Berezin Symbols and Operator Theory

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A Thesis
Submitted to the
University of Manitoba
in partial fulfilment of the
requirements for the degree of
Master of Science

Department of Mathematics

0-612-53208-9

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES
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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

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Acknowledgements

I would like to thank the staff and faculty of the department for their help and support during my studies. I am especially grateful to Nina Zorboska for the encouragement, instruction, and direction (not to mention patience) given throughout all the lectures, reading courses, and months and months of thesis supervision. My special thanks to all those who have taught me over the years: professors, ministers, friends, family, and Mom and Dad. My love and thanks to you all.

A student is not above his teacher, but everyone who is fully trained will be like his teacher.

A special acknowledgement also goes to the Natural Sciences and Engineering Research Council of Canada for providing the financial backing that supported these studies.

Abstract

Let H be a standard analytic functional Hilbert space over a bounded domain $\Omega \subset \mathbf{C}$. We examine the Berezin symbols $\tilde{A}$ of bounded operators $A \in \mathcal{B}(H)$ and characterize the compact operators $\mathcal{K}(H)$ by Berezin symbol behavior. We show that $A \in \mathcal{K}(H)$ iff the Berezin symbol of every unitary conjugate of A is in $C_0(\Omega)$ (Nordgren and Rosenthal, 1994). Special attention is also given to examples and the theory of Berezin symbols on the Bergman and Hardy space. We show a characterization (Axler and Zheng, 1998) of compact Toeplitz operators on the Bergman space that generalizes to Hankel operators. The condition A is compact iff $\widetilde{A^*A}(z) \to 0$ as $|z| \to 1^-$ holds for all Toeplitz, Hankel, and composition operators on both the Bergman and Hardy spaces.

Contents

Chapter 1

Introduction

Operators on complex Hilbert spaces are wonderful things that have many, many talents. They have an algebraic structure. They have multiple natural topologies. They transform other objects and can mathematically model "change". Operators are useful in many different realms of mathematics, and like any frequent traveller, they carry a lot of baggage. In particular, we note a segment of canonical literature devoted to set functions of Hilbert space operators and their properties. Among these we would find the kernel and the range (functions into subsets of the Hilbert space itself), and the spectrum and numerical range (functions into subsets of the complex plane). We survey in this paper another item in the operator's baggage: a "function" function. We will define a map from bounded operators on a standard analytic functional Hilbert space to bounded continuous functions and examine some of the operator theoretic results from this association.

Let k_z be normalized reproducing kernels for a Hilbert space H. Then for the bounded operator A on H, the *Berezin symbol* of A is the bounded continuous function

$$\tilde{A}(z) = \langle Ak_z, k_z \rangle$$

and the mapping $A \longmapsto \tilde{A}$ is called the *Berezin transform*.

The Berezin symbol of an operator was first introduced by F. A. Berezin [4] as an extension of Wick symbols on the Fock space[1]. Several topics branch from his original work. One branch uses the transform as an algebraic isomorphism to formulate function spaces with a non-commutative (non-pointwise) product which is useful in the quantization of physical systems. see [5]. Another branch asks operator theoretic questions about how properties of the Berezin symbol $\tilde{A}$ are related to the properties of A. Among today's authors working in the fields of Toeplitz. Hankel. and composition operators. the Berezin symbol has become another item of baggage carried by operators that is useful in the characterization of operator classes.

Our goal in this thesis is to present the Berezin symbol $\tilde{A}$ from this operator theoretic point of view and demonstrate its relationship to the compactness of A. Specifically. in Chapter 2 we demonstrate Nordgren and Rosenthal's compactness characterization for general operators on standard functional Hilbert spaces using the Berezin symbols of unitary conjugates:

Theorem 1 *Let H be a standard functional Hilbert space over the domain Ω. Let $A \in \mathcal{B}(H)$. Then the compactness of A is characterized by the continuous extension to 0 on $\partial\Omega$ of all Berezin symbols of conjugates of A. That is*

$$A \in \mathcal{K}(H) \quad \Longleftrightarrow \quad \forall U \text{ unitary and } z_n \to z \in \partial\Omega. \lim_{n\to\infty} \widetilde{A^U}(z_n) = 0.$$

where $A^U = U^*AU$.

Subsequently, in Chapter 4. we study the compactness of Toeplitz. Hankel. and composition operators on the Bergman or Hardy spaces via the Berezin symbol function. The common condition to all the cases proven here is

$$A \text{ is compact } \Longleftrightarrow \widetilde{A^*A}(z) \to 0 \text{ as } z \to \partial\mathbf{D}.$$

where $\mathbf{D}$ is the unit disk. Chapter 3 presents the preliminaries required for this theory and shows by example that the above condition is not true in general. The centre of this analysis is a theorem by Axler and Zheng:

[1]See [4].

Theorem 2 *Let $\mathcal{T}$ be the set of all linear combinations of finite products of Toeplitz operators on the Bergman space, L^2_a. Then the compact operators in $\mathcal{T}$ are characterized by vanishing Berezin symbols. In fact, for $A \in \mathcal{T}$, the following are equivalent:*

(i). $A \in \mathcal{K}(L^2_a) \cap \mathcal{T}$

(ii). $\lim_{|z| \to 1^-} \tilde{A}(z) = 0$

(iii). $\lim_{|z| \to 1^-} \|(U_z^* A U_z)1\|_p = 0, \quad \forall\ 1 < p < \infty$

where the U_z are Möbian unitary operators as defined in section 3.1.

Rifling through this item of operator baggage lets you uncover many interesting relationships. The compactness of an operator is related to the boundary behavior of bounded continuous functions. Möbius transformations on $\mathbf{D}$ define an often-used class of self-adjoint unitary operators related to a change of variable in Berezin symbols (sections 3.1 and 3.2). Berezin symbols connect with integral operator theory (section 4.1), the theory of function spaces like L^p, BMO_∂, VMO_∂, (section 4.2.2) and classical results from complex function theory like the Weierstrauss approximation theorem (section 4.2.3), the Littlewood subordination theorem, and the Littlewood-Paley identity (section 4.3.1). And, of course, it does not stop there. Questions raised by this account of Berezin symbol theory are discussed in Chapter 5.

An outline of the topics covered in the paper is as follows:

Chapter 2: Reproducing kernels. Notation and terminology. The definition of the Berezin symbol. Compact operators (on standard H) have symbols that continuously extend to 0 on the boundary. Unitary conjugates. Essential numerical range. Theorem 1.

Chapter 3: Möbius automorphisms. The Bergman space. The Hardy space. Möbius change of variables and Möbian unitaries. Berezin symbols of Toeplitz operators, composition operators, and Möbian unitaries. Non-compact operators with Berezin symbols that vanish on $\partial\mathbf{D}$.

Chapter 4: Compact Toeplitz operators and Berezin symbols on L_a^2. Theorem 2. Compact Toeplitz operators and Berezin symbols on H^2. Compact Hankel operators and Berezin symbols on L_a^2. Berezin symbols, Hankel operators and the BMO_∂ and VMO_∂ spaces. Compact Hankel operators and Berezin symbols on H^2. Compact composition operators and Berezin symbols on H^2. Nevanlinna function. Compact composition operators and Berezin symbols on L_a^2. Generalized Nevanlinna function.

Chapter 5: Remarks and further questions.

Chapter 2

Berezin Symbols on General H

2.1 Basic Definitions and Properties

Here we will introduce the definitions and notation required for our consideration of Berezin symbols. Let $\Omega \subset \mathbf{C}$ be an open simply-connected domain for a collection of complex-valued functions. Let H be the Hilbert space of these functions under the usual pointwise vector operations and some inner product $\langle \cdot, \cdot \rangle$. (Then H is complete in the natural inner product norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$.) If the point evaluation $\epsilon_z(f) = f(z)$ for $z \in \Omega$ is a bounded linear functional on H, then by virtue of the Reisz Representation Theorem of Hilbert spaces there exists a function K_z in H with the reproducing property, $\epsilon_z(f) = \langle f, K_z \rangle$. Given a Hilbert space where point evaluations are bounded for every $z \in \Omega$, we call H a *functional Hilbert space*. The *normalized reproducing kernels* of H are the functions $k_z = K_z / \|K_z\|$, which have norm equal to one. ($\|K_z\|^2 = K_z(z)$.) We will denote the algebra of bounded linear operators on H by the symbol $\mathcal{B}(H)$, and the Hermitian adjoint of an operator $A \in \mathcal{B}(H)$ by A^*. The symbols $\mathbf{D}$ and $\partial\mathbf{D}$ will be used to represent the unit disk and its boundary respectively. The overbar $\bar{z}$ on a complex number z will be indicate the complex conjugate of z, while the set closure of Ω will be noted by $\overline{\Omega}$ where necessary.

Assume Ω to be bounded. We will often be considering function behavior in the limit near the boundary, $\partial\Omega$, of Ω. The notation

$$\lim_{z\to\partial\Omega} f(z)$$

will consistently mean the convergence of f in the limit as the distance $d(z,\partial\Omega)$ goes to zero uniformly in the Euclidean metric. Specifically considering $\Omega = \mathbf{D}$, this will typically be written

$$\lim_{|z|\to 1^-} f(z).$$

If these limits converge to 0, we will say "f vanishes on $\partial\Omega$" and reserve that phrase for such cases.

Let H be a functional Hilbert space over some bounded domain $\Omega \subset \mathbf{C}$. As defined in [11], we call H *standard* if the normalized reproducing kernels k_z satisfy the convergence property

$$z_n \to z \in \partial\Omega \quad \Longrightarrow \quad k_{z_n} \to 0 \text{ weakly.} \tag{2.1}$$

H will be *uniformly standard* if it satisfies the slightly stronger condition

$$z \to \partial\Omega \quad \Longrightarrow \quad k_z \to 0 \text{ weakly.} \tag{2.2}$$

We will most often consider Hilbert spaces of analytic functions on a bounded domain Ω, or *analytic* Hilbert spaces. When an analytic space H is a closed subspace of some larger Hilbert space H', the orthogonal projection from H' onto H will be written P. We will oftentimes be using functions from the classical $L^p(\Omega)$ spaces. These will carry their usual $\|\cdot\|_p$ norms, $1 \leq p \leq \infty$, with respect to the usual normalized Lebesgue measure dm, ie: $m(\Omega) = 1$.

Now, let us make the fundamental definition of the Berezin symbol of an operator in $\mathcal{B}(H)$.

Definition 2.1 Let H be a standard functional Hilbert space over $\Omega \subset \mathbf{C}$. Then for $A \in \mathcal{B}(H)$, the *Berezin symbol of* A, written $\widetilde{A}$, is the function defined by

$$\widetilde{A}(z) := \langle Ak_z, k_z\rangle \quad : \quad z \in \Omega.$$

The following shows that Berezin symbols of operators on analytic spaces are nicely-behaved bounded continuous functions.

Proposition 2.2 *Let H be a standard analytic functional Hilbert space over the symmetric domain Ω. The Berezin symbol of an operator $A \in \mathcal{B}(H)$ satisfies the following:*

(i). $\sup_{z\in\Omega} \left|\tilde{A}(z)\right| \leq \|A\|$. *(ie: $\tilde{A} \in L^\infty(\Omega)$).*

(ii). *$\tilde{A}$ is real analytic (ie: in $C^\infty(x,y)$) and hence is continuous.*

(iii). $\widetilde{A^*} = \overline{\tilde{A}}$.

Proof.

(i). From the definition, $\left|\tilde{A}(z)\right| = |\langle Ak_z, k_z\rangle| \leq \|Ak_z\| \, \|k_z\|$ by the Cauchy-Schwarz inequality. Since $\|k_z\| = 1$ for all z, it follows that $\left|\tilde{A}(z)\right| \leq \|A\|$ independent of z.

(ii). Let $w, z \in \Omega$ and let K_w and K_z be the reproducing kernels in H for these points. Consider the function in two complex variables defined by $\dot{A}(w,z) := \langle AK_{\overline{w}}, K_z\rangle$. Since Ω is symmetric (ie: $w \in \Omega$ iff $\overline{w} \in \Omega$), $AK_{\overline{w}} \in H$ is analytic, whereby we note that $\dot{A}$ is analytic in z because of the reproducing property of K_z. Considering the equality $\dot{A}(w,z) = \overline{\langle A^* K_z, K_{\overline{w}}\rangle}$, we similarly get that $\dot{A}$ is analytic in w. Therefore $\dot{A}$ is analytic on $\Omega \times \Omega$. Considering the special case of A equalling the identity, we also get that $\langle K_{\overline{w}}, K_z\rangle$ is analytic on $\Omega \times \Omega$. The quotient function defined by

$$Q(w,z) := \frac{\langle AK_{\overline{w}}, K_z\rangle}{\langle K_{\overline{w}}, K_z\rangle} = \frac{\dot{A}(w,z)}{K_{\overline{w}}(z)} \tag{2.3}$$

is therefore analytic wherever $\langle K_{\overline{w}}, K_z\rangle \neq 0$. Recall that a function is real analytic on Ω if it is expressible as a power series in the real and imaginary coordinates that converges absolutely and uniformly on compact subsets of Ω. It is sufficient, then, that a function be analytic in z and $\overline{z}$ for it to be real analytic in (x,y). It follows that $\tilde{A}(z) = Q(\overline{z}, z)$ is real analytic by the subsitution $w = \overline{z}$ in (2.3). Hence $\tilde{A}$ has infinitely many derivatives with respect to (x,y) (or $(z,\overline{z})$) and is continuous in z. (That $\tilde{A}$ is a restriction to the antidiagonal $\overline{w} = z$ of $\Omega \times \Omega$ of the analytic function $Q(w, z)$ shows also that the map $A \longmapsto \tilde{A}$ is one-to-one.)

(iii). The identity follows from $\widetilde{A^*}(z) = \langle A^* k_z, k_z \rangle = \langle k_z, Ak_z \rangle = \overline{\langle Ak_z, k_z \rangle} = \overline{\tilde{A}}(z)$. □

The mapping $A \longmapsto \tilde{A}$, called the *Berezin transform*, is also well-behaved. It is a continuous linear and injective map from $\mathcal{B}(H)$ to $C^{\infty}(\Omega)$. Property (iii) of the above also shows that the transform preserves involution. We note, however, that $\widetilde{A_1 A_2} \neq \widetilde{A_1}\widetilde{A_2}$, so the transform is not multiplicative.

The Berezin transform also has several good operator theoretic properties which we now consider. Recall that the definition of the *numerical range*, $W(A)$, of the operator $A \in \mathcal{B}(H)$ is

$$W(A) = \{\langle Af, f \rangle : \|f\| = 1\}.$$

Clearly $\tilde{A}(\Omega)$ is a subset of $W(A)$, since the k_z's are of unit norm. Therefore, the Berezin transform inherits some properties of $W(A)$, namely,

(i). If A is positive, $A \geq 0$, then $\tilde{A} \geq 0$.

(ii). If A is self-adjoint, $A^* = A$, then $\tilde{A}$ is real.

(iii). If A is scalar, $A = \lambda I$, then $\tilde{A}$ is the constant λ.

Each of these proofs are trivial uses of the inner product, and so we will not include them here. But of particular interest to our discussion is the behavior of the Berezin symbol when the operator A is compact.

Recall that an operator $A \in \mathcal{B}(H)$ is compact if the image of the open unit ball of H has compact closure. This is equivalent to the property that A maps all weakly converging sequences into strongly converging ones. (See [7].)

Proposition 2.3 *Let A be a compact operator on a standard functional Hilbert space H. Then*

$$\lim_{z_n \to z \in \partial\Omega} \tilde{A}(z_n) = 0.$$

Proof. Using the "standardness" of H, (2.1), and the property that a compact A maps weakly converging sequences into strongly converging ones, the result directly follows from the Cauchy-Schwarz inequality:

$$\left|\tilde{A}(z_n)\right| \leq \|Ak_{z_n}\| \, \|k_{z_n}\| = \|Ak_{z_n}\| \to 0. \quad \Box$$

If $\tilde{A}$ is also continuous, as it is when H is analytic, then the above is equivalent to A compact $\Rightarrow \tilde{A} \in C_0(\Omega)$.

We are interested to know if the converse to Proposition 2.3 is also true. It is not, however, as we will show with some counterexamples later in sections 3.1 and 3.2, but there is a stronger condition on Berezin symbol boundary behavior that gives necessary and sufficient conditions for the compactness of an operator. We demonstrate this in the next section.

2.2 General Compactness Condition

The set of compact operators forms the largest two-sided ideal in the algebra $\mathcal{B}(H)$. Denote this ideal $\mathcal{K}(H)$. The image of this ideal under the Berezin transform must be characterized by some function theoretic property — but what property? We have remarked that having $\tilde{A}$ continuously extend to zero on $\partial\Omega$ is not sufficient to make A compact. We proceed now to work toward a result due to Nordgren and Rosenthal [11] which characterizes the compactness of a general operator in $\mathcal{B}(H)$ in terms of the Berezin symbol boundary behavior of all operators unitarily equivalent to it. For notational purposes, we introduce an algebraic notation for unitary equivalence.

Definition 2.4 Let H be a standard functional Hilbert space and let U be a bounded unitary operator on H. Then for any $A \in \mathcal{B}(H)$ the *conjugate of A by U* is defined as

$$A^U := U^* A U. \tag{2.4}$$

By definition, then, B is unitarily equivalent to A iff B is a conjugate of A. Note also that conjugates behave well under adjoints and products, since

$$\begin{aligned} \left(A^U\right)^* &= (A^*)^U, \\ (A_1 A_2)^U &= A_1^U A_2^U. \end{aligned}$$

Now, in general, $\widetilde{A}(z) = \langle Ak_z, k_z\rangle \neq \widetilde{A^U}(z) = \langle AUk_z, Uk_z\rangle$, and in some cases their boundary behavior can be strikingly different (see Example 3.16). This lack of uniformity in symbol behavior is what prevents Proposition 2.3 from characterizing the image of the ideal $\mathcal{K}(H)$ in the Berezin symbols: the function theoretic property of compactness must be shared by $\widetilde{A}$ and $\widetilde{A^U}$.

To help in the proof of the general compactness condition, we observe the following definition from [9]:

Definition 2.5 The *essential numerical range* of an operator $A \in \mathcal{B}(H)$ is the set $W_e(A) \subset \mathbf{C}$ given by the closed numerical range common to all compact perturbations of A. That is

$$W_e(A) = \bigcap \{\overline{W(A+K)} : K \in \mathcal{K}(H)\}$$

where $W(A)$ is the numerical range of A. Equivalently, $\lambda \in W_e(A)$ is characterized (Corollary to Theorem 5.1 of [9]) by

$$\langle Ah_n, h_n\rangle \to \lambda \text{ for some weak null sequence } \{h_n\},\ \|h_n\| = 1. \tag{2.5}$$

or

$$\langle Ag_n, g_n\rangle \to \lambda \text{ for some orthonormal sequence } \{g_n\}. \tag{2.6}$$

This is relevant to our pursuit by its appearance in the following useful descriptions of compactness.

Proposition 2.6 (Compactness) *For an operator $A \in \mathcal{B}(H)$, the following are equivalent:*

(i). *A is compact.*

(ii). *A^* is compact.*

(iii). *A^*A is compact.*

(iv). *There exists a sequence A_n of compact operators such that $A_n \to A$ (in the operator norm topology).*

(v). *The essential numerical range of A is trivial, ie: $W_e(A) = \{0\}$.*

References: The equivalence of (i) - (iv) is fundamental and can be found in any text like [18]. That (v) is equivalent to (i) follows directly from the Calkin algebra characterization of the essential numerical range.

We are now able to present the proof of the general theorem characterizing the compactness of A in function theoretic terms.

Theorem 1 (Nordgren and Rosenthal, 1994) *Let H be a standard functional Hilbert space over Ω. Let $A \in \mathcal{B}(H)$. Then the compactness of A is characterized by the continuous extension to 0 on $\partial\Omega$ of the Berezin symbols of all conjugates of A. That is*

$$A \in \mathcal{K}(H) \quad \Longleftrightarrow \quad \forall U \text{ unitary and } z_n \to z \in \partial\Omega, \; \lim_{n\to\infty} \widetilde{A^U}(z_n) = 0.$$

The proof is done by demonstrating that the boundary cluster values of $\widetilde{A^U}$ determine the essential numerical range, ie:

$$W_e(A) = \left\{\lambda = \lim_{n\to\infty} \widetilde{A^U}(z_n) : z_n \to z \in \partial\Omega, U \text{ unitary}\right\}. \tag{2.7}$$

Then the essential numerical range compactness condition in Proposition 2.6 extends to give the conclusion. When the unitary operator U is given, the proof of inclusion ($\supset$) is quite simple. However, the other direction requires us to produce from the condition (2.6) a unitary operator that will give us the right Berezin symbol. For this, we make use of the following lemma.

Lemma 2.7 (Dixmier) *Let $\{f_n\}$ be a weak null sequence of unit vectors in a Hilbert space. Then there exists a subsequence which is "approximately orthonormal". That is, for any sequence of positive numbers δ_m, we can find an orthonormal sequence $\{h_m\}$ and a subsequence $\{f_{n_m}\}$ such that $\|f_{n_m} - h_m\| < \delta_m \; \forall m$.*

Proof of Lemma. We use a Grahm-Schmidt orthonormalization process. Start the induction process at $n_1 = 1$ and $h_1 = f_1$, and proceed from the mth stage by defining the vectors

$$g_{m,k} = \frac{f_k - P_m(f_k)}{\|f_k - P_m(f_k)\|}$$

where $P_m = \sum_{j=1}^{m} \langle \cdot, h_j \rangle h_j$ is the projection onto the subspace spanned by $\{h_1, \ldots, h_m\}$. Note that since the f_k's are weakly converging to zero, then $P_m f_k \to 0$ as k increases. Using $R_m f_k := f_k - P_m f_k$ for notational simplicity, we have that,

$$\lim_{k\to\infty} \|R_m f_k\| = \lim_{k\to\infty} \|f_k\| = 1$$

and so,

$$\begin{aligned} \|f_k - g_{m,k}\| &= \frac{\|(\|R_m f_k\| - 1) f_k + P_m f_k\|}{\|R_m f_k\|} \\ &\leq \frac{1 - \|R_m f_k\|}{\|R_m f_k\|} \|f_k\| + \frac{\|P_m f_k\|}{\|R_m f_k\|} \to 0. \end{aligned}$$

So we pick sufficently large K validating $\|f_K - g_{m,K}\| < \delta_{m+1}$ and set $n_{m+1} = K$ and $h_{m+1} = g_{m,K}$ to fully describe the next stage. $\square$

Proof of Theorem 1.

Let $A \in \mathcal{B}(H)$ be given. Recall we wish to prove

$$W_e(A) = \left\{ \lambda = \lim_{n\to\infty} \widetilde{A^U}(z_n) : z_n \to z \in \partial\Omega, U \text{ unitary} \right\}.$$

($\supset$) Let $\lambda \in \mathbf{C}$ and let U be unitary such that for a sequence z_n converging to $\partial\Omega$ we have

$$\lambda = \lim_{n\to\infty} \widetilde{A^U}(z_n) = \lim_{n\to\infty} \langle AUk_{z_n}, Uk_{z_n}\rangle$$

Now, since H is assumed standard, the sequence $\{k_{z_n}\}$ is weakly converging to zero. Then the functions Uk_{z_n} also weakly converges to zero (U is continuous) and so by applying the first Fillmore-Stampfli-Williams characterization of the essential numerical range (2.5) to the above, we have $\lambda \in W_e(A)$.

($\subset$) Now, let $\lambda \in W_e(A)$ where, by (2.6), $\{g_n\}$ is an orthonormal sequence such that $\lim_{n\to\infty} \langle Ag_n, g_n\rangle = \lambda$. Let also $\{z_n\}$ be a sequence converging to a point $z \in \partial\Omega$ so that we have the weak null sequence $\{f_n = k_{z_n}\}$. From Dixmier's lemma generate an orthonormal sequence $\{h_n\}$ "approximate" to the reproducing kernels. Without loss of generality, let us assume (by passing to subsequences and relabeling) that

$$\lim_{n\to\infty} \|h_n - f_n\| = 0. \tag{2.8}$$

and also assume that both the sequences $\{g_n\}$ and $\{h_n\}$ have an infinite orthocomplement. The unitary operator matching $Uh_n = g_n$ and the orthocomplements is then well defined.

So a calculation shows

$$\lim_{n\to\infty} \left\langle A^U h_n, h_n \right\rangle = \lim_{n\to\infty} \langle Ag_n, g_n\rangle = \lambda.$$

But the h_n approximation to $f_n = k_{z_n}$ is good enough, for by (2.8),

$$\begin{aligned} \left|\left\langle A^U h_n, h_n\right\rangle - \left\langle A^U f_n, f_n\right\rangle\right| &= \left|\left\langle A^U (h_n - f_n), h_n\right\rangle - \left\langle A^U f_n, (f_n - h_n)\right\rangle\right| \\ &\leq \|A\| \cdot \|h_n - f_n\| + \|A\| \cdot \|h_n - f_n\| \to 0, \end{aligned}$$

giving that λ is also a boundary cluster value of $\widetilde{A^U}$ at $z \in \partial\Omega$ as required:

$$\lim_{n\to\infty} \left\langle A^U k_{z_n}, k_{z_n}\right\rangle = \lim_{n\to\infty} \left\langle A^U f_n, f_n\right\rangle = \lambda.$$

Thus the equality (2.7) is proven. $\square$

This theorem therefore identifies the ideal $\mathcal{K}(H)$ with the functional theoretic property that the family of functions $\widetilde{A^U}$ all be continuously extendable to 0 on $\partial\Omega$. ie: a behavior invariant under unitary conjugation. By reformulating the expression of Theorem 1 with the operator A^*A we get

$$A \in \mathcal{K}(H) \iff \forall U \text{ unitary and } z_n \to z \in \partial\Omega. \lim_{n\to\infty} \widetilde{(A^*A)^U}(z_n) = 0.$$

which in turn can be written as

$$A \in \mathcal{K}(H) \iff \forall U \text{ unitary and } z_n \to z \in \partial\Omega. \lim_{n\to\infty} \|AUk_{z_n}\| = 0.$$

In this way, the collection of weak null sequences $\{\{Uk_{z_n}\} : U \text{ unitary and } z_n \to z \in \partial\Omega\}$. is sufficient to determine the behavior of all weak null sequences.

We will leave the general case with a summary of the general compactness condition.

Corollary 2.8 *Let H be a standard functional Hilbert space. Let $A \in \mathcal{B}(H)$. Then the following are equivalent.*

(i). *A is compact.*

(ii). $\lim_{n\to\infty} \widetilde{(A^*A)^U}(z_n) = 0$. *$\forall U$ unitary and $z_n \to z \in \partial\Omega$.*

(iii). $\lim_{n\to\infty} \|AUk_{z_n}\| = 0$. *$\forall U$ unitary and $z_n \to z \in \partial\Omega$.*

(iv). $\lim_{n\to\infty} \widetilde{A^U}(z_n) = 0$. *$\forall U$ unitary and $z_n \to z \in \partial\Omega$.*

If H is analytic. then (iv) is equivalent to $\{\widetilde{A^U} : U \text{ unitary}\} \subset C_0(\Omega)$.

Chapter 3

Berezin Symbols on the Bergman and Hardy Spaces

We want to examine the Berezin symbol behavior of some concrete operators on two classical examples of standard analytic functional Hilbert spaces over the unit disk $\mathbf{D}$. These are the Bergman space L_a^2, and the Hardy space H^2. In the following, we define these spaces and develop the theory of Berezin symbols on them from the definition of their reproducing kernels. We also present some concrete examples of symbol behavior at the boundary $\partial\mathbf{D}$.

First, let us note some properties of automorphisms on $\mathbf{D}$ that will be an important technique in our analysis. The following properties of Möbius transformations are easily found in texts such as [7, 18] or can be proved by simple technical calculations.

Definition 3.1 The *Möbius Transformations* φ_a defined for each $a \in \mathbf{D}$ as

$$\varphi_a(w) = \frac{a - w}{1 - \bar{a}w}$$

satisfy the following properties:

(i). φ_a is analytic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$.

(ii). $\varphi_a(0) = a$ and $\varphi_a(a) = 0$.

(iii). φ_a is injective, $\varphi_a(\mathbf{D}) = \mathbf{D}$ and $\varphi_a(\partial\mathbf{D}) = \partial\mathbf{D}$. Furthermore, $\varphi_a^{-1} = \varphi_a$.

(iv). Any disk automorphism ψ can be written as $\psi = \lambda\varphi_a$ for some $\lambda \in \partial\mathbf{D}$, and $a \in \mathbf{D}$.

(v). $1 - \bar{a}\varphi_a(w) = (1 - |a|^2)/(1 - \bar{a}w)$.

(vi). $|\varphi_a(w)| = |\varphi_w(a)|$ and

$$1 - |\varphi_a(w)|^2 = \frac{(1 - |a|^2)(1 - |w|^2)}{|1 - \bar{a}w|^2}. \tag{3.1}$$

We present in the first section the development for L_a^2, and where the Hardy space proofs are similar, they will be omitted in section 3.2.

3.1 The Bergman Space

Definition 3.2 The *Bergman space*, L_a^2, is the closed subspace of analytic functions in $L^2(\mathbf{D})$. That is

$$L_a^2 = \left\{ f \text{ analytic on } \mathbf{D} \ : \ \int_{\mathbf{D}} |f|^2 \, dm < \infty \right\},$$

where dm is the normalized Lebesgue measure on $\mathbf{D}$, and the inner product is the usual

$$\langle f, g \rangle = \int_{\mathbf{D}} f(w)\overline{g(w)}\, dm(w).$$

That each f in L_a^2 is analytic, of course, implies it has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which converges absolutely and uniformly on compact subsets of $\mathbf{D}$. The following properties of L_a^2 are fundamental and easily proved, see [18].

Proposition 3.3 *The following are fundamental properties of the Bergman space.*

(i). *L_a^2 has the canonical orthonormal basis*

$$\{e_n(z) = \sqrt{n+1}\, z^n : n = 0, 1 \ldots\}.$$

(ii). *For every z in $\mathbf{D}$, L_a^2 has the reproducing kernel $K_z(w) = 1/(1 - \bar{z}w)^2$.*

(iii). *Let $L^p(\mathbf{D})$ be the classical Banach space of Lebesgue-p-integrable functions and let $L_a^p(\mathbf{D})$ be the subset of analytic L^p functions. Then the* Bergman projection *$P : L^p(\mathbf{D}) \longmapsto L_a^p(\mathbf{D})$ has the integral form*

$$Pf(z) = \int_{\mathbf{D}} f(w)\overline{K_z(w)}\, dm(w)$$

and $\|Pf\|_p < c_p \|f\|_p$ for all $1 < p < \infty$. (See specifically [18, page 55].)

The use of the L^p space and norm in the last property will be useful in later sections, but unless otherwise specified we will always be considering the Bergman projection between the Hilbert spaces L^2 and L_a^2.

It follows directly from the above definition of the reproducing kernels that the normalized Bergman reproducing kernels k_z are defined by

(3.2) $$k_z(w) = \frac{1 - |z|^2}{(1 - \bar{z}w)^2} \qquad \text{for every } z \in \mathbf{D}.$$

The following properties of normalized reproducing kernels will be used repeatedly in our calculations. The first property proves that L_a^2 is in fact uniformly standard.

Proposition 3.4 *The Bergman normalized reproducing kernels, $k_z \in L_a^2$, satisfy the following:*

(i). *$k_z \to 0$ weakly as $|z| \to 1^-$. (ie: L_a^2 is uniformly standard.)*

(ii). *Let φ_z be a Möbius transformation. Then $\varphi_z' = -k_z$.*

(iii). *$k_z \circ \varphi_z(w) = k_z(w)^{-1}$.*

(iv). *$k_z(w) = (1 - |z|^2) \sum_{n=0}^{\infty} (n+1)\, \bar{z}^n w^n$.*

Proof.

(i). Let $f \in L_a^2$ be arbitrary and let $\epsilon > 0$. f is the L^2 limit of its MacLaurin series, so there exists a polynomial p_n to within $\epsilon/2$ of f in the norm. Now $\langle p_n, k_z\rangle = (1-|z|^2)\, p_n(z)$

is bounded by the finite value $(1-|z|^2)\,\|p_n\|_\infty$. Therefore picking $|z|$ close enough to one, we get

$$|\langle f, k_z\rangle| = |\langle f - p_n, k_z\rangle + \langle p_n, k_z\rangle| \le \|f - p_n\| + (1-|z|^2)\,\|p_n\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

(ii). We proceed by direct calculation using the quotient rule. Let $z \in \mathbf{D}$.

$$\begin{aligned}\varphi_z'(w) = \frac{d}{dw}\frac{z-w}{(1-\bar{z}w)} &= \frac{(1-\bar{z}w)(-1)-(z-w)(-\bar{z})}{(1-\bar{z}w)^2} \\ &= \frac{-1+|z|^2}{(1-\bar{z}w)^2} = -k_z(w).\end{aligned}$$

(iii). Differentiate the Möbius identity $w = \varphi_z(\varphi_z(w))$. Then using the substitution from the preceeding property we get

$$1 = \varphi_z'(\varphi_z(w))\varphi_z'(w) = k_z \circ \varphi_z(w)k_z(w).$$

(iv). This follows directly from the MacLaurin expansion of $(1-\bar{z}w)^{-2}$. □

It will be useful to note that properties (ii) and (iii) combine to make the identity

$$\varphi_a' \circ \varphi_a(w) = -\frac{(1-\bar{a}w)^2}{1-|a|^2}. \tag{3.3}$$

Using the propositions stated, we can now show how the inner product of the Bergman space reacts to an automorphic change of variables.

Lemma 3.5 (Möbius Change of Variables) *Let $f, g \in L_a^2$ and let φ_a be the Möbius transformation for $a \in \mathbf{D}$. Then*

$$\langle f \circ \varphi_a, g \circ \varphi_a\rangle = \langle k_a f, k_a g\rangle. \tag{3.4}$$

Proof. Let f, g and φ_a be as given. Then

$$\begin{aligned}\langle f \circ \varphi_a, g \circ \varphi_a\rangle &= \int_{\mathbf{D}} f \circ \varphi_a(w)\overline{g \circ \varphi_a(w)}\, dm(w) \\ &= \int_{\mathbf{D}} f(z)\overline{g(z)}\,|\varphi_a'(z)|^2\, dm(z)\end{aligned}$$

by the change of variable $z = \varphi_a(w)$. Therefore, using (ii) from the previous Proposition we get

$$\begin{aligned}\langle f \circ \varphi_a, g \circ \varphi_a \rangle &= \int_{\mathbf{D}} f(z)\overline{g(z)}\,|-k_a(z)|^2 \, dm(z) \\ &= \int_{\mathbf{D}} k_a(z) f(z) \overline{k_a(z) g(z)} \, dm(z) = \langle k_a f, k_a g \rangle\end{aligned}$$

as desired. □

The Toeplitz operators on L^2_a will be an important class of operators for us to study. Here we give their definition and note some useful properties.

Definition 3.6 Let $u \in L^\infty(\mathbf{D})$, and let P be the Bergman projection. Then u induces a *Toeplitz operator* T_u on L^2_a defined by $T_u f := P(uf)$ for all $f \in L^2_a$. $T_u \in \mathcal{B}(L^2_a)$ and satisfies the following:

(i). $\|T_u\| = \|u\|_\infty$.

(ii). $T_{\alpha u + \beta v} = \alpha T_u + \beta T_v$ for $u, v \in L^\infty(\mathbf{D})$ and $\alpha, \beta \in \mathbf{C}$.

(iii). $T_u^* = T_{\overline{u}}$.

(iv). T_u has the integral form

$$(T_u f)(z) = \int_{\mathbf{D}} u(w) f(w) \overline{K_z(w)} \, dm(w).$$

Reference: pages 105–106, [18].

Now, let us work out our first example of a Berezin symbol. The Berezin symbols of Toeplitz operators are so often of interest, $\widetilde{T_u}$ is predominantly written shorthand in the literature as $\tilde{u}$.

Example 3.7 (Berezin Symbol of T_u) Let T_u be a Toeplitz operator on the Bergman space. Then,

$$\tilde{u}(z) = \int_{\mathbf{D}} u \circ \varphi_z(w) \, dm(w) = \widetilde{u \circ \varphi_z}(0).$$

For $\tilde{u}(z) = \langle uk_z, k_z\rangle = \langle u \circ \varphi_z, 1\rangle = \langle Pu \circ \varphi_z k_0, k_0\rangle$ by the change of variable formula (3.4). □

Next we examine the class of composition operators which will also be important for both the theory and providing concrete examples.

Definition 3.8 Let $\phi : \mathbf{D} \longmapsto \mathbf{D}$ be analytic. Then ϕ induces a *composition operator* C_ϕ on L^2_a, defined by $C_\phi f := f \circ \phi$ for all $f \in L^2_a$. The composition operator C_ϕ satisfies the following:

(i). $C_\phi \in \mathcal{B}(L^2_a)$.

(ii). $C_{\phi_2} C_{\phi_1} = C_{\phi_1 \circ \phi_2}$.

(iii). $C_\phi(fg) = C_\phi f \cdot C_\phi g$.

Reference: page 117 of [8].

The definition of C_ϕ makes it very easy to calculate $\widetilde{C_\phi}$, and also demonstrates how the boundary behavior of ϕ is important to the Berezin symbol behavior.

Example 3.9 (Berezin Symbol of C_ϕ) Let C_ϕ be a composition operator on the Bergman space. Then,

$$\widetilde{C_\phi}(z) = \left(\frac{1 - |z|^2}{1 - \bar{z}\phi(z)}\right)^2 .$$

For $\widetilde{C_\phi}(z) = \langle k_z \circ \phi, k_z\rangle = (1 - |z|^2) k_z \circ \phi(z)$ by the reproducing property of the kernel K_z. □

The following class of weighted composition operators have been a principal tool in proofs involving Berezin symbols on L^2_a, see [19] and [2] among others. These unitary operators are fundamental in how they arise from the Möbius change of variables and their relation to normalized reproducing kernels, equation (3.4).

Proposition 3.10 (Mõbian Unitary Operators) *Let $a \in \mathbf{D}$. The weighted composition operator on L_a^2 defined by $U_a f := k_a C_{\varphi_a} f$ satisfies the following:*

(i). *U_a is a self-adjoint unitary operator.*

(ii). $U_a K_z = \overline{k_a(z)} K_{\varphi_a(z)}$.

(iii). $U_a k_z = \alpha k_{\varphi_a(z)}$ *where* $\alpha = \alpha_a(z) \in \partial\mathbf{D}$.

Proof.

(i). Let f, g be arbitrary elements of L_a^2. First, we will prove that U_a is self-adjoint.

$$\langle U_a^* f, g\rangle = \langle f, U_a g\rangle = \langle f, k_a g \circ \varphi_a\rangle = \langle k_a f \circ \varphi_a, k_a(k_a \circ \varphi_a g)\rangle = \langle U_a f, (k_a k_a \circ \varphi_a) g\rangle$$

where we have used the change of variable (3.4) and the identity $\varphi_a \circ \varphi_a(w) = w$. Now, by Proposition 3.4, $(k_a k_a \circ \varphi_a)(w) = w$ and thus $\langle U_a^* f, g\rangle = \langle U_a f, g\rangle$ from the above.

That U_a is unitary follows similarily. Simply rewrite Lemma 3.5 replacing f by $f \circ \varphi_a$, and simplify the result with the identity (iii) from Proposition 3.4.

Combining these results yields

$$U_a^{-1} = U_a^* = U_a \tag{3.5}$$

proving (i).

(ii). Let $f \in L_a^2$ be arbitrary. Then as Axler did in [2, Proposition 2.26], we calculate.

$$\begin{aligned} \langle f, U_a K_z\rangle &= \langle U_a f, K_z\rangle = U_a f(z) = k_a(z) f(\varphi_a(z)) \\ &= k_a(z) \left\langle f, K_{\varphi_a(z)} \right\rangle = \left\langle f, \overline{k_a(z)} K_{\varphi_a(z)} \right\rangle. \end{aligned}$$

proving the identity.

(iii). This identity comes from adjusting the former with the appropriate norms.

$$U_a k_z = \frac{1}{\|K_z\|} U_a K_z = \frac{1}{\|K_z\|} \overline{k_a(z)} K_{\varphi_a(z)} = \left(\frac{\left\|K_{\varphi_a(z)}\right\|}{\|K_z\|} \overline{k_a(z)} \right) k_{\varphi_a(z)}.$$

Setting $\alpha = \alpha_a(z) = \left\|K_{\varphi_a(z)}\right\| \|K_z\|^{-1}\overline{k_a(z)}$ we get the desired identity. To prove $\alpha_a(z) \in \partial\mathbf{D}$ we use (3.1) and (3.2) and calculate

$$\alpha_a(z) = \frac{1-|z|^2}{1-|\varphi_a(z)|^2}\frac{1-|a|^2}{(1-a\bar{z})^2} = \frac{|1-a\bar{z}|^2}{(1-a\bar{z})^2}. \quad \square \tag{3.6}$$

The following discussion serves as a counterexample to the converse of Proposition 2.3. It proves that the Berezin symbol of many non-compact operators vanish at the boundary.

Example 3.11 (Berezin Symbol of Möbian Unitaries) Consider the Möbian unitary operator U_a. The Berezin symbol $\widetilde{U_a}(z)$ is then

$$\begin{aligned} \widetilde{U_a}(z) &= \langle U_a k_z, k_z\rangle = \left\langle \alpha_a(z) k_{\varphi_a(z)}, k_z \right\rangle \\ &= \alpha_a(z)\left\langle k_{\varphi_a(z)}, k_z\right\rangle = \alpha_a(z) k_{\varphi_a(z)}(z)/\|K_z\|. \end{aligned} \tag{3.7}$$

Now, since

$$\begin{aligned} 1-\bar{z}\varphi_a(z) &= \frac{1-\bar{a}z-\bar{z}(a-z)}{1-\bar{a}z} = \frac{1-\bar{a}z-\bar{z}a+|z|^2}{1-\bar{a}z} \\ &= \frac{1-|a|^2+(|a|^2-\bar{a}z-\bar{z}a+|z|^2)}{1-\bar{a}z} = \frac{1-|a|^2+|\bar{a}-z|^2}{1-\bar{a}z} \end{aligned}$$

we observe that

$$|1-\bar{z}\varphi_a(z)|^{-1} \le \frac{|1-\bar{a}z|}{1-|a|^2}$$

implying that for a fixed a in $\mathbf{D}$,

$$\left|\frac{1-|\varphi_a(z)|^2}{(1-\overline{\varphi_a(z)}z)^2}\right| \le (1-|\varphi_a(z)|^2)\left(\frac{|1-\bar{a}z|}{1-|a|^2}\right)^2 \le \frac{4}{1-|a|^2}. \tag{3.8}$$

This proves that, for fixed $a \in \mathbf{D}$, $k_{\varphi_a(z)}(z)$ is bounded above since the far left hand side of (3.8) is $\left|k_{\varphi_a(z)}(z)\right|$. Therefore, from (3.7), we observe that the Berezin symbol $\widetilde{U_a}$ vanishes on the boundary since

$$\lim_{|z|\to 1^-}\left|\widetilde{U_a}(z)\right| \le |\alpha_a(z)|\cdot|k_{\varphi_a(z)}(z)|\cdot\|K_z\|^{-1} \le const.\,\|K_z\|^{-1}$$

and $\|K_z\|^{-1} = (1-|z|^2)$ converges to zero as $z \to \partial\mathbf{D}$. However U_a is unitary and highly non-compact. $\square$

Theorem 1 proves that for any operator $A \in \mathcal{K}(L^2_a)$, the family of Berezin symbols created by unitary conjugation will satisfy $\{\widetilde{A^U}\} \subset C_0(\mathbf{D})$. The following demonstrates that the Möbian conjugates are too well-mannered to be important in the theorem. The identity makes for a very useful technique in calculations, and so we specifically name the operators A^{U_z} for $z \in \mathbf{D}$ the *Möbian conjugates of* A. Due to the relationship between Möbian unitaries and automorphic changes of variable and the precedent of Example 3.7, it may come as no surprise that they induce a change of variable in the Berezin symbol.

Proposition 3.12 (Berezin Symbols of Möbian Conjugates) *Let $A \in \mathcal{B}(L^2_a)$. Let U_z be the Möbian unitary for $z \in \mathbf{D}$. Then*

$$\widetilde{A^{U_z}} = \tilde{A} \circ \varphi_z \tag{3.9}$$

Proof. We use property (ii) of Proposition 3.10 to calculate

$$\begin{aligned}\widetilde{A^{U_z}}(w) &= \left\langle A^{U_z}k_w, k_w\right\rangle = \langle AU_zk_w, U_zk_w\rangle = |\alpha_z(w)|^2 \left\langle Ak_{\varphi_z(w)}, k_{\varphi_z(w)}\right\rangle \\ &= \tilde{A}(\varphi_z(w)). \quad \square\end{aligned}$$

Clearly if $\tilde{A}(w) \to 0$ as $|w| \to 1^-$, then so does $\widetilde{A^{U_z}}$ since, for each $z \in \mathbf{D}$, it follows from the continuity of φ_z that $|\varphi_z(w)| \to 1^-$ too. (ie: $\tilde{A} \in C_0(\mathbf{D}) \Rightarrow \{\widetilde{A^{U_z}} : z \in \mathbf{D}\} \subset C_0(\mathbf{D})$.)

3.2 The Hardy Space

The second standard analytic functional Hilbert space we will study is the Hardy space H^2.

Definition 3.13 The *Hardy space*, $H^2 = H^2(\mathbf{D})$, is the Hilbert space of analytic functions on $\mathbf{D}$ satisfying the growth condition

$$\sup_{0<r<1} \int_0^{2\pi} \left|f(re^{i\theta})\right|^2 \frac{d\theta}{2\pi} < \infty,$$

where the inner product is defined by

$$\langle f, g\rangle = \lim_{r\to 1^-} \int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\frac{d\theta}{2\pi}.$$

Reference: pages 9–12 of [8].

All functions $f \in H^2$ have a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with coefficients a_n from l^2, ie: $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is possible to consider H^2 as a subspace of $L^2(\partial\mathbf{D})$ by looking at the extension of these series onto $\partial\mathbf{D}$ (ie: $H^2(\partial\mathbf{D})$). There, the inner product has the form

$$\langle f, g \rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}.$$

Just as with the Bergman space, we will begin with a list of properties of H^2.

Proposition 3.14 *The following are fundamental properties of H^2.*

(i). *H^2 has the canonical orthonormal basis $\{e_n(z) = z^n : n = 0, 1, \ldots\}$.*

(ii). *For every z in $\mathbf{D}$, H^2 has the reproducing kernel $K_z(w) = 1/(1 - \bar{z}w)$.*

(iii). *H^2 is a subspace of harmonic functions.*

$$H^2(\mathbf{D}) \subset \mathcal{H} = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n + b_{n+1}\bar{z}^{n+1} : \sum_n |a_n|^2 + |b_{n+1}|^2 < \infty \right\}.$$

(iv). *Let $L^p(\partial\mathbf{D})$ be the classical Banach spaces and let $H^p(\mathbf{D})$ be the classical Hardy spaces. Then the Szëgo projection $P : L^p(\partial\mathbf{D}) \to H^p(\mathbf{D})$ has the integral form*

$$Pf(z) = \int_0^{2\pi} f(e^{i\theta})\overline{K_z(e^{i\theta})}\frac{d\theta}{2\pi}.$$

and $\|Pf\|_p < c_p \|f\|_p$ for all $1 < p < \infty$. (See [18, pg. 164].)

Reference: See Zhu [18].

It follows directly from the above definition that the normalized Hardy reproducing kernels are defined by

$$k_z(w) = \frac{\sqrt{1 - |z|^2}}{1 - \bar{z}w} \qquad \text{for every } z \in \mathbf{D}. \tag{3.10}$$

The following results on Hardy space normalized reproducing kernels show that they are very similar in behavior to the ones we studied on the Bergman space, so the proofs which require only trivial adaptation are omitted. This does not indicate, however, that H^2 has nothing to add to the discussion. The final statement in the following connects the Hardy Berezin symbol with the Poisson transform on $L^2(\partial\mathbf{D})$.

Proposition 3.15 (i). *$k_z \to 0$ weakly as $|z| \to 1^-$. (ie: H^2 is uniformly standard.)*

(ii). $k_z \circ \varphi_z(w) = k_z(w)^{-1}$.

(iii). $k_z(w) = \sqrt{1-|z|^2}\sum_{n=0}^{\infty} \bar{z}^n w^n$.

(iv). $\left|k_z(e^{i\theta})\right|^2 = \mathrm{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z}$ *the Poisson kernel of* $\mathbf{D}$.

We observed in the preceeding section an example of a non-compact operator whose Berezin symbol did vanish on the boundary. Here we present another example of this behavior in a positive non-unitary operator on H^2 due to Axler, [11].

Example 3.16 Let A be the projection from H^2 to the space spanned by the canonical vectors $\{e_{2^n} : n = 0, 1, \ldots\}$. Since the sequence $\{Ae_{2^n}\} = \{e_{2^n}\}$ has no converging subsequence, A is not compact. Now $\tilde{A}$ can be calculated as

$$\begin{aligned} \tilde{A}(z) = \langle Ak_z, k_z\rangle &= (1-|z|^2)\langle AK_z, K_z\rangle \\ &= (1-|z|^2)\sum_{n=0}^{\infty} \bar{z}^{2^n} e_{2^n}(z) = (1-|z|^2)\sum_{n=0}^{\infty}(|z|^2)^{2^n}. \end{aligned} \tag{3.11}$$

Since $t = |z|^2 < 1$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} t^{2^n} &= \sum_{n=0}^{N} t^{2^n} + \sum_{n=N+1}^{\infty} t^{2^n} = \sum_{n=0}^{N} t^{2^n} + \sum_{j=1}^{\infty}(t^{2^N})^{2^j} \\ &\le \sum_{n=0}^{N} t^{2^n} + \sum_{j=1}^{\infty}(t^{2^N})^j = \sum_{n=0}^{N} t^{2^n} + \frac{t^{2^N}}{1-t^{2^N}} \\ &\le N+1+\frac{t^{2^N}}{1-t^{2^N}}. \end{aligned}$$

Equation (3.11) is therefore bounded above by a finite sum and a geometric series. It is then easy to see that

$$\lim_{|z|\to 1^-} \tilde{A}(z) \leq \lim_{|z|\to 1^-} \left((1-|z|^2)(N+1) + \frac{(1-|z|^2)}{1-(|z|^2)^{2^N}} \right) \leq \frac{1}{2^N},$$

where the limit of the second term is evaluated using l'Hôpital's rule. Since N can be chosen arbitrarily large, we have $\tilde{A} \to 0$ as $|z|$ increases to 1.

Given the general theorem, now, there must exist a unitary V on H^2 such that A^V does not have a Berezin symbol that vanishes on $\partial\mathbf{D}$. We can determine a suitable V in the following way.

Define a unitary operator V by mapping

$$Ve_{2n} = e_{2^n},\ n = 0, \ldots$$

and mapping the odd basis vectors onto a basis for the orthocomplement of the e_{2^n}'s. Then

$$\begin{aligned} \widetilde{A^V}(z) &= (1-|z|^2)\sum_{n=0}^{\infty} \bar{z}^n \left\langle V^{-1}AVw^n, K_z \right\rangle = (1-|z|^2)\sum_{j=0}^{\infty} \bar{z}^{2j} \left\langle w^{2j}, K_z \right\rangle, \\ &= (1-|z|^2)\sum_{j=0}^{\infty} (|z|^4)^j = \frac{1-|z|^2}{1-|z|^4} = \frac{1}{1+|z^2|} \end{aligned}$$

which is nowhere vanishing. □

A similar projection operator on the Bergman space can be constructed, see [2].

We noted in the last section how the Möbius transformations on **D** were connected to Bergman reproducing kernels. We find the same is true for H^2.

Lemma 3.17 (Möbius Change of Variables) *Let $f, g \in H^2$ and let φ_a be the Möbius transformation for $a \in \mathbf{D}$. Then*

$$\langle f\circ\varphi_a, g\circ\varphi_a\rangle = \langle k_a f, k_a g\rangle \tag{3.12}$$

Proof. Let f, g and φ_a be in H^2. We will proceed to calculate their inner product in the $H^2(\partial\mathbf{D})$ form. Then

$$\langle f \circ \varphi_a, g \circ \varphi_a \rangle = \int_0^{2\pi} f \circ \varphi_a(e^{i\theta})\overline{g \circ \varphi_a(e^{i\theta})}\frac{d\theta}{2\pi}.$$

Now consider the change of variable $e^{it} = \varphi_a(e^{i\theta})$, giving $e^{it}dt = \varphi_a'(e^{i\theta})e^{i\theta}d\theta$ as the appropriate substitution. With the self inverse of φ_a, then, we get

$$\langle f \circ \varphi_a, g \circ \varphi_a \rangle = \int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\frac{e^{it}}{\varphi_a'(\varphi_a(e^{it}))\varphi_a(e^{it})}\frac{dt}{2\pi}. \tag{3.13}$$

But the fraction in the integrand simplifies (using the identity (3.3)) as

$$\begin{aligned} \frac{e^{it}}{\varphi_a'(\varphi_a(e^{it}))\varphi_a(e^{it})} &= -\frac{1-|a|^2}{(1-\overline{a}e^{it})^2}\left(\frac{1-\overline{a}e^{it}}{a-e^{it}}\right)e^{it} \\ &= \frac{(1-|a|^2)}{(1-\overline{a}e^{it})}\left(\frac{-1}{ae^{-it}-1}\right) \\ &= \frac{1-|a|^2}{|1-\overline{a}e^{it}|^2} = \left|k_a(e^{it})\right|^2 \end{aligned} \tag{3.14}$$

which when reorganized in (3.13) yields (3.12). □

The Toeplitz operators on H^2 are defined in exact analogy to the L_a^2 case (however, their behavior can be quite different). For simplicity, we will always consider a Toeplitz operator to use $H^2 = H^2(\partial\mathbf{D})$ in its domain and $H^2 = H^2(\mathbf{D})$ in its range.

Definition 3.18 Let $u \in L^\infty(\partial\mathbf{D})$, and let $P : L^\infty(\partial\mathbf{D}) \longmapsto H^2(\mathbf{D})$ be the Szëgo projection. Then u induces a *Toeplitz operator* T_u on H^2 defined by $T_u f := P(uf)$ for all $f \in H^2$. $T_u \in \mathcal{B}(H^2)$ and satisfies the following:

(i). $\|T_u\| = \|u\|_\infty$.

(ii). $T_{\alpha u+\beta v} = \alpha T_u + \beta T_v$ for $u, v \in L^\infty(\partial\mathbf{D})$ and $\alpha, \beta \in \mathbf{C}$.

(iii). $T_u^* = T_{\overline{u}}$.

(iv). T_u has the form

$$(T_u f)(z) = \int_0^{2\pi} u(e^{i\theta})f(e^{i\theta})\overline{K_z(e^{i\theta})}\frac{d\theta}{2\pi}.$$

Reference: pages 193-194 of [18].

The Berezin symbols of Toeplitz operators on H^2 actually are a well studied classical object: the Poisson integral.

Example 3.19 Let T_u be a Toeplitz operator on the Hardy space. Then using the $H^2(\partial\mathbf{D})$ inner product we have that

$$\tilde{u}(z) = \int_0^{2\pi} u \circ \varphi_z(e^{i\theta}) \frac{d\theta}{2\pi} = \widetilde{u \circ \varphi_z}(0).$$

and

$$\widetilde{T_u} = P[u]$$

where $P[u]$ is the Poisson integral (or harmonic extension) of u over $\mathbf{D}$. The first equation comes from the change of variable formula applied to $\langle uk_z, k_z\rangle$. (The same argument as the Bergman case.) The second comes from the correspondence of the Hardy kernels with the Poisson kernel $|k_z(w)|^2 = P_z(w)$ as in Proposition 3.15.(iv). $\square$

The class of composition operators will also be important in the H^2 analysis.

Definition 3.20 Let $\phi : \mathbf{D} \mapsto \mathbf{D}$ be analytic. Then ϕ induces a *composition operator* C_ϕ on H^2, defined by $C_\phi f := f \circ \phi$ for all $f \in H^2$. The composition operator C_ϕ satisfies the following:

(i). $C_\phi \in \mathcal{B}(H^2)$.

(ii). $C_{\phi_2} C_{\phi_1} = C_{\phi_1 \circ \phi_2}$.

(iii). $C_\phi(fg) = C_\phi f \cdot C_\phi g$.

Reference: page 117 of [8].

The composition operators on H^2 have a Berezin symbol structure very similar to the L_a^2 case, as we note here.

Example 3.21 (Berezin Symbol of C_ϕ) Let C_ϕ be a composition operator on the Hardy space. Then.

$$\widetilde{C_\phi}(z) = \frac{1 - |z|^2}{1 - \bar{z}\phi(z)}.$$

For $\widetilde{C_\phi}(z) = \langle k_z \circ \phi, k_z \rangle = \sqrt{1 - |z|^2}\, k_z \circ \phi(z)$ by the reproducing property of the kernel K_z. □

Considering the depth of their application in the Bergman Space. we present the analogous definition of Möbian unitary operators on H^2.

Proposition 3.22 (Möbian Unitary Operators) *Let $a \in \mathbf{D}$. The weighted composition operator on H^2 defined by $U_a f := k_a C_{\varphi_a} f$ satisfies*

(i). *U_a is a self-adjoint unitary operator.*

(ii). *$U_a K_z = \overline{k_a(z)} K_{\varphi_a}$. and $U_a k_z = \alpha k_{\varphi_a(z)}$ where $\alpha = \alpha_a(z) \in \partial\mathbf{D}$.*

Proof. The method of proof is identical to that used in Proposition 3.10 using the results of Propositions 3.1. and 3.15 and the change of variable. equation (3.12). For the H^2 case. we find that

$$\alpha_a(z) = \frac{|1 - a\bar{z}|}{(1 - a\bar{z})}. \tag{3.15}$$ □

These Hardy Möbian unitaries also have Berezin symbols that vanish on $\partial\mathbf{D}$.

Example 3.23 (Berezin Symbol of Möbian Unitaries) The proof is identical to the Bergman case, Example 3.11. with the modification that the left hand side of (3.8) is now $\left|k_{\varphi_a(z)}(z)\right|^2$ using the Hardy reproducing kernel. Similarly. now $\|K_z\|^{-1} = \sqrt{1 - |z|^2} \to 0$ as $|z| \to 1^-$. which still squeezes $\widetilde{U_a}(z)$ to zero. □

Möbian conjugates in H^2 also do nothing more than provide a change of variable. The proof is the same as Proposition 3.12.

Proposition 3.24 (Berezin Symbols of Möbian Conjugates) *Let $A \in \mathcal{B}(H^2)$. Let U_a be the Möbian unitary for $a \in \mathbf{D}$. Then*

$$\widetilde{A^{U_a}} = \tilde{A} \circ \varphi_a. \tag{3.16}$$

As with the projection operator of Example 3.16, there must exist some conjugate of U_a with a non-vanshing Berezin symbol in order to satisfy Theorem 1. The case of general $a \in \mathbf{D}$ is rather difficult, so we will consider the simplest case $a = 0$.

Example 3.25 Our goal is to determine a unitary operator V such that

$$\lim_{|z| \to 1^-} \widetilde{U_0^V} \neq 0.$$

Note that $U_0 f(w) = f(-w)$ and U_0 is a diagonal operator in the canonical H^2 basis with diagonal entries $(-1)^n$. Define the operator V on the H^2 canonical basis as the block diagonal matrix with identical blocks V_0

$$V = \begin{pmatrix} V_0 & 0 & 0 & \cdots \\ 0 & V_0 & 0 & \cdots \\ 0 & 0 & V_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad : \quad V_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.17}$$

V is a unitary operator on H^2 since $V_0^* V_0$ is the 2×2 identity matrix. It is a simple matter of matrix computation, then, that in the canonical H^2 basis,

$$U_0^V = V^* U_0 V = \begin{pmatrix} B_0 & 0 & 0 & \cdots \\ 0 & B_0 & 0 & \cdots \\ 0 & 0 & B_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.18}$$

where

$$B_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Algebraically, then, we have

$$U_0^V w^{2m} = w^{2m+1} \quad \text{and} \quad U_0^V w^{2m+1} = w^{2m}. \tag{3.19}$$

If we calculate the Berezin symbol of this conjugate of U_0 using the even-odd expansion $k_z(w) = \sum_{m=0}^{\infty} (\bar{z}^{2m} w^{2m} + \bar{z}^{2m+1} w^{2m+1})$, we get

$$\widetilde{U_0^V}(z) = \left\langle U_0^V k_z, k_z \right\rangle$$

$$
\begin{aligned}
&= \sqrt{1-|z|^2}\sum_{m=0}^{\infty}\left(\overline{z}^{2m}\left\langle U_0^V w^{2m}, k_z\right\rangle + \overline{z}^{2m+1}\left\langle U_0^V w^{2m+1}, k_z\right\rangle\right)\\
&= \sqrt{1-|z|^2}\sum_{m=0}^{\infty}\left(\overline{z}^{2m}\left\langle w^{2m+1}, k_z\right\rangle + \overline{z}^{2m+1}\left\langle w^{2m}, k_z\right\rangle\right)\\
&= (1-|z|^2)\sum_{m=0}^{\infty}\left(\overline{z}^{2m}z^{2m+1} + \overline{z}^{2m+1}z^{2m}\right)\\
&= (1-|z|^2)\sum_{m=0}^{\infty}\left((|z|^2)^{2m}(z+\overline{z})\right)\\
&= (1-|z|^2)\frac{z+\overline{z}}{1-|z|^4} = \frac{z+\overline{z}}{1+|z|^2}
\end{aligned}
$$

which vanishes only at $\pm i$. □

Chapter 4

Special Cases of the Compactness Condition

As we see from Examples 3.11, 3.16, and 3.25 in the previous chapter, given a standard functional Hilbert space H, the ideal of compact operators $\mathcal{K}(H)$ is not simply identified under the Berezin transform as the set of operators with symbols vanishing on the boundary. However, it is quite possible that such a characterization exists for smaller classes of operators. We examine here some results involving Toeplitz, Hankel, and composition operators.

4.1 Toeplitz Operators

4.1.1 Compact Toeplitz Operators on L_a^2

Toeplitz operators as defined in section 3.1 have been a staple topic in operator theory, and work linking compactness and the Berezin symbol has attracted the interest of multiple authors. Since 1988, K. Zhu, B. Korenblum, K. Stroethoff, S. Axler, and D. Zheng have all contributed to characterizations that state T_u is compact iff $\widetilde{T_u}$ vanishes on $\partial\mathbf{D}$ provided we make some special assumption on the symbol u. (See [2] for references.)

The most general result is due to Axler and Zheng (1998). on which we will focus the bulk of this section. Before stating the theorem. we include a lemma that summarizes some properties of AK_w and gives the integral form of A.

Lemma 4.1 *Given arbitrary $f \in L^2_a$ and $A \in \mathcal{B}(L^2_a)$. we have the following:*

$$A^* K_z(w) = \langle A^* K_z, K_w \rangle = \overline{\langle AK_w, K_z \rangle} = \overline{AK_w(z)} \tag{4.1}$$

$$Af(z) = \langle Af, K_z \rangle = \langle f, A^* K_z \rangle = \int_{\mathbf{D}} f(w) AK_w(z)\, dm(w) \tag{4.2}$$

thereby writing A as an integral operator with kernel $\check{A}(z, w) = AK_w(z)$.

In the following we will use $\mathcal{T}$ to represent the set of all finite sums of finite products of Toeplitz operators on the Bergman space.

Theorem 2 (Axler and Zheng, 1998) *Let $\mathcal{T}$ be the algebra generated by the Toeplitz operators on L^2_a. Then the compact operators in $\mathcal{T}$ are characterized by vanishing Berezin symbols. In fact. for $A \in \mathcal{T}$. the following are equivalent:*

(i). $A \in \mathcal{K}(L^2_a) \cap \mathcal{T}$.

(ii). $\lim_{|z| \to 1^-} \tilde{A}(z) = 0$.

(iii). $\lim_{|z| \to 1^-} \left\| A^{U_z} 1 \right\|_p = 0 \quad \forall\ 1 < p < \infty$.

In particular. a Toeplitz operator T_u on L^2_a is compact iff $\tilde{u}$ (or $\widetilde{T_u^* T_u}$) vanish on $\partial\mathbf{D}$.

Discussion of the Proof. The implication (i) $\Rightarrow$ (ii) is the general result for standard functional Hilbert spaces (Proposition 2.3). A lemma for $A \in \mathcal{T}$ shows (ii) $\Rightarrow$ (iii). The most difficult step. (iii) $\Rightarrow$ (i). uses Lemma 4.1 and techniques of integral operators to demonstrate that $A \in \mathcal{T}$ with the given property implies A is a limit of a sequence of compact operators as described below.

From the integral form of A (equation (4.2)). define for every positive $r < 1$ the Bergman space operator $A_{[r]}$ by the equation

$$A_{[r]} f(z) = \int_{\mathbf{D}} f(w) AK_w(z) \chi_r(w)\, dm(w) \tag{4.3}$$

where χ_r is the characteristic function of the disk $r\mathbf{D}$. To complete the proof of the theorem we will show:

1. *$A_{[r]}$ is compact for every $0 < r < 1$.*

2. *If $A \in \mathcal{T}$ such that* (iii) *holds, then $\lim_{r \to 1^-} A_{[r]} = A$ in the operator norm.*

Let us deal with the technicalities in a bunch of lemmas and proceed with the necessary proofs.

The most important property particular to operators in $\mathcal{T}$ used in Axler and Zheng's proof is the following.

Lemma 4.2 **([2])** *Let $A \in \mathcal{T}$ be arbitrary. Then the set $\{A^{U_z}1 \mid z \in \mathbf{D}\}$ is bounded in the $L^p(\mathbf{D}, dm)$ norm for all $1 < p < \infty$. That is,*

$$\sup_{z \in \mathbf{D}} \left\| A^{U_z} 1 \right\|_p < \infty.$$

Proof. Consider first only the finite product $A = T_{u_1} T_{u_2} \dots T_{u_n}$, where the u_i are bounded functions on $\mathbf{D}$. Then

$$A^{U_z} = U_z T_{u_1} T_{u_2} \dots T_{u_n} U_z = (U_z T_{u_1} U_z)(U_z T_{u_2} U_z) \dots (U_z T_{u_n} U_z). \tag{4.4}$$

and for every f, g in L^2_a,

$$\begin{aligned} \langle U_z T_v U_z f, g \rangle &= \langle P v U_z f, U_z g \rangle = \langle v U_z f, P U_z g \rangle \\ &= \langle U_z v \circ \varphi_z f, U_z g \rangle = \langle v \circ \varphi_z f, g \rangle = \langle T_{v \circ \varphi_z} f, g \rangle . \end{aligned} \tag{4.5}$$

Thus $T_v^{U_z} = T_{v \circ \varphi_z}$. Making a replacement in (4.4) we therefore have

$$U_z (T_{u_1} T_{u_2} \dots T_{u_n}) U_z = T_{u_1 \circ \varphi_z} T_{u_2 \circ \varphi_z} \dots T_{u_n \circ \varphi_z}.$$

Now, the Bergman projection operator $P : L^p(\mathbf{D}) \longmapsto L^p_a(\mathbf{D})$ is bounded for each p, that is $\|Pf\|_p \le c_p \|f\|_p$, (see Proposition 3.3.(iii)), so

$$\sup_{z \in \mathbf{D}} \left\| (T_{u_1} T_{u_2} \dots T_{u_n})^{U_z} 1 \right\|_p = \sup_{z \in \mathbf{D}} \| T_{u_1 \circ \varphi_z} T_{u_2 \circ \varphi_z} \dots T_{u_n \circ \varphi_z} 1 \|_p$$

$$
\begin{aligned}
&\le \sup_{z\in\mathbf{D}} c_p \left\|u_1 \circ \varphi_z T_{u_2\circ\varphi_z} \dots T_{u_n\circ\varphi_z} 1\right\|_p \\
&\le \sup_{z\in\mathbf{D}} c_p \left\|u_1 \circ \varphi_z\right\|_\infty \left\|T_{u_2\circ\varphi_z} \dots T_{u_n\circ\varphi_z} 1\right\|_p \\
&\le \sup_{z\in\mathbf{D}} c_p^n \prod_{k=1}^{n} \left\|u_k \circ \varphi_z\right\|_\infty
\end{aligned}
$$

$$
\le \prod_{k=1}^{n} c \left\|u_k\right\|_\infty \tag{4.6}
$$

which is bounded independent of z.

For general $A \in \mathcal{T}$, the previous result extends by the triangle inequality to prove the lemma. □

Lemma 4.3 *Given an operator A bounded on L_a^2, there exists a constant $c < \infty$ such that*

$$
\int_{\mathbf{D}} \frac{|(AK_w)(z)|}{\sqrt{1-|z|^2}}\, dm(z) \le \frac{c\left\|A^{U_w}1\right\|_6}{\sqrt{1-|w|^2}} \tag{4.7}
$$

for all $w \in \mathbf{D}$, and

$$
\int_{\mathbf{D}} \frac{|(AK_w)(z)|}{\sqrt{1-|w|^2}}\, dm(w) \le \frac{c\left\|(A^*)^{U_z}1\right\|_6}{\sqrt{1-|z|^2}} \tag{4.8}
$$

for all $z \in \mathbf{D}$. (The particular use of the $L^6(\mathbf{D})$ norm is by convenience.)

Proof. We note that since $U_w 1 = k_w = (1-|w|^2)K_w$,

$$
AK_w = \frac{AU_w 1}{1-|w|^2} = \frac{U_w A^{U_w} 1}{1-|w|^2} = \frac{k_w (A^{U_w}1)\circ\varphi_w}{1-|w|^2}.
$$

Rewriting our integral we get

$$
\int_{\mathbf{D}} \frac{|(AK_w)(z)|}{\sqrt{1-|z|^2}}\, dm(z) = \frac{1}{1-|w|^2}\int_{\mathbf{D}} \frac{\left|k_w(z)(A^{U_w}1)\circ\varphi_w(z)\right|}{\sqrt{1-|z|^2}}\, dm(z).
$$

Applying the change of variable $z = \varphi_w(\lambda)$ and some manipulation of reproducing kernels and Möbius transformations this becomes

$$
\frac{1}{1-|w|^2}\int_{\mathbf{D}} \frac{\left|k_w\circ\varphi_w(\lambda)(A^{U_w}1)(\lambda)\right|}{\sqrt{1-|\varphi_w(\lambda)|^2}} |k_w(\lambda)|^2\, dm(\lambda)
$$

$$= \frac{1}{1-|w|^2}\int_{\mathbf{D}} \left|(A^{U_w}1)(\lambda)\right| \frac{|k_w(\lambda)|}{\sqrt{1-|\varphi_w(\lambda)|^2}}\, dm(\lambda)$$

$$(4.9) \qquad = \frac{1}{\sqrt{1-|w|^2}}\int_{\mathbf{D}} \frac{\left|(A^{U_w}1)(\lambda)\right|}{|1-\overline{w}\lambda|\sqrt{1-|\lambda|^2}}\, dm(\lambda).$$

Then by applying Hölder's inequality with $p = 6$ to the numerator and $q = 6/5$ to the denominator we get

$$\int_{\mathbf{D}} \frac{|(AK_w)(z)|}{\sqrt{1-|z|^2}}\, dm(z) \le \frac{\left\|(A^{U_w}1)\right\|_6}{\sqrt{1-|w|^2}} \left(\int_{\mathbf{D}} \frac{dm(\lambda)}{|1-\overline{w}\lambda|^{6/5}(1-|\lambda|^2)^{3/5}}\right)^{5/6}.$$

We are motivated to use this $p = 6$ case because the integral on the right is known to be bounded independent of z by a result from Axler (Lemma 4 [1]), but any p leaving an integral bounded over z is sufficient. Therefore, taking $c^{6/5}$ as an upper bound to the integral we arrive at (4.7).

It would perhaps be simplest to take $p = q = 2$ in this final stage, but unfortunately the integral

$$\int_{\mathbf{D}} \frac{dm(\lambda)}{|1-\overline{w}\lambda|^2(1-|\lambda|^2)}$$

does not converge for any $w \in \mathbf{D}$. (Otherwise, it would always be true that A is compact iff $\tilde{A}$ vanishes on $\partial\mathbf{D}$.)

To determine inequality (4.8), use equation (4.1) to write

$$\int_{\mathbf{D}} \frac{|(AK_w)(z)|}{\sqrt{1-|w|^2}}\, dm(w) = \int_{\mathbf{D}} \frac{\left|\overline{(AK_w)(z)}\right|}{\sqrt{1-|w|^2}}\, dm(w) = \int_{\mathbf{D}} \frac{|(A^*K_z)(w)|}{\sqrt{1-|w|^2}}\, dm(w).$$

So by replacing A with A^* and switching the roles of z and w in inequality (4.7), we actually obtain the desired result by applying the above identity. □

We now proceed with the proof of the theorem.

Proof of Theorem 2.

(ii) $\Rightarrow$ (iii).

Claim: *Let $A \in \mathcal{T}$ and let $\tilde{A}$ vanish on $\partial\mathbf{D}$. Then $\left\|A^{U_z}1\right\|_p \to 0$ as $z \to \partial\mathbf{D}$ for any $p > 1$.*

Proof. First, we would like to observe the behavior of the vectors $\{A^{U_z}1\}$ under the hypotheses. Let us fix a positive $r < 1$. Using the equality of Proposition 3.12, we observe, using power expansions of k_λ, that the integral

$$
\begin{aligned}
\int_{r\mathbf{D}} \frac{\tilde{A}(\varphi_z(\lambda))\overline{\lambda}^n}{(1-|\lambda|^2)^2}\,dm(\lambda) &= \int_{r\mathbf{D}} \frac{\widetilde{A^{U_z}}(\lambda)\overline{\lambda}^n}{(1-|\lambda|^2)^2}\,dm(\lambda) = \int_{r\mathbf{D}} \frac{\left\langle A^{U_z}k_\lambda, k_\lambda\right\rangle \overline{\lambda}^n}{(1-|\lambda|^2)^2}\,dm(\lambda) \\
&= \sum_{l,m=0}^{\infty} (l+1)(m+1) \int_{r\mathbf{D}} \left\langle A^{U_z}w^l, w^m\right\rangle \overline{\lambda}^{l+n}\lambda^m\,dm(\lambda) \\
&= \sum_{l,m=0}^{\infty} (l+1)(m+1) \left\langle A^{U_z}w^l, w^m\right\rangle \int_{r\mathbf{D}} \overline{\lambda}^{l+n}\lambda^m\,dm(\lambda) \\
&= \sum_{l=0}^{\infty}(l+1)\left\langle A^{U_z}w^l, w^{l+n}\right\rangle r^{2l+2n+2} \\
&= r^{2n+2}\left(\left\langle A^{U_z}1, w^n\right\rangle + \sum_{l=1}^{\infty}(l+1)\left\langle A^{U_z}w^l, w^{l+n}\right\rangle r^{2l}\right).
\end{aligned}
$$

So we can make, for any positive $r < 1$, the estimate

$$
\begin{aligned}
\left|\left\langle A^{U_z}1, w^n\right\rangle\right| \;\leq\; & \frac{1}{r^{2n+2}}\left|\int_{r\mathbf{D}} \frac{\tilde{A}(\varphi_z(\lambda))\overline{\lambda}^n}{(1-|\lambda|^2)^2}\,dm(\lambda)\right| \\
& + \left|\sum_{l=1}^{\infty}(l+1)\left\langle A^{U_z}w^l, w^{l+n}\right\rangle r^{2l}\right|.
\end{aligned}
\tag{4.10}
$$

By the hypothesis of our lemma, for any fixed $r < 1$, the integral will converge to 0 as $|z| \to 1^-$ makes $|\varphi_z(\lambda)|$ increase to 1. (The integrand is bounded by $\|A\|\,(1-r^2)^{-2}$ so the limit can pass inside the integral sign.) The sum in equation 4.10 can also be chosen arbitrarily small independent of z for

$$
\left|\sum_{l=1}^{\infty}(l+1)\left\langle A^{U_z}w^l, w^{l+n}\right\rangle r^{2l}\right| \leq \|A\| \sum_{l=1}^{\infty}(l+1)\left\|w^l\right\|\left\|w^{l+n}\right\| r^{2l} \leq \|A\|\, r^2 \sum_{l=0}^{\infty} r^{2l}
$$

which converges to zero as r approaches 0.

Since we have proved that $\left\langle A^{U_z}1, w^n\right\rangle$ converges to zero for all integers n, we have that as $|z| \to 1^-$, the vectors $A^{U_z}1$ weakly converge to 0 in L^2_a, or, equivalently, that they uniformly converge to zero on compact subsets of $\mathbf{D}$. (For details see page 74 of [18].) This enables us to prove our claim as follows.

Divide the p-norm integral of $A^{U_z}1$ over a closed disk and an open annulus, as

$$\left\|A^{U_z}1\right\|_p^p = \int_{\mathbf{D}\setminus\overline{r\mathbf{D}}} \left|(A^{U_z}1)(w)\right|^p dm(w) + \int_{\overline{r\mathbf{D}}} \left|(A^{U_z}1)(w)\right|^p dm(w).$$

For any choice of $r < 1$ the second integral will converge to zero because the integrand vanishes uniformly on the compact set $\overline{r\mathbf{D}}$ as $|z| \to 1^-$. It suffices, then, to show that the first integral can be chosen arbitrarily small uniformly in z. To do this we use the substitution $\left|(A^{U_z}1)(w)\right|^p = 1 \cdot \left|(A^{U_z}1)(w)\right|^p$ and Hölder's inequality (with the case $p = q = 2$) to bound the integral as follows:

$$\begin{aligned}
&\int_{\mathbf{D}\setminus\overline{r\mathbf{D}}} \left|(A^{U_z}1)(w)\right|^p dm(w) \\
&\quad\le \left(\int_{\mathbf{D}\setminus\overline{r\mathbf{D}}} 1\, dm(w)\right)^{1/2} \left(\int_{\mathbf{D}\setminus\overline{r\mathbf{D}}} \left|(A^{U_z}1)(w)\right|^{2p} dm(w)\right)^{1/2} \\
&\quad\le (1-r^2)^{1/2} \left\|A^{U_z}1\right\|_{2p}^p \le (1-r^2)^{1/2} C
\end{aligned}$$

where we bound the $2p$-norm of $A^{U_z}1$ by Lemma 4.2. Taking r sufficiently close to 1 we can therefore make the integral converge to 0 uniformly in z. *Claim Q.E.D.*

(iii) $\Rightarrow$ (i). Recall that the proof comes in two parts.

Claim 1: *Given $A \in \mathcal{B}(L_a^2)$, the operator $A_{[r]}$ defined in (4.3) is compact for every $0 < r < 1$.*

Proof of Claim 1. We will show, in fact, that $A_{[r]}$ is in the Hilbert-Schmidt class S_2. We do this by calculating ([18, pages 39–41])

$$\begin{aligned}
\int_{\mathbf{D}}\int_{\mathbf{D}} |AK_w(z)\chi_r(w)|^2\, dm(z)\, dm(w) &= \int_{r\mathbf{D}}\int_{\mathbf{D}} |AK_w(z)|^2\, dm(z)\, dm(w) \\
= \int_{r\mathbf{D}} \|AK_w\|^2\, dm(w) \le \|A\|^2 \int_{r\mathbf{D}} \|K_w\|^2\, dm(w) &= \|A\|^2 \frac{r^2}{(1-r^2)} < \infty.
\end{aligned}$$

Claim 1 Q.E.D.

Claim 2: *Given $A \in \mathcal{T}$ and $\tilde{A}(z) \to 0$ as $|z| \to 1^-$, then $A_{[r]} \to A$ as $|z| \to 1^-$.*

Proof of Claim 2. We wish to demonstrate that $\left\|A - A_{[r]}\right\| \to 0$ as r increases to 1, but note that $A - A_{[r]}$ is an integral operator with kernel

$$AK_w(z)(1 - \chi_r(w)).$$

Therefore we can apply the Schur test (see §3.2 and Notes, [18]) on the nonnegative measurable function of two variables $|AK_w(z)(1-\chi_r(w))|$. We observe, then, that the relations from Lemma 4.3 can be applied in such a way that

$$\begin{aligned}
&\int_{\mathbf{D}} |AK_w(z)(1-\chi_r(w))| \left(\frac{1}{\sqrt{1-|w|^2}}\right) dm(w) \\
&\leq \int_{\mathbf{D}} |AK_w(z)| \left(\frac{1}{\sqrt{1-|w|^2}}\right) dm(w) \\
&\leq \left(\frac{c\left\|(A^*)^{U_z}1\right\|_6}{\sqrt{1-|z|^2}}\right) = c\sup_{z\in\mathbf{D}}\left\|(A^*)^{U_z}1\right\|_6 \left(\frac{1}{\sqrt{1-|z|^2}}\right)
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&\int_{\mathbf{D}} |AK_w(z)(1-\chi_r(w))| \left(\frac{1}{\sqrt{1-|z|^2}}\right) dm(z) \\
&\leq (1-\chi_r(w)) \int_{\mathbf{D}} |AK_w(z)| \left(\frac{1}{\sqrt{1-|z|^2}}\right) dm(z) \\
&\leq (1-\chi_r(w)) \left(\frac{c\left\|A^{U_w}1\right\|_6}{\sqrt{1-|w|^2}}\right) = c\sup_{r\leq|w|<1}\left\|A^{U_w}1\right\|_6 \left(\frac{1}{\sqrt{1-|w|^2}}\right).
\end{aligned} \tag{4.12}$$

Applying the Schur's test, we get the bound

$$\left\|A - A_{[r]}\right\|^2 \leq c^2 \left(\sup_{z\in\mathbf{D}}\left\|(A^*)^{U_z}1\right\|_6\right)\left(\sup_{r\leq|w|<1}\left\|A^{U_w}1\right\|_6\right).$$

The first supremum over $\mathbf{D}$ is bounded using Lemma 4.2 applied to A^*. However, by the hypothesis of (iii), we have that

$$\left\|A^{U_w}1\right\|_6 \to 0 \text{ as } w \to \partial\mathbf{D}.$$

so the second supremum converges to 0 as r increases to 1. Hence A is the limit to the compact operators $A_{[r]}$, and A is compact. *Claim 2 Q.E.D.*

This completes the proof of Theorem 2. □

The following theorem is a direct result of our characterization. It nicely expresses the condition for compactness in a form similar to the hypothesis condition in Lemma 4.2.

Theorem 3 (Zheng, 1989 [16]) *Let T_u be a Toeplitz operator on L_a^2. Then T_u is compact iff*

$$\|T_{u\circ\varphi_z}1\|_p = \|P(u\circ\varphi_z)\|_p \to 0 \text{ as } |z| \to 1^-,$$

for all $p \in (1,\infty)$.

Proof. Apply condition (iii) of Theorem 2 to the operator $A = T_u$ and use the result of Lemma 4.2 that $T_u^{U_z} = T_{u\circ\varphi_z}$. □

Note that the bounded supremum property of Lemma 4.2 is the only special property of elements of $\mathcal{T}$ used in the proof. So it is natural to extend this characterization to the class of operators $\mathcal{T}'$ defined as all $A \in \mathcal{B}(L_a^2)$ for which $\sup_{z\in\mathbf{D}} \left\|A^{U_z}1\right\|_p < \infty$. However, we do not know what is in $\mathcal{T}' \setminus \mathcal{T}$ or even if it is nonempty. But it does follow from our previous observations that not all bounded operators can be included in $\mathcal{T}'$.

Example 4.4 Since the Möbian unitary operator U_0 of Example 3.11 is non-compact and has a Berezin symbol that vanishes on $\partial\mathbf{D}$, it must not satisfy some hypothesis of Theorem 2. Let us calculate $\left\|U_0^{U_z}1\right\|_p$.

$$\begin{aligned}
\|U_zU_0U_z1\|_p^p &= \|k_zC_{\varphi_z}U_0k_z\|_p^p = \|k_zk_z\circ(-\varphi_z)\|_p^p \\
&= \int_{\mathbf{D}} |k_z(w)|^p\,|k_z(-\varphi_z(w))|^p\,dm(w) \\
&= \int_{\mathbf{D}} \left|\frac{1-|z|^2}{(1-\bar z w)^2}\right|^p \left|\frac{1-|z|^2}{(1+\bar z\varphi_z(w))^2}\right|^p dm(w) \\
&= (1-|z|^2)^{2p} \int_{\mathbf{D}} \frac{1}{|(1-\bar z w + \bar z(z-w))^2|^p} \\
&= \frac{(1-|z|^2)^{2p}}{(1+|z|^2)^{2p}} \int_{\mathbf{D}} \frac{1}{|1-\bar a w|^{2p}}dm(w)
\end{aligned}$$

where $a = 2z/(1+|z|^2) \in \mathbf{D}$. The last integral on the right is known to have an asymptotic relationship as $|z| \to 1^-$, [18, page 53],

$$\lim_{|z|\to 1^-} \int_{\mathbf{D}} \frac{1}{|1-\bar a w|^{2p}}dm(w) \geq \lim_{|z|\to 1^-} c\frac{1}{(1-|a|)^{2p-2}}.$$

Remark that $(1+|z|^2)^2(1-|a|^2) = (1+|z|^2)^2 - 4|z|^2 = (1-|z|^2)^2$. and so

$$\frac{1}{(1-|a|)^{2p-2}} = \frac{(1+|z|^2)^{4(p-1)}}{(1-|z|^2)^{4(p-1)}}.$$

. Thus we see that for $p > 2$,

$$\sup_{z\in\mathbf{D}} \left\|U_0^{U_z}1\right\|_p \geq \lim_{|z|\to 1^-} \|U_zU_0U_z1\|_p \geq \lim_{|z|\to 1^-} \left(\frac{(1+|z|^2)^{2(p-2)}}{(1-|z|^2)^{2(p-2)}}\right)^{1/p} \to \infty.$$

and U_0 does not satisfy Lemma 4.2. □

In summary, we formulate the following identification of $\mathcal{K}(L_a^2)\cap\mathcal{T}$.

Corollary 4.5 *Let $\mathcal{T}$ be the set of all finite sums of finite products of Toeplitz operators on L_a^2. Let $A\in\mathcal{T}$. Then the following are equivalent.*

(i). $A\in\mathcal{K}(L_a^2)\cap\mathcal{T}$.

(ii). $\tilde{A}(z)\to 0$ *as* $|z|\to 1^-$.

(iii). $\widetilde{A^*A}(z)\to 0$ *as* $|z|\to 1^-$.

(iv). $\|Ak_{z_n}\|\to 0$ *for every sequence* z_n *where* $|z_n|\to 1^-$.

Specifically, we note that T_u *is compact iff* $\tilde{u}(z)$ *or* $\widetilde{T_u^*T_u}(z)\to 0$ *on* $\partial\mathbf{D}$. *or* $\|T_{u\circ\varphi_z}1\|_p\to 0$ *as* $|z|\to 1^-$ *for all* $p\in(1,\infty)$.

4.1.2 Compact Toeplitz Operators on H^2

The question of characterizing compact Toeplitz operators on the Hardy space is a trivial one: by a classical result only T_0 is compact. In this way the analog of Theorem 2 is true for Toeplitz operators on H^2, for

$$T_u \text{ on } H^2(\partial\mathbf{D}) \text{ is compact} \iff u=0 \ a.e. \iff \widetilde{T_u} \text{ vanishes on } \partial\mathbf{D}.$$

The second equivalence above comes from the relationship between $\widetilde{T_u}$ and the Poisson integral. Recall that for $u\in L^\infty(\partial\mathbf{D})$, the Berezin symbol $\widetilde{T_u} = P[u]$ is the harmonic extension of u onto $\mathbf{D}$, Example 3.7. It follows from the theory of the Poisson integral and harmonic function theory (see Zhu [18]) that (i). $\lim_{r \to 1^-} P[u](re^{i\theta}) = u(e^{i\theta})$ almost everywhere θ.

(ii). (Mean Value Theorem) $P[u](z) = P[u] \circ \varphi_z(0) = \int_0^{2\pi} u(\varphi_z(e^{i\theta}))d\theta/2\pi$.

Clearly $\tilde{u} = P[u] = 0$ iff $u = 0$ a.e., completing our remark.

In fact, more can also be said parallel to the Bergman results.

Theorem 4 *Let $u \in L^\infty(\partial\mathbf{D})$. Then the compact Toeplitz operators T_u on H^2 are characterized by*

$$T_u \text{ compact} \quad \Longleftrightarrow \quad \widetilde{T_u^* T_u} \text{ vanishes on } \partial\mathbf{D}.$$

Proof. The forward direction is the general result for standard functional Hilbert spaces (Proposition 2.3). However, the Cauchy-Schwarz inequality is enough to make the backward direction work.

$$|\tilde{u}(z)|^2 = |\langle T_u k_z, k_z\rangle|^2 \le \|T_u k_z\|^2 = \widetilde{T_u^* T_u}(z).$$

and so if $\widetilde{T_u^* T_u}$ vanishes on $\partial\mathbf{D}$, so does $\tilde{u}$. By the above remark, T_u is compact. □

We have not, unfortunately, even approached in the Hardy space case the generality of Theorem 2. The Axler and Zheng characterization also succeeded for operators A that are finite sums of finite products of Toeplitz operators, and there are definitely nontrivial compact operators of this type on H^2. (Consider, for example, the operator $A = T_1 - T_z T_{\bar{z}} = P_0$, a projection of rank one.) But it is not yet established whether all compact operators that are finite sums of finite products of Toeplitz operators on H^2 can be characterized by vanishing Berezin symbols.

4.2 Hankel Operators

4.2.1 Compact Hankel Operators on L_a^2

We have considered Toeplitz operators on the Bergman space and the behaviour of their Berezin symbols under compactness. We also want to study a related linear transformation on L_a^2: the ("large") Hankel operator.

Definition 4.6 Let $P : L^2(\mathbf{D}) \mapsto L_a^2$ be the Bergman projection. Then for every $u \in L^\infty(\mathbf{D})$ the Hankel operator $H_u : L_a^2 \mapsto (L_a^2)^\perp$ is defined by

$$H_u f := (I - P)(uf)$$

for every f in L_a^2. It follows that H_u is bounded and $\|H_u\| \leq \|u\|_\infty$.

Definition 4.6 makes Hankel operators distinct among the operators we will consider because H_u is not a linear self-transformation on a standard functional Hilbert space of analytic functions. It is a transformation from an analytic subspace of L^2 to its orthocomplement, making $H_u f$ necessarily orthogonal to f whenever f is analytic. It follows that $\widetilde{H_u} = 0$ for all H_u. However $H_u^* H_v$ is an operator from L_a^2 into L_a^2, and there is a well known relationship between this and Toeplitz operators:

$$\begin{aligned}\langle (H_u^* H_v) f, g\rangle &= \langle H_v f, (I-P)ug\rangle = \langle H_v f, ug\rangle = \langle vf - T_v f, ug\rangle \\ &= \langle (\overline{u}v - \overline{u}T_v) f, g\rangle = \langle (T_{\overline{u}v} - T_{\overline{u}} T_v) f, g\rangle \qquad (4.13)\end{aligned}$$

which demonstrates that $H_u^* H_v$ can be written as a finite sum of finite products of Toeplitz operators.

Using the Axler-Zheng theorem of the previous section and the relationship (4.13), therefore, we can determine a compactness criterion for Hankel operators on the Bergman space using Berezin symbols.

Theorem 5 *Let $u \in L^\infty(\mathbf{D})$. Then the compact Hankel operators H_u on L_a^2 are characterized by*

$$(4.14) \qquad H_u \text{ compact} \iff \widetilde{H_u^* H_u} = \widetilde{|u|^2} - \widetilde{T_{\overline{u}} T_u} \text{ vanishes on } \partial\mathbf{D}.$$

Proof. Using (4.13) we have $H_u^* H_u \in \mathcal{T}$, and so this product is compact iff its Berezin symbol vanishes by Theorem 2. The proof is finished by noting that in order for H_u to be compact, it is necessary and sufficient that $H_u^* H_u$ be compact. $\square$

Hankel operators on the Bergman space have been extensively studied and several characterizations of compact H_u have already been proven. In this subsection, we will build some required preliminaries and state these characterizations, and then demonstrate how the result (4.14) can be used as an alternate means of proof for them.

The first collection of results characterize compact H_u using L^p norm behavior which involves the following lemma, the analog to Lemma 4.2.

Lemma 4.7 *Let $u \in L^\infty(\mathbf{D})$ and $z \in \mathbf{D}$. Then for arbitrary $1 < q < \infty$,*

$$\sup_{z \in \mathbf{D}} \|H_{u \circ \varphi_z} 1\|_q < \infty.$$

Proof. We have from 3.3 that the Bergman projection P is bounded for every $1 < q < \infty$. Then $(I - P)$ is also bounded, and so,

$$\|H_{u \circ \varphi_z} 1\|_q = \|(I - P) u \circ \varphi_z\|_q \leq c_q \|u \circ \varphi_z\|_q \leq c_q \|u\|_\infty < \infty. \quad \square$$

Theorem 6 *Let $u \in L^\infty(\mathbf{D})$ and let $1 < p < \infty$. Then the following are equivalent:*

(i). *H_u is compact.*

(ii). (Zheng, 1989 [16]). $\|H_u k_z\|_2 \to 0$ *as* $|z| \to 1^-$.

(iii). (Stroethoff, 1990 [14]). $\|u \circ \varphi_z - P(u \circ \varphi_z)\|_p \to 0$ *as* $|z| \to 1^-$.

Proof. To show (i) is equivalent to (ii), we simply note that

$$\widetilde{H_u^* H_u}(z) = \langle H_u^* H_u k_z, k_z \rangle = \langle H_u k_z, H_u k_z \rangle = \|H_u k_z\|_2^2.$$

The equivalence follows from Theorem 5.

To show (i) is equivalent to (iii), we use the Toeplitz form of $H_u^*H_u$ to calculate

$$(H_u^*H_u)^{U_z} = H_{u\circ\varphi_z}^* H_{u\circ\varphi_z}$$

by steps similar to the proof of Lemma 4.2. So, according to property (iii) of Theorem 2, H_u is compact iff $\left\|H_{u\circ\varphi_z}^* H_{u\circ\varphi_z} 1\right\|_p \to 0$ as $z \to \partial\mathbf{D}$. Now,

$$\|H_{u\circ\varphi_z}1\|_2^2 \leq \left\|H_{u\circ\varphi_z}^* H_{u\circ\varphi_z} 1\right\|_2 \leq \left\|H_{u\circ\varphi_z}^*\right\| \|H_{u\circ\varphi_z}1\|_2$$

so H_u is compact iff $\|H_{u\circ\varphi_z}1\|_2$ vanishes on $\partial\mathbf{D}$. But a calculation shows that

$$\|H_{u\circ\varphi_z}1\|_2 = \|(I-P)u\circ\varphi_z\|_2 = \|u\circ\varphi_z - P(u\circ\varphi_z)\|_2 .$$

Hölder's inequality and Lemma 4.7 are then enough to generalize to arbitrary $p > 1$. For observe that in light of the p-boundedness of $\|H_{u\circ\varphi_z}1\|$ (Lemma 4.7) we have by Hölder's inequality,

$$\|H_{u\circ\varphi_z}1\|_p^p \leq \|H_{u\circ\varphi_z}1\|_2 \cdot \|H_{u\circ\varphi_z}1\|_{2(p-1)}^{p-1} \leq c\,\|H_{u\circ\varphi_z}1\|_2 ,$$

so $\|H_{u\circ\varphi_z}1\|_2 \to 0$ implies $\|H_{u\circ\varphi_z}1\|_p \to 0$ as well. In fact they are equivalent, for, if $p \geq 2$ then $\|H_{u\circ\varphi_z}1\|_p \geq \|H_{u\circ\varphi_z}1\|_2$ directly, and if $1 < p < 2$, then another application of Hölder's inequality yields $\|H_{u\circ\varphi_z}1\|_2 \leq \|H_{u\circ\varphi_z}1\|_p \|H_{u\circ\varphi_z}1\|_q$ where $q = p/(p-1) > 2$. Therefore $\widetilde{H_u^*H_u}(z)$ is equivalent to $\|H_{u\circ\varphi_z}1\|_p$, and by Theorem 5 we have an equivalence with the compactness of H_u. □

Since we have completed the primary focus of our study, we formulate here a summary of the function theoretic identifications of compact Hankel operators on L_a^2:

Corollary 4.8 *Let $u \in L^\infty(\mathbf{D})$. Let H_u be the Hankel operator on L_a^2 with symbol u. Then the following are equivalent.*

(i). *H_u is compact.*

(ii). *$\widetilde{H_u^*H_u}(z) \to 0$ as $|z| \to 1^-$.*

(iii). $\|H_{u\circ\varphi_z}1\|_p \to 0$ *as* $|z| \to 1^-$ *for all* $1 < p < \infty$.

(iv). $\|H_u k_{z_n}\| \to 0$ *for every sequence* z_n *where* $|z_n| \to 1^-$.

4.2.2 $\widetilde{H_u^* H_u}$ and Function Spaces

There are some other points of interest regarding Berezin symbols and compact Hankel operators which touch on the structure of function spaces. This is really no surprise, as we have been studying a function theoretic characterization of an ideal. We would like to include the following exposition to note how spaces with "well-mannered oscillation" are related to the compactness of Hankel operators and Berezin symbol behavior near $\partial\mathbf{D}$. First, a discussion of integral averages of L^2 functions over Bergman disks is necessary to define the BMO_∂ and VMO_∂ spaces.

Definition 4.9 The *Bergman disk of radius* r *and centre* z is the Euclidean disk in $\mathbf{D}$ defined by

$$D(z,r) := \{w \in \mathbf{D} \mid \beta(z,w) < r\}$$

where β is the Bergman metric on $\mathbf{D}$,

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

Bergman disks provide a natural neighborhood on which to average a function. Note that even though $D(z,r)$ is an Euclidean disk, z and r are neither the centre or radius in this geometric sense. Instead, because the Bergman metric β is Möbius invariant, the disks behave nicely under Möbian automorphisms: $\varphi_a(D(z,r)) = D(\varphi_a(z),r)$ for every Möbius transformation φ_a.

Now, let $f \in L^\infty(\mathbf{D})$. We define the *average of* f *over* $D(z,r)$, $\widehat{f}_r(z)$, as the integral average

$$\widehat{f}_r(z) = \frac{1}{m(D(z,r))}\int_{D(z,r)} f(w)\, dm(w), \quad z \in \mathbf{D}.$$

The oscillation of f from $\widehat{f}_r$ on the disk $D(z,r)$, or the *mean oscillation of f at z*, is then logically defined as

$$\begin{aligned} MO_r(f)(z) &= \left(\frac{1}{m(D(z,r))}\int_{D(z,r)} \left|f(w)-\widehat{f}_r(z)\right|^2 \, dm(w)\right)^{1/2} \\ &= \left(\widehat{|f|^2}_r(z) - \left|\widehat{f}_r(z)\right|^2\right)^{1/2}, \end{aligned}$$

where the second expression comes from expanding the integrand.

The behavior of the mean oscillation $MO_r(f)$ is the source of the following spaces of functions.

Definition 4.10 Let $f \in L^\infty(\mathbf{D})$. We make the following definitions for the BMO_∂ and VMO_∂ spaces. For details consult [18].

(i). The semi-norm $\|f\|_{MO_r} := \sup_{z\in\mathbf{D}} MO_r(f)(z)$ is bounded independent of r. Therefore the space BMO_∂ of functions of *bounded mean oscillation at the boundary* is defined as

$$BMO_\partial = \{f \in L^\infty(\mathbf{D}) : \|f\|_{MO_r} < \infty \text{ for some } r > 0\}.$$

(ii). The Banach space of functions of *vanishing mean oscillation at the boundary*, VMO_∂, is defined as

$$VMO_\partial = \{f \in BMO_\partial : \lim_{|z|\to 1^-} MO_r(f)(z) \to 0 \text{ for some } r > 0\}.^1$$

The definitions of these spaces naturally relate them to Berezin symbols by the following correspondence which we leave without proof. Details can be found in chapter 7 of [18].

[1]These two spaces are, in fact, examples of using Carleson measures to define function classes. Let the function f define the positive measure μ_f on Bergman disks by the definition $\mu_f(D(z,r)) := \int_{D(z,r)} \left|f(w) - \widehat{f}_r(z)\right|^2 dm(w)$. Then BMO_∂ is the collection of functions that induce "big-oh" Carleson measures, ie: measures satisfying $\sup_{z\in\mathbf{D}} \mu_f(D(z,r))/m(D(z,r) < \infty$, and VMO_∂ is the collection that induce "little-oh" Carleson measures, ie: $\limsup_{|z|\to 1} \mu_f(D(z,r))/m(D(z,r) \to 0$.

Proposition 4.11 *Let $f \in L^\infty(\mathbf{D})$. Then $MO_r(f)(z)$, the mean oscillation of f over the Bergman disk $D(z,r)$, is comparable to a calculation of Berezin symbols:*

$$MO_r(f)(z) = \left(\widehat{|f|^2}_r(z) - \left|\hat{f}_r(z)\right|^2\right)^{1/2} \sim \left(\widetilde{|f|^2}(z) - \left|\tilde{f}(z)\right|^2\right)^{1/2}.$$

Thus BMO_∂ and VMO_∂ are also characterized in terms of Berezin symbols by the following:

$$f \in BMO_\partial \iff \sup_{z\in\mathbf{D}}\left(\widetilde{|f|^2}(z) - \left|\tilde{f}(z)\right|^2\right) < \infty.$$
$$f \in VMO_\partial \iff \limsup_{|z|\to 1^-}\left(\widetilde{|f|^2}(z) - \left|\tilde{f}(z)\right|^2\right) = 0.$$

From this, we can show that VMO_∂ generates compact Hankel operators. We will write $MO(f)(z)$ for $\left(\widetilde{|f|^2}(z) - \left|\tilde{f}(z)\right|^2\right)^{1/2}$.

Theorem 7 (Zhu, 1987 [17]) *Let $u \in L^\infty(\mathbf{D})$. Then H_u and $H_{\overline{u}}$ are compact iff $u \in VMO_\partial$.*

Proof. We will use the result (4.14) to establish the theorem, but first we require some identities. Let $u \in L^\infty(\mathbf{D})$ and let $\tilde{u}$ be the Berezin symbol of its Toeplitz operator. Let $r \in (0,1)$ be arbitrary. Then

$$\begin{aligned}\|(u - \tilde{u}(z))k_z\|^2 &= \langle (u-\tilde{u}(z))k_z, (u-\tilde{u}(z))k_z\rangle \\ &= \langle \overline{u}uk_z, k_z\rangle - \tilde{u}(z)\langle k_z, uk_z\rangle - \overline{\tilde{u}(z)}\langle uk_z, k_z\rangle + |\tilde{u}(z)|^2\langle k_z, k_z\rangle \\ &= \widetilde{|u|^2}(z) - |\tilde{u}(z)|^2 = (MO(u)(z))^2. \end{aligned} \tag{4.15}$$

We will also need the following.

$$\begin{aligned} P\overline{H_{\overline{u}\circ\varphi_z}1} &= P(\overline{\overline{u}\circ\varphi_z - P\overline{u}\circ\varphi_z}) \\ &= Pu\circ\varphi_z - P\overline{P\overline{u}\circ\varphi_z} \\ &= Pu\circ\varphi_z - \overline{P\overline{u}\circ\varphi_z(0)} \\ &= Pu\circ\varphi_z - Pu\circ\varphi_z(0) \end{aligned}$$

where the equality between the second and third lines comes from the fact that the only analytic part of the anti-analytic function $\overline{P\overline{u} \circ \varphi_z}$ is its constant $\overline{P\overline{u} \circ \varphi_z}(0)$. That constant is the same as the constant $\overline{P\overline{u} \circ \varphi_z}(0) = Pu \circ \varphi_z(0)$ which makes the final equality.

We want to prove that $MO(u)(z) \to 0$ as $|z| \to 1^-$, and therefore $u \in VMO_\partial$, iff the Berezin symbols $\widetilde{H_u^*H_u}$ and $\widetilde{H_{\overline{u}}^*H_{\overline{u}}}$ vanish on $\partial\mathbf{D}$. Starting from equation (4.15), we make a change of variable with the Möbius transformation φ_z yielding

$$\begin{aligned}(MO(u)(z))^2 &= \|(u - \tilde{u}(z))k_z\|^2 = \|u \circ \varphi_z - \tilde{u}(z)\|^2 \\ &= \|u \circ \varphi_z - P(u \circ \varphi_z) + P(u \circ \varphi_z) - Pu \circ \varphi_z(0) + Pu \circ \varphi_z(0) - \tilde{u}(z)\|^2 \\ &= \left\|H_{u\circ\varphi_z}1 - P\overline{H_{\overline{u}\circ\varphi_z}1} + Pu \circ \varphi_z(0) - \tilde{u}(z)\right\|^2 .\end{aligned}$$

Now $\tilde{u}(z) = \widetilde{u \circ \varphi_z}(0) = \langle T_{u\circ\varphi_z}k_0, k_0\rangle = Pu \circ \varphi_z(0)$, so in fact,

$$(MO(u)(z))^2 = \left\|H_{u\circ\varphi_z}1 - P\overline{H_{\overline{u}\circ\varphi_z}1}\right\|^2 = \|H_{u\circ\varphi_z}1\|^2 + \left\|P\overline{H_{\overline{u}\circ\varphi_z}1}\right\|^2 . \tag{4.16}$$

It follows that

$$(MO(u)(z))^2 \leq \|H_u k_z\|^2 + \|H_{\overline{u}} k_z\|^2 = \widetilde{H_u^*H_u} + \widetilde{H_{\overline{u}}^*H_{\overline{u}}}$$

and

$$\widetilde{H_u^*H_u} = \|H_u k_z\|^2 \leq (MO(u)(z))^2 .$$

Since $MO_r(u)(z) = MO_r(\overline{u})(z)$, the last inequality also holds with u replaced by $\overline{u}$ proving the equivalence. □

4.2.3 Compact Hankel Operators on H^2

The open question of the last section regarding an analagous "Theorem 2" for finite sums of finite products of Toeplitz operators on the Hardy space would have consequences for Hankel operators, just as it did on the Bergman space. Since the Toeplitz representation

$H_u^* H_u = T_{|u|^2} - T_{\overline{u}} T_u$, equation (4.13), also holds for H^2 (with a similar proof), it is possible that similar arguments for current characterizations of compact H_u could be reduced to Berezin symbol analysis. We can, however, as we did in the H^2 Toeplitz case, use a current theorem to prove that vanishing Berezin symbols do characterize compact Hankel operators.

We will begin by observing that the Hardy space has BMO and VMO spaces as well.

Definition 4.12 The following are the definitions for $BMO = BMO(\partial\mathbf{D})$ and $VMO = VMO(\partial\mathbf{D})$. Let I be an arc length in $\partial\mathbf{D}$. The *average of f over I* is $\hat{f}_I = \frac{1}{m(I)} \int_I f(\theta) d\theta$.

(i). The collection of functions in $L^2(\partial\mathbf{D})$ of *bounded mean oscillation*, BMO, are those satisfying

$$\sup_I \frac{1}{m(I)} \left(\int_I \left| f(\theta) - \hat{f}_I(\theta) \right|^2 d\theta \right)^{1/2} < \infty.$$

(ii). The collection of functions in BMO of *vanishing mean oscillation*, VMO, are those satisfying

$$\lim_{m(I)\to 0} \frac{1}{m(I)} \int_I \left| f(\theta) - \hat{f}_I(\theta) \right|^2 d\theta < \infty.$$

These spaces also have a characterization in terms of Berezin symbols, [18, page 175, 183]:

$$f \in BMO \iff \sup_{z\in\mathbf{D}} \left(\widetilde{|f|^2}(z) - \left| \tilde{f}(z) \right|^2 \right) < \infty.$$
$$f \in VMO \iff \limsup_{|z|\to 1^-} \left(\widetilde{|f|^2}(z) - \left| \tilde{f}(z) \right|^2 \right) = 0.$$

The collection of all analytic VMO functions is called $VMOA$. The following lemma is proven in [18].

Lemma 4.13 *$VMOA$ is the Szëgo projection of all continuous functions on $\partial\mathbf{D}$.*

We can now prove the following.

Theorem 8 *Let $u \in L^\infty(\partial\mathbf{D})$. Then the compact Hankel operators H_u on H^2 are characterized by*

$$H_u \text{ compact} \iff \widetilde{H_u^* H_u} \text{ vanishes on } \partial\mathbf{D}.$$

Proof. As usual, only the proof of the backward direction is necessary. Considering u as an $L^2(\partial\mathbf{D})$ function, we can split it into analytic and anti-analytic parts, $u = u_1 + \overline{u_2}$, where both u_i are analytic. Now, by the definiton of H_u, the analytic part u_1 does not affect the behavior of the Hankel operator. We have that $H_u = H_{\overline{u_2}}$ and $\widetilde{H_u^* H_u} = \widetilde{H_{\overline{u_2}}^* H_{\overline{u_2}}}$. The calculation of $MO(u_2)$ carries over directly from the previous subsection (equation (4.16)), making

$$\widetilde{|u_2|^2}(z) - |\widetilde{u_2}(z)|^2 = \|H_{\overline{u}_2 \circ \varphi_z} 1\|^2 + \left\|P\overline{H_{u_2 \circ \varphi_z} 1}\right\|^2 .$$

By the analyticity of u_2, this simplifies to

$$\widetilde{|u_2|^2}(z) - |\widetilde{u_2}(z)|^2 = \|H_{\overline{u}_2 \circ \varphi_z} 1\|^2 = \widetilde{H_{\overline{u_2}}^* H_{\overline{u_2}}}(z).$$

Therefore our hypothesis of vanishing $\widetilde{H_u^* H_u}$ implies $u_2 \in VMOA$.

Using our lemma above, then, there is a continuous function g on $\partial\mathbf{D}$ such that $Pg = u_2$. Now the Weierstrauss approximation theorem [7] generates a sequence of polynomials $p_n(z, \overline{z})$ that converge to g in the $L^\infty(\partial\mathbf{D})$ norm. From this, we can see that $H_{\overline{g}}$ is the limit of a sequence of compact operators. For the polynomial $p_n(z, \overline{z})$ has a finite order r, and we have that $(I - P)\overline{p_n} z^m = 0$ for all $m \geq r$. So, at most, there are r independent vectors[2] in the range of $H_{\overline{p_n}}$, making the operator of finite rank. Further, the operators $H_{\overline{p_n}}$ converge to $H_{\overline{g}}$, since

$$\begin{aligned} \|H_{\overline{p_n}} f - H_{\overline{g}} f\| &= \|(I-P)(\overline{p_n} - \overline{g}) f\| \\ &\leq \|\overline{p_n} - \overline{g}\|_\infty \|f\| . \end{aligned}$$

Thus $H_{\overline{g}}$ is compact, and, therefore, so is $H_u = H_{\overline{u_2}} = H_{\overline{Pg}} = H_{\overline{g}}$, where the last inequality follows because $\overline{(I-P)g}$ is analytic and ignorable in a Hankel operator. □

[2]Specifically, the vectors $\{H_{\overline{p_n}} z^m : m = 0, 1, \ldots, r-1\}$

To close, we mention a Hardy space theorem parallel to Theorem 7. A proof may be found in Chapter 9 of [18].

Theorem 9 (Sarason, 1975) *Let $u \in L^\infty(\mathbf{D})$. Then H_u and $H_{\overline{u}}$ are compact iff $u \in VMO$.*

4.3 Composition Operators

Unlike our previous cases, no characterization of compact composition operators C_ϕ via Berezin symbol methods has appeared in the literature. It is apparent from current theory, however, that such an approach is possible. We will present in this section some current conditions for the compactness of C_ϕ on H^2 and L_a^2 that are linked to Berezin symbol calculations and Toeplitz operators.

4.3.1 Compact Composition operators on H^2

We break our trend and consider the Hardy space case first. This imitates the original consideration by Shapiro [12] and will clearly motivate the formulation for the Bergman space that will follow in section 4.3.2.

To begin our study of composition operators we require an alternate norm for the Hardy space and a few definitions from value distribution theory. For the rather consuming proofs of the results, we refer the reader to the appropriate literature.

Lemma 4.14 (Littlewood-Paley Identity, 1931) *Let f, g be functions on the Hardy space $H^2(\mathbf{D})$. Then the inner product $\langle f, g\rangle$ is equal to the calculation*

$$\langle f, g\rangle = f(0)\overline{g(0)} + \langle\langle f, g\rangle\rangle$$

where $\langle\langle f, g\rangle\rangle$ is defined as a weighted integral over $\mathbf{D}$ of the derivatives:

$$\langle\langle f, g\rangle\rangle = \int_{\mathbf{D}} f'(w)\overline{g'(w)} \log \frac{1}{|w|^2}\, dm(w).$$

Proof. See [10]. □

Definition 4.15 Let $\phi : \mathbf{D} \mapsto \mathbf{D}$ be an analytic function. Then the *Nevanlinna counting function of ϕ* is the function N_ϕ defined on $\mathbf{D} \setminus \{\phi(0)\}$ as

$$N_\phi(z) = \sum \left\{ \log \frac{1}{|w|} : \phi(w) = z \right\} \tag{4.17}$$

and we assume that if the set on the right is empty, N_ϕ is given the value 0. Now, since the analytic function $\phi(w) - z$ can have at most countably many roots, the number of elements in the right hand set is countable and we will typically write this as an indexed sum, $N_\phi(z) = \sum_j -\log|w_j(z)|$, where the $w_j(z)$ are the pre-images of z under ϕ. Considering that the $w_j(z)$'s are the elements of the zero sequence for an analytic function, the sum is also guaranteed to converge. (See [13].)

The importance of the Nevanlinna counting function is its appearance in the non-bijective change of variable formula below.

Lemma 4.16 (Properties of N_ϕ) *Let $\phi : \mathbf{D} \mapsto \mathbf{D}$ be a nonconstant analytic function. Then the Nevanlinna function N_ϕ satisfies the following:*

(i). **(Change of Variable)** *Let g be analytic on $\mathbf{D}$. Then*

$$\int_{\mathbf{D}} |g \circ \phi(w)|\, |\phi'(w)|^2 \log \frac{1}{|w|}\, dm(w) = \int_{\mathbf{D}} |g(z)|\, N_\phi(z)\, dm(z).$$

Equally, taking $g = (f')^2$,

$$\langle\langle f \circ \phi, f \circ \phi \rangle\rangle = 2 \int_{\mathbf{D}} |f'(z)|^2 N_\phi(z)\, dm(z). \tag{4.18}$$

(ii). **(Littlewood's Inequality, 1925)** $N_\phi(z) \leq -\log|\varphi_a(z)|$ *where $a = \phi(0)$.*

(iii). **(Sub-Mean-Value Property)** *N_ϕ is a limit of subharmonic functions and when $r < |\varphi_z(\phi(0))|$,*

$$N_\phi(z) = N_\phi \circ \varphi_z(0) \leq \frac{1}{r^2} \int_{r\mathbf{D}} N_\phi \circ \varphi_z(\lambda)\, dm(\lambda).$$

(iv). $N_\phi \in L^1(\mathbf{D})$.

Proof. (i) - (iii) Equation (4.18) comes from comparing the definition of $\langle\langle \phi, \phi \rangle\rangle$ to the first change of variable formula and using the identity $-\log|w|^2 = -2\log|w|$. For the rest, see Zhu [18, Chapter 10] for the proofs.

(iv) Substitute $f(z) = z$ into equation (4.18):

$$\langle\langle \phi, \phi \rangle\rangle = 2 \int_{\mathbf{D}} N_\phi(z)\, dm(z).$$

Since $\|\phi\|_\infty \leq 1$ and $\langle\langle \phi, \phi \rangle\rangle = \|\phi\|_2 - |\phi(0)| \leq \|\phi\|_\infty - |\phi(0)|$, we have that N_ϕ is in L^1, and in fact $\|N_\phi\|_1 \leq 1/2$. □

The Möbius disk automorphisms $\psi = \lambda\varphi_a$, being invertible, have the Nevanlinna functions $N_\psi(z) = -\log|\varphi_a(z)|$ where $a = \psi(0) \in \mathbf{D}$. According to Littlewood's Inequality, these are in fact the maximal ones.

The classical characterization of compact composition operators on the Hardy space is the following, due to J. Shapiro [12].

Theorem 10 (Shapiro, 1987) *Let $\phi : \mathbf{D} \mapsto \mathbf{D}$ be analytic. Then the composition operator C_ϕ on $H^2(\mathbf{D})$ is compact (ie: in $\mathcal{K}(H^2)$) iff*

$$\lim_{|z| \to 1^-} \frac{N_\phi(z)}{-\log|z|} = 0.$$

Proof. ($\Rightarrow$) Assume C_ϕ is compact. Then $\|C_\phi k_z\| \to 0$ as $|z| \to 1^-$ since H^2 is uniformly standard. Applying the Littlewood-Paley identity and the change of variable formula to the norm calculation $\|C_\phi k_z\|^2$, we have

$$\|C_\phi k_z\|^2 = |k_z(\phi(0))|^2 + \langle\langle C_\phi k_z, C_\phi k_z \rangle\rangle$$

and

$$\begin{aligned} \langle\langle C_\phi k_z, C_\phi k_z \rangle\rangle &= 2 \int_{\mathbf{D}} |k_z'(w)|^2 N_\phi(w)\, dm(w) \\ &= \frac{2|z|^2}{1 - |z|^2} \int_{\mathbf{D}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} N_\phi(w)\, dm(w) \end{aligned}$$

by explicitly calculating k_z'. Recall that $\varphi_z' = -(1 - |z|^2)/(1 - \bar{z}w)^2$. Making a substitution and Möbius change of variable, the above becomes

$$\langle\langle C_\phi k_z, C_\phi k_z \rangle\rangle = \frac{2|z|^2}{(1 - |z|^2)} \int_{\mathbf{D}} N_\phi(\varphi_z(w))\, dm(w).$$

Using the sub-mean value theorem over $r\mathbf{D}$ as an estimate, where $r = |\varphi_z(\phi(0))|$, we get

$$\langle\langle C_\phi k_z, C_\phi k_z\rangle\rangle \geq \frac{2|z|^2}{(1-|z|^2)} N_\phi(\varphi_z(0))\,|\varphi_z(\phi(0))|^2 .$$

We therefore have the inequality

$$\|C_\phi k_z\|^2 \geq \frac{2|z|^2}{(1-|z|^2)} N_\phi(z)\,|\varphi_z(\phi(0))|^2 \tag{4.19}$$

which as the norm converges to zero is equivalent in the limit of $|z| \to 1^-$ to

$$\lim_{|z|\to 1^-} \frac{N_\phi(z)}{-\log|z|} = 0.$$

($\Leftarrow$) Let $\epsilon > 0$. Let f_n be a sequence weakly converging to zero in H^2. This implies that $\sup_n \|f_n\| < \infty$ and f_n uniformly converges to zero on compact subsets of $\mathbf{D}$ (see [18, page 189]). It follows from classical complex analysis that $f_n' \to 0$ uniformly on compact subsets too (see [7, page 151]). Now making a calculation with the Littlewood-Paley identity and equation (4.18),

$$\|C_\phi f_n\|^2 = |f_n(\phi(0))|^2 + 2\int_{\mathbf{D}} |f_n'(z)|^2 N_\phi(z)\,dm(z).$$

The first term is simply $\left|\left\langle f_n, K_{\phi(0)}\right\rangle\right|^2$ and so will converge to zero as $n \to \infty$ since the f_n's are a weak null sequence. Now pick an $r \in (0,1)$ such that by our hypothesis $N_\phi(z) < \epsilon \log|1/z|$ for all $|z| > r$. Then dividing the integral above over the compact set $\overline{r\mathbf{D}}$ and the open annulus $r < |z| < 1$, we can make the estimate

$$\begin{aligned}
&\int_{\overline{r\mathbf{D}}} |f_n'(z)|^2 N_\phi(z)\,dm(z) + \int_{r<|z|<1} |f_n'(z)|^2 N_\phi(z)\,dm(z) \\
&\quad\leq \sup_{\overline{r\mathbf{D}}} |f_n'(z)|^2 \int_{\overline{r\mathbf{D}}} N_\phi(z)\,dm(z) \\
&\qquad + \sup_{r<|z|<1} \frac{N_\phi(z)}{-\log|z|} \int_{r<|z|<1} |f_n'(z)|^2 (-\log|z|)\,dm(z) \\
&\quad\leq \epsilon\,\|N_\phi\|_1 + \epsilon\,\langle\langle f_n, f_n\rangle\rangle
\end{aligned}$$

for large enough n. Therefore C_ϕ is compact. □

As a Corollary to the Shapiro theorem, we can show that Berezin symbols behave nicely for compact composition operators on H^2.

Corollary 4.17 *Let $\phi : \mathbf{D} \mapsto \mathbf{D}$ be analytic. Then the compact composition operator C_ϕ on $H^2(\mathbf{D})$ is compact iff $\widetilde{C_\phi^* C_\phi}$ vanishes on $\partial\mathbf{D}$.*

Proof. As usual, we are only required to prove that the vanishing Berezin symbol implies C_ϕ is compact. Using Theorem 10, it is enough to show

$$\lim_{|z|\to 1^-} \widetilde{C_\phi^* C_\phi}(z) = 0 \Longrightarrow \lim_{|z|\to 1^-} \frac{N_\phi(z)}{-\log|z|} = 0.$$

Observe, however, from equation (4.19) that

$$\widetilde{C_\phi^* C_\phi}(z) = \|C_\phi k_z\|^2 \geq \frac{2|z|^2}{(1-|z|^2)} N_\phi(z)\,|\varphi_z(\phi(0))|^2 .$$

This gives the necessary implication. □

In our previous section we observed the characterization for Hankel operators on L_a^2 stemmed from a result for Toeplitz operators. Things are not so different here. By making the non-univalent change of variable (4.18), we introduce a multiplication function into an integral, but the integral does not correspond to either L_a^2 or H^2. However, if we write

$$\begin{aligned} \langle\langle f\circ\phi, f\circ\phi\rangle\rangle &= 2\int_{\mathbf{D}} |f'(z)|^2 N_\phi(z)\,dm(z) \\ &= 2\int_{\mathbf{D}} |f'(z)|^2 \frac{N_\phi(z)}{-\log|z|}(-\log|z|)\,dm(z) \\ &= \langle\langle \tau_\phi(z) f, f\rangle\rangle \end{aligned}$$

we see we would say $C_\phi^* C_\phi = T_{\tau_\phi}$ in this $\langle\langle\cdot,\cdot\rangle\rangle$ inner product, where τ_ϕ is defined

$$\tau_\phi(z) = \frac{N_\phi(z)}{-\log|z|} \geq 0.$$

This demonstrates a connection between Hardy composition operators and Toeplitz operators on weighted Bergman spaces. At least in an interpretive sense, the Shapiro characterization is a result from a Berezin symbol theory for Toeplitz operators on a weighted Bergman space. However further consideration of this goes well beyond the scope of this paper. For a discussion about Berezin symbols and positive Toeplitz operators on weighted Bergman spaces, see [18, §6.4].

4.3.2 Compact Composition operators on L_a^2

The theory of the previous subsection is robust enough to transfer even to the Bergman space. Shapiro generalized the classical definitions of the Littlewood-Paley identity and Nevanlinna function in [12] in the following way. For brevity, we will omit all proof.

Lemma 4.18 (Littlewood-Paley Identity on L_a^2) *Let f, g be functions on the Hardy space L_a^2. Then the inner product $\langle f, g\rangle$ on L_a^2 is equal to the calculation*

$$\langle f, g\rangle = f(0)\overline{g(0)} + \langle\langle f, g\rangle\rangle$$

where $\langle\langle f, g\rangle\rangle$ is defined as a weighted integral over $\mathbf{D}$ of the derivatives:

$$\langle\langle f, g\rangle\rangle = \int_{\mathbf{D}} f'(w)\overline{g'(w)}\frac{1}{2}(\log\frac{1}{|w|^2})^2\, dm(w).$$

Definition 4.19 Let $\phi : \mathbf{D} \mapsto \mathbf{D}$ be an analytic function. Then the *Bergman Nevanlinna counting function of ϕ* is the function $N_{\phi,2}$ defined on $\mathbf{D} \setminus \{\phi(0)\}$ as

$$N_{\phi,2}(z) = \sum_j (-\log|w_j(z)|)^2 \quad : \quad \phi(w_j(z)) = z \tag{4.20}$$

and we assume that $N_{\phi,2}$ is 0 if there are no pre-images $w_j(z)$.

The Bergman Nevanlinna function also contains all the necessary change of variable information.

Lemma 4.20 (Properties of $N_{\phi,2}$) *Let $\phi : \mathbf{D} \to \mathbf{D}$ be a nonconstant analytic function. Then the Nevanlinna function $N_{\phi,2}$ satisfies the following:*

(i). **(L_a^2 Change of Variable)** *Let g be analytic on $\mathbf{D}$. Then*

$$\int_{\mathbf{D}} |g\circ\phi(w)|\,|\phi'(w)|^2\,\frac{1}{2}(-2\log|w|)^2\, dm(w) = 2\int_{\mathbf{D}} |g(z)|\, N_{\phi,2}(z)\, dm(z).$$

Equally, taking $g = |f'|^2$,

$$\langle\langle f\circ\phi, f\circ\phi\rangle\rangle = 2\int_{\mathbf{D}} |f'(z)|^2\, N_{\phi,2}(z)\, dm(z) \tag{4.21}$$

(ii). (**L^2_a Littlewood's Inequality**) $N_{\phi,2}(z) \leq (-\log|\varphi_a(z)|)^2$ *where* $a = \phi(0)$.

(iii). (**L^2_a Sub-Mean-Value Property**) $N_{\phi,2}$ *is a limit of subharmonic functions and when* $r < |\varphi_z(\phi(0))|$.

$$N_{\phi,2}(z) = N_{\phi,2} \circ \varphi_z(0) \leq \frac{1}{r^2} \int_{r\mathbf{D}} N_{\phi,2} \circ \varphi_z(\lambda)\, dm(\lambda).$$

(iv). $N_{\phi,2} \in L^1(\mathbf{D})$.

The characterization of compact composition operators on L^2_a using $N_{\phi,2}$ is the following.

Theorem 11 (Shapiro) *Let* $\phi : \mathbf{D} \mapsto \mathbf{D}$ *be analytic. Then the compact composition operator* C_ϕ *on* L^2_a *is compact (ie: in* $\mathcal{K}(L^2_a)$*) iff*

$$\lim_{|z|\to 1^-} \frac{N_{\phi,2}(z)}{(-\log|z|)^2} = 0.$$

The corollary for Theorem 11 follows the exact lines of the Hardy space consideration.

Corollary 4.21 *Let* $\phi : \mathbf{D} \mapsto \mathbf{D}$ *be analytic. Then the compact composition operator* C_ϕ *on* L^2_a *is compact iff* $\widetilde{C_\phi^* C_\phi}$ *vanishes on* $\partial\mathbf{D}$ *iff* $\widetilde{\tau_{\phi,2}}$ *vanishes on* $\partial\mathbf{D}$. *where* $\tau_{\phi,2} = N_{\phi,2}(z)/(-\log|z|)^2$ *is a symbol for a Toeplitz operator on a weighted Bergman space* (A^2_2).

Chapter 5

Further Questions

We have demonstrated in this paper that, on a standard analytic Hilbert space, the Berezin symbol of a bounded operator is related to operator compactness by several kinds of boundary behavior. Expressing these conditions in the case of the Bergman space, they are:

(i). A is compact iff $\widetilde{A^U} \in C_0(\mathbf{D})$ for all unitary U.

(ii). For a class of operators $\mathcal{T}'$ defined by the condition of Lemma 4.2, A is compact iff $\left\|A^{U_z}1\right\|_p \to 0$ as $|z| \to 1^-$ for all $p \in (1, \infty)$. This applies to all finite sums of finite products of Toeplitz operators.

(iii). For a class of operators $\mathcal{V}$, which includes all Toeplitz, Hankel, and composition operators, A is compact iff $\widetilde{A^*A}$ vanishes on the boundary.

We record here some questions for further investigation:

1. What are the contents of $\mathcal{T}'$? (ie: for which operators A is $\sup_{z\in\mathbf{D}} \left\|A^{U_z}1\right\|_p < \infty$?) Are there any other necessary and sufficient conditions? Does the class $\mathcal{T}'_{H^2}$ have any meaning for operators on the Hardy space? If so, what are its contents and equivalent characterizations?

2. What are the contents of the class $\mathcal{V}$? (ie: for which operators A is A compact iff $\widetilde{A^*A}$ vanishes on the boundary?) In the Hardy space class $\mathcal{V}_{H^2}$ in particular, are finite sums of finite products of Toeplitz operators in $\mathcal{V}_{H^2}$? What characterization is there? Can every operator in $\mathcal{V}$ and $\mathcal{V}_{H^2}$ be written as a Toeplitz operator on a weighted Bergman space? Or an operator in the algebra generated by the Toeplitz operators on a weighted Bergman space?

3. How do $\mathcal{T}'$ and $\mathcal{V}$ compare? If $A \in \mathcal{T}'$, according to our proof of Theorem 2, then $A \in \mathcal{K}(L_a^2)$ iff $\widetilde{A^*A}$ converges to zero on $\partial\mathbf{D}$. That is, $A \in \mathcal{V}$. We therefore observe that $\mathcal{T}' \subset \mathcal{V}$. The reverse inclusion appears false, because composition operators are in $\mathcal{V}$ but the Möbian unitary $U_0 = C_{-z}$ of Example 4.4 is not included in $\mathcal{T}'$. How does a class defined by the condition $\sup_{z\in\mathbf{D}} \left\|(A^*A)^{U_z}1\right\|_p < \infty$ compare?

4. What connection is there between arbitrary unitary conjugates and the classes $\mathcal{T}'$ and $\mathcal{V}$? What is it about the projection operator of Example 3.16 that excludes it from $\mathcal{V}$? Does the polar decomposition have some role to be considered?

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