally disconnected space.*

The following theorem, proved by Stoilow, gives a topological characterisation of holomorphic mappings up to topological equivalence (see [12, p.121], [14, p.103]).

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^2$ be an open domain, and $f : \Omega \to \mathbb{R}^2$ be an open, light continuous mapping. Then there exists a holomorphic mapping $h : U_1 \to U_2$, and a pair of homeomorphisms $s_1 : \Omega \to U_1$ and $s_2 : U_2 \to s_2(U_2) \subset \mathbb{R}^2$ such that $f = s_2 h s_1$.*

2.4 Extension theorems

In the proof of the main result we use several extension theorems, here we only state the most important known results, and two technical lemmas which will be proved in Chapter 3. First theorem we need is Schoenflies Theorem, which is a significant sharpening of the Jordan curve theorem.

Proposition 2.6 (Schoenflies Theorem, see [11], p.65)**.** *Let* $S^1 = \{z \in \mathbb{R}^2 : |z| = 1\}$*, and let* $\Gamma : S^1 \to \mathbb{R}^2$ *be some closed Jordan curve. Then there exists a homeomorphism* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $f|_{S^1} = \Gamma$*.*

A proof of this theorem can be found in [11, pp. 65-71].

This theorem is only valid in two dimensions, the most simple counterexample in dimension 3 being the Alexander's horned sphere. Note, however, that the following piecewise linear analogue of this theorem holds (see [11, Ch. 3, Ch. 17]).

Proposition 2.7 (PL Schoenflies Theorem)**.** *Let* $n \in \{2, 3\}$*,* Δ *be a nondegenerate simplex in* $\mathbb{R}^n$*, and* $f : \partial\Delta \to \mathbb{R}^n$ *be a piecewise linear (PL) homeomorphism. Then there exists a PL homeomorphism* $h : \Delta \to \mathbb{R}^n$ *such that* $h|_{\partial\Delta} = f$*.*

A proof of this theorem can also be found in [11].

Here are two extension results of a more technical nature.

Lemma 2.1 (Annulus extension)**.** *Let* $R > \rho > 0$*, and denote* $S = \{x \in \mathbb{R}^2 : |x| = R\}$*. Let a mapping* $\psi : O_\rho(S) \to \mathbb{R}^2$ *be a* C^k*-diffeomorphism such that* $0 \notin \psi(O_\rho(S))$ *and suppose that in polar coordinates* (r, φ)*, where* ψ *is written as* $\psi(r, \varphi) = (\psi^r(r, \varphi), \psi^\varphi(r, \varphi))$*, the inequalities* $\partial\psi^r/\partial r > 0$ *and* $\partial\psi^\varphi/\partial\varphi > 0$ *hold. Then for some* $\varepsilon \in (0, \rho)$ *there exists a* C^k*-diffeomorphism* $\Psi : B_R(0) \cup O_\varepsilon(S) \to \mathbb{R}^2$ *such that* $\Psi|_{O_\varepsilon(S)} = \psi|_{O_\varepsilon(S)}$*.*

Lemma 2.2 (Locally univalent extension)**.** *Let* Ω *be a bounded domain with Jordan piecewise smooth boundary and a continuous mapping* $f : \overline{\Omega} \to \mathbb{R}^2$ *be locally one-to-one. Then there exists an open set* $U \supset \overline{\Omega}$ *and a locally one-to-one continuous mapping* $g : U \to \mathbb{R}^2$ *such that* $g|_{\overline{\Omega}} = f$*.*

Chapter 3

Main Part

3.1 Auxiliary results

3.1.1 Piecewise linear approximation

We first prove a piecewise linear approximation result for locally univalent mappings. The result is similar to the PL homeomorphism approximation theorem [11, pp. 46-51], and the proof relies on PL Schoenflies Theorem.

Before stating the result, we need some definitions. The combinatorial distance between vertices of a simplicial 1-complex is the minimal number of edges in any path connecting these vertices. Denote by $\operatorname{cdist}(v_i, v_j)$ the combinatorial distance between v_i and v_j. For sets of vertices U, V define $\operatorname{cdist}(U, V) := \max_{u \in U, v \in V} \operatorname{cdist}(u, v)$. Combinatorial diameter of U is then given by $\operatorname{cdist}(U, U)$.

Lemma 3.1. *Let $\mathcal{K}^1$ be a (homogeneous) simplicial 1-complex, and let $d \geq 2$ be an integer. Suppose that a continuous mapping $f : |\mathcal{K}^1| \to \mathbb{R}^2$ is one-to-one on every subcomplex of $\mathcal{K}^1$ of combinatorial diameter $\leq d$. Then for each $\varepsilon > 0$ there exists a piecewise linear mapping $g : |\mathcal{K}^1| \to \mathbb{R}^2$ such that g is one-to-one on every subcomplex of $\mathcal{K}^1$ with combinatorial diameter $\leq d$, g satisfies $\|f - g\|_{|\mathcal{K}^1|} < \varepsilon$, and $f(v) = g(v)$ for each vertex v of $\mathcal{K}^1$.*

Proof. For any two points v, w, define $\overline{vw}$ to be the edge with endpoints v and w. Take a subdivision $\mathcal{L}^1$ of $\mathcal{K}^1$ such that $\operatorname{diam}(f(\overline{v_i v_j})) < \varepsilon/3$ for any edge $v_i v_j$ of $\mathcal{L}^1$.

Let v_i be vertices of $\mathcal{L}^1$, $w_i = f(v_i)$, and $A_{ij} = f(\overline{v_i v_j})$. Then $\mathrm{diam}(A_{ij}) < \varepsilon/3$, and hence for all $x, y \in O_{\varepsilon/3}(A_{ij})$ the inequality $\mathrm{dist}(x, y) < \varepsilon$ holds. Extend the function $\mathrm{cdist} = \mathrm{cdist}_{\mathcal{K}^1}$ to the vertices of $\mathcal{L}^1$ in the following way. Let v and w be some vertices of $\mathcal{L}^1$. If v and w are vertices of $\mathcal{K}^1$, then define $\mathrm{cdist}_{\mathcal{L}^1}(v, w) = \mathrm{cdist}_{\mathcal{K}^1}(v, w)$. If v belongs to the edge $\overline{v_1 v_2}$ of $\mathcal{K}^1$, and w belongs to the edge $\overline{w_1 w_2}$ of $\mathcal{K}^1$, we define $\mathrm{cdist}_{\mathcal{L}^1}(v, w) = \max_{i,j} \mathrm{cdist}_{\mathcal{K}^1}(v_i, w_i)$. In similar fashion the distance is defined in two other cases. This extended distance satisfies an important property: if a subcomplex of $\mathcal{K}^1$ has combinatorial diameter d, then the corresponding subcomplex of $\mathcal{L}^1$ also has combinatorial diameter d (with respect to the new distance).

Define $N_i = B_{\varepsilon_i}(w_i)$, where ε_i are small enough, so that:

(i) $\overline{N_i} \cap \overline{N_j} = \emptyset$ whenever $\mathrm{cdist}(v_i, v_j) \le d$;

(ii) $\varepsilon_i < \varepsilon/3$;

(iii) For any three vertices v_i, v_k, v_j such that $\mathrm{cdist}(v_i, v_j) \le d$, $\mathrm{cdist}(v_i, v_k) \le d$ and $\overline{v_j v_k}$ is an edge, we have that $\overline{N_i} \cap A_{kj} \ne \emptyset$ exactly when $v_k = v_i$ or $v_j = v_i$.

Let x_{ij} be the last point of A_{ij} (in the order from w_i) that lies in $\overline{N_i}$. Let x'_{ij} be the first point of A_{ij} that follows x_{ij} and belongs to $\overline{N_j}$. Let A'_{ij} be the arc from x_{ij} to x'_{ij} in A_{ij}. Then arcs A'_{ij} and A'_{kl} are disjoint whenever $\mathrm{cdist}(\overline{v_i v_j}, \overline{v_k v_l}) \le d$. Next, take δ-neighbourhoods of A'_{ij} with $\delta < \varepsilon/3$, so that they are disjoint whenever corresponding A'_{ij} are disjoint. Then for each edge $\overline{v_i v_j}$ there exists a broken line B_{ij} in $O_\delta(A'_{ij})$ that joins x_{ij} and x'_{ij} (cf. [11, Th.6.1]). Therefore, B_{ij} and B_{kl} are disjoint whenever $\mathrm{cdist}(\{v_i, v_j\}, \{v_k, v_l\}) \le d$.

Let y_{ij} be the last point of B_{ij} that lies in $\overline{N_i}$, and y'_{ij} be the first point of B_{ij} that follows y_{ij} and belongs to $\overline{N_j}$. Now define B'_{ij} to be a part of broken line B_{ij} from y_{ij} to y'_{ij}. Finally, let $B''_{ij} = w_i y_{ij} \cup B'_{ij} \cup y'_{ij} w_j$. The broken line B''_{ij} connects w_i to w_j. The edges $\overline{v_i v_j}$ and $\overline{v_k v_l}$ can only intersect at endpoints when $\mathrm{cdist}(\{v_i, v_j\}, \{v_k, v_l\}) \le d$. Therefore, the broken lines B''_{ij} also have this property. We also have that $B''_{ij} \subset O_{\varepsilon/3}(A_{ij})$.

Define the mapping $g : |\mathcal{K}^1| \to \mathbb{R}^2$ on each edge $\overline{v_i v_j}$ to be a piecewise linear homeomorphism that sends $\overline{v_i v_j}$ to B''_{ij}, v_i to w_i, and v_j to w_j. If $\mathcal{K}'$ is any subcomplex of $\mathcal{K}^1$ of combinatorial diameter d, then the distance between any two vertices of $\mathcal{L}^1$

is at most d. Therefore, the corresponding broken lines B''_{ij} can only intersect at the endpoints. This implies that g is univalent on each subcomplex of combinatorial diameter $\leq d$. The mapping g also interpolates f at the vertices of $\mathcal{L}^1$. To show that g is an ε-approximation for f, notice that for $x \in \overline{v_i v_j}$ both $f(x)$ and $g(x)$ lie in $O_{\varepsilon/3}(A_{ij})$, so that $|f(x) - g(x)| < \varepsilon$. $\square$

Using this result, we can prove the next lemma.

Lemma 3.2. *Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^2$ and $f : |\mathcal{K}| \to \mathbb{R}^2$ be a locally one-to-one continuous mapping. Then for each $\varepsilon > 0$ there exists a locally one-to-one piecewise linear mapping $g : |\mathcal{K}| \to \mathbb{R}^2$ such that $\|f - g\|_{|\mathcal{K}|} < \varepsilon$.*

Proof. First, for every point $x \in |\mathcal{K}|$ consider a neighbourhood $B_\varepsilon(x)$ such that $f|_{B_\varepsilon(x)}$ is one-to-one. The family of these neighbourhoods covers $|\mathcal{K}|$, therefore by compactness there exists a finite subcover. Let δ be the Lebesgue number of this cover. Let $\mathcal{L}$ be a subdivision of $\mathcal{K}$ such that for every triangle $\sigma \in \mathcal{L}$ we have $\operatorname{diam}(\sigma) < \delta/3$ and $\operatorname{diam}(f(\sigma)) < \varepsilon/3$. For a triangle $\sigma \in \mathcal{L}$ let $D(\sigma)$ be the set of vertices of $\mathcal{L}$ at combinatorial distance not more than 3 from σ excluding the vertices of σ. Let $\theta_\sigma = \frac{1}{3} \min \{\varepsilon, \operatorname{dist}(f(\sigma), f(D(\sigma)))\}$. Finally, define $\delta_1 = \min_\sigma \theta_\sigma$.

Consider $g_1 : |\mathcal{L}^1| \to \mathbb{R}^2$ to be a δ_1-approximation to f as given by Lemma 3.1 with $d = 3$. Using PL Schoenflies theorem, extend this function to $g : |\mathcal{L}| \to \mathbb{R}^2$ in such a way that it is one-to-one on each triangle. It remains to prove that g satisfies the required properties.

We first prove that g is an ε-approximation for f. For every triangle σ in $\mathcal{L}$ it is true that $\operatorname{diam}(f(\sigma)) < \varepsilon/3$. Since $\delta_1 < \varepsilon/3$, we have $g(\partial\sigma) \subset O_{\varepsilon/3}(f(\sigma))$. The restriction $g|_\sigma$ is a homeomorphism, therefore $g(\sigma) \subset O_{\varepsilon/3}(f(\sigma))$. Thus, for any $x \in \sigma$, $f(x)$ and $g(x)$ both lie in $O_{\varepsilon/3}(f(\sigma))$, hence $|f(x) - g(x)| < \varepsilon$.

Next, we prove that g is locally univalent. Let σ_1 and σ_2 be two different triangles of $\mathcal{L}$ that have a non-empty intersection. If $\sigma_1 = \sigma_2$, then there is nothing to prove, since the restriction of g to σ_1 is a homeomorphism. If $\sigma_1 \neq \sigma_2$ then the combinatorial diameter of $\sigma_1 \cup \sigma_2$ is equal to 2, therefore g is one-to-one on $\partial\sigma_1 \cup \partial\sigma_2$. Thus, $g(\partial\sigma_1) \cap g(\partial\sigma_2) = g(\partial\sigma_1 \cap \partial\sigma_2)$. This implies that either $g(\sigma_1) \subset g(\sigma_2)$ or

$g(\sigma_1) \cap g(\sigma_2) = g(\sigma_1 \cap \sigma_2)$. In the latter case, g is a homeomorphism of $\sigma_1 \cup \sigma_2$. Suppose that $g(\sigma_1) \subset g(\sigma_2)$. Let x be one of the vertices of σ_1 that is not a vertex of σ_2. Then $g(x) \in g(\sigma_1) \subset g(\sigma_2) \subset O_{\delta_1}(f(\sigma_2))$, and therefore

$$\operatorname{dist}(f(D(\sigma_2)), f(\sigma_2)) \leq \operatorname{dist}(f(x), f(\sigma_2)) < \operatorname{dist}(g(x), f(\sigma_2)) + \delta_1 < 2\delta_1.$$

This contradicts the choice of δ_1, hence $g(\sigma_1 \cap \sigma_2) = g(\sigma_1) \cap g(\sigma_2)$, and g is locally univalent. $\square$

3.1.2 Extension lemmas

Now we shall prove the extension lemmas from Chapter 2.

Lemma 2.1 (Annulus extension)**.** *Let $R > \rho > 0$, and denote $S = \{x \in \mathbb{R}^2 : |x| = R\}$. Let a mapping $\psi : O_\rho(S) \to \mathbb{R}^2$ be a C^k-diffeomorphism such that $0 \notin \psi(O_\rho(S))$ and suppose that in polar coordinates (r, φ), where ψ is written as $\psi(r, \varphi) = (\psi^r(r, \varphi), \psi^\varphi(r, \varphi))$, the inequalities $\partial\psi^r/\partial r > 0$ and $\partial\psi^\varphi/\partial\varphi > 0$ hold. Then for some $\varepsilon \in (0, \rho)$ there exists a C^k-diffeomorphism $\Psi : B_R(0) \cup O_\varepsilon(S) \to \mathbb{R}^2$ such that $\Psi|_{O_\varepsilon(S)} = \psi|_{O_\varepsilon(S)}$.*

Proof. What follows is a rather technical proof, but the idea is simple. The mapping ψ behaves nicely on circles $\{z : |z| = r\}$, and on rays $\{z : Arg(z) = \phi\}$, so we start "untwisting" this mapping, first on circles, and then on rays, until we get to the identity mapping of a smaller annulus. Since the identity mapping of annulus can be extended (trivially) to the identity mapping of a disk, we get the desired extension.

Without loss of generality assume that $R = 1$, and $\rho < 1/10$. Put $\Psi_1(r, \varphi) = (\psi^r(r, \varphi)\omega_1(r), \psi^\varphi(r, \varphi))$ and $\Psi_2(r, \varphi) = \Psi_1(\omega_2(r), \varphi)$, where $\omega_i(t)$ are C^∞-smooth functions such that the following holds:

$\omega_1(1 - \rho) = 0$; $\omega_1'(t) > 0$ for $t \in (1 - \rho, 1 - \rho/2)$; $\omega_1(t) = 1$ for $t > 1 - \rho/2$;
$\omega_2'(t) > 0$ for $t > 0$; $\omega_2(t) = t$ for $t > 1 - \rho/2$; $\omega_2(0) = 1 - \rho$. Then

$$J_{\Psi_1}(r, \varphi) = \omega_1(r) J_\psi(r, \varphi) + \omega_1'(r)\psi^r \frac{\partial\psi^\varphi}{\partial\varphi} > 0,$$

and

$$J_{\Psi_2}(r, \varphi) = J_{\Psi_1}(\omega_2(r), \varphi)\omega_2'(r) > 0.$$

We first show that Ψ_1 is one-to-one on the ring $\{(r,\varphi) : 1-\rho < r < 1\}$. Indeed, let $\alpha \in \mathbb{R}$, then by virtue of $\partial\psi^\varphi/\partial\varphi > 0$ we have that for any $r \in (1-\rho, 1)$ there exists a unique number $f(r) \pmod{2\pi}$ such that $\psi^\varphi(r, f(r)) = \alpha$. Since ψ is a diffeomorphism, f must also be continuous. Moreover, $\psi^r(r, f(r))$ is continuous and injective, therefore it is strictly increasing in r (it is not decreasing, since otherwise ψ would map outer boundary onto inner, and this contradicts $\partial\psi^r/\partial r > 0$). If for some r_1, φ_1 and r_2, φ_2 we have $\Psi_1(r_1, \varphi_1) = \Psi_1(r_2, \varphi_2)$, then $\psi^\varphi(r_1, \varphi_1) = \psi^\varphi(r_2, \varphi_2) = \alpha$, therefore $\varphi_1 = f(r_1)$, $\varphi_2 = f(r_2)$. This implies that $\psi^r(r_1, f(r_1))\omega_1(r_1) = \psi^r(r_2, f(r_2))\omega_2(r_2)$, hence $r_1 = r_2$, since $\psi^r(r, f(r))\omega_1(r)$ is monotone. All this implies that Ψ_2 is a C^k-diffeomorphism of some punctured neighbourhood of 0, and it extends ψ, and also satisfies inequalities $\partial\Psi_2^r/\partial r > 0$ and $\partial\Psi_2^\varphi/\partial\varphi > 0$. Moreover, Ψ_2 is a homeomorphism on $B_1(0)$ and $\Psi_2(0) = 0$.

Next, let $d < 1/10$ be such that $B_d(0) \cap \psi(O_\rho(C)) = \emptyset$. Define $\alpha(r,\varphi)$ by equality $\Psi_2^r(\alpha(r,\varphi),\varphi) = r, r < d$. There is a unique solution, because $\partial\Psi_2/\partial r > 0$. Moreover, an application of the implicit function theorem to $F(\alpha, r, \varphi) = \Psi_2^r(\alpha(r,\varphi),\varphi) - r$ shows that the mapping α is C^k-smooth and increases in r. Consider

$$\beta(r,\varphi) = \int_0^r \Big(\frac{\partial\alpha}{\partial r}(s,\varphi)\omega_3(s) + c(\varphi)(1-\omega_3(s)) + (1-c(\varphi))\omega_4(s)\Big) ds,$$

where $\omega_3(t) = 1$ for $t < d/2$, $\omega_3(t) = 0$ for $t > d$, $\omega_4(t) = 1$ for $t > 1-\rho/2$, $\omega_4(t) = 0$ for $t < 1-\rho$, and $c(\varphi)$ is chosen in such a way that $\beta(1,\varphi) \equiv 1$. Since we also have that $\alpha(d,\varphi) < 1-\rho$, it is easily verified that $c(\varphi) > 0$ and $\partial\beta/\partial r > 0$. Then the mapping Ψ_3 defined by $\Psi_3(r,\varphi) = \Psi_2(\beta(r,\varphi),\varphi)$ is a C^k-diffeomorphism of punctured neighbourhood of 0, which extends ψ and satisfies $\Psi_3^r(r,\varphi) = r$ for $r < d/2$.

Define $\gamma(r,\varphi)$ by the equation $\Psi_3^\varphi(r,\gamma(r,\varphi)) = \varphi$ for $r < d/3$ (it is only defined modulo 2π, but we can choose some continuous branch). Extend this mapping to all real φ by $\gamma(r, \varphi + 2l\pi) = \gamma(r,\varphi) + 2l\pi$. Consider $\delta(r,\varphi) = \gamma(r,0) + \omega_5(r)(\gamma(r,\varphi) - \gamma(r,0)) + (1-\omega_5(r))\varphi$, where $\omega_5(t)$ is a C^∞-smooth function such that $\omega_5(t) \in [0,1]$, $\omega_5(t) = 0$ for $t \in [0, d/5]$ and $\omega_5(t) = 1$ for $t \geq d/4$. Next, define Δ by $\Delta(r,\varphi) = \gamma(r,0)\omega_6(r) + \varphi$, for $r < d/6$ and $\Delta(r,\varphi) = \delta(r,\varphi), r \geq d/6$, where $\omega_6(t) = 0, t < d/8$ and $\omega_6(t) = 1, t > d/7$. Then the mapping Ψ defined by $\Psi(r,\varphi) = \Psi_3(r, \Delta(r,\varphi))$ is a C^k-diffeomorphism of $B_1(0) \cup O_\rho(C) \setminus 0$ and is identity for $r < d/8$, therefore it is a C^k-diffeomorphism of $B_1(0) \cup O_\rho(C)$, and Ψ extends ψ. □

Lemma 2.2 (Locally univalent extension)**.** *Let Ω be a bounded domain with Jordan piecewise smooth boundary and a continuous mapping $f : \overline{\Omega} \to \mathbb{R}^2$ be locally one-to-one. Then there exists an open set $U \supset \overline{\Omega}$ and a locally one-to-one continuous mapping $g : U \to \mathbb{R}^2$ such that $g|_{\overline{\Omega}} = f$.*

Proof. It is clear that we only need to construct extension through each component of the boundary. Therefore, by the virtue of Schoenflies Theorem, we may assume that the mapping is defined on $\overline{\Omega} = \{z : 1-\varepsilon \leq |z| \leq 1\}$, and we need to extend the mapping to the set $\{z : 1-\varepsilon \leq |z| \leq 1+\delta\}$.

By decreasing ε if necessary, we may assume that for some big enough $N \in \mathbb{N}$, the mapping f is univalent on each sector $S_k \cap \overline{\Omega}$, where $S_k = \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [2\pi k/(3N), 2\pi(k+3)/(3N)]\}$.

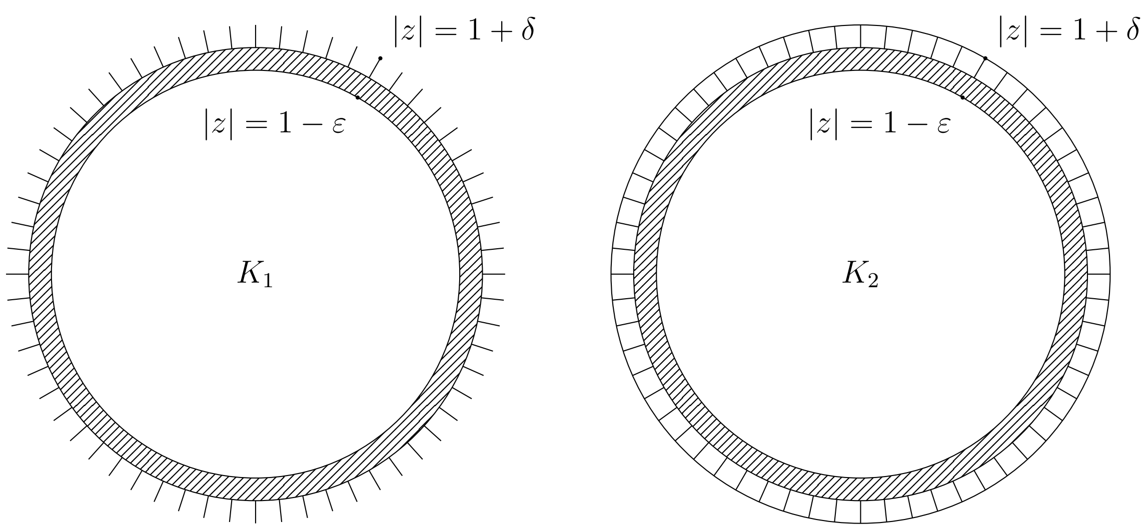

Figure 1. The sets K_1 and K_2 for $N = 20$.

Let $K_1 = \overline{\Omega} \cup \{z : |z| \in [1, 1+\delta], 3N\arg(z)/2\pi \in \mathbb{Z}\}$ and extend f so that it is still one-to-one on the new sectors $\{z \in K_1 : \arg(z) \in [2\pi k/(3N), 2\pi(k+3)/(3N)]\}$. For this, put $C_k = \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [2\pi(k-1)/(3N), 2\pi(k+1)/(3N)]\}$ and $z_k = f(e^{2\pi ik/(3N)}), k = 0, \ldots, 3N-1$ and define Jordan curves $\gamma_k : [0,1] \to \mathbb{R}^2$ in such a way that $\gamma_k(0) = z_k$ and $\gamma_k([0,1]) \cap f(C_k \cap \overline{\Omega}) = \{z_k\}$. Let $\varepsilon_1 > 0$ be such that sets $\gamma_k([0,\varepsilon_1])$ do not intersect, whenever the corresponding z_k do not coincide. Then we take the extension given by $f((1+t)e^{2\pi ik/(3N)}) = \gamma_k(t\varepsilon_1/\delta)$.

Next, we denote $K_2 = K_1 \cup \{z : |z| = 1+\delta\}$ and extend f to K_2, in such a way that f is one-to-one on $C_k \cap K_2$. We construct the extension on the set $F_k = \{z \in \mathbb{C} : |z| = 1, \arg(z) \in [2\pi k/(3N), 2\pi(k+1)/(3N)]\}$ by induction on $k = 1, \ldots, 3N$. Denote $t_k = f((1+\delta)e^{2\pi ik/(3N)})$. On the set F_1 we construct the extension, requiring it to be one-to-one on $S_0 \cap K_2$ and for the set $f(C_1 \cap K_2)$ to lie in the unbounded component of $\mathbb{R}^2 \setminus f(C_1 \cap K_2 \cap \{|z| \geq 1\})$. Suppose that the extension is constructed for $F_1, \ldots, F_l, l < 3N-1$. The set $\mathbb{R}^2 \setminus f(C_{l+1})$ has two linear components (see, for example, Th. 1.2.14 in [13]), and the point t_{l+1} belongs to the unbounded component, therefore there exists a path connecting it to the point t_l, such that $f(C_l \cap K_2)$ belongs to the unbounded component of $\mathbb{R}^2 \setminus f(C_l \cap K_2 \cap \{|z| \geq 1\})$ — this is our extension on F_{l+1}. On the second last part we must also require that extension is one-to-one on S_{-1}. On the last part, we connect t_0 with t_1, requiring that extension is one-to-one on S_{-1}, which is possible, since the mapping was one-to-one on S_{-1}.

Finally, extend the mapping to the whole $\{z : 1-\varepsilon \leq |z| \leq 1+\delta\}$ by applying the Schoenflies Theorem to each sector $\{z \in K_2 : |z| \in [1, 1+\delta], \arg(z) \in [2\pi k/(3N), 2\pi(k+1)/(3N)]\}$. The resulting mapping is one-to-one. Indeed, in all new points, in which $3N\arg(z)/2\pi \notin \mathbb{Z}$, it is locally one-to-one by Schoenflies Theorem. For all the other points, that is for points on the common boundary of two neighboring sectors it is locally one-to-one, since the interiors of the curves $f(\{z \in K_2 : |z| \in [1, 1+\delta], \arg(z) \in [2\pi k/(3N), 2\pi(k+1)/(3N)]\})$ do not intersect. $\square$

3.2 Necessary condition

We now prove Theorem 1.2. The result is more general, as the proof works in any dimension n.

Theorem 1.2. *If the mapping* $f : \overline{\Omega} \to \mathbb{R}^2$ *can be approximated by* C^1*-smooth mappings with nonnegative Jacobian, then for every open subset* $U \subset \Omega$ *and any* $p \in f(U) \setminus f(\partial U)$*, we have* $\deg(f, U, p) \geq 0$*.*

Proof. For a set $U \subset \Omega$ and any $p \in f(U) \setminus f(\partial U)$ put $\varepsilon = \operatorname{dist}(p, f(\partial U))/2$. Let $g : \overline{\Omega} \to \mathbb{R}^n$ be a smooth mapping with nonnegative Jacobian such that $\|f - g\|_{\overline{\Omega}} < \varepsilon$. Then by property (P4) we have $\deg(f, U, p) = \deg(g, U, p)$. Since the set of regular values of a smooth mapping is dense, there exists a regular value p' in the same component of $\mathbb{R}^n \setminus g(\partial U)$ as p, therefore by (P5) we have $\deg(g, U, p) = \deg(g, U, p')$. Finally, it follows from the definition of topological degree of a smooth mapping that $\deg(g, U, p') \geq 0$. □

3.3 Sufficient condition

Before we can prove Theorem 1.1, we need some auxiliary results.

Lemma 3.3. *Let* $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ *be linear maps. Suppose that* $\det(A) > 0$*,* $\det(B) > 0$*, and there exists a nonzero vector* x *such that* $Ax = Bx$*. Then for any* $\alpha > 0$*,* $\beta > 0$ *we have* $\det(\alpha A + \beta B) > 0$ *.*

Proof: The proof is straightforward. □

Lemma 3.4. *Let* $\mathcal{K}$ *be a simplicial complex in* $\mathbb{R}^2$*,* $f : |\mathcal{K}| \to \mathbb{R}^2$ *be a locally one-to-one piecewise linear mapping. Then for each* $\varepsilon > 0$ *there exists a* C^1*-smooth mapping* $g : |\mathcal{K}| \to \mathbb{R}^2$ *with nonzero Jacobian such that* $\|f - g\|_{|\mathcal{K}|} < \varepsilon$*.*

Proof: It is easily proved that we can extend the map f to some simplicial complex $\mathcal{K}' \supset \mathcal{K}$, so that $|\mathcal{K}|$ is contained in the interior of $|\mathcal{K}'|$ and the extension of f is locally one-to-one. Let $a > 0$ be such that for every $x \in |\mathcal{K}|$ the ball $B_a(x)$ is contained in the interior of $|\mathcal{K}'|$ (this number exists due to the compactness of $|\mathcal{K}|$).

Recall from Chapter 2 the standard nonnegative C^∞-function with compact support given by:

$$\omega(x) = \begin{cases} c\exp(-\frac{1}{1-|x|^2}), & \text{if } |x| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\omega_\delta(x) = \delta^{-2}\omega\left(\frac{x}{\delta}\right).$$

For any $0 < \delta < a$, the convolution $f * \omega_\delta$, defined by

$$f * \omega_\delta(x) = \int_{B_\delta(x)} f(y)\omega_\delta(x-y)dy,$$

is a C^∞-smooth mapping that uniformly converges to f, as $\delta \to 0$. As

$$\int_{\mathbb{R}^2} x\omega(x,y)dxdy = \int_{\mathbb{R}^2} y\omega(x,y)dxdy = 0,$$

we see that if f is linear on $B_\delta(x)$, then $f * \omega_\delta(x) = f(x)$. Note that if $B_\delta(x)$ only intersects two triangles of $\mathcal{K}$, then $D(f*\omega_\delta)(x)$ is a convex combination of two linear maps, and it is easy to see that these maps satisfy conditions of Lemma 3.3. Therefore, in this case $J_{f*\omega_\delta}(x) > 0$.

Let $\delta_1 > 0$ be such that for every vertex v_i of $\mathcal{K}$ we have $\operatorname{diam}(f(B_{\delta_1}(v_i))) < \varepsilon$, $B_{\delta_1}(v_i)$ are pairwise disjoint, and $B_{\delta_1}(v_i)$ intersects $\mathcal{K}^1$ (the set of edges of all triangles in $\mathcal{K}$) only at edges that have v_i as an endpoint. Let δ be such that $|f(x)-f(y)| < \varepsilon$ whenever $|x-y| < \delta$. Let $\delta_2 < \delta$ be such that for $x \in \mathcal{K} \setminus \bigcup_{v_i \in K^0} B_{\delta_1/2}(v_i)$ we have that $B_{\delta_2}(x)$ intersects $\mathcal{K}$ at not more than two triangles.

Let us show now that for an arbitrary vertex $v \in \mathcal{K}^0$ the function $g = f * \omega_{\delta_2}$ satisfies conditions of Lemma 2.1 on the annulus $B_{\delta_1}(v) \setminus B_{\delta_1/2}(v)$. For simplicity, assume that $v = f(v) = 0$. First, we show that g is one-to-one on $U = B_{\delta_1}(v) \setminus B_{\delta_1/2}(v)$. Indeed, suppose that $g(x) = g(y)$. Then either $B_{\delta_2}(x)$ or $B_{\delta_2}(y)$ must intersect some edge of $\mathcal{K}^1$, because otherwise we would have $f(x) = g(x) = g(y) = f(y)$, which contradicts the fact that f is one-to-one on each triangle. By relabelling x and y we may assume that $B_{\delta_2}(x)$ intersects some edge. By decreasing the value of δ_2 if necessary, we may also assume that $g(\sigma \cap U) \cap g(\sigma' \cap U) = \emptyset$ for any pair of triangles σ, σ' with $\sigma \cap \sigma' = \{0\}$. Therefore, it suffices to consider the case when x and y lie in the union of two neighboring triangles σ_1 and σ_2 with a common vertex at 0. Once again, we decrease the value of δ_2 to meet the condition $g(O_{\delta_2}(e)\cap U)\cap g(\sigma\cap U)=\emptyset$ for each pair of an edge e and a triangle σ satisfying $\sigma\cap e=\{0\}$. Let e be the common edge of the triangles σ_1 and σ_2, and e' be the common edge of the triangles $f(\sigma_1)$ and $f(\sigma_2)$. Introduce Cartesian coordinate systems (u,w) and (u',w') such that the direction of u coincides with the direction of e and direction of u' coincides with the direction of e', and write $g(u,v)=(g^{u'}(u,v),g^{v'}(u,v))$. Simple calculation shows that $g^{v'}$ depends only on v, and is monotone with respect to v, hence the v-coordinates of x and y coincide. Moreover, for a fixed v, the function $g^{u'}$ is monotone with respect to u, so the u-coordinates of x and y also coincide. In a similar fashion one also checks that the inequalities $\partial g^r/\partial r>0$ and $\partial g^\varphi/\partial\varphi>0$ hold on the corresponding annulus.

Therefore, using Lemma 2.1 for each vertex v we can extend $g=f*\omega_{\delta_2}$ from $B_{\delta_1}(v)\setminus B_{\delta_1/2}(v)$ to $B_{\delta_1}(v)$. Since the Jacobian of an orientation-preserving diffeomorphism is positive, we get $J_g(x)>0$ for all x. Moreover, by definition of δ, for $x\in|\mathcal{K}|\setminus\bigcup B_{\delta_1/2}(v_i)$ we have the inequality $|f(x)-g(x)|=|f(x)-f*\omega_\delta(x)|<\varepsilon$. For a point $x\in B_{\delta_1}(v_i)$ the extension property gives $g(x),f(x)\in f(B_{\delta_1}(v_i))$, and using $diam(f(B_{\delta_1}(v_i)))<\varepsilon$, we get $|f(x)-g(x)|<\varepsilon$, hence g gives the necessary approximation. $\square$

Now we prove Theorem 1.1 and Theorem 1.3.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain. If a continuous mapping $f:\overline{\Omega}\to\mathbb{R}^2$ is locally one-to-one, then for each $\varepsilon>0$ there exists a polynomial mapping $p:\overline{\Omega}\to\mathbb{R}^2$ with nonzero Jacobian such that $\|f-p\|_{\overline{\Omega}}<\varepsilon$.*

Proof. First, consider the case when $\overline{\Omega}$ is equal to $|\mathcal{K}|$ for some simplicial complex $\mathcal{K}$. From Lemma 3.2 and Lemma 3.4 we see that there exists a C^1-smooth mapping $g:\overline{\Omega}\to\mathbb{R}^2$ with nonzero Jacobian such that $\|f-g\|_{\overline{\Omega}}<\varepsilon/2$. Since $\overline{\Omega}$ is a compact set, there exists $\delta>0$ such that $|J_g(x)|>\delta$ for all $x\in\overline{\Omega}$. Define $M=\max\{\|\frac{\partial g_i}{\partial x_j}\|_{\overline{\Omega}}:1\le i,j\le 2\}$. Since $\overline{\Omega}=|\mathcal{K}|$ satisfies the conditions of Th.2 of [1], there exists a polynomial map $p:\overline{\Omega}\to\mathbb{R}^2$ such that $\|g-p\|<\varepsilon/2$ and $\|\frac{\partial g_i}{\partial x_j}-\frac{\partial p_i}{\partial x_j}\|_{\overline{\Omega}}<\max(2M,\frac{\delta}{8M})$. Then it is easy to see that $\|f-p\|_{\overline{\Omega}}<\varepsilon$ and

$|J_p(x)| > 0$.

Now we turn to the general case. By Lemma 2.2 there exists an open set $U \supset \overline{\Omega}$ and a locally one-to-one mapping $f' : U \to \mathbb{R}^2$ such that $f'|_{\overline{\Omega}} = f$. Then we can build a simplicial complex $\mathcal{K}$ such that $U \supset |\mathcal{K}| \supset \overline{\Omega}$ and by applying the result of the previous paragraph to f' and $|\mathcal{K}|$ we get the desired approximation. $\square$

Theorem 1.3. *Let Δ be a bounded domain and $f : \Delta \to \mathbb{R}^2$ be a light continuous mapping. Suppose that for every open set U with $\overline{U} \subset \Delta$ and any $p \in f(U) \setminus f(\partial U)$ we have* $\deg(f, U, p) > 0$.

Then for any domain Ω such that $\overline{\Omega} \subset \Delta$ and each $\varepsilon > 0$, there exists a C^∞-smooth mapping $g : \overline{\Omega} \to \mathbb{R}^2$ with nonnegative Jacobian such that $\|f - g\|_{\overline{\Omega}} < \varepsilon$.

If additionally Δ is simply connected, then there exists a polynomial mapping g with the above properties.

Proof. Recall that a mapping $F : X \to \mathbb{R}^2$ is called *quasi-open* (see [14, p.110]) if for any $y \in F(X)$ and any open set V containing a compact component of $F^{-1}(y)$, y is interior to $F(V)$. Note that our mapping f is quasi-open. Indeed, for every $x \in \Delta$ and V containing a compact component of $f^{-1}(f(x))$, there is some open subset $V_0 \subset V$ such that $\partial V_0 \cap f^{-1}(f(x)) = \emptyset$. Then we have that $\deg(f, V_0, f(x)) > 0$. Therefore, by properties (P5) and (P6) of the topological degree we have that $f(V_0)$ contains some neighbourhood of $f(x)$. Hence, f is quasi-open. Since f is also light, we have (see [14, pp.110-113]) that f is open.

Since f is open and light, by Stoilow theorem, f is topologically equivalent to a complex analytic mapping $h : U_1 \to U_2 \subset \mathbb{R}^2$. This means that there exists a pair of homeomorphisms $s_1 : \Delta \to U_1$ and $s_2 : U_2 \to s_2(U_2) \subset \mathbb{R}^2$ such that $f = s_2 h s_1$. By Theorem 1.1 there are polynomial mappings p_1, p_2 with nonzero Jacobian such that $\|s_2 h s_1 - s_2 h p_1\|_{\overline{\Omega}} < \varepsilon/2$ and $\|s_2 - p_2\|_{h(p_1(\overline{\Omega}))} < \varepsilon/2$ (note that we only need the "simplicial" case of Theorem 1.1). Then $\|f - p_2 h p_1\|_{\overline{\Omega}} < \varepsilon$. Since f preserves orientation, p_1 and p_2 must have Jacobians of the same sign. Therefore, $p_2 h p_1$ is a C^∞-smooth mapping with nonnegative Jacobian.

In case when Δ is simply connected, U_1 is also simply connected, and hence by Runge's theorem (see [2, p.198]) there exists a polynomial $p_3(z)$ of one complex variable that approximates h on $\overline{\Omega}$. Since $p_3(z)$ as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ has nonnegative Jacobian, we get a polynomial mapping $p_2p_3p_1$ with a nonnegative Jacobian that approximates the continuous mapping f. $\square$

3.4 Counterexamples

Here we prove two negative results that show the difference between approximation by mappings with nonnegative Jacobian and approximation by mappings with strictly positive Jacobian.

The first result relies on the fact that mappings with positive Jacobian are open, and hence satisfy the maximum modulus principle.

Lemma 3.5. *Let $f_n : \overline{\Omega} \to \mathbb{R}^2$ be a sequence of mappings with positive Jacobians that converges uniformly to a continuous mapping $f : \overline{\Omega} \to \mathbb{R}^2$. Then f satisfies the maximum modulus principle, i.e. for any nonempty open set $U \subset \Omega$ we have*

$$\max_{x\in\overline{U}} |f(x)| = \max_{x\in\partial U} |f(x)|.$$

Proof. Let $U \subset \Omega$ be an open set. By the above remark each f_n satisfies the maximum modulus principle. Let $\varepsilon > 0$ be a positive number, and let n_0 be such that $\|f_{n_0} - f\|_{\overline{\Omega}} \leq \varepsilon$. Then we have

$$\max_{x\in\overline{U}} |f(x)| \leq \max_{x\in\overline{U}} |f_{n_0}(x)| + \varepsilon = \max_{x\in\partial U} |f_{n_0}(x)| + \varepsilon \leq \max_{x\in\partial U} |f(x)| + 2\varepsilon, \tag{3.1}$$

and since ε was arbitrary, by taking limits we get

$$\max_{x\in\overline{U}} |f(x)| \leq \max_{x\in\partial U} |f(x)|,$$

and since the inverse inequality is obviously true, we have

$$\max_{x\in\overline{U}} |f(x)| = \max_{x\in\partial U} |f(x)|.$$

$\square$

The next theorem provides an example of a C^1-smooth mapping with nonnegative Jacobian that does not satisfy the maximum modulus principle, and hence cannot be approximated by mappings with positive Jacobian.

Theorem 1.4. *Define the mapping* $f : \overline{B_1(0)} \to \mathbb{R}^2$ *by the formula*

$$f(x,y) = \begin{cases} \left(8(\frac{1}{2} - \sqrt{x^2+y^2})^2, 0\right), & \text{if } x^2+y^2 \le \frac{1}{4} \\ \left(x\frac{(\sqrt{x^2+y^2}-1/2)^2}{x^2+y^2}, y\frac{(\sqrt{x^2+y^2}-1/2)^2}{x^2+y^2}\right), & \text{if } \frac{1}{4} < x^2+y^2 \le 1. \end{cases}$$

Then f *is* C^1*-smooth, has nonnegative Jacobian, and cannot be approximated by smooth mappings with positive Jacobian.*

Proof. The mapping is differentiable outside $S = \{(x,y) : x^2+y^2 = 1/4\}$, and partial derivatives go to 0, as x^2+y^2 goes to $1/4$, so it is sufficient to prove that f is differentiable on S and the derivative is equal to 0.

Let $v = (x,y)$ be a point of S, and let w be a point of $S_+ = \{(x,y) : x^2+y^2 > 1/4\}$. Then $f(v) = 0$ and $f(w) = w\frac{(|w|-1/2)^2}{|w|^2}$. Then

$$|f(w) - f(v)| = \frac{(|w|-1/2)^2}{|w|} \le 2|w-v|^2 = o(|w-v|).$$

Similarly, if w is a point of $S_- = \{(x,y) : x^2+y^2 < 1/4\}$, then $f(w) = (8(|w| - 1/2)^2, 0)$. We have

$$|f(w) - f(v)| = 8(|w|-1/2)^2 \le 8|w-v|^2 = o(|w-v|),$$

therefore the derivative of f at v is 0. The Jacobian of f on $S \cup S_-$ is equal to 0. If $v \in S_+$, then $J_f(v) = (|v|-1/2)^2(1+2|v|) > 0$. Finally, $|f(0)| = 2$, while for every v with $|v| = 1$ we have $|f(v)| = 1/4$, therefore f does not satisfy the maximum modulus principle, and hence cannot be approximated by C^1-smooth mappings with positive Jacobians. □

Note that the above example can be made C^∞-smooth, if we take the auxiliary function to be $\exp(-1/(|w|-1/2)^2)$ instead of $(|w|-1/2)^2$.

The second negative result shows that there are open mappings (hence they satisfy the maximum modulus principle automatically) that cannot be approximated by mappings with positive Jacobian. The proof requires one simple fact from covering space theory (see [6, Ch. 1.3]).

Definition 3.1. *Let B be a topological space. A covering space of B is a pair (E, π) of a topological space E, and a continuous surjective mapping $\pi : E \to B$ with the following property. For every $b \in B$ there exists an open neighbourhood U of b, discrete space I, and a homeomorphism $f : U \times I \to \pi^{-1}(U)$, such that $\pi \circ f = \pi_U$, where $\pi_U : U \times I \to U$ is the natural projection.*

The cardinality of the corresponding I is independent of $b \in B$. Any open set U that satisfies the above property is called ***evenly-covered***. If B itself is evenly-covered, the covering space (E, π) is called trivial, in this case E is homeomorphic to $B \times I$, and π is the projection onto B.

Lemma 3.6. *If Ω is a simply connected domain in $\mathbb{R}^n$, then every covering space of Ω is trivial.*

Proof. This result follows from the classification of covering spaces (see [6, Th. 1.38]). $\square$

Theorem 1.5. *Let the mapping $f : \overline{B_1(0)} \to \mathbb{C}$ be defined by the formula $f(x, y) = (x^2 - y^2, 2xy)$. Then for any C^1-smooth mapping $g : \overline{B_1(0)} \to \mathbb{R}^2$ with strictly positive Jacobian we have $\|f - g\|_{\overline{B_1(0)}} \geq 1/4$.*

Proof. Suppose that there exists a mapping g such that $\|f - g\|_{\overline{B_1(0)}} < 1/4$. Consider the homotopy $H_t(x) : [0, 1] \times \overline{B_{1/2}(0)} \to \mathbb{R}^2$ given by $H_t(x) = (1 - t)f(x) + tg(x)$. For $x \in \partial B_1(0)$ we have

$$|H_t(x)| = |f(x) - t(f(x) - g(x))| \geq |f(x)| - |f(x) - g(x)| = 1 - |f(x) - g(x)| > 1/2.$$

Since $H_0 = f$, and $H_1 = g$, using the homotopy invariance (P1) of the topological degree, we get that for every $x \in \overline{B_{1/2}(0)}$ the degrees $\deg(g, B_1(0), x)$ and $\deg(f, B_1(0), x)$ are equal. Since for all $x \in \overline{B_{1/2}(0)}$ the degree $\deg(f, B_1(0), x) = 2$, it follows that $\deg(g, B_1(0), x) = 2$ for same values of x. The mapping g has a positive Jacobian, therefore there are exactly two different solutions y_1, y_2 to $g(y) = x$, for x in $\overline{B_{1/2}(0)}$.

Consider the set $U = g^{-1}(B_{1/2}(0))$. We see that $g : U \to B_{1/2}(0)$ is a covering map and U is a double cover of $B_{1/2}(0)$. Since $B_{1/2}(0)$ is simply connected, we get by

Lemma 3.6 that U is a disjoint union of two homeomorphic copies of $B_{1/2}(0)$. Then $U = U_1 \cup U_2$ and g maps U_i homeomorphically onto $B_{1/2}(0)$, so we can define two inverse maps y_1, y_2 from $B_{1/2}(0)$ to U.

From $\|f - g\|_{\overline{B_1(0)}} < 1/4$ we have $|y_i^2(z) - z| < 1/4$. Consider the map $\gamma(\phi) = e^{i\phi}/(2+\varepsilon)$, where $\varepsilon < 1/10$. Then the points $y_i(\gamma(\phi))$ are contained in a disjoint union of disks $B_{1/2}(e^{i\phi/2}/\sqrt{2+\varepsilon}))$ and $B_{1/2}(e^{i(\phi/2+\pi)}/\sqrt{2+\varepsilon}))$. It is easy to verify that each disk contains exactly one of the points $y_1(\gamma(\phi)), y_2(\gamma(\phi))$. Then, as we continuously change ϕ from 0 to 2π, $y_1(\gamma(\phi))$ ends up in a different disk, so we must have $y_1(\gamma(0)) = y_1(\gamma(2\pi)) = y_2(\gamma(0))$. This contradiction concludes the proof. $\square$

The mapping f is just the complex analytic function $h(z) = z^2$ written as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, so it provides an example of a polynomial mapping with nonnegative Jacobian that cannot be approximated by C^1-smooth mappings with positive Jacobian. This example is easily generalized to mappings of $\mathbb{R}^n$ by taking

$$f(x_1, x_2, x_3, \ldots, x_n) = (x_1^2 - x_2^2, 2x_1x_2, x_3, \ldots, x_n).$$

Bibliography

[1] T. Bagby, L. Bos and N. Levenberg, Multivariate simultaneous approximation, Constructive Approximation, Vol. 18, No. 4, pp. 569-577 (2002).

[2] J.B. Conway, Functions of One Complex Variable, Second edition, Springer-Verlag (1977).

[3] S.K. Donaldson, D.P. Sullivan, Quasiconformal 4-maniflods, Acta Math., No 163, pp. 181-252 (1989).

[4] I. Fonseka, W. Gangbo, Degree Theory in Analysis and Applications, Oxford Lecture Series in Mathematics and Its Applications, No 2, (1995).

[5] P. Franklin, N. Wiener, Analytic Approximations to Topological Transformations, Transactions of the American Mathematical Society, Vol. 28, No. 4, pp. 762-785 (1926).

[6] A. Hatcher, Algebraic Topology, Cambridge University Press (2002).

[7] E. Heinz, An elementary analytic theory of the degree of mapping in n-dimensional space, Journal of Mathematics and Mechanics, No. 8, pp. 231-247 (1959).

[8] W. Heubsch, M. Morse, Schoenflies extensions without interior differential singularities, Annals of Mathematics, Vol. 76, No. 1, pp. 18-54 (1962).

[9] W. Heubsch, M. Morse, An explicit solution of the Schoenflies extension problem, Journal of the Mathematical Society of Japan, Vol. 12, No. 3, pp. 271-289 (1960).

[10] M. Hirsch, Differential Topology, Springer-Verlag (1976).

[11] Edwin E. Moise, Geometric Topology in Dimensions 2 and 3, New York, Springer-Verlag (1977).

[12] S. Stoilow, Leçons sur les principles topologique de la théorie des fonctions analytique, 2nd edition, Paris (1956).

[13] Donal O'Regan, Yeol Je Cho and Yu Qing Chen, Topological Degree Theory and Applications, Chapman and Hall/CRC, New York (2006).

[14] G. Whyburn, Topological analysis, Princeton University Press, Princeton (1964).