# The Generalization of Stop Loss Transforms and Its Applications in Ruin Probabilities

A Master's Thesis

Submitted by Yu Cheng

Supervisor : Dr. Jeffrey Pai

The Warren Centre for Actuarial Studies & Research

University of Manitoba

0-612-32076-6

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

COPYRIGHT PERMISSION

# THE GENERALIZATION OF STOP LOSS TRANSFORMS AND ITS APPLICATIONS IN RUIN PROBABILITIES

BY

YU CHENG

A Thesis/Practicum submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

## Abstract

This thesis generalized the concept of stop-loss transforms, an important concept in risk theory [6], to the $n$th stop-loss transforms. Some useful properties of the $n$th stop-loss transforms were discovered and a recursion formula for the $n$th stop-loss transforms was established. Also, the maintenance properties of the $n$th stop-loss order under convolution, compound and mixture operations were proved. Finally the results mentioned above were applied to the study of losses $L_i \, (i = 1, 2, \cdots)$, maximal aggregate loss $L$ and ruin probability $\psi(u)$. Some inequalities for the expectation of $L_i$ and $L$ were given, and a relationship between the claim amount random variable and ruin probability was found.

**Key Words:** classical risk model, homogeneous Poisson processes, surplus process, stop-loss transform, stop-loss order, ruin probability.

## Acknowledgements

I wish to express my gratitude to my supervisor Dr. Jeffrey Pai for his invaluable guidance during the preparation of this thesis. Dr. J. Pai provided me with much support and encouragement, and made many suggestions that led to the completion and improvement of this thesis. I have benefited greatly through my association with him.

I would like to thank Assistant Director Kevin Shand and Director Michael Byrne, for their enthusiastic help in various ways. My thanks also go out to my friends for their constant encouragement.

Finally, I wish to thank my family and colleagues who supported me in so many special ways.

The work on this thesis was partially supported by the Natural Sciences and Engineering Research Council of Canada through a grant OGP0195676.

# Contents

# List of Figures

# 0 Introduction

Decision making is essentially a problem about how to put actions in order. Generally there are some actions called alternatives for a decision-maker to choose. What the decision-maker needs to do is to choose one action that he/she prefers mostly from the alternatives. If there is an order of preference among the alternatives beforehand, then to obtain the most preferred action is extremely easy.

Each action corresponds to a perspective. In fact, a decision-maker prefers an action according to whether he/she prefers the perspective corresponding to that action. The perspectives are generally uncertain, especially in economic problems. Since we usually express perspectives by random variables, so the order between the alternatives can be reduced to the order between random variables.

The theory of partial order on a family of random variables is useful broadly, in queuing theory, reliability theory, economics as well as in actuarial science. For an insurance company, each contract of insurance brings a risk with it. A claim may occur some time in the future and the amount of the claim is a nonnegative random variable which is called a risk. The partial orders on a family of risks are called risk orders. The theory of risk orders is useful in risk theory and in optimal reinsurance. The stop-loss order discussed in this thesis is one type of risk orders.

This thesis is based upon the works of Goovaerts et al. [6] and Gerber et al. [4]. Many kinds of partial order were discussed in [6]. The stop-loss order is one of them. The classical risk model was studied in, but not limited to [2]–[5]. Two results from [4] will be quoted in this thesis. In this thesis, the concept of stop-loss transforms in [6] will be generalized to the $n$th stop-loss transforms ($n = 0, 1, 2, \cdots$), denoted by $\Pi_X^{(n)}(x)$, $\Pi_H^{(n)}(x)$ etc., with the stop-loss transforms in [6] as a special case when $n = 1$. Many interesting properties of the $n$th stop-loss transforms and a recursion formula for $\Pi_X^{(n)}(x)$ will be given. In this thesis the $n$th stop-loss order on a family of random variables will be studied. We denote that random variable $X$ is less than the random variable $Y$ in the meaning of $n$th stop-loss order by $X <_{sl(n)} Y$. In practical applications we concern the distribution function or survival function of a random variable, rather than the random variable itself. From the practical viewpoint there is no difference between two random variables with the same distribution or survival function. Consequently, a partial order on a family of distribution functions(or survival functions) brings a partial order to a family of random variables, and vice versa. From now on we will not distinguish these partial orders. For example, if we denote the distribution function, survival function of $X$ and $Y$ by $F(x)$, $\overline{F}(x) = 1 - F(x)$ and $G(x)$, $\overline{G}(x) = 1 - G(x)$ respectively, then we consider the following three inequalities to be equivalent to each other: $X <_{sl(n)} Y$, $F <_{sl(n)} G$ and $\overline{F} <_{sl(n)} \overline{G}$ . Based on the final symbol, the concept of the stop-loss order can be generalized to a family of monotonous decrease functions.

In this thesis the maintenance properties of the $n$th stop-loss order under the operations of convolution, compound and mixture will be proved. In [6] the maintenance properties of the $1^{st}$ stop-loss order were proved. But the method used in this thesis for proving the properties is different from that in [6].

Finally, the results mentioned above will be used to study the losses $L_i (i = 1, 2, \cdots)$

(for the definition of $L_i$ see (2.12)), maximal aggregate loss $L$ and ruin probability $\psi(u)$ (see (2.5)). The concepts of stop-loss transform and stop-loss order will be introduced to a family of nonnegative monotonous decrease functions. We will derive the relationship between the stop-loss transform of $X$ and $L_i$ (see (2.28)), some inequalities for the expectation of $L_i$ and $L$ (see (2.26), (2.27) , (2.33), (2.34)), and the relationship between claim random variable and the ruin probability (see theorem (2.13)).

# 1 Stop-Loss Transforms and Stop-Loss orders

## 1.1 Stop-loss transforms and recursion formula

The concept of stop-loss transforms and its properties play an important role in this thesis. At first we generalize the concept of stop-loss transforms in [6] as follows.

**Definition 1.1** Suppose random variable $X$ is nonnegative with its distribution function being $F(x)$, its survival function being $\overline{F}(x) = 1 - F(x)$, and $E(X^n) < \infty$. Let

$$\Pi^{(n)}(u) = E\{[(X-u)_+]^n\}, \quad u \geq 0, n = 1, 2, \cdots, \tag{1.1}$$

where

$$(x-u)_+ = \begin{cases} 0, & \text{for } x \leq u, \\ x-u, & \text{for } x > u, \end{cases}$$

$$\Pi^{(0)}(u) = \overline{F}(u) = 1 - F(u). \tag{1.2}$$

As a function of $u$, $\Pi^{(n)}(u), n = 1, 2, \cdots$ will have domain $[0, \infty)$. We call function $\Pi^{(n)}(u)$ the $n$th stop-loss transform of $X$ (or its distribution function $F(x)$ and survival function $\overline{F}(x)$). When we need to indicate of which $n$th stop-loss transform it is, we will add subscript to it. For example, the $n$th stop-loss transforms of $X$ ( its distribution function and survival function are $F(x)$ and $\overline{F}(x)$ respectively) is denoted by $\Pi^{(n)}_X(u)$, $\Pi^{(n)}_F(u)$ or $\Pi^{(n)}_{\overline{F}}(u)$ (the three are considered no difference ). It is easy to see that the concept of stop-loss transform in [6] ( see page 25, definition 3.1.4 in [6] ) is the special case of definition 1.1 when $n = 1$.

The following corollary then becomes obvious.

**Remark 1.2** $\Pi^{(n)}(0) = E(X^n)$, $n = 1, 2, \cdots$ *and* $\Pi^{(0)}(0) = 1$.

**Example 1.3** Prove that

$$\Pi^{(1)}(u) = \int_u^\infty \overline{F}(x)dx. \tag{1.3}$$

**Proof.** Let $n = 1$ in (1.1) and take integration by parts, we have

$$\begin{aligned}
\Pi^{(1)}(u) &= E[(X-u)_+] \\
&= \int_u^\infty (x-u)dF(x) = -\int_u^\infty (x-u)d\overline{F}(x) \\
&= -(x-u)\overline{F}(x)\mid_{x=u}^\infty + \int_u^\infty \overline{F}(x)dx \\
&= \int_u^\infty \overline{F}(x)dx. \quad \blacksquare
\end{aligned}$$

Note that in the above proof we used the following equation:

$$\lim_{x\to\infty}(x-u)\overline{F}(x) = 0.$$

When $E(X) < \infty$, the above equation always holds (see proposition 1.4. Letting $n = 1$ in proposition 1.4, we get the above equation). For convenience to use later, we prove a more general result as follows:

**Proposition 1.4** If nonnegative random variable $X$ has a finite $n$th moment, then

$$\lim_{x\to\infty}(x-u)^n\overline{F}(x) = 0, \quad \forall u \geq 0, \tag{1.4}$$

where $\overline{F}(x)$ is the survival function of $X$.

**Proof.** Because the $n$th moment of $X$ is finite, we have

$$\lim_{x\to\infty}\int_x^\infty y^n dF(y) = 0.$$

Hence

$$\lim_{x\to\infty}(x-u)^n\overline{F}(x) \le \lim_{x\to\infty} x^n\overline{F}(x) \le \lim_{x\to\infty}\int_x^\infty y^n dF(y) = 0. \blacksquare$$

**Example 1.5** Suppose $E(X^2) < \infty$, then

$$E(X^2) = 2\int_0^\infty \Pi_X^{(1)}(u)du. \tag{1.5}$$

**Proof.**

$$E(X^2) = \int_0^\infty x^2 dF_X(x) = -\int_0^\infty x^2 d\overline{F}_X(x).$$

By using integration by parts and then (1.4) ( let $u = 0$ in (1.4)), we have

$$\begin{aligned} E(X^2) &= 2\int_0^\infty x\overline{F}_X(x)dx = 2\int_0^\infty [\overline{F}_X(x)\int_0^x dy]dx \\ &= 2\int_0^\infty \int_y^\infty \overline{F}_X(x)dxdy = 2\int_0^\infty \Pi_X^{(1)}(y)dy. \end{aligned}$$

In the above proof we interchange the order of integration and use our results from Example 1.3 to complete the proof. $\blacksquare$

The following recursion formula for the $n$th stop-loss transforms is significant for some later results.

**Theorem 1.6**

$$\Pi^{(n)}(u) = n\int_u^\infty \Pi^{(n-1)}(x)dx, \quad n = 1, 2, \cdots. \tag{1.6}$$

To prove this theorem we need the following lemma which has its own meaning and can be used in other occasions.

**Lemma 1.7** Suppose $F(x)$ is a distribution function. If function $f(x, y)$ satisfies the following conditions:

(a) $\dfrac{\partial f(x,y)}{\partial y}$ exists,

(b) When $\triangle u$ is in some neighborhood of 0, say $(-\alpha, \alpha)$ , we have

$$\left| \frac{f(x, u + \triangle) - f(x, u)}{\triangle u} \right| \leq g(x),$$

where nonnegative function $g(x)$ is Stieltjes integrable on $[0, u]$ with respect to the distribution function $F(x)$, i.e.

$$\int_0^u g(x) dF(x) < \infty.$$

( c ) $\lim_{|x-y| \to 0} \left| \frac{f(x, y)}{x - y} \right| = 0$. Then

$$\frac{d}{du}\left[\int_0^u f(x, u) dF(x)\right] = \int_0^u \frac{\partial f(x, u)}{\partial u} dF(x). \tag{1.7}$$

That is, we could place the derivative of the left-hand side of (1.7) into the integration of which the upper limit is variable $u$.

**Proof.** According to the definition of derivative we have

$$\begin{aligned}
& \frac{d}{du}\left[\int_0^u f(x, u) dF(x)\right] \\
= \;& \lim_{\triangle u \to 0} \frac{1}{\triangle u}\left\{\int_0^{u+\triangle u} f(x, u + \triangle u) dF(x) - \int_0^u f(x, u) dF(x)\right\} \\
= \;& \lim_{\triangle u \to 0} \int_0^u \frac{f(x, u + \triangle u) - f(x, u)}{\triangle u} dF(x) + \lim_{\triangle u \to 0} \int_u^{u+\triangle u} \frac{f(x, u + \triangle u)}{\triangle u} dF(x) \\
= \;& A + B,
\end{aligned}$$

where $A$ and $B$ express the first and the second limit above, respectively. From condition (b) we know that the integrand in $A$ satisfies the condition of Lebesgue's convergence theorem, so the limit can be taken into the integration, that is

$$A = \int_0^u \frac{\partial f(x, u)}{\partial u} dF(x).$$

From condition (c) we have

$$\begin{aligned}
\mid B \mid &= \mid \lim_{\Delta u \to 0} \int_u^{u+\Delta u} \frac{f(x, u+\Delta u)}{\Delta u} dF(x) \mid \\
&\leq \lim_{\Delta u \to 0} \int_u^{u+\Delta u} \mid \frac{f(x, u+\Delta u)}{\Delta u} \mid dF(x) \\
&= \lim_{\Delta u \to 0} \int_u^{u+\Delta u} \mid \frac{f(x, u+\Delta u)}{u+\Delta u - x} \mid\mid \frac{u+\Delta u - x}{\Delta u} \mid dF(x) \\
&\leq \lim_{\Delta u \to 0} \int_u^{u+\Delta u} \mid \frac{f(x, u+\Delta u)}{u+\Delta u - x} \mid dF(x) = 0. \blacksquare
\end{aligned}$$

Now we prove theorem 1.6.

Because of $E(X^n) < \infty$, we know

$$\lim_{u \to \infty} \int_u^{\infty} (x-u)^n dF(x) \leq \lim_{u \to \infty} \int_u^{\infty} x^n dF(x) = 0.$$

So,

$$\Pi^{(n)}(\infty) = \lim_{u \to \infty} \int_u^{\infty} (x-u)^n dF(x) = 0. \qquad (1.8)$$

If the following equation holds,

$$\frac{d}{du}[\Pi^{(n)}(u)] = -n\Pi^{(n-1)}(u), \qquad (1.9)$$

integrating both sides of (1.9) from $u$ to $\infty$, we would then have

$$\int_u^{\infty} \frac{d}{dx}\Pi^{(n)}(x)dx = -n \int_u^{\infty} \Pi^{(n-1)}(x)dx.$$

That is,

$$\Pi^{(n)}(\infty) - \Pi^{(n)}(u) = -n \int_u^{\infty} \Pi^{(n-1)}(x)dx.$$

By (1.8) we have

$$\Pi^{(n)}(u) = n \int_u^{\infty} \Pi^{(n-1)}(x)dx.$$

We need only then to prove (1.9) true.

At first we prove (1.9) for $n > 1$. We set $f(x,y) = (x-y)^n$ in lemma 1.7. Then the condition (a) of lemma 1.7 holds. Furthermore,

$$
\begin{aligned}
& \left| \frac{f(x,u+\Delta u) - f(x,u)}{\Delta u} \right| = \left| \frac{(x-u-\Delta u)^n - (x-u)^n}{\Delta u} \right| \\
= & \left| \frac{1}{\Delta u}[-C_n^1 (x-u)^{n-1}\Delta u + C_n^2 (x-u)^{(n-2)}(\Delta u)^2 + \cdots \right. \\
& \left. +(-1)^k C_n^k (x-u)^{(n-k)}(\Delta u)^k + \cdots + (-1)^n (\Delta u)^n] \right| \\
\leq & \mid n(x-u)^{n-1} \mid + \sum_{k=2}^{n} C_n^k \mid x-u \mid^{(n-k)} \mid \Delta u \mid^{(k-1)},
\end{aligned}
$$

where $C_n^k = \dfrac{n!}{k!(n-k)!}$.

If $x$ and $u$ both take values in finite intervals, without loss of generality, we assume the interval is $[0, A]$, and $\mid \Delta u \mid \leq 1$, then the right-hand side of above equation is bounded. If we let $G$ denote this bound, then we can take $G$ as $g(x)$ in lemma 1.7 and the condition (b) of lemma 1.7 holds. Furthermore,

$$
\lim_{|x-y| \to 0} \left| \frac{f(x,y)}{x-y} \right| = \lim_{|x-y| \to 0} \mid x-y \mid^{n-1} = 0 \quad \text{for } n > 1.
$$

So the condition (c) also holds.

In the following we use lemma 1.7 to prove formula (1.9) for $n > 1$. In lemma 1.7, the interval of integration is 0 to $u$, but now we need the interval of integration to be $u$ to $\infty$. We begin as follows:

$$
\begin{aligned}
\Pi^{(n)}(u) &= \int_u^{\infty} (x-u)^n dF(x) \\
&= \int_0^{\infty} (x-u)^n dF(x) - \int_0^u (x-u)^n dF(x) \\
&= \int_0^{\infty} \sum_{k=0}^{n} (-1)^k C_n^k x^{n-k} u^k dF(x) - \int_0^u (x-u)^n dF(x)
\end{aligned}
$$

$$= \sum_{k=0}^{n}(-1)^k C_n^k u^k E(X^{n-k}) - \int_0^u (x-u)^n dF(x)$$
$$= I(u) - J(u),$$

where $I(u)$ denotes the sum and $J(u)$ denotes the integral at the right-hand side above . Taking derivative of $I(u)$ and $J(u)$ respectively, we have

$$\begin{aligned} \frac{d}{du}I(u) &= \sum_{k=1}^{n}(-1)^k C_n^k k u^{k-1} E(X^{n-k}) \\ &= -n\sum_{i=0}^{n-1}(-1)^i C_{n-1}^i u^i E(X^{n-1-i}) \\ &= -n\int_0^\infty (x-u)^{n-1} dF(x). \end{aligned} \tag{1.10}$$

And by lemma 1.7,

$$\frac{d}{du}J(u) = \frac{d}{du}[\int_0^u (x-u)^n dF(x)] = -n\int_0^u (x-u)^{n-1} dF(x). \tag{1.11}$$

Combine (1.10) and (1.11) we have

$$\begin{aligned} \frac{d}{du}\Pi^{(n)}(u) &= -n[\int_0^\infty (x-u)^{n-1} dF(x) - \int_0^u (x-u)^{n-1} dF(x)] \\ &= -n\int_u^\infty (x-u)^{n-1} dF(x) = -n\Pi^{(n-1)}(u). \end{aligned}$$

That is formula (1.9) which holds for $n = 2, 3, \cdots$. So theorem 1.6 also holds for $n = 2, 3, \cdots$. In the following we check theorem 1.6 directly for $n = 1$. Taking integration by parts,

$$\begin{aligned} \Pi^{(1)}(u) &= \int_u^\infty (x-u) dF(x) = \{-(x-u)\overline{F}(x)\} \mid_{x=u}^{\infty} + \int_u^\infty \overline{F}(x) dx \\ &= \int_u^\infty \Pi^{(0)}(x) dx. \end{aligned}$$

Thus theorem 1.6 holds for $n = 1$. The proof of theorem 1.6 is complete. ∎

**Corollary 1.8** A distribution function $F(x)$ (or survival function $\overline{F}(x)$) and its $n$th stop-loss transform (n is an arbitrary nonnegative integer) are determined by each other.

**Proof**. When $n = 0$, $\Pi_F^{(0)}(x) = \overline{F}(x) = 1 - F(x)$. Corollary 1.8 becomes true. When $n \geq 1$, from (1.6) we know that $\Pi_F^{(n)}(x)$ is determined by $\Pi_F^{(n-1)}(x)$. And by (1.9), we have

$$\Pi_F^{(n-1)}(x) = -\frac{1}{n}\frac{d}{dx}\Pi_F^n(x).$$

Then we arrive at our conclusion by induction. ▮

## 1.2 Stop-loss orders and their properties

**Definition 1.9.** We say that $X$ is less than $Y$ in the meaning of the $n$th stop-loss order, denoted by $X <_{sl(n)} Y$, if

$$E(X^k) \leq E(Y^k), \quad k = 1, 2, \cdots, n-1. \tag{1.12}$$

$$\Pi_X^{(n)}(u) \leq \Pi_Y^{(n)}(u), \quad \forall u \geq 0. \tag{1.13}$$

When $n = 0$, the formula (1.12) disappears and formula (1.13) becomes

$$\overline{F}_X(u) \leq \overline{F}_Y(u), \quad \forall u \geq 0.$$

When $n = 1$, then formula (1.12) is trivial and formula (1.13) becomes

$$\int_u^\infty \overline{F}_X(x)dx \leq \int_u^\infty \overline{F}_Y(x)dx, \quad \forall u \geq 0.$$

Now we study a class of functions with certain properites. Suppose function $u(x)$, $-\infty < x < \infty$ satisfies: $u^{(n+1)}(x)$ exists except at a finite number of points, and

$$(-1)^{k-1}u^{(k)}(x) \geq 0, \quad \forall x, k = 1, 2, \cdots, n+1. \tag{1.14}$$

Let

$$U_n = \{u(x) : u(x) \text{ satisfies } (1.14)\}, n = 0, 1, 2, \cdots.$$

Obviously, $U_{n+1} \subset U_n$, that is, classes of functions decrease with respect to $n$, $n = 0, 1, 2, \cdots$.

Inequality (1.14) implies that

$$u^{(k)}(x) \geq 0, \text{ when k is odd,}$$

$$u^{(k)}(x) \leq 0, \text{ when k is even.}$$

Let

$$w(x) = -u(-x),\ u \in U_n.$$

Then for an arbitrary real number $x$ and nonnegative integer $k \leq n+1$, we have

$$w^{(k)}(x) = (-1)^{(k+1)}u^{(k)}(-x) \geq 0. \tag{1.15}$$

Let

$$W_n = \{w(x) :\ w(x) \text{ satisfies } (1.15)\}.$$

It is easy to see that if we let $u(x) = -w(-x)$, where $w(x) \in W_n$, then

$$u^{(k)}(x) = (-1)^{k+1}w^{(k)}(-x),$$

$$(-1)^{(k-1)}u^{(k)}(x) = (-1)^{2k}w^{(k)}(-x) \geq 0,$$

so $u(x) \in U_n$. Hence we reach a conclusion that there is an one to one correspondence between the elements of $U_n$ and $W_n$.

The following theorem and its proof are similar to that of theorem 4.2.1 in [6]. But here we add one sufficient and necessary condition, (1.17), and the proof becomes more clear than that in [6].

**Theorem 1.10** $X <_{sl(n)} Y$, if and only if

$$E[u(-X)] \geq E[u(-Y)], \quad \forall u \in U_n, \tag{1.16}$$

if and only if

$$E[w(X)] \leq E[w(Y)], \quad \forall w \in W_n. \tag{1.17}$$

**Proof.** First we prove the equivalence of (1.16) and (1.17). Suppose inequality (1.16) holds, we want to prove (1.17) holds. Let $u(x) = -w(-x)$, then $u(x) \in U_n$. Hence by(1.16) we have

$$E[u(-X)] \geq E[u(-Y)].$$

That is

$$E[-w(X)] \geq E[-w(Y)].$$

Thus

$$E[w(X)] \leq E[w(Y)].$$

Hence inequality (1.17) holds. Similarly we can deduce (1.16) from (1.17).

In the following we prove $X <_{sl(n)} Y \Longleftrightarrow$ (1.17).

($\Leftarrow$): Suppose (1.17) holds. Let

$$w(x) = [(x-u)_+]^k, \; u \geq 0, 1 \leq k \leq n.$$

Then $\forall i \leq k, -\infty < x < \infty$,

$$w^{(i)}(x) = \begin{cases} k(k-1)\cdots(k-i+1)(x-u)^{(n-i)}, & \text{for } x > u, \\ 0, & \text{for } x < u. \end{cases}$$

and $\forall i > k, -\infty < x < \infty, \;\; w^{(i)}(x) = 0$.

Since $w^{(k)}(x) \geq 0$ for all positive integer $k$, we have $w(x) \in W_n$. By the assumption of (1.17) we have

$$E[w(X)] \leq E[w(Y)].$$

That is

$$E\{[(X-u)_+]^k\} \le E\{[(Y-u)_+]^k\}.$$

Let $k$ take value from 1 to $n-1$, and let $u=0$, we see that the inequalities (1.12) hold; let $k=n$, we go to (1.13). So, $X <_{sl(n)} Y$ by definition.

($\Rightarrow$ ) Suppose $w^{(k)}(x) \ge 0, \forall k = 1, 2, \cdots n+1$, then we have the following expansion of $w(x)$

$$w(x) = \sum_{k=0}^{n} \frac{w^{(k)}(0)}{k!} x^k + \int_0^x \frac{(x-t)^n}{n!} dw^{(n)}(t). \tag{1.18}$$

We prove formula (1.18) at first. Taking integration by parts, we have

$$\begin{aligned}
&\int_0^x \frac{(x-t)^n}{n!} dw^{(n)}(t) \\
=&[\frac{(x-t)^n}{n!} w^{(n)}(t)] \,|_{t=0}^{x} - \int_0^x w^{(n)}(t) \frac{d}{dt}[\frac{(x-t)^n}{n!}] \\
=&- \frac{x^n}{n!} w^{(n)}(0) + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} w^{(n)}(t) dt \\
=&- \frac{x^n}{n!} w^{(n)}(0) + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} dw^{(n-1)}(t) \\
=&\cdots = - \sum_{k=1}^{n} \frac{w^{(n)}(0)}{k!} x^k + \int_0^x dw(t) \\
=&- \sum_{k=1}^{n} \frac{w^{(n)}}{k!} x^k + w(x) - w(0).
\end{aligned}$$

Removing the terms on the right-hand side except $w(x)$, we go to (1.18). Now suppose $X <_{sl(n)} Y$ we want to prove $E[w(X)] \le E[w(Y)]$. By (1.18) we have

$$\begin{aligned}
E[w(X)] &= E[\sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} X^k + \int_0^X \frac{(X-u)^n}{n!} dw^{(n)}(u)] \\
&= \sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} E(X^k) + E[\int_0^X \frac{(X-u)_+^n}{n!} dw^{(n)}(u)] \\
&= \sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} E(X^k) + \int_0^\infty \frac{E\{[(X-u)_+]^n\}}{n!} dw^{(n)}(u).
\end{aligned}$$

(On the right-hand side above the upper limit of integration can be expanded from $X$ to $\infty$, because $(x-u)_+ = 0$ when $u > x$).

By $X <_{sl(n)} Y$, we know that

$$E(X^k) \leq E(Y^k), \quad k = 0, 1, \cdots n,$$

and

$$E\{[(X-u)_+]^n\} \leq E\{[(Y-u)_+]^n\}, \quad \forall u \geq 0.$$

So,

$$E[w(X)] \leq \sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} E(Y^k) + \int_0^\infty \frac{E\{[(Y-u)_+]^n\}}{n!} dw^{(n)}(u).$$

From the above we see that the right-hand side of the final inequality is just $E[w(Y)]$.

We then have

$$E[w(X)] \leq E[w(Y)]. \blacksquare$$

**Proposition 1.11.** Suppose $E(X) = E(Y)$. If $X <_{sl(1)} Y$ then

$$var(X) \leq var(Y).$$

**Proof.** From (1.5) we know

$$E(X^2) = 2\int_0^\infty \Pi_X^{(1)}(y)dy \leq 2\int_0^\infty \Pi_Y^{(1)}(y)dy = E(Y^2).$$

Hence, by $E(X) = E(Y)$,

$$var(X) = E(X^2) - [E(X)]^2 \leq E(Y^2) - [E(Y)]^2 = var(Y). \blacksquare$$

**Theorem 1.12.** If $X <_{sl(n)} Y$, then

$$X <_{sl(m)} Y, \quad \forall m > n.$$

**Proof.** Because of the decreasing property of $U_n$ with respect to $n$, when $m > n$, we have $U_m \subset U_n$. By theorem 1.10 we arrive at our desired conclusion. ∎

But if $X <_{sl(m)} Y$, then for $n < m$, it is not required to have $X <_{sl(n)} Y$. This can be seen from the following example.

**Example 1.13.** Suppose

$$Pr(X = 1/2) = Pr(X = 1) = 0.5,$$

$$Pr(Y = 1/3) = 0.2, \quad Pr(Y = 1) = 0.8.$$

then we have

$$\overline{F}_X(u) = \begin{cases} 1, & \text{for } 0 \le u < 1/2, \\ 0.5, & \text{for } 1/2 \le u < 1, \\ 0, & \text{for } 1 \le u, \end{cases}$$

and

$$\overline{F}_Y(u) = \begin{cases} 1, & \text{for } 0 \le u < 1/3, \\ 0.8, & \text{for } 1/3 \le u < 1, \\ 0, & \text{for } 1 \le u, \end{cases}$$

(see figure 1.1). We see that $\overline{F}_X(u) > \overline{F}_Y(u)$, when $1/3 < u < 1/2$. We can conclude that $X <_{sl(0)} Y$ does not hold. On the other hand, we have

$$\int_u^\infty \overline{F}_X(x)dx = \int_u^\infty \overline{F}_Y(x)dx = 0, \quad \text{when } u \ge 1,$$
$$\int_u^\infty \overline{F}_X(x)dx \le \int_u^\infty \overline{F}_Y(x)dx, \quad \text{when } 1/2 \le u < 1,$$

and when $1/3 \le u < 1/2$,

$$\int_u^\infty \overline{F}_X(x)dx = \frac{1}{2} - u + 0.5 \times \frac{1}{2} = \frac{3}{4} - u,$$
$$\int_u^\infty \overline{F}_Y(x)dx = 0.8 \times (1-u) = \frac{4}{5} - \frac{4}{5}u > \int_u^\infty \overline{F}_X(x)dx.$$

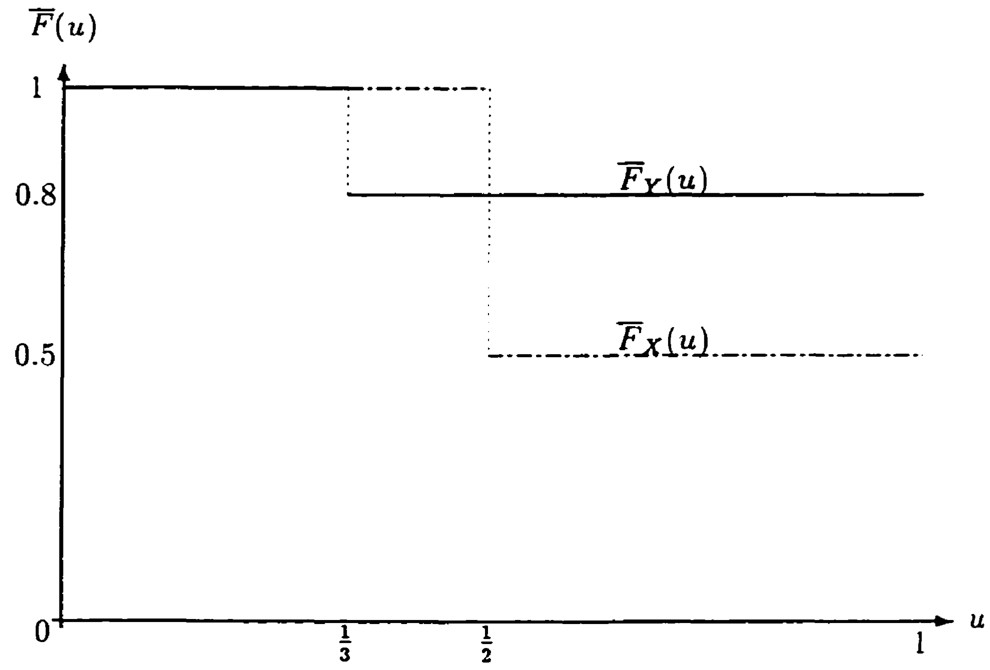



Figure 1.1: The survival functions of $X$ and $Y$ in Example 1.13

When $0 \le u < 1/3$, we have

$$\int_u^\infty \overline{F}_X(x)dx \le \int_u^\infty \overline{F}_Y(x)dx.$$

So $\int_u^\infty \overline{F}_X(x)dx \le \int_u^\infty \overline{F}_Y(x)dx$, $\forall u \ge 0$, and $X <_{sl(1)} Y$ by definition. We see that $X <_{sl(1)} Y$ holds, but $X <_{sl(0)} Y$ does not hold in this example.

**Proposition 1.14.** If $E(X) \le E(Y)$ and $\exists\, c \ge 0$ such that

$$F_X(x) \le F_Y(x), \quad \text{for } x \le c, \tag{1.19}$$

$$F_X(x) \ge F_Y(x), \quad \text{for } x > c. \tag{1.20}$$

Then $X <_{sl(1)} Y$.

**Proof.** Let

$$h(x) = \Pi_Y^{(1)}(x) - \Pi_X^{(1)}(x) = \int_x^\infty \overline{F}_Y(u)du - \int_x^\infty \overline{F}_X(u)du,$$

then we have

$$h'(x) = -\overline{F}_Y(x) - [-\overline{F}_X(x)] = F_Y(x) - F_X(x).$$

And by conditions (1.19) and (1.20) we have

$$h'(x) \ge 0, \quad \text{for } x \le c,$$

$$h'(x) \le 0, \quad \text{for } x > c,$$

(See figure 1.2). We then have

$$h(0) = \int_0^\infty \overline{F}_Y(u)du - \int_0^\infty \overline{F}_X(u)du = E(Y) - E(X) \ge 0,$$

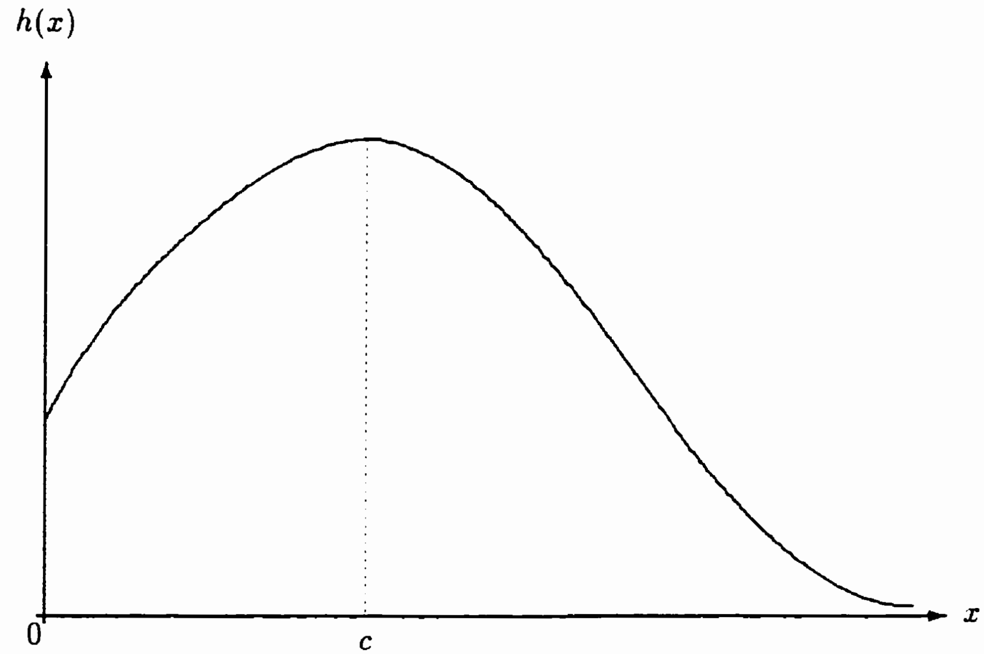



Figure 1.2: A plot of $\Pi_Y^{(1)}(x) - \Pi_X^{(1)}(x)$ in Proposition 1.14

and

$$h(\infty) = \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \int_x^\infty \overline{F}_Y(u)du - \lim_{x\to\infty} \int_x^\infty \overline{F}_X(u)du = 0.$$

From the above figure of $h(x)$ we conclude that $h(x) \geq 0$, $\forall x \geq 0$. Otherwise, if $h(x) < 0$ for some $x_1$, then there must be an intersection point of $h(x)$ with the $x$-axis, say, at point $x_0$, $x_0 < x_1$, and $h'(x) \leq 0$ must hold for $\forall x \geq x_0$, that means $h(\infty) = 0$ can not be held. See figure 1.2 for the graph of $h(x)$. Now from $h(x) \geq 0$, $\forall x \geq 0$ , we have $\Pi_X^{(1)}(x) \leq \Pi_Y^{(1)}(x)$, $\forall x \geq 0$. So we have $X <_{sl(1)} Y$ by definition. ∎

We can interpret proposition 1.14 more easily by diagram (see figure 1.3). By conditions (1.19) and (1.20) we know that the curves of $\overline{F}_X(x) = 1 - F_X(x)$ and $\overline{F}_Y(x) = 1 - F_Y(x)$ intersect at $x = c$, as shown in figure 1.3. We know also that $E(X)$ equals the area under the curve of $\overline{F}_X(x)$ and $E(Y)$ equals the area under the curve of $\overline{F}_Y(u)$. Therefore, by the condition of $E(X) \leq E(Y)$, we can conclude that the area of A in the figure 1.3 must be less than the area of B. Hence, for arbitrary $u \geq 0$, the area on the right-hand side of $x = u$ and under the curve of $\overline{F}_X(x)$ must be less than that under the curve of $\overline{F}_Y(x)$ (marked by a shadow). That is $\Pi_X^{(1)}(u) \leq \Pi_Y^{(1)}(u)$, $\forall u \geq 0$, which is desired for proposition 1.14.

**Proposition 1.15.** If $E(X) \leq E(Y)$, and $\exists\, a, b, 0 \leq a \leq b < \infty$ such that

$$dF_X(x) \leq dF_Y(x), \quad \text{for } x \leq a \text{ or } x \geq b, \tag{1.21}$$

$$dF_X(x) \geq dF_Y(x), \quad \text{for } a < x < b. \tag{1.22}$$

Then $X <_{sl(1)} Y$.

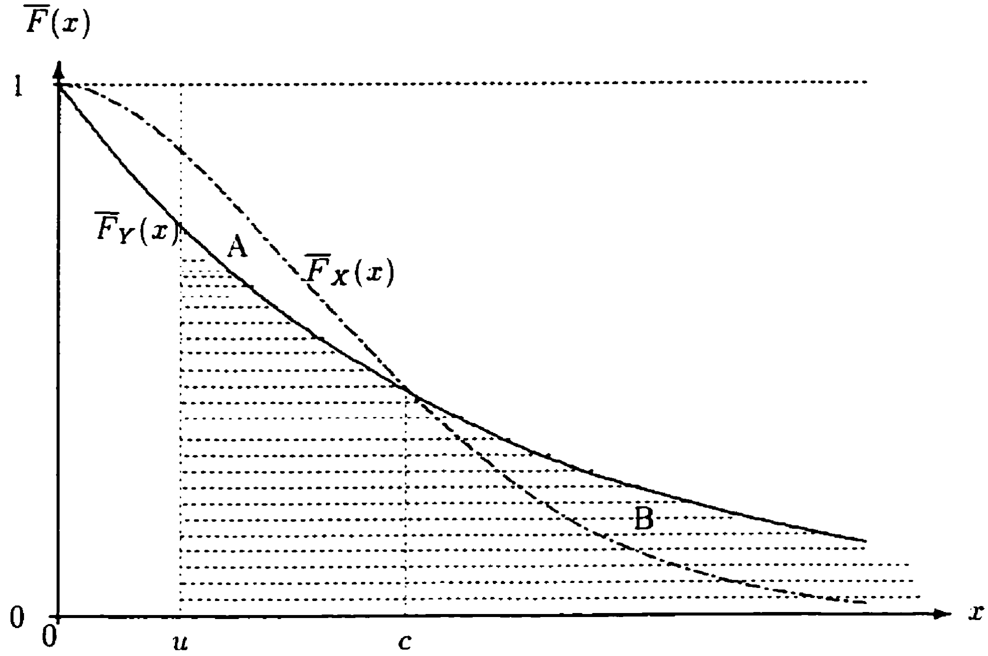



Figure 1.3: The survival functions of $X$ and $Y$ in Proposition 1.14

**Proof.** Similar to the proof of proposition 1.14, we need to show

$$h(x) = \Pi_Y^{(1)}(x) - \Pi_X^{(1)}(x) \geq 0.$$

We have

$$h'(x) = -\overline{F}_Y(x) - [-\overline{F}_X(x)] = F_Y(x) - F_X(x) = \int_0^x [dF_Y(x) - dF_X(x)].$$

By conditions (1.21) and (1.22) we know that when $x \leq a$, $h'(x) \geq 0$ and $h'(x)$ monotonously increases; when $a < x < b$, $h'(x)$ monotonously decreases; when $x \geq b$, $h'(x)$ increases again, and

$$\lim_{x \to \infty} h'(x) = \int_0^\infty dF_Y(x) - \int_0^\infty dF_X(x) = 1 - 1 = 0,$$

(The graph of $h'(x)$ is shown in figure 1.4.). There must be a point $c$, such that $a < c < b$, and $h'(x) \geq 0$, $\forall x \leq c$; $h'(x) \leq 0$ $\forall x > c$. Furthermore, as we have seen in the proposition 1.14, we have

$$h(0) = E(Y) - E(X) \geq 0,$$

and

$$\lim_{x \to \infty} h(x) = 0.$$

The figure of $h(x)$ is the same as that in the proposition 1.14. Hence we have $X <_{sl(1)} Y$ as in the proposition 1.14. ∎

When $X$ and $Y$ are both continuous, denoting the distribution density function by $f_X(x)$ and $f_Y(x)$ respectively, then the conditions (1.21) and (1.22) are equivalent to:

$$f_X(x) \leq f_Y(x), \quad \text{for } x \leq a \text{ or } x \geq b,$$

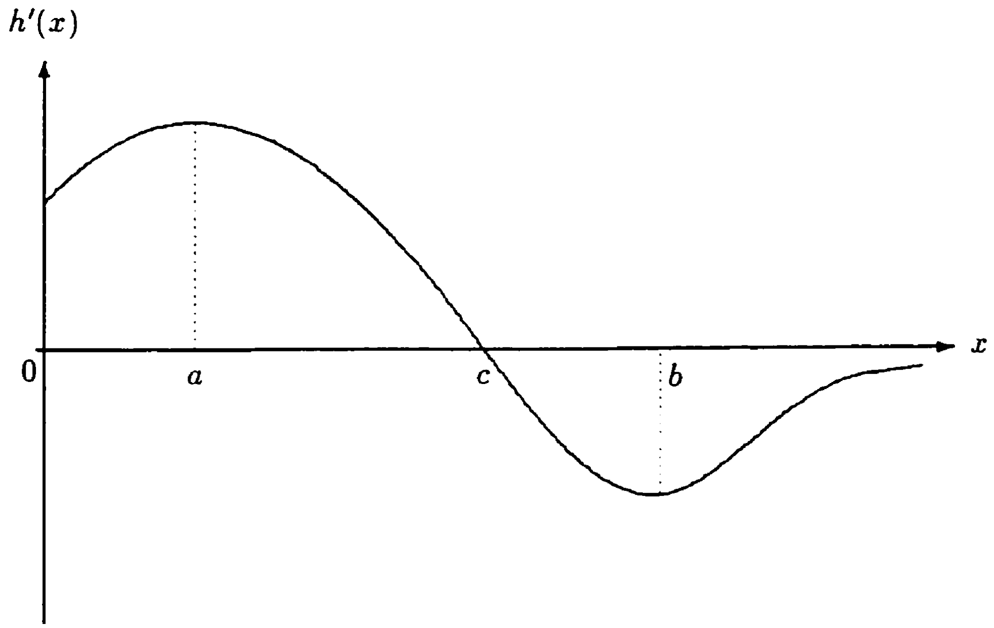



Figure 1.4: A plot of $F_Y(x) - F_X(x)$ in Proposition 1.15

and

$$f_X(x) \geq f_Y(x), \quad \text{for } a < x < b.$$

When $X$ and $Y$ both are discrete, assuming their domain is $\{x_i,\ i = 1, 2, \cdots\}$ and their probability functions are $P_X(x_i)$ and $P_Y(x_i)$ respectively, then conditions (1.21) and (1.22) are equivalent to

$$P_X(x_i) \leq P_Y(x_i), \quad \text{for } x_i \leq a \text{ or } x_i \geq b,$$

$$P_X(x_i) \geq P_Y(x_i), \quad \text{for } a < x_i < b.$$

**Example 1.16.** Suppose $X_i$ has a $Binomial(1, p_i)$ distribution, $i = 1, 2,\ 0 < p_1 < p_2 < 1$. Denote $d = p_2 - p_1$. And suppose $X_1(\alpha)$ has the distribution of a $Binomial(1, p_1 + \alpha)$, $X_2(\alpha)$ has the distribution of a $Binomial(1, p_2 - \alpha)$, $X_1(\alpha)$ and $X_2(\alpha)$ are independent, where $0 \leq \alpha < \frac{d}{2}$. Let

$$X(\alpha) = X_1(\alpha) + X_2(\alpha). \tag{1.23}$$

Then $X(\alpha)$ is monotonously increasing with respect to $\alpha$ in the $1^{st}$ stop-loss order meaning. That is, if $0 \leq \alpha_1 \leq \alpha_2 < d/2$, then

$$X(\alpha_1) <_{sl(1)} X(\alpha_2) \tag{1.24}$$

**Proof.** By (1.23) we have

$$E(X(\alpha)) = E(X_1(\alpha)) + E(X_2(\alpha))$$

$$= p_1 + \alpha + p_2 - \alpha = p_1 + p_2.$$

Hence we have

$$E(X(\alpha_1)) = E(X(\alpha_2)).$$

The probability distribution of $X(\alpha)$ is:

$$Pr(X(\alpha)=0) = Pr(X_1(\alpha)=0, X_2(\alpha)=0) = [1-(p_1+\alpha)][1-(p_2-\alpha)]$$

$$= (1-\alpha-p_1)(1+\alpha-p_2). \quad (1.25)$$

$$Pr(X(\alpha)=1) = Pr(X_1(\alpha)=0, X_2(\alpha)=1) + Pr(X_1(\alpha)=1, X_2(\alpha)=0)$$

$$= (1-\alpha-p_1)(p_2-\alpha) + (p_1+\alpha)(1+\alpha-p_2). \quad (1.26)$$

$$Pr(X(\alpha)=2) = Pr(X_1(\alpha)=1, X_2(\alpha)=1) = (p_1+\alpha)(p_2-\alpha). \quad (1.27)$$

With the condition of $0 \le \alpha \le d/2$, by(1.25)-(1.27) it is easy to verify that $Pr(X(\alpha)=0)$ and $Pr(X(\alpha)=2)$ are increasing in $\alpha$ , $Pr(X(\alpha)=1)$ is decreasing in $\alpha$. Therefore, formula (1.24) is obtained by proposition 1.15. ∎

Combining with proposition 1.11, the example 1.16 says: if the sum of success probabilities in two Bernoulli experiments is a constant, i.e. $p_1+p_2=$ constant (this means the expectation of success number in the two experiments is a constant), then we can conclude that the closer of $p_1$ and $p_2$, the bigger of the variance of success number of the two experiments (this number is a random variable).

**Example 1.17**. Suppose $F_X(x)$ and $F_Y(x)$ are two life distribution functions, the corresponding force of mortality is denoted by $m_X(x)$ and $m_Y(x)$ respectively. If there is a real number $c$ such that

$$m_X(x) < m_Y(x), \quad \text{for } x \le c, \quad (1.28)$$

$$m_X(x) > m_Y(x), \quad \text{for } x > c, \quad (1.29)$$

and

$$E(X) \le E(Y). \quad (1.30)$$

Then $X <_{sl(1)} Y$.

**Proof.** Let

$$H(x) = \overline{F}_Y(x) - \overline{F}_X(x).$$

We shall show that there exists a real number $s$ such that

$$\begin{aligned} H(x) &\le 0, \quad \text{for } x \le s, \\ H(x) &\ge 0, \quad \text{for } x > s. \end{aligned} \tag{1.31}$$

Then the conclusion desired follows from proposition 1.14.

Using the relationship between the survival function and its force of mortality we have

$$H(x) = \overline{F}_Y(x) - \overline{F}_X(x) = \exp(-\int_0^x m_Y(t)dt) - \exp(-\int_0^x m_X(t)dt).$$

From this and (1.28) we know that $H(x) < 0$ when $x \le c$.

From (1.30) we have

$$\int_0^\infty H(x)dx = \int_0^\infty \overline{F}_Y(x)dx - \int_0^\infty \overline{F}_X(x)dx = E(Y) - E(X) \ge 0. \tag{1.32}$$

Hence $H(x)$ can not be negative forever and must become positive at some point, therefore we know that there is at least one point $t \ge c$ such that $H(t) \ge 0$. Let

$$s = \inf\{t : \ H(t) \ge 0\}. \tag{1.33}$$

In the following we show this $s$ satisfies (1.31).

Since $H(x)$ is a continuous function, we have $s > c$ and $H(s) \ge 0$ by the definition of $s$. That is

$$\exp\{-\int_0^s m_Y(t)dt\} \ge \exp\{-\int_0^s m_X(t)dt\}.$$

Suppose $x > s$, from (1.29) and $s > c$, we then have

$$\exp\{-\int_s^x m_Y(t)dt\} > \exp\{-\int_s^x m_X(t)dt\}.$$

Therefore

$$\begin{aligned} H(x) =& \exp\{-\int_0^s m_Y(t)dt\}\exp\{-\int_s^x m_Y(t)dt\} \\ &- \exp\{-\int_0^s m_X(t)dt\}\exp\{-\int_s^x m_X(t)dt\} > 0, \quad \forall x > s. \end{aligned}$$

We can see the features of $H(x)$ in figure 1.5 where $H(x)$ is shown as the difference $\overline{F}_Y(x) - \overline{F}_X(x)$. In addition, according to the definition of $s$ we know that $H(x) \leq 0$ for $x \leq s$. ∎

The following example shows that the $2^{nd}$ stop-loss order will order random variables more widely than the $1^{st}$ stop-loss order. In general, if $m > n$ then the $m$th stop-loss order will order random variables more widely than the $n$th stop-loss order. This is confirmed by theorem 1.12.

**Example 1.18.** Let $X$ and $Y$ be two random variables with probability distributions as follows

$$Pr(X = 2) = \frac{5}{6}, \quad Pr(X = 6) = \frac{1}{6},$$

and

$$Pr(Y = 1) = \frac{1}{6}, \quad Pr(Y = 3) = \frac{5}{6}.$$

We have,

$$E[(X-u)_+] = \begin{cases} \frac{8}{3} - u, & \text{for } u < 2, \\ 1 - \frac{u}{6}, & \text{for } 2 \leq u < 6, \\ 0, & \text{for } u \geq 6. \end{cases}$$

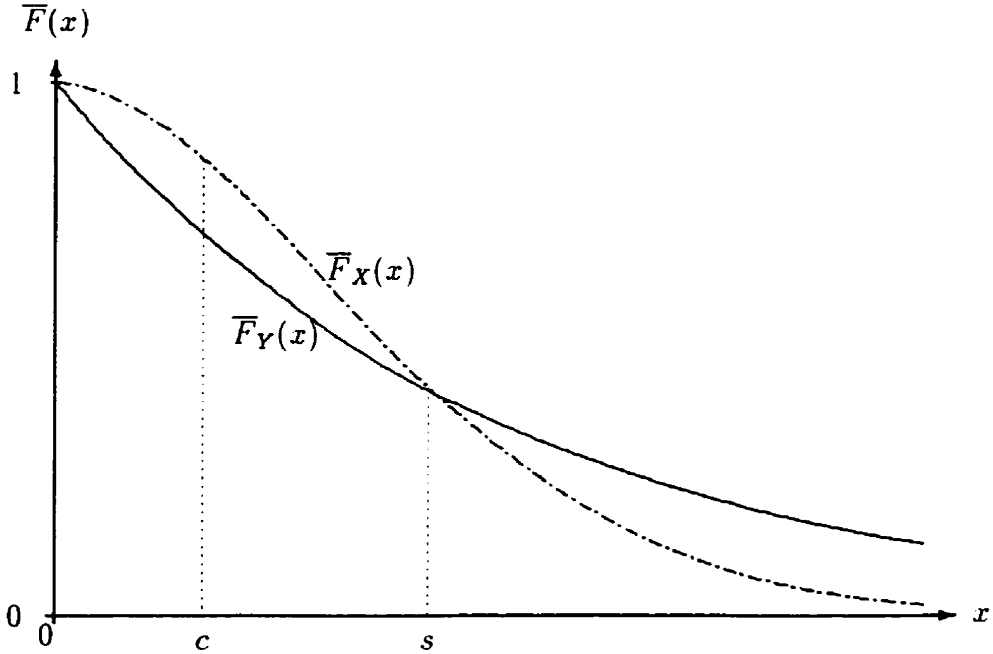



Figure 1.5: The survival function of $X$ and $Y$ in Example 1.17

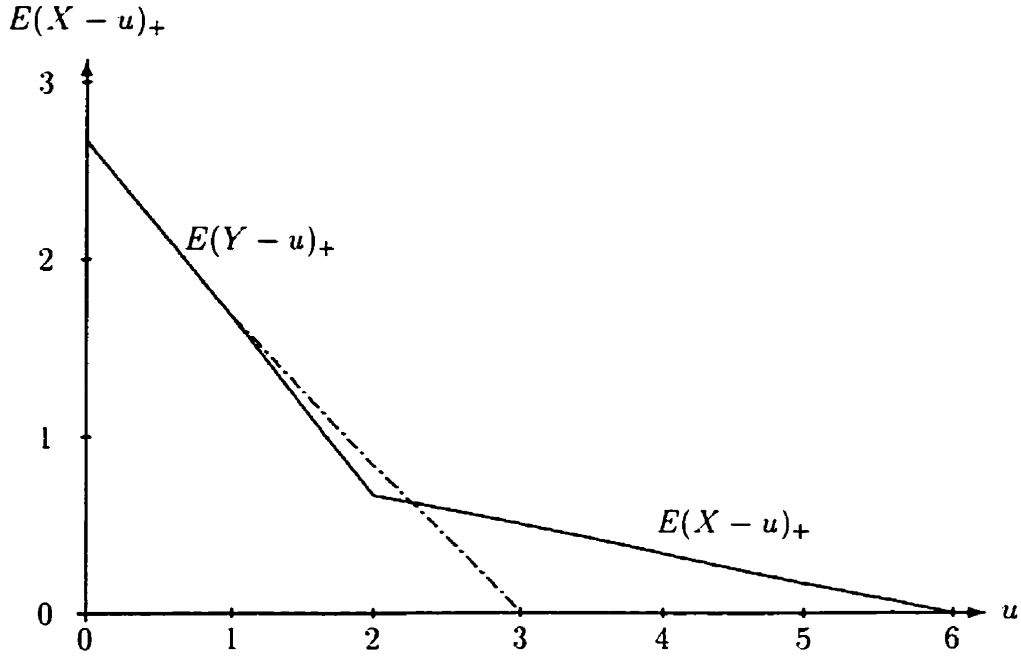



Figure 1.6: The $1^{st}$ stop-loss transforms of $X$ and $Y$ in Example 1.18

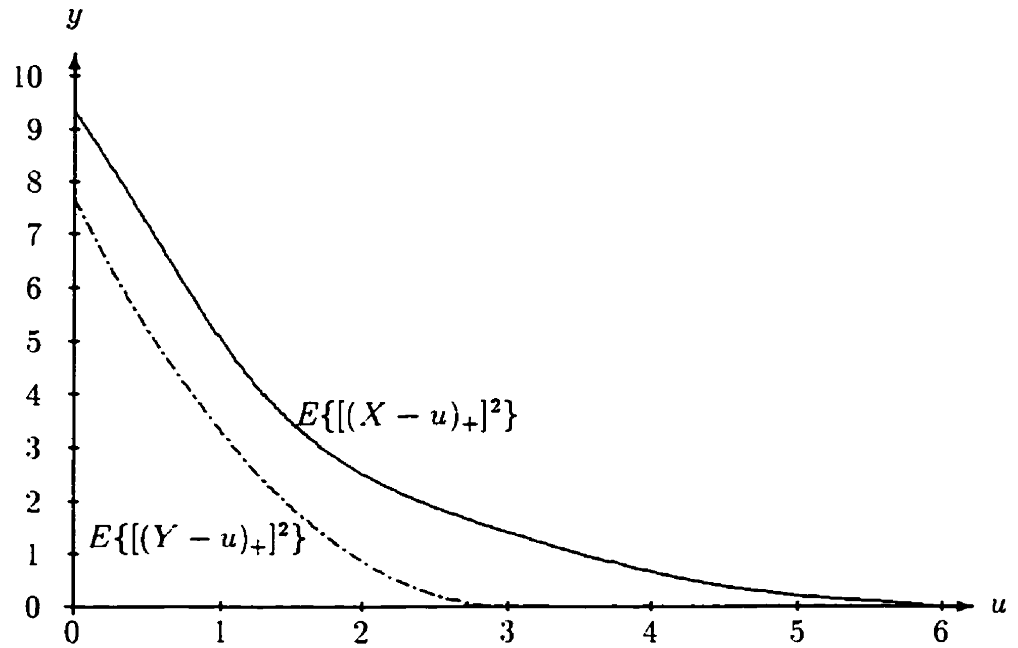



Figure 1.7: The $2^{nd}$ stop-loss transforms of $X$ and $Y$ in Example 1.18

$$E[(Y-u)_+] = \begin{cases} \dfrac{8}{3} - u, & \text{for } u < 1, \\ \dfrac{5}{2} - \dfrac{5u}{6}, & \text{for } 1 \le u < 3, \\ 0, & \text{for } u \ge 3. \end{cases}$$

We can see that

$$E[(X-2)_+] = \frac{2}{3} < \frac{5}{6} = E[(Y-2)_+].$$

We also can see that

$$E[(X-3)_+] > E[(Y-3)_+] = 0.$$

Hence neither $X <_{sl(1)} Y$ nor $Y <_{sl(1)} X$ holds. On the other hand, we have

$$E\{[(X-u)_+]^2\} = \begin{cases} u^2 - \dfrac{16}{3}u + \dfrac{28}{3}, & \text{for } u < 2, \\ \dfrac{1}{6}u^2 - 2u + 6, & \text{for } 2 \le u < 6, \\ 0, & \text{for } u \ge 6, \end{cases}$$

and

$$E\{[(Y-u)_+]^2\} = \begin{cases} u^2 - \dfrac{16}{3}u + \dfrac{23}{3}, & \text{for } u < 1, \\ \dfrac{5}{6}u^2 - 5u + \dfrac{15}{2}, & \text{for } 1 \le u < 3, \\ 0, & \text{for } u \ge 3. \end{cases}$$

We can check that $E\{[(Y-u)_+]^2\} \le E\{[(X-u)_+]^2\}$, $\forall u \ge 0$ (see Figure 1.6 and 1.7). In addition, since $E(X) = E(Y) = \frac{8}{3}$ we can show that $Y <_{sl(2)} X$, by definition 1.9. Therefore we can conclude that the $2^{nd}$ stop-loss order can order random variables $X$ and $Y$ but the $1^{st}$ stop-loss order can not.

Next we show the maintenance properties of the $n$th stop-loss order.

**Theorem 1.19.** The $n$th stop-loss order is maintained under the summation of independent random variables. That is, if

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots k,$$

where $k$ is a positive integer, then

$$\sum_{i=1}^{k} X_i <_{sl(n)} \sum_{i=1}^{k} Y_i, \quad n = 0, 1, 2, \cdots. \tag{1.34}$$

It was proved in [6] that the $1^{st}$ stop-loss order is maintained under the summation of independent random variables (see page 30 of [6], theorem 3.2.2. ). Theorem 1.19 is its generalization and the method used here for proving the theorem is completely different from the method in [6].

**Proof.** We first prove theorem 1.19 for $k = 2$.

Suppose $X_1$ and $X_2$ are independent, $Y_1$ and $Y_2$ are independent and

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \; n \geq 0.$$

We now use theorem 1.10 to prove (1.34). By theorem 1.10, $\forall\, w(x) \in W_n$, we need only to prove

$$E[w(X_1 + X_2)] \leq E[w(Y_1 + Y_2)].$$

Let

$$w_1(x, t) = w(x + t), \tag{1.35}$$

where $t$ is a real number. Since $w(x) \in W_n$, from the definition of $W_n$ we have

$$\frac{d^k}{dx^k} w_1(x, t) = w^{(k)}(x + t) \geq 0, \quad k = 1, \cdots, n + 1.$$

Again by the definition of $W_n$, we know that for a fixed $t$, $w_1(x, t)$ is a function of $x$ and belongs to $W_n$. From $X_1 <_{sl(n)} Y_1$, and by theorem 1.10, we can have that

$$\int_0^\infty w(x + t) dF_{X_1}(x) = E[w_1(X_1, t)] \leq E[w_1(Y_1, t)] = \int_0^\infty w(x + t) dF_{Y_1}(x). \tag{1.36}$$

Further, let

$$w_2(x) = E[w_1(Y_1, x)] = \int_0^\infty w(y+x)dF_{Y_1}(y). \tag{1.37}$$

Since $w^{(k)}(x) \geq 0$, we have

$$w_2^{(k)}(x) = \int_0^\infty w^{(k)}(y+x)dF_{Y_1}(x) \geq 0, \quad k = 1, 2, \cdots, n+1.$$

Hence $w_2(x) \in W_n$. From this and the condition $X_2 <_{sl(n)} Y_2$ we have

$$\begin{aligned} &\int_0^\infty [\int_0^\infty w(y+x)dF_{Y_1}(y)]dF_{X_2}(x) = \int_0^\infty w_2(x)dF_{X_2}(x) = E[w_2(X_2)] \\ &\leq E[w_2(Y_2)] = \int_0^\infty [\int_0^\infty w(y+x)dF_{Y_1}(y)]dF_{Y_2}(x). \end{aligned} \tag{1.38}$$

Taking the integration of the both sides of (1.36) with the distribution function $dF_{X_2}(t)$, we have

$$\int_0^\infty [\int_0^\infty w(y+t)dF_{X_1}(y)]dF_{X_2}(t) \leq \int_0^\infty [\int_0^\infty w(y+t)dF_{Y_1}(y)]dF_{X_2}(t). \tag{1.39}$$

Combine (1.39) and (1.38) to arrive at

$$\int_0^\infty [\int_0^\infty w(y+t)dF_{X_1}(y)]dF_{X_2}(t) \leq \int_0^\infty [\int_0^\infty w(y+x)dF_{Y_1}(y)]dF_{Y_2}(t).$$

This is simply $E[w(X_1 + X_2)] \leq E[w(Y_1 + Y_2)]$. Next by mathematical induction we can conclude that (1.34) holds. ∎

**Theorem 1.20.** The $n$th stop-loss order is maintained under a compound operation. That is, suppose $X_1, X_2, \cdots, Y_1, Y_2, \cdots$, and integer valued, $N_1, N_2$ are all independent random variables. In addition, $N_1$ and $N_2$ have identical probability distributions. Let

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

If

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots,$$

then

$$S_1 <_{sl(n)} S_2. \tag{1.40}$$

**Proof.** According to theorem 1.10, it is sufficient to prove that

$$\forall\, w \in W_n, \quad E[w(S_1)] \leq E[w(S_2)].$$

In fact we have,

$$\begin{aligned}
E[w(S_1)] &= E[E[w(S_1) \mid N_1]] \\
&= \sum_{n=0}^{\infty} E[w(S_1) \mid N_1 = n] Pr(N_1 = n) \\
&= \sum_{n=0}^{\infty} E[w(X_1 + X_2 + \cdots + X_n) \mid N_1 = n] Pr(N_1 = n) \\
&= \sum_{n=0}^{\infty} E[w(X_1 + X_2 + \cdots + X_n)] Pr(N_1 = n).
\end{aligned}$$

The last equation holds because $X_1, X_2, \cdots, X_n$ and $N_1$ are independent. Next, using theorem 1.19 we have

$$E[X_1 + X_2 + \cdots + X_n] \leq E[Y_1 + Y_2 + \cdots + Y_n].$$

Notice $N_1$ and $N_2$ have identical probability distributions, so we have

$$\begin{aligned}
E[w(S_1)] &\leq \sum_{n=0}^{\infty} E[w(Y_1 + Y_2 + \cdots + Y_n)] Pr(N_1 = n) \\
&= \sum_{n=0}^{\infty} E[w(Y_1 + Y_2 + \cdots + Y_n)] Pr(N_2 = n) \\
&= E[w(S_2)]. \blacksquare
\end{aligned}$$

**Theorem 1.21.** The $n$th stop-loss order is maintained under a mixture operation. That is, suppose random variable $\alpha \geq 0$ has distribution function $H(\alpha)$. When $\alpha$

is given, $X_\alpha$ and $Y_\alpha$ have distribution function $F(t,\alpha)$ and $G(t,\alpha)$ respectively and satisfy $X_\alpha <_{sl(n)} Y_\alpha$. Let

$$F(t) = \int_0^\infty F(t,\alpha)dH(\alpha),$$
$$G(t) = \int_0^\infty G(t,\alpha)dH(\alpha).$$

Then

$$F(t) <_{sl(n)} G(t). \tag{1.41}$$

**Proof.** When $\alpha$ is given, the stop-loss transforms of $X_\alpha$ is

$$\Pi_{X_\alpha}^{(k)}(u) = \int_u^\infty (x-u)^k dF(x,\alpha).$$

Taking integration by parts, the equation above becomes

$$\Pi_{X_\alpha}^{(k)}(u) = k\int_u^\infty (x-u)^{k-1}\overline{F}(x,\alpha)dx. \tag{1.42}$$

From $X_\alpha <_{sl(n)} Y_\alpha$, $\forall\alpha \geq 0$, for $\forall\alpha \geq 0$, we have

$$\Pi_{X_\alpha}^{(k)}(0) \leq \Pi_{Y_\alpha}^{(k)}(0), \quad k = 1, 2, \cdots, n-1, \tag{1.43}$$

$$\Pi_{X_\alpha}^{(k)}(u) \leq \Pi_{Y_\alpha}^{(k)}(u), \quad \forall u \geq 0. \tag{1.44}$$

Using expression (1.42), for $u = 0$, (1.43) becomes

$$k\int_0^\infty x^{k-1}\overline{F}(x,\alpha)dx \leq k\int_0^\infty x^{k-1}\overline{G}(x,\alpha)dx.$$

Integrating the above formula with distribution function $H(\alpha)$, we have

$$k\int_0^\infty\int_0^\infty x^{k-1}\overline{F}(x,\alpha)dxdH(\alpha) \leq k\int_0^\infty\int_0^\infty x^{k-1}\overline{G}(x,\alpha)dxdH(\alpha).$$

Exchanging the order of integration, we have

$$\int_0^\infty kx^{k-1}[\int_0^\infty \overline{F}(x,\alpha)dH(\alpha)]dx \le \int_0^\infty kx^{k-1}[\int_0^\infty \overline{G}(x,\alpha)dH(\alpha)]dx.$$

Note that,

$$\int_0^\infty \overline{F}(x,\alpha)dH(\alpha) = 1 - \int_0^\infty F(x,\alpha)dH(\alpha) = 1 - F(x) = \overline{F}(x).$$

Similarly, we have

$$\int_0^\infty \overline{G}(x,\alpha)dH(\alpha) = \overline{G}(x).$$

Then we have

$$\int_0^\infty kx^{k-1}\overline{F}(x)dx \le \int_0^\infty kx^{k-1}\overline{G}(x)dx.$$

That is

$$\Pi_F^{(k)}(0) \le \Pi_G^{(k)}(0), \quad k = 1, 2, \cdots, n-1. \tag{1.45}$$

Repeating the steps above for (1.44), we have

$$\Pi_F^{(n)}(u) \le \Pi_G^{(n)}(u), \quad \forall\, u \ge 0. \tag{1.46}$$

Combine (1.45) and (1.46), we have $F(t) <_{sl(n)} G(t)$. ∎

An important special case is when $\alpha$ takes values in $\{1, 2, \cdots, n\}$. Denote

$$I_i = \begin{cases} 1, & \text{for } \alpha = i, \\ 0, & \text{for } \alpha \ne i. \end{cases}$$

and $Pr(\alpha = i) = p_i$, where $i = 1, 2, \cdots, n$, $0 \le p_i \le 1$, $\sum_{i=1}^n p_i = 1$. From theorem 1.21 we know that if $X_i <_{sl(n)} Y_i$, $i = 1, \cdots, n$, then

$$\sum_{i=1}^n p_i F(x,i) <_{sl(n)} \sum_{i=1}^n p_i G(x,i), \tag{1.47}$$

where $F(x, i)$ and $G(x, i)$ are the distribution functions of $X_i$ and $Y_i$ respectively. The formula (1.47) can also be written as

$$\sum_{i=1}^{n} I_i X_i <_{sl(n)} \sum_{i=1}^{n} I_i Y_i. \tag{1.48}$$

## 1.3 Generalization

Now let's generalize the concept of the $n$th stop-loss transforms (given by definition 1.1) and the concept of the $n$th stop-loss order (given by definition 1.9) to the class of general nonnegative monotonous decreasing functions on $[0, \infty)$. ( For monotonous increasing function, assuming it has a finite limit at $\infty$, the same approach would be derived.)

Suppose function $H(x) \geq 0$ $(0 \leq x < \infty)$ is monotonously decreasing, $H(0) > 0$ and $H(x)$ is continuous from the right side. Without loss of generality, we assume $\lim_{x \to \infty} H(x) = 0$ (otherwise, replace $H(x)$ by $H^*(x) = H(x) - H(\infty)$). Then $H(x)$ has similar properties as that of a survival function : nonnegative, monotonously decreasing and $\lim_{x \to \infty} H(x) = 0$. There maybe only one difference between them, that is, $H(0) \leq 1$ does not hold all the time. But there is no trouble with this difference when we generalize the stop-loss transforms of survival functions to the stop-loss transforms of $H(x)$, and furthermore, generalize the stop-loss order of the family of survival functions to the family of nonnegative and monotonously decreasing functions.

Looking back at the definition 1.1, for $n \geq 1$ we have

$$\Pi^{(n)}_{F_X}(u) = \int_u^\infty (x-u)^n dF_X(x) = -\int_u^\infty (x-u)^n d\overline{F}_X(x). \tag{1.49}$$

In this integration, the measure introduced by $F_X(x)$ or by $\overline{F}_X(x)$ is used. Because $\overline{F}_X(0) \leq 1$, the measure of set $(0,\infty)$ is equal to or less than 1. The measure of $(0,\infty)$ introduced by $H(x)$ equals to $H(0)$. This may be greater than 1, but at least it is finite. We can define the stop-loss transforms of $H(x)$ similar to definition 1.1. According to (1.49), replacing $\overline{F}_X(x)$ in (1.49) by $H(x)$, leads us to the following definition:

**Definition 1.22.** Suppose function $H(x) \geq 0$, $0 \leq x < \infty$ is monotonous decreasing, and $\lim_{x\to\infty} H(x) = 0$. $\forall\, u \geq 0$ for nonnegative integer $k$, let

$$\Pi^{(k)}_H(u) = -\int_u^\infty (x-u)^k dH(x), \tag{1.50}$$

assuming the integral of the right-hand side of (1.50) is finite. $\Pi^{(k)}_H(u)$ is called the $k$th stop-loss transform of $H(x)$. Similar to theorem 1.6, we can prove the following theorem related to $\Pi^{(n)}_H(u)$.

**Theorem** 1.6′

$$\Pi^{(n)}_H(u) = n\int_u^\infty \Pi^{(n-1)}_H(x)dx, \quad n = 1, 2, \cdots.$$

(To prove this theorem we only need to note that the function $H(x)$ is corresponding to the function $\overline{F}(x)$ in theorem 1.6.).

Now we generalize the concept of the $n$th stop-loss order to the family of nonnegative monotonous decreasing functions. Let

$$\Omega = \{H(x),\, x \geq 0:\ H(x) \geq 0 \text{ monotonous decreasing and } \lim_{x\to\infty} H(x) = 0\}. \tag{1.51}$$

**Definition 1.23.** Suppose $H(x), G(x) \in \Omega$. We say that $H(x)$ is less than $G(x)$ in the meaning of $k$th stop-loss order, denoted by $H(x) <_{sl(k)} G(x)$, if the $k$th stop-loss transforms of $H(x)$ and $G(x)$ exist, and

$$\int_0^\infty x^i dH(x) \geq \int_0^\infty x^i dG(x), \quad i = 1, 2, \cdots, k-1, \tag{1.52}$$

$$\Pi_H^{(k)}(u) \leq \Pi_G^{(k)}(u), \quad \forall\, u \geq 0. \tag{1.53}$$

(In formula (1.52) we note that the direction of the inequality is opposite to that in definition 1.9, since here both of the integrations on both sides of (1.52) are negative.)

At first we look for the relationship between the stop-loss transforms of $\Pi_X^{(n)}(u)$ and those of $X$.

Suppose $X \geq 0$, $E(X^n) < \infty$. From (1.1) we know that $\Pi_X^{(n)}(u) \in \Omega$. Then from (1.50) we have

$$\Pi_{\Pi_X^{(n)}}^{(k)}(u) = -\int_u^\infty (x-u)^k d[\Pi_X^{(n)}(x)], \tag{1.54}$$

assuming the integral at the right-hand side of (1.54) is finite. When $n = 0$, we have

$$\begin{aligned}\Pi_{\Pi_X^{(0)}}^{(k)}(u) &= -\int_u^\infty (x-u)^k d[\Pi_X^{(0)}(x)] = -\int_u^\infty (x-u)^k d\overline{F}(x) \\ &= E\{[(X-u)_+]^k\} = \Pi_X^{(k)}(u).\end{aligned} \tag{1.55}$$

When $k = 0$, we have

$$\Pi_{\Pi_X^{(n)}}^{(0)}(u) = -\int_u^\infty d[\Pi_X^{(n)}(x)] = \Pi_X^{(n)}(u). \tag{1.56}$$

In general, the relationship between the transforms of $\Pi_X^{(n)}$ and those of $X$ are stated in the following theorem.

**Theorem 1.24.** Suppose $E(X^{n+k}) < \infty$, then

$$\Pi^{(k)}_{\Pi^{(n)}_X}(u) = \frac{1}{C^n_{n+k}}\Pi^{(n+k)}_X(u), \quad \forall\, u \geq 0. \tag{1.57}$$

**Proof.** By formula (1.54) and (1.9), we have

$$\begin{aligned}
&\Pi^{(k)}_{\Pi^{(n)}_X}(u)\\
=&-\int_u^\infty (x-u)^k d[\Pi^{(n)}_X(x)] = n\int_u^\infty (x-u)^k \Pi^{(n-1)}_X(x)dx\\
=&\frac{n}{k+1}\int_u^\infty \Pi^{(n-1)}_X(x)d[(x-u)^{k+1}]\\
=&\frac{n}{k+1}\{[\Pi^{(n-1)}_X(x)(x-u)^{k+1}]\,|^\infty_{x=u} - \int_u^\infty (x-u)^{k+1}d[\Pi^{(n-1)}_X(x)]\}\\
=&-\frac{n}{k+1}\int_u^\infty (x-u)^{k+1}d[\Pi^{(n-1)}_X(x)]\\
=&\cdots\\
=&-\frac{n!}{(k+1)\cdots(k+n)}\int_u^\infty (x-u)^{n+k}d\overline{F}_X(x)\\
=&\frac{1}{C^n_{n+k}}\Pi^{(n+k)}_X(u).
\end{aligned}$$

In the fourth equation above the following expression is used

$$\lim_{x\to\infty}\Pi^{(n-1)}_X(x)(x-u)^{k+1} = 0, \quad \forall\, u \text{ to be fixed }. \tag{1.58}$$

The formula (1.58) holds when $E(X^{n+k}) < \infty$, that is

$$\int_0^\infty y^{n+k}dF_X(y) < \infty.$$

It gives us

$$\lim_{x\to\infty}\int_x^\infty y^{n+k}dF_X(y) = 0.$$

Hence, we have

$$\lim_{x\to\infty} \Pi_X^{(n-1)}(x)(x-u)^{k+1}$$
$$\leq \lim_{x\to\infty} x^{k+1}\int_x^\infty (y-x)^{n-1}dF_X(y)$$
$$\leq \lim_{x\to\infty} \int_x^\infty y^{n+k}dF_X(y) = 0. \blacksquare$$

**Corollary 1.25.** $X <_{sl(n+k)} Y$ if and only if

$$E(X^j) \leq E(Y^j), \quad j = 1, 2, \cdots, n,$$

and

$$\Pi_X^{(n)} <_{sl(k)} \Pi_X^{(n)}, \tag{1.59}$$

$n, k = 0, 1, \cdots$.

**Proof.** By definition 1.22 we have

$$\Pi_X^{(n)} <_{sl(k)} \Pi_Y^{(n)},$$

if and only if

$$\int_0^\infty x^i d[\Pi_X^{(n)}(x)] \geq \int_0^\infty x^i d[\Pi_Y^{(n)}(x)] \quad i = 1, 2, \cdots, k-1, \tag{1.60}$$

and

$$\Pi^{(k)}_{\Pi_X^{(n)}}(u) \leq \Pi^{(k)}_{\Pi_Y^{(n)}}(u), \quad \forall\ u \geq 0. \tag{1.61}$$

Taking $k = i,\ u = 0$, in formula (1.57), we have

$$\Pi^{(i)}_{\Pi_X^{(n)}}(0) = \frac{1}{C^n_{n+i}}\Pi_X^{(n+i)}(0) = \frac{1}{C^n_{n+i}}E(X^{n+i}).$$

Similarly we have

$$\Pi^{(i)}_{\Pi_Y^{(n)}}(0) = \frac{1}{C^n_{n+i}}\Pi_Y^{(n+i)}(0) = \frac{1}{C^n_{n+i}}E(Y^{n+i}).$$

Therefore formula (1.60) is equivalent to

$$E(X^{n+i}) \le E(Y^{n+i}), \quad i = 1, 2, \cdots, k-1.$$

Since $E(X^j) \le E(Y^j), \quad j = 1, 2, \cdots, n$, we have

$$E(X^j) \le E(Y^j), \quad j = 1, 2, \cdots, n+k-1. \tag{1.62}$$

On the other hand, by theorem 1.23 we have

$$\Pi^{(k)}_{\Pi^{(n)}_X}(u) = \frac{1}{C^n_{n+k}} \Pi^{(n+k)}_X(u),$$

and

$$\Pi^{(k)}_{\Pi^{(n)}_Y}(u) = \frac{1}{C^n_{n+k}} \Pi^{(n+k)}_Y(u).$$

Formula (1.61) holds if and only if

$$\Pi^{(n+k)}_X(u) \le \Pi^{(n+k)}_Y(u), \quad \forall\, u \ge 0. \tag{1.63}$$

By definition, $X <_{sl(n+k)} Y$ if and only if (1.62) and (1.63) hold. ∎

The concept of weak $n$th stop-loss order is given as follows:

**Definition 1.26.** Suppose $H(x), G(x) \in \Omega$. We say that $H(x)$ is less than $G(x)$ in the meaning of weak $n$th stop-loss order, denoted by $H <_{wsl(n)} G$ if

$$\Pi^{(n)}_H(u) \le \Pi^{(n)}_G(u), \quad \forall\, u \ge 0,\ n = 0, 1, \cdots. \tag{1.64}$$

We can see that if condition (1.52) is removed in the definition of $n$th stop-loss order, then the definition of weak $n$th stop-loss order follows. The weak $0^{th}$ stop-loss order and weak $1^{st}$ stop-loss order are no differente from the $0^{th}$ and $1^{st}$ stop-loss order, respectively.

**Proposition 1.27.** If $H <_{wsl(n)} G$, then $H <_{wsl(m)} G$, $\forall\, m > n$.

**Proof.** By definition, we need only from (1.64) to reason that

$$\Pi_H^{(m)}(u) \leq \Pi_G^{(m)}(u), \quad \forall\, u \geq 0. \tag{1.65}$$

First we prove (1.65) for $m = n + 1$. Using theorem 1.6 and (1.64) we have

$$\begin{aligned} \Pi_H^{(n+1)}(u) &= (n+1) \int_u^\infty \Pi_H^{(n)}(x)dx \\ &\leq (n+1) \int_u^\infty \Pi_G^{(n)}(x)dx \\ &= \Pi_G^{(n+1)}(u), \;\; \forall\, u \geq 0. \end{aligned}$$

By mathematical induction we can get proposition 1.27. ∎

# 2 The applications of stop-loss order in ruin probability

## 2.1 Surplus processes and the distribution of deficit

We denote insurer's surplus at time $t$ by $U(t)$, $t \geq 0$ and assume that the premium rate is a constant $c$ and paid continuously. Furthermore, we let the insurer's initial surplus be $U(0) = u \geq 0$. Let $S(t)$ denote the aggregate claims up to time $t$. Then the basic model for surplus processes is as follows:

$$U(t) = u + ct - S(t), \quad t \geq 0. \tag{2.1}$$

We say the model (2.1) is basic because in this model the time value of money ( the interest factor) and other factors that could influence the insurer's surplus (such as expenses, dividends, etc. ) are ignored.

The aggregate claims up to time $t$, $S(t)$, are determined by the number of claims that occurred in $[0, t)$ denoted by $N(t)$, and the amount of each claim. In classical risk theory, $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process with constant parameter $\lambda$. $\{N(t),\ t \geq 0\}$ is called the claim number process. We now denote the amount of the $i$th claim by $X_i$ and assume $X_i$, $i = 1, 2, \cdots$, are independent and identically distributed. This thesis is based on these assumptions. As a result, $\{S(t),\ t \geq 0\}$, called the aggregate claims process, can be expressed as :

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)} = \sum_{i=1}^{N(t)} X_i. \tag{2.2}$$

From the assumptions above we know that $\{S(t),\ t \geq 0\}$ is a compound Poisson process. Formula (2.1) can be rewritten as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0. \tag{2.3}$$

Let

$$T = \min\{t : \ t \geq 0, \ U(t) < 0\}. \tag{2.4}$$

If the set in (2.4) is empty, that is $U(t) \geq 0, \ \forall \, t \geq 0$, we let $T = \infty$. That is, we let $\min \emptyset = \infty$ by convention. The random variable $T$ is called ruin time and may take the value of $\infty$. Let

$$\psi(u) = Pr(T < \infty \mid U(0) = u). \tag{2.5}$$

This is the probability of ruin when the initial surplus equals $u$.

Because $\{N(t), \ t \geq 0\}$ is a homogeneous Poisson process, and the amount of claims are *i.i.d.*, the aggregate claims in every one unit time interval are *i.i.d.*, and have the same distribution as that of

$$S(1) = \sum_{i=1}^{N(1)} X_i, \tag{2.6}$$

where the random variable $N(1)$ has a Poisson distribution with parameter $\lambda$. As a result we have

$$E[S(1)] = \lambda E(X). \tag{2.7}$$

Suppose $c > \lambda E(X)$. That is, the premium paid in one unit time is greater than the expected value of claims in the same period. We denote

$$\theta = \frac{c}{\lambda E(X)} - 1, \tag{2.8}$$

and therefore

$$c = (1 + \theta)\lambda E(X). \tag{2.9}$$

where $\theta$ is security loading. From $c > \lambda E(X)$ we know $\theta > 0$. Taking the expectation of the both sides of (2.3) and using the properties of a compound Poisson process, we have

$$E[U(t)] = u + ct - \lambda t E(X) = u + \theta t \lambda E(X). \tag{2.10}$$

By $\theta > 0$ we can see that

$$\lim_{t \to \infty} E[U(t)] = \infty. \tag{2.11}$$

Though (2.11) holds, ruin may occur in a finite period. We illustrate this by figure 2.1. In this figure, $T_1, T_2, \cdots$ are the times at which claims occur. $T_4$ is the first time when the surplus is below zero. That is, $T_4$ is the ruin time defined by formula (2.4). At a ruin time $T$ we always have $U(T-0) \geq 0$ and $U(T) < 0$.

Now we study the surplus process in the situation of $u = 0$. By moving the $t$-axis up $u$ units in figure 2.1, we get figure 2.2. In figure 2.2, $T_1, T_2, \cdots$ denote the times at which claims occur. But now $T_2$ is the ruin time at which a deficit occurs. Let's denote $T_1^* = T_2$. In figure 2.2 we also see $U(T_3) < U(T_2)$. So, $T_3$ is the time at which a new deficit occurs. Denote $T_2^* = T_3$. Similarly we have $T_3^* = T_4$, and so on. We call $T_i^*$ the time of the $i$th new deficit occurrence, $i = 1, 2, \cdots$. Denote $T_0^* = 0$ and let

$$L_i = U(T_{i-1}^*) - U(T_i^*), \quad \text{if } T_i^* < \infty, \; i = 1, 2, \cdots. \tag{2.12}$$

Then $L_i$ is the difference between $\mid U(T_i^*) \mid$ and $\mid U(T_{i-1}^*) \mid$. We call $L_i$ the $i$th deficit. We should also note that $L_i$ is defined only when $T_i^* < \infty$. Therefore, when we speak of the probability (or expectation ) of $L_i$ later, we will mean the conditional probability (or expectation ) under the condition of $T_i^* < \infty$.

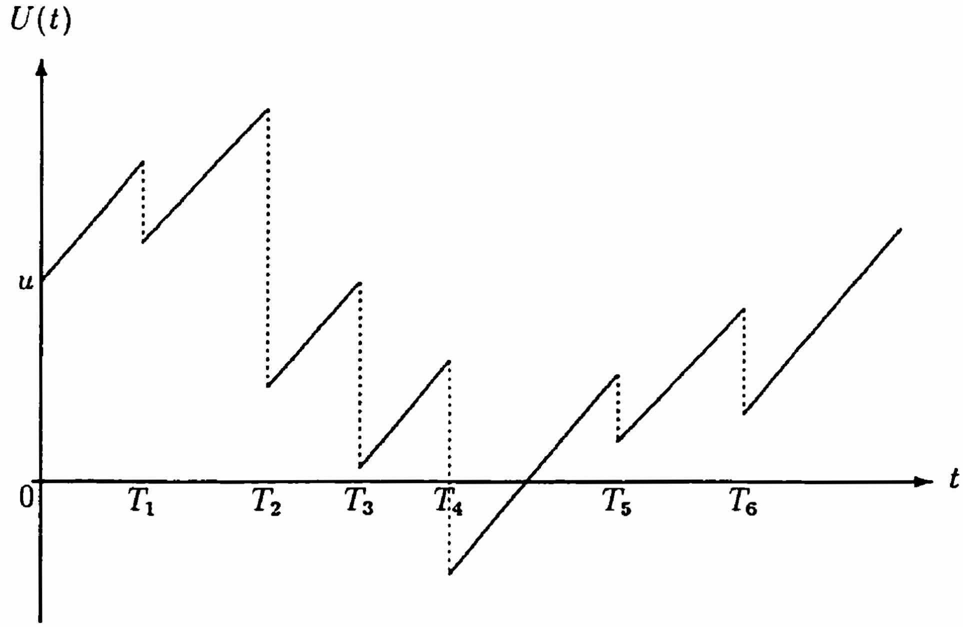



Figure 2.1: A sample path of $U(t)$ with initial surplus $u$

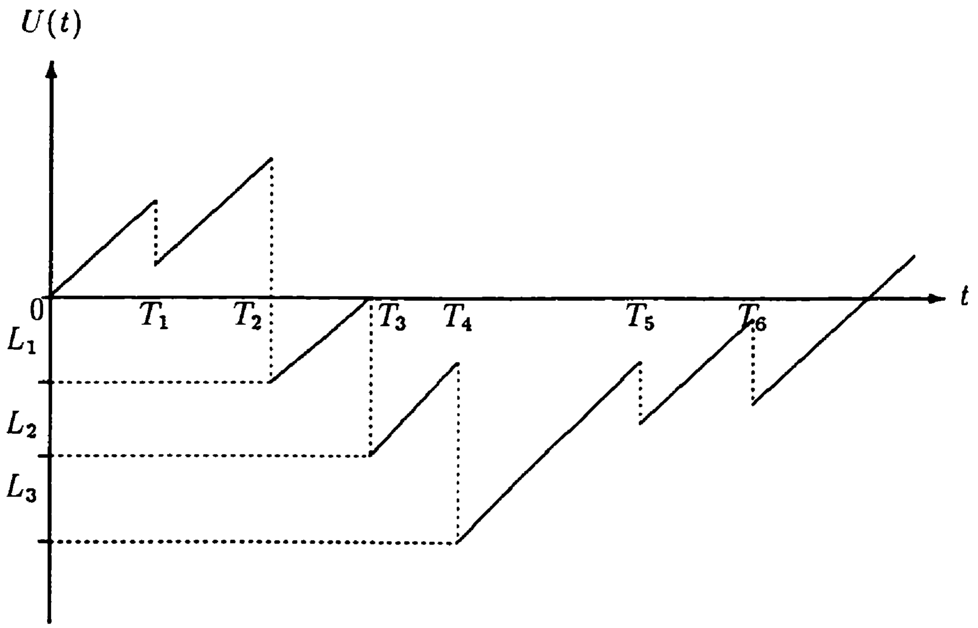



Figure 2.2: A plot of $L_i$ with initial surplus $u = 0$

From the assumption of $\{N(t),\ t \geq 0\}$ being a homogeneous Poisson process, we know that the length of time intervals between two claims are independent and have a common exponential distribution. In addition, by the assumption of the classical risk model, the sequence of the claim amounts are also a series of independent and identically distributed random variables. Hence, the surplus process beginning at the time when a new deficit occurs is independent of the process before this time, and has the same probability law as the process beginning at $t = 0$. If we move the co-ordinate original point to $O'(T_1^*, U(T_1^*))$ ( see figure 2.3. The new co-ordinate axes are marked by dotted lines), and view the surplus process beginning at $T_1^*$ in the new co-ordinate system(now the initial surplus is zero), then from the analysis above we know that this process is independent of the process before $T_1^*$ and that they have the same probability law. Therefore in the two co-ordinate systems, the new one and the original one, both the time at which the first deficit occurs and the amount of claim at this time are independent and identically distributed. Hence $T_2^* - T_1^*$ and $T_1^* = T_1^* - T_0^*$ are independent and identically distributed, as are $L_1$ and $L_2$. We can analyze the process beginning at $T_2^*$ in a similar fashion. We can now conclude that:

$$T_n^* - T_{n-1}^*, \quad n = 1, 2, \cdots, i.i.d. \tag{2.13}$$

and

$$L_1, L_2, \cdots, i.i.d. \tag{2.14}$$

((2.14) means that if $T_m^* < \infty$ then $L_1, L_2, \cdots, L_m i.i.d.$), and that

$$\psi(0) = Pr(T_n^* - T_{n-1}^* < \infty \mid U(0) = 0, T_{n-1}^* < \infty). \tag{2.15}$$

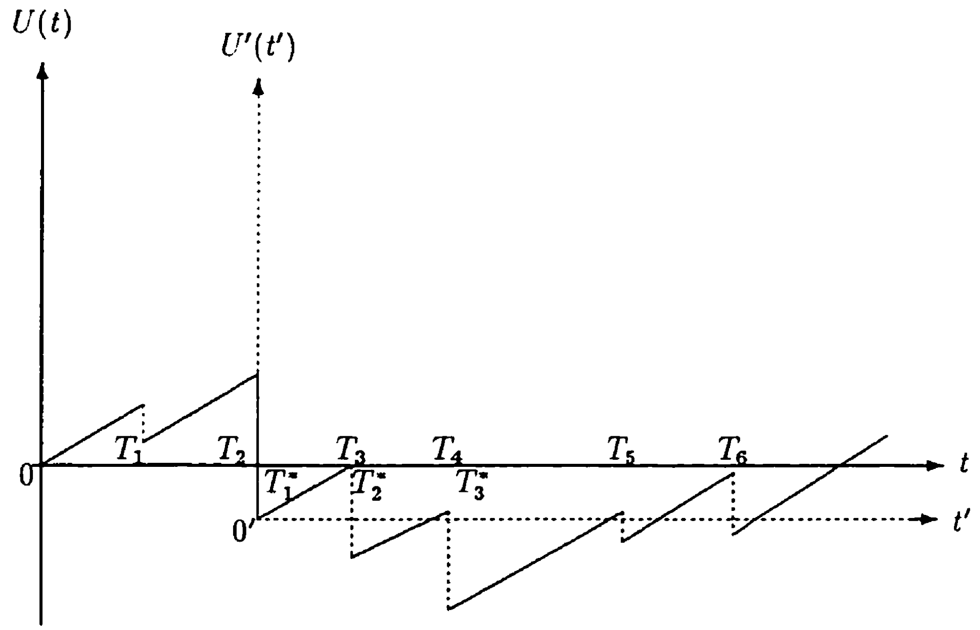



Figure 2.3: Sample paths of $U(t)$ and $U'(t')$ with initial surplus $u = 0$($t'$ is measured from the time when the first deficit occurs)

We introduce another new random variable, the total number of deficits, denoted by $M$

$$M = \max\{n : T_n^* < \infty\}. \tag{2.16}$$

By the following proposition we know that random variable $M$ has a geometric distribution with parameter $\psi(0)$.

**Proposition 2.1.**

$$Pr(M = m) = [\psi(0)]^m[1 - \psi(0)], \quad m = 0, 1, \cdots. \tag{2.17}$$

**Proof.**

$$\begin{aligned}
&Pr(M = m) \\
=&Pr(T_1^* < \infty,\ T_2^* - T_1^* < \infty,\ \cdots, T_m^* - T_{m-1}^* < \infty,\ T_{m+1}^* = \infty \mid U(0) = 0) \\
=&\prod_{k=1}^{m} Pr(T_k^* - T_{k-1}^* < \infty \mid U(0) = 0) Pr(T_{m+1}^* = \infty \mid U(0) = 0) \\
=&[\psi(0)]^m[1 - \psi(0)].
\end{aligned}$$

The second equation above holds since $T_n^* - T_{n-1}^*$, $n = 1, 2, \cdots$ are *i.i.d.*. ∎

From the fact of (2.14) we know that to get the distributions of $L_i$, $i = 1, 2, \cdots$, we only need to study $L_1 = U(T_1^*)$. Dickson, D.C.M., and Water, H.R. (1992) [2], Gerber, H.U., Goovaerts, M.J. and Kaas, R. (1987) [4], Dufresne, F. and Gerber, H.U. (1986) [3], etc. studied the distributions of $U(T_1^* - 0)$ and $U(T_1^*)$. Here we quote two results from [4]. By our notations the two results can be expressed as follows:

(a).

$$\psi(0) = Pr(T_1^* < \infty \mid U(0) = 0) = \frac{1}{1 + \theta}, \tag{2.18}$$

where $\theta$ is the security loading.

It should be noted that $\psi(0)$ is determined by $\theta$ only. That is, $\psi(0)$ does not depend on the claim random variable $X$ (the amount of each claim has the same probability distribution as $X$ ) and the claim number process $\{N(t),\ t \geq 0\}$ when $\theta$ is given.

By using formula (2.18) we can rewrite the probability distribution of $M$, (2.17), as follows:

$$Pr(M = m) = (\frac{1}{1+\theta})^m \frac{\theta}{1+\theta}, \quad m = 0, 1, 2, \cdots. \tag{2.19}$$

The second result from [4] is

(b). $L_1, L_2, \cdots$ are *i.i.d.* and independent of $M$, the common (conditional) survival function of $L_i$ is

$$Pr(L_1 > x \mid T_1^* < \infty) = \frac{1}{E(X)} \int_x^\infty \overline{F}(y) dy, \tag{2.20}$$

here $X$ is claim random variable. Denote the $n$th stop-loss transform of $L_i$ under the condition of $T_i^* < \infty$ by $\Pi_{L_i}^{(n)}(x \mid T_i^* < \infty)$, that is

$$\Pi_{L_i}^{(n)}(x \mid T_i^* < \infty) = E\{[(L_i - x)_+]^n \mid T_i^* < \infty\}, \quad n = 1, 2, \cdots.$$

Since $L_1, L_2, \cdots$ are *i.i.d.* , their stop-loss transforms are the same as that of $L_1$. The formula (2.20) can be rewritten as

$$\Pi_{L_1}^{(0)}(x \mid T_1^* < \infty) = \frac{1}{E(X)} \Pi_X^{(1)}(x). \tag{2.21}$$

Since the deficit depends on the claim random variable, we will mark it when it is needed. For example, if the claim random variable is $X$, then we denote the $n$th deficit by $L_n(X)$.

**Example 2.2.** Denote the random variable having an exponential distribution with parameter $\mu$ by $e_\mu$. We say that if $T_1^* < \infty$, then $L_1(e_\mu)$ has the same distribution as

$e_\mu$.

**Proof.** By using the formula (2.20) we have

$$Pr(L_1(e_\mu) > x \mid T_i^* < \infty) = \frac{1}{E(e_\mu)} \int_x^\infty \overline{F}_{e_\mu}(y)dy = \mu \int_x^\infty e^{-\mu y} dy = e^{-\mu y}. \blacksquare$$

**Proposition 2.3.** Suppose $E(X) = \frac{1}{\mu}$. If

$$\Pi_X^{(1)}(x) \leq \Pi_{e_\mu}^{(1)}(x), \quad \forall\, x \geq 0,$$

then

$$\frac{1}{\mu^2} \leq E(X^2) \leq \frac{2}{\mu^2}. \tag{2.22}$$

If

$$\Pi_X^{(1)}(x) \geq \Pi_{e_\mu}^{(1)}(x), \quad \forall\, x \geq 0,$$

then

$$E(X^2) \geq \frac{2}{\mu^2}. \tag{2.23}$$

**Proof.** The left-hand side of (2.22) is reduced from the inequality of

$$E(X^2) \geq [E(X)]^2.$$

On the right side, the condition $\Pi_X^{(1)}(x) \leq \Pi_{e_\mu}^{(1)}(x)$, $\forall\, x \geq 0$, implies

$$\Pi_X^{(1)}(x) = \int_x^\infty \overline{F}_X(u)du \leq \Pi_{e_\mu}^{(1)}(x) = \int_x^\infty e^{-\mu u} du = \frac{1}{\mu} e^{-\mu x}. \tag{2.24}$$

By taking integration of both sides from 0 to $\infty$, and using formula (1.5) we get (2.22). The proof of (2.23) is similar. $\blacksquare$

**Proposition 2.4.**

$$E[L_1(X) \mid T_1^* < \infty] = \frac{E(X^2)}{2E(X)}. \tag{2.25}$$

**Proof.** From (2.20) and (1.5) we

$$
\begin{aligned}
&E[L_1(X) \mid T_1^* < \infty] \\
=&\int_0^\infty P\{L_1(X) > x \mid T_1^* < \infty\}dx \\
=&\int_0^\infty \frac{1}{E(X)}\int_x^\infty \overline{F}_X(y)dydx \\
=&\frac{1}{E(X)}\int_0^\infty \Pi_X^{(1)}(x)dx = \frac{E(X^2)}{2E(X)}.
\end{aligned}
$$

In the last equation the formula (1.5) is used. ∎

**Corollary 2.5.** Suppose $E(X) = \dfrac{1}{\mu}$. If

$$\Pi_X^{(1)}(x) \le \Pi_{e_\mu}^{(1)}(x), \quad \forall\, x \ge 0,$$

then

$$\frac{1}{2\mu} \le E[L_1(X) \mid T_1^* < \infty] \le \frac{1}{\mu}. \tag{2.26}$$

If

$$\Pi_X^{(1)}(x) \ge \Pi_{e_\mu}^{(1)}(x), \quad \forall\, x \ge 0,$$

then

$$E[L_1(X) \mid T_1^* < \infty] \ge \frac{1}{\mu}. \tag{2.27}$$

**Proof.** The inequality (2.26) can be achieved by (2.25) and (2.22) . The inequality (2.27) can be achieved by (2.25) and (2.23). ∎

**Proposition 2.6.**

$$\Pi_{L_1}^{(n)}(x \mid T_1^* < \infty) = \frac{\Pi_X^{(n+1)}(x)}{(n+1)E(X)}, \quad \forall\, x \ge 0,\ n = 0, 1, 2, \cdots. \tag{2.28}$$

**Proof.** When $n = 0$, from (2.21) we have

$$\Pi_{L_i}^{(0)}(x \mid T_1^* < \infty) = \frac{\Pi_X^{(1)}(x)}{E(X)}, \quad \forall\, x \ge 0.$$

This means that (2.28) holds. In the following we use mathematical induction to prove that (2.28) holds for an arbitrary nonnegative integer $n$.

Assume that (2.28) holds for $n = k$, that is

$$\Pi_{L_1}^{(k)}(x \mid T_1^* < \infty) = \frac{\Pi_X^{(k+1)}(x)}{(k+1)E(X)}, \quad \forall\, x \geq 0. \tag{2.29}$$

When $n = k + 1$, according to theorem 1.6, we have

$$\Pi_{L_1}^{(k+1)}(x \mid T_1^* < \infty) = (k+1)\int_x^\infty \Pi_{L_1}^{(k)}(y \mid T_1^* < \infty)dy. \tag{2.30}$$

Substituting (2.29) into the right-hand side of (2.30) gives us

$$\begin{aligned}\Pi_{L_1}^{(k+1)}(x \mid T_1^* < \infty) &= \frac{k+1}{(k+1)E(X)}\int_x^\infty \Pi_X^{(k+1)}(y)dy \\ &= \frac{\Pi_X^{(k+2)}(x)}{(k+2)E(X)}, \quad \forall\, x \geq 0.\end{aligned}$$

For the last equation we use theorem 1.6. According to the principle of mathematical induction we conclude that (2.28) holds for an arbitrary nonnegative integer. ∎

## 2.2 The relationship between the order in claims and the order in ruin probabilities

We will now apply the results we achieved for $u = 0$ to the study of general ruin probability $\psi(u)$, where $u \geq 0$. For this purpose we need to review figure 2.1 again. For clarity, we mark $L_1, L_2, \cdots$ in figure 2.1, then figure 2.1 becomes figure 2.4. In figure 2.4 we have $u > 0$, $T_1^*$ is not a ruin time but rather the first time at which a loss occurs. That is,

$$U(t) \geq u, \text{ when } t < T_1^*, \text{ but } U(T_1^*) < u.$$

If $0 \le U(T_1^*)$, then $L_1$ is not a deficit, it is a loss defined as $L_1 = u - U(T_1^*)$. $L_2, L_3, \cdots$ are losses that occur successively after the first loss $L_1$. For example, $L_2 = U(T_1^*) - U(T_2^*)$. If $U(T_2^*) \ge 0$ then $L_2$ is a new loss after $L_1$, but ruin (and hence deficit) does not occur at this time.

Let

$$L = \begin{cases} 0, & T_1^* = \infty, \\ \sum_{i=1}^{M} L_i, & T_1^* < \infty, \end{cases} \tag{2.31}$$

where $M$, the total number of deficits occurring in the situation of $u = 0$. $M$ now becomes the total number of the losses that makes the surplus process reach a lower point than it has been ever before ( in figure 2.4, $M = 3$). Thus $L$ is the maximal aggregate loss. We also note that

$$Pr(L = 0) = Pr(T_1^* = \infty) = 1 - \psi(0) = \frac{\theta}{1+\theta} > 0.$$

Thus $L$ has a positive probability at zero.

**Proposition 2.7.**

$$E(L) = \frac{E(X^2)}{2\theta E(X)}. \tag{2.32}$$

**Proof.** Because $L_1, L_2, L_3, \cdots$ are *i.i.d.* and independent of $M$, given $T_1^* < \infty$, and when $T_1^* = \infty$, $L = 0$, we have

$$\begin{aligned} E(L) &= Pr(T_1^* < \infty) E[\sum_{i=1}^{M} L_i \mid T_1^* < \infty] \\ &= Pr(T_1^* < \infty) E(M \mid T_1^* < \infty) E(L_1 \mid T_1^* < \infty). \end{aligned}$$

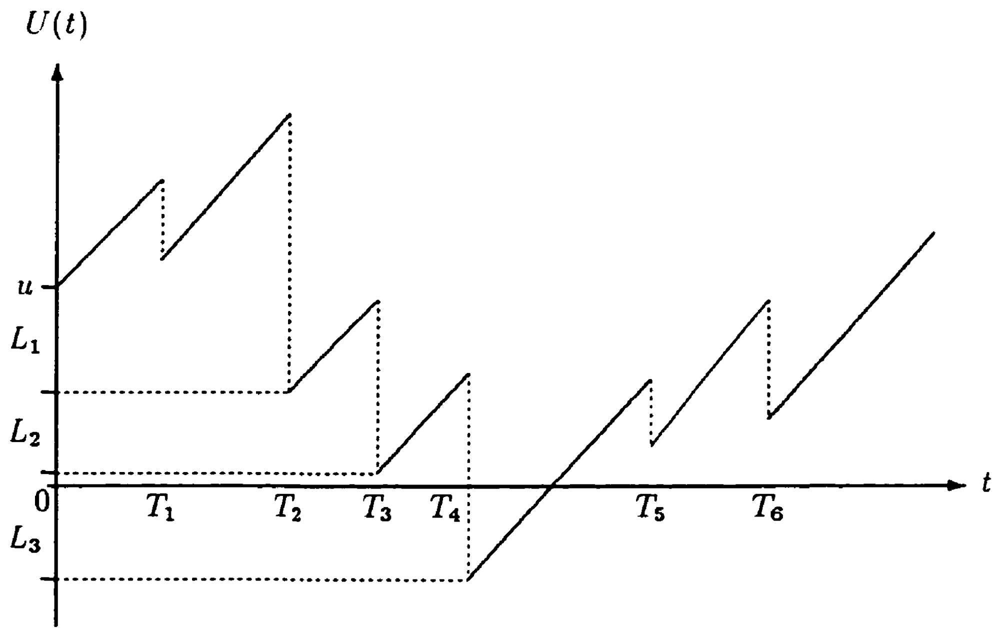



Figure 2.4: Losses occuring in the sample path of $U(t)$ with initial surplus $u$

From (2.20), we have

$$E(L_1 \mid T_1^* < \infty) = \int_0^\infty Pr(L_1 > x \mid T_1^* < \infty)dx$$
$$= \frac{1}{E(X)} \int_0^\infty \Pi_X^{(1)}(x)dx = \frac{E(X^2)}{2E(X)}.$$

(At the last equation (1.5) is applied). We also note that

$$Pr(M \geq 1) = Pr(T_1^* < \infty) = \frac{\theta}{1+\theta},$$

and using (2.19) we have

$$E(M \mid T_1^* < \infty) = E(M \mid M \geq 1) = \frac{1+\theta}{\theta}.$$

Hence, we can see that

$$E(L) = \frac{1}{1+\theta} \frac{1+\theta}{\theta} \frac{E(X^2)}{2E(X)} = \frac{E(X^2)}{2\theta E(X)}. \blacksquare$$

**Corollary 2.8.** Suppose $E(X) = \frac{1}{\mu}$. If

$$\Pi_X^{(1)}(x) \leq \Pi_{e_\mu}^{(1)}(x), \quad \forall\, x \geq 0,$$

then

$$\frac{1}{2\theta\mu} \leq E(L) \leq \frac{1}{\theta\mu}. \tag{2.33}$$

If

$$\Pi_X^{(1)}(x) \geq \Pi_{e_\mu}^{(1)}(x), \quad \forall\, x \geq 0,$$

then

$$E(L) \geq \frac{1}{\theta\mu}. \tag{2.34}$$

**Proof.** By proposition 2.7 and formula (2.22) the corollary 2.8 follows immediately.

$\blacksquare$

Under the condition of $u \geq 0$, ruin occurs if and only if $L > u$. As a result we have

$$\psi(u) = Pr\{L > u \mid U(0) = u\}, \tag{2.36}$$

where $\psi(u)$ is a function on $[0, \infty)$. From now on we will call $\psi(u)$ the ruin probability function. Ruin probability also depends on the claim random variable. We will mark it when needed. For example, if the claim random variable is $X$, then we denote the ruin probability function as $\psi_X(u)$.

From (2.36) we see that $\psi(u) \geq 0$, monotonous decreasing and $\lim_{u\to\infty} \psi(u) = 0$. So $\psi(u)$ belongs to $\Omega$ ($\Omega$ is defined by (1.51)). Thus we can define the $n$th stop-loss order and weak $n$th stop-loss order on the family of ruin probability functions. Limiting the definitions 1.22 and 1.26 on the family of ruin probability functions and considering only the influence of the claim random variable on ruin ( suppose the initial surplus and the claim number process are the same), we introduce the following two definitions:

**Definition 2.9.** For nonnegative integer $k$, assume $\mid \int_0^\infty x^k d\psi_Y(x) \mid < \infty$. We say that ruin probability function $\psi_X(u)$ is less than $\psi_Y(u)$ in the meaning of the $k$th stop-loss order, denoted by $\psi_X <_{sl(k)} \psi_Y$, if (2.37) and (2.38) hold.

$$\int_0^\infty x^i d\psi_X(x) \geq \int_0^\infty x^i d\psi_Y(x), \quad i = 0, 1, \cdots, k-1, \tag{2.37}$$

and

$$\Pi_{\psi_X}^{(k)}(u) \leq \Pi_{\psi_Y}^{(k)}(u), \quad \forall\, u \geq 0. \tag{2.38}$$

**Definition 2.10.** For nonnegative integer $k$, assume $\mid \int_0^\infty x^k d\psi_Y(x) \mid < \infty$. We say that ruin probability function $\psi_X(u)$ is less than $\psi_Y(u)$ in the meaning of weak $k$th stop-loss order, denoted by $\psi_X <_{wsl(k)} \psi_Y$, if (2.38) holds.

At first we illustrate the meaning of introducting stop-loss order and weak stop-loss order on the family of ruin probability functions.

**Proposition 2.11.** Suppose $\psi_i \in \Omega,\ i = 1, 2$, and $\psi_1(0) = \psi_2(0)$, functions $\psi_1(x)$ and $\psi_2(x)$ intersect at finite points, denoted by $x_1 < x_2 < \cdots < x_k$. If there is an integer $n \geq 0$ such that $\psi_X <_{wsl(n)} \psi_Y$, then the following inequality holds:

$$\psi_1(x) < \psi_2(x), \quad \forall\, x > x_k. \tag{2.39}$$

**Proof.** We use the method of reduction to absurdity to prove this proposition. If (2.39) does not hold, since $x_k$ is the largest intersection point of $\psi_1(x)$ and $\psi_2(x)$, we can say that

$$\psi_1(x) > \psi_2(x), \quad \forall\, x > x_k.$$

So, for $u \geq x_k$ we have

$$\Pi^{(0)}_{\psi_1}(u) = \psi_1(u) > \psi_2(u) = \Pi^{(0)}_{\psi_2}(u),$$

and

$$\Pi^{(1)}_{\psi_1}(u) = \int_u^\infty \psi_1(x)dx > \int_u^\infty \psi_2(x)dx = \Pi^{(1)}_{\psi_2}(u).$$

By induction and use theorem 1.6$'$ we have

$$\Pi^{(n)}_{\psi_1}(u) = n\int_u^\infty \Pi^{(n-1)}_{\psi_1}(x)dx > n\int_u^\infty \Pi^{(n-1)}_{\psi_2}(x)dx = \Pi^{(n)}_{\psi_2}(u).$$

This is contrary to $\psi_X <_{wsl(n)} \psi_Y$. ∎

Applying proposition 2.11 to ruin probability functions, we can assume that the initial surplus is $u > 0$ and two ruin probability functions $\psi_1(u)$ and $\psi_2(u)$ satisfy the conditions of proposition 2.11. Only if the initial surplus satisfies $u \geq x_k$ we have $\psi_1(u) < \psi_2(u)$.

In the following we compare the two surplus processes. Suppose there are two surplus processes defined as:

$$U_1(t) = u + ct - \sum_{i=1}^{N_1(t)} X_i, \tag{2.40}$$

and

$$U_2(t) = u + ct - \sum_{i=1}^{N_2(t)} X_i. \tag{2.41}$$

And suppose $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ are homogeneous Poisson processes with the same parameter $\lambda$. For process $U_1(t)$, the claims $X_1, X_2, \cdots$, are *i.i.d.* and distributed as $X$; for process $U_2(t)$, the claims $Y_1, Y_2, \cdots$, are *i.i.d.* and distributed as $Y$. For process $U_i(t)$, $i = 1, 2$, denote the first time of loss occurrence, the losses, the total number of losses and the maximal aggregate loss by $T^{i*}$, $L_n^i(n = 1, 2, \cdots)$, $M_i$ and $L^i$ respectively. We prove the following lemma first:

**Lemma 2.12.** Suppose two surplus processes $U_1(t)$ and $U_2(t)$ as defined by (2.40) and (2.41) satisfy $E(X) = E(Y)$. If

$$X <_{sl(n)} Y, \tag{2.42}$$

then

$$L^1 <_{sl(n-1)} L^2, \quad n = 1, 2, \cdots. \tag{2.43}$$

**Proof.** Let $\psi_i(u)$ be the ruin probability of $U_i(t)$ and $\theta_i$ be the security loading, $i = 1, 2$ . Since $E(X) = E(Y)$ and the two claim number processes have the same parameter $\lambda$, we have

$$\theta_1 = \frac{c}{\lambda E(X)} - 1 = \frac{c}{\lambda E(Y)} - 1 = \theta_2 \overset{\text{say}}{=} \theta. \tag{2.44}$$

Furthermore,

$$\psi_1(0) = \frac{1}{1+\theta_1} = \frac{1}{1+\theta_2} = \psi_2(0). \tag{2.45}$$

By (2.19), we know that $M_i\,(i = 1, 2)$ are distributed identically. By (2.31), we have

$$L^1 = \begin{cases} 0, & T^{1*} = \infty, \\ \sum_{m=1}^{M_1} L_m^1, & T^{1*} < \infty, \end{cases}$$

$$L^2 = \begin{cases} 0, & T^{2*} = \infty, \\ \sum_{m=1}^{M_2} L_m^2, & T^{2*} < \infty. \end{cases}$$

Since

$$Pr(L^1 = 0) = Pr(L^2 = 0) = \frac{\theta}{1+\theta},$$

we need only to prove that under the conditions of $T^{i*} < \infty$, $i = 1, 2$,

$$\sum_{m=1}^{M_1} L_m^1 <_{sl(n-1)} \sum_{m=1}^{M_2} L_m^2.$$

By theorem 1.9, we need only to prove that $L_m^1 <_{sl(n-1)} L_m^2$, $m = 1, 2, \cdots$ (under the conditions of $T^{i*} < \infty$, $i = 1, 2$; otherwise, there would be no $L_m^i$ to sum). Further, we need only to prove $L_1^1 <_{sl(n-1)} L_1^2$, since $L_m^i$ has the same distribution as $L_1^i$, $i = 1, 2$. By (2.28) and (2.31) we have

$$\Pi_{L_1^1}^{(k)}(x \mid T^{1*} < \infty) = \frac{\Pi_X^{(k+1)}(x)}{(k+1)E(X)}, \quad k = 0, 1, 2, \cdots, \tag{2.46}$$

$$\Pi_{L_1^2}^{(k)}(x \mid T^{2*} < \infty) = \frac{\Pi_Y^{(k+1)}(x)}{(k+1)E(Y)}, \quad k = 0, 1, 2, \cdots. \tag{2.47}$$

The condition $X <_{sl(n)} Y$ implies that

$$\Pi_X^{(k)}(0) = E(X^k) \leq E(Y^k) = \Pi_Y^{(k)}(0), \quad k = 0, 1, 2, \cdots, \tag{2.48}$$

and

$$\Pi_X^{(n)}(x) \leq \Pi_Y^{(n)}(x), \quad \forall\, x \geq 0. \tag{2.49}$$

By formulas (2.46)-(2.49) and $E(X) = E(Y)$ we have

$$\begin{aligned} E[(L_1^1)^k \mid T^{1*} < \infty] &= \Pi_{L_1^1}^{(k)}(0 \mid T^{1*} < \infty) = \frac{\Pi_X^{(k+1)}(0)}{(k+1)E(X)} \\ &\leq \frac{\Pi_Y^{(k+1)}(0)}{(k+1)E(Y)} = \Pi_{L_1^2}^{(k)}(0 \mid T^{2*} < \infty) \\ &= E[(L_1^2)^k \mid T^{2*} < \infty],\ k = 0, 1, 2, \cdots, n-2, \end{aligned} \tag{2.50}$$

and

$$\Pi_{L_1^1}^{(n-1)}(x \mid T^{1*} < \infty) = \frac{\Pi_X^{(n)}(x)}{nE(X)} \leq \frac{\Pi_Y^{(n)}(x)}{nE(Y)} = \Pi_{L_1^2}^{(n-1)}(x \mid T^{2*} < \infty). \tag{2.51}$$

Formulas (2.50) and (2.51) imply that $L_m^1 <_{sl(n-1)} L_m^2$. ∎

**Theorem 2.13.** Suppose $E(X) = E(Y)$. If

$$X <_{sl(n)} Y, \tag{2.52}$$

then

$$\psi_1(u) <_{sl(n-1)} \psi_2(u), \tag{2.53}$$

here $\psi_i(u)$ is the ruin probability function of $U_i(t)$, $i = 1, 2$.

**Proof.** Notice that

$$\psi_i(u) = Pr(L^i > u),$$

our conclusion can be arrived at immediately. ∎

In [6] (pages 65-66) it is said that $X <_{sl(1)} Y$ implies $\psi_1(u) \leq \psi_2(u), \forall u \geq 0$, but $X <_{sl(2)} Y$ does not imply $\psi_1(u) \leq \psi_2(u), \forall u \geq 0$. From theorem 2.13 we see that $X <_{sl(2)} Y$ does imply $\psi_1(u) <_{sl(1)} \psi_2(u)$. Therefore theorem 2.13 generalized the result of [6].

## 2.3 Further remark

Ruin probability is an important topic in risk theory. Various transformations of a random variable (or equivalently its distribution function and survival function) and the theory of partial order are interesting and useful in many fields.

This thesis began by introducing the concept of the $n$th stop-loss transforms of non-negative monotonous decrease function, and examined extensively their properties. As a result, the relationship between a claim random variable and ruin probability was established. The relationship between the $n$th stop-loss order and other kinds of order that appeared in economics, queuing theory and reliability theory was not discussed. These topics as well as the introducing of new transforms and their orders are worth further examination.

# References


[1] Bowers, N.L., Gerber, H.U., et al. (1997), *Actuarial Mathematics*, SOA.

[2] Dickson, D.C.M. and Water, H.R. (1992), The Probability and Severity of Ruin in Finite and Infinite Time, **Astin Bulletin, 23**, 177–190

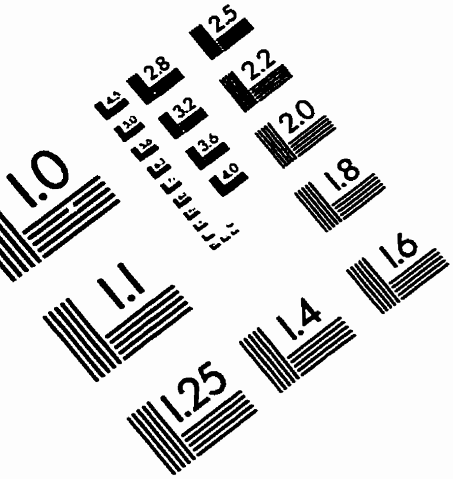
[3] Dufresne, F., and Gerber, H.U. (1988), The Surpluses Immediately before and at Ruin, and the Amount of the Claim Causing Ruin, **Insurance: Mathematics and Economics, 7**, 193–199.

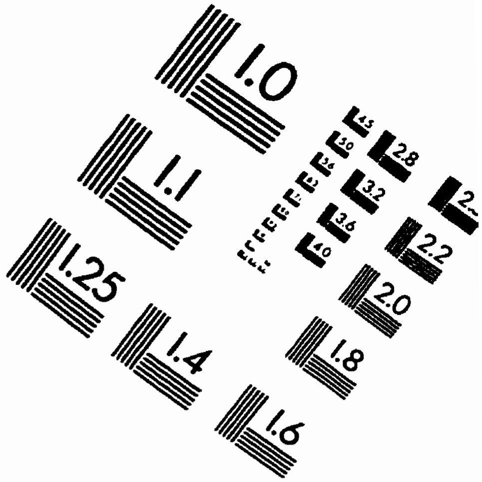
[4] Gerber, H.U. Goovaerts, M.J., and Kaas, R. (1987), On the Probability and Severity of Ruin, **Astin Bulletin, 17**, 151–163.

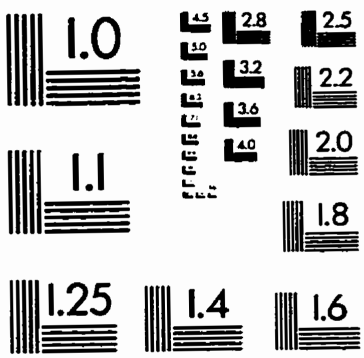
[5] Gerber, H.U. (1979), *Introduction to Mathematical Risk Theory*, S.S. Huebner Foundation for Insurance Education, University of Pennsylvania.

[6] Goovaerts, M.J., Kaas, R., et al. (1990), *Effective Actuarial Methods*, North-Holland.

# IMAGE EVALUATION TEST TARGET (QA-3)

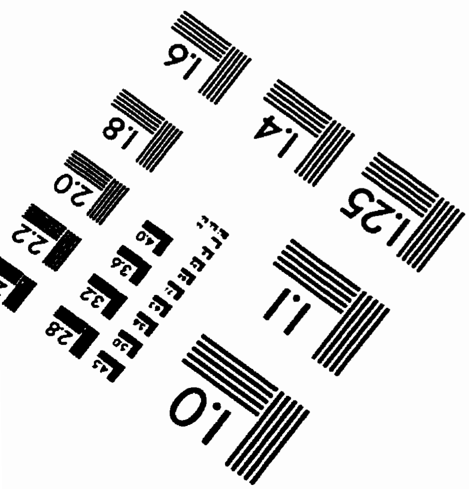


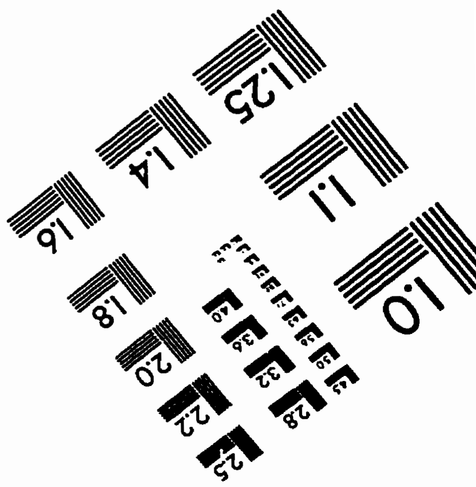





150mm

6″





APPLIED IMAGE, Inc
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989