Initial Boundary Value Problems Associated with a Spinning String

by

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

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Abstract

A model is developed for the transverse vibrations of a spinning string. This model involves a pair of coupled second order partial differential equations for the components of the transverse displacement along the string. Analytic solutions are presented for various choices of initial and boundary conditions, in the case of semi-infinite strings and finite length strings (with fixed or variable length). Techniques used involve an extension of the standard d'Alembert solution of the one-dimensional wave equation and Fourier analysis. Graphical illustrations are presented.

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Finally, the author wishes to dedicate this dissertation to his beloved wife and parents, whose sustained love and inspiring encouragement helped him to achieve this success.

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CHAPTER 1

A Mathematical Model for a Spinning String

1.1 Introduction

A tethered scientific research rocket, the first of several in the Canadian Space Agency's Oedipus Project, was launched in northern Norway on January 30, 1989. During the flight of the rocket, the spin-stabilized payload separated into two subpayloads (referred to as the forward and aft payload), each with its own complement of scientific instruments, control, power and telemetry systems. Throughout the flight, the two payloads remained connected by a flexible, electrical-conducting tether, whose purpose included the direct measurement of the electric potential gradient along the earth's magnetic field lines. Flight data from the Oedipus-A mission indicated that the forward payload experienced "coning" motion as a result of separation and tether deployment dynamics and a post-flight dynamics investigation found that the likely cause of the observed dynamic behavior was the tether interaction with the subpayloads. Based on this observation and on the fact that the Oedipus-A payload did not incorporate systems to record the dynamics of the tether or the tether-payload interactions, a comprehensive tether dynamics experiment was included in the second Oedipus mission, which was referred to as Oedipus-C (the proposed Oedipus-B mission was never flown). The primary objectives of this experiment were to develop a comprehensive understanding of the dynamics of a spin-stabilized, tethered, two-body rocket payload with flexible appendages. The study of the tether deployment phase of such a flight constitutes an important component in the development of future tethered space systems.

In this thesis, under some assumptions regarding the tether dynamics, the tether will be represented as a string spinning about its longitudinal axis. A system of two second order partial differential equations will be derived to describe the transverse vibrations of such a spinning tether. In particular, the cases of strings of a constant length or a linearly varying length will be studied. We will obtain some analytical solutions of the equations, including some special "travelling wave" solutions, and the general solution. These will be used to investigate the behaviour of infinite, semi-infinite, and finite strings with various boundary and initial conditions. The theoretical results will be illustrated through some examples and their corresponding graphs.

1.2 Development of the Model

The model in this thesis will be developed in a right-handed coordinate system (x, u_1, u_2) as shown below:

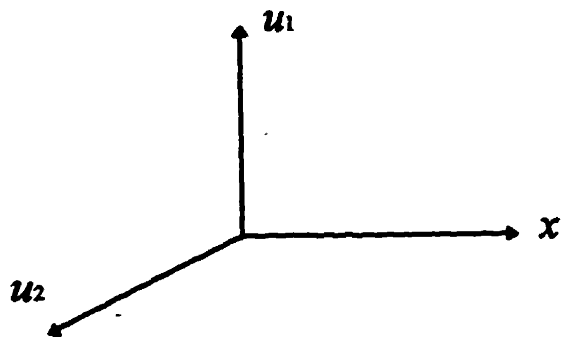


Diagram 1.2.1

In addition, it will be assumed that when the string we are studying lies in its rest (equilibrium) position, it coincides with the x-axis.

The "forced" one-dimensional wave equation, corresponding to the planar vibration of a string, is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}), \tag{1.2.1}$$

where $u = u(x, t)$ denotes the transverse displacement from its equilibrium position of the point on the string at position x at time t, and the function $f(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t})$ denotes the transverse force per unit mass acting on the string. The positive constant c physically denotes the velocity of wave propagation along the string, and is equal to $\sqrt{\frac{T}{\rho}}$, where T is the longitudinal tension and ρ is the linear mass density of the string [7, p. 13].

The two-dimensional wave equation, analogous to (1.2.1), can be written in the form of a vectorial extension of the usual one-dimensional wave equation as

$$\frac{\partial^2 \vec{u}}{\partial t^2} = c^2 \frac{\partial^2 \vec{u}}{\partial x^2} + \vec{f}(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \frac{\partial \vec{u}}{\partial t}), \tag{1.2.2}$$

where, in analogy with the above, $\vec{u} = \vec{u}(x, t) = u_1(x, t)\hat{j} + u_2(x, t)\hat{k}$ denotes the vectorial transverse displacement from its equilibrium position of the point on the string at position x at time t. The vectorial function $\vec{f} = f_y\hat{j} + f_z\hat{k}$ denotes the transverse force per unit mass acting on the string [2].

The derivation of the models (1.2.1-2) is based on the following indispensable assumptions:

(1) Each point on the tether is allowed to move only within a transverse plane, i.e., in a plane perpendicular to the equilibrium line. In other words, only transverse motion is permitted. Loosely, this may be interpreted to mean that the longitudinal component of the tension is constant along the tether.

(2) Deflections are small in comparison with the total length of the tether, so that their squares or higher powers can be neglected.

(3) The slope at any point of the deflected tether is small compared with unity.

The above assumptions give rise to the following intuitive mathematical interpretations:

(a) The difference between the length of the tether and the distance between its end-points is negligible.

(b) The tension is constant along the tether as well as constant in time.

The most distinctive feature of the Oedipus payload involves the rotation of the complete system (including the tether) about an axis passing through the two centres of mass of the subpayloads at the ends of the tether. In order to incorporate this spin about the longitudinal axis, we replace the operation of differentiation of any (generally 3-dimensional) vector $\vec{v} = \vec{v}(x, t)$ with respect to time by the "rotational" time derivative

$$D_t \vec{v} \equiv \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{\omega} \times \vec{v}, \tag{1.2.3}$$

where $\frac{\partial \vec{v}}{\partial t}$ denotes the usual time-derivative of $\vec{v}$, $\times$ denotes the usual vector cross-product operator, and $\vec{\omega} \times \vec{v}$ represents the "rotational velocity" at position x at time t as a result of the local spin vector $\overrightarrow{\omega} = \overrightarrow{\omega}(x,t)$ at that location and time. In this way, we generalize system (1.2.2) to the form

$$\frac{D^2\overrightarrow{u}}{Dt^2} = c^2\frac{\partial^2\overrightarrow{u}}{\partial x^2} + \overrightarrow{f}(x, t, \overrightarrow{u}, \frac{\partial \overrightarrow{u}}{\partial x}, \frac{\partial \overrightarrow{u}}{\partial t}). \tag{1.2.4}$$

To display the dependence of (1.2.4) on the spin vector, which is written in component form as $\overrightarrow{\omega} = \overrightarrow{\omega}(x,t) = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$, we observe the following:

(i) The cross-product $\overrightarrow{\omega} \times \overrightarrow{u}$ can be expanded in the form of a determinant in the usual fashion as

$$\begin{aligned}\overrightarrow{\omega} \times \overrightarrow{u} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ u_x & u_y & u_z \end{vmatrix} \\ &= (\omega_y u_z - \omega_z u_y)\hat{i} + (\omega_z u_x - \omega_x u_z)\hat{j} + (\omega_x u_y - \omega_y u_x)\hat{k},\end{aligned} \tag{1.2.5}$$

where u_x, u_y and u_z are the components of the vector $\overrightarrow{u}$.

(ii) Using the definition in (1.2.3), we have

$$\frac{D\overrightarrow{u}}{Dt} = \frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{\omega} \times \overrightarrow{u}, \tag{1.2.6}$$

and

$$\begin{aligned}&\frac{D^2\overrightarrow{u}}{Dt^2} \\ =\ &\frac{D}{Dt}(\frac{D\overrightarrow{u}}{Dt}) \\ =\ &\frac{\partial}{\partial t}(\frac{D\overrightarrow{u}}{Dt}) + \overrightarrow{\omega} \times \frac{D\overrightarrow{u}}{Dt}.\end{aligned} \tag{1.2.7}$$

Substitution of (1.2.6) into (1.2.7) gives

$$\begin{aligned}
\frac{D^2\vec{u}}{Dt^2} &= \frac{\partial}{\partial t}\left(\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u}\right) + \vec{\omega} \times \left(\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u}\right) \\
&= \frac{\partial^2 \vec{u}}{\partial t^2} + \frac{\partial}{\partial t}(\vec{\omega} \times \vec{u}) + \vec{\omega} \times \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times (\vec{\omega} \times \vec{u}) \\
&= \frac{\partial^2 \vec{u}}{\partial t^2} + \frac{\partial \vec{\omega}}{\partial t} \times \vec{u} + \vec{\omega} \times \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times (\vec{\omega} \times \vec{u}) \\
&= \frac{\partial^2 \vec{u}}{\partial t^2} + \frac{\partial \vec{\omega}}{\partial t} \times \vec{u} + 2\left(\vec{\omega} \times \frac{\partial \vec{u}}{\partial t}\right) + \vec{\omega} \times (\vec{\omega} \times \vec{u}), \qquad (1.2.8)
\end{aligned}$$

where $\frac{\partial \vec{\omega}}{\partial t} = \frac{\partial \omega_x}{\partial t}\hat{i} + \frac{\partial \omega_y}{\partial t}\hat{j} + \frac{\partial \omega_z}{\partial t}\hat{k}$ denotes the rotational angular acceleration.

(iii) Using (1.2.5), the expression $\vec{\omega} \times (\vec{\omega} \times \vec{u})$ can be expanded as

$$\begin{aligned}
\vec{\omega} \times (\vec{\omega} \times \vec{u}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ (\omega_y u_z - \omega_z u_y) & (\omega_z u_x - \omega_x u_z) & (\omega_x u_y - \omega_y u_x) \end{vmatrix} \\
&= [\omega_y(\omega_x u_y - \omega_y u_x) - \omega_z(\omega_z u_x - \omega_x u_z)]\hat{i} \\
&\quad +[\omega_z(\omega_y u_z - \omega_z u_y) - \omega_x(\omega_x u_y - \omega_y u_x)]\hat{j} \\
&\quad +[\omega_x(\omega_z u_x - \omega_x u_z) - \omega_y(\omega_y u_z - \omega_z u_y)]\hat{k} \\
&= [-(\omega_y^2 + \omega_z^2)u_x + \omega_x(\omega_y u_y + \omega_z u_z)]\hat{i} \\
&\quad +[-(\omega_x^2 + \omega_z^2)u_y + \omega_y(\omega_x u_x + \omega_z u_z)]\hat{j} \\
&\quad +[-(\omega_x^2 + \omega_y^2)u_z + \omega_z(\omega_x u_x + \omega_y u_y)]\hat{k}.
\end{aligned}$$

The dot product $\vec{\omega} \cdot \vec{u}$ can be expanded as

$$\vec{\omega} \cdot \vec{u} = \omega_x u_x + \omega_y u_y + \omega_z u_z,$$

and specifically

$$\vec{\omega} \cdot \vec{\omega} = \omega_x^2 + \omega_y^2 + \omega_z^2,$$

so that

$$
\begin{aligned}
\omega_y u_y + \omega_z u_z &= \overrightarrow{\omega} \cdot \overrightarrow{u} - \omega_x u_x, \\
\omega_x u_x + \omega_y u_y &= \overrightarrow{\omega} \cdot \overrightarrow{u} - \omega_z u_z, \\
\omega_x u_x + \omega_z u_z &= \overrightarrow{\omega} \cdot \overrightarrow{u} - \omega_y u_y,
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_x^2 + \omega_y^2 &= \overrightarrow{\omega} \cdot \overrightarrow{\omega} - \omega_z^2, \\
\omega_x^2 + \omega_z^2 &= \overrightarrow{\omega} \cdot \overrightarrow{\omega} - \omega_y^2, \\
\omega_y^2 + \omega_z^2 &= \overrightarrow{\omega} \cdot \overrightarrow{\omega} - \omega_x^2.
\end{aligned}
$$

Therefore, we may write

$$
\begin{aligned}
&\overrightarrow{\omega} \times (\overrightarrow{\omega} \times \overrightarrow{u}) \\
= \ & [(\omega_x^2 - \overrightarrow{\omega} \cdot \overrightarrow{\omega})u_x + \omega_x(\overrightarrow{\omega} \cdot \overrightarrow{u} - \omega_x u_x)]\widehat{i} \\
& +[(\omega_y^2 - \overrightarrow{\omega} \cdot \overrightarrow{\omega})u_y + \omega_y(\overrightarrow{\omega} \cdot \overrightarrow{u} - \omega_y u_y)]\widehat{j} \\
& +[(\omega_z^2 - \overrightarrow{\omega} \cdot \overrightarrow{\omega})u_z + \omega_z(\overrightarrow{\omega} \cdot \overrightarrow{u} - \omega_z u_z)]\widehat{k} \\
= \ & [-(\overrightarrow{\omega} \cdot \overrightarrow{\omega})u_x + \omega_x(\overrightarrow{\omega} \cdot \overrightarrow{u})]\widehat{i} \\
& +[-(\overrightarrow{\omega} \cdot \overrightarrow{\omega})u_y + \omega_y(\overrightarrow{\omega} \cdot \overrightarrow{u})]\widehat{j} \\
& +[-(\overrightarrow{\omega} \cdot \overrightarrow{\omega})u_z + \omega_z(\overrightarrow{\omega} \cdot \overrightarrow{u})]\widehat{k} \\
= \ & -(\overrightarrow{\omega} \cdot \overrightarrow{\omega})(u_x\widehat{i} + u_y\widehat{j} + u_z\widehat{k}) + (\overrightarrow{\omega} \cdot \overrightarrow{u})(\omega_x\widehat{i} + \omega_y\widehat{j} + \omega_z\widehat{k}) \\
= \ & -(\overrightarrow{\omega} \cdot \overrightarrow{\omega})\overrightarrow{u} + (\overrightarrow{\omega} \cdot \overrightarrow{u})\overrightarrow{\omega} \\
= \ & -\omega^2\overrightarrow{u} + (\overrightarrow{\omega} \cdot \overrightarrow{u})\overrightarrow{\omega} \qquad (1.2.9)
\end{aligned}
$$

where $\omega^2 = \overrightarrow{\omega} \cdot \overrightarrow{\omega}$.

Substitution of (1.2.9) into (1.2.8) yields

$$\frac{D^2\vec{u}}{Dt^2} = \frac{\partial^2\vec{u}}{\partial t^2} + \frac{\partial\vec{\omega}}{\partial t} \times \vec{u} + 2(\vec{\omega} \times \frac{\partial\vec{u}}{\partial t}) - \omega^2\vec{u} + (\vec{\omega}\cdot\vec{u})\vec{\omega}. \qquad (1.2.10)$$

Incorporation of (1.2.10) into (1.2.4) yields the vectorial differential equation

$$\begin{aligned} &\frac{\partial^2\vec{u}}{\partial t^2} + \frac{\partial\vec{\omega}}{\partial t} \times \vec{u} + 2(\vec{\omega} \times \frac{\partial\vec{u}}{\partial t}) - \omega^2\vec{u} + (\vec{\omega}\cdot\vec{u})\vec{\omega} \\ = \quad & c^2\frac{\partial^2\vec{u}}{\partial x^2} + \vec{f}(x, t, \vec{u}, \frac{\partial\vec{u}}{\partial x}, \frac{\partial\vec{u}}{\partial t}). \qquad (1.2.11)\end{aligned}$$

Various simplifying assumptions based on the physical nature of the problem may now be made, in an effort to make this system more manageable.

(a) According to assumption (1), only transverse displacements of the tether are allowed, so that the longitudinal component of $\vec{u}(x,t)$ is zero, i.e.,

$$u_x = 0. \qquad (1.2.12)$$

(b) For simplicity, we further assume the angular velocity $\vec{\omega}$ is constant, so that the angular acceleration is zero, i.e.,

$$\frac{\partial\vec{\omega}}{\partial t} = \vec{0}. \qquad (1.2.13)$$

(c) Since the spin of the system is about the longitudinal axis of the system, and since all tether displacements are assumed to be small, the local spin vector $\vec{\omega} = \vec{\omega}(x,t)$ may be written in the form

$$\vec{\omega} = \omega\hat{i}, \qquad (1.2.14)$$

in which it is assumed that the "spin scalar" $\omega = \omega(x,t)$ is a constant both in position and in time. In addition, it is noted that $\vec{\omega}$ and $\vec{u}$ are perpendicular at each point of the tether, so that

$$\vec{\omega} \cdot \vec{u} = 0. \tag{1.2.15}$$

(d) Moreover, since the longitudinal tension in the tether is constant, there can be no longitudinal external forces acting along the tether, so that

$$f_x = 0. \tag{1.2.16}$$

By virtue of (1.2.12-16), the equation (1.2.11) is simplified to

$$\frac{\partial^2 \vec{u}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{u}}{\partial t} - \omega^2 \vec{u} = c^2 \frac{\partial^2 \vec{u}}{\partial x^2} + \vec{f}, \tag{1.2.17}$$

where $\vec{f} = f_y \hat{j} + f_z \hat{k}$ is the external transverse force per unit length.

For consistency of notation we now write $\vec{u} = u_y \hat{j} + u_z \hat{k} = u_1 \hat{j} + u_2 \hat{k}$ and $\vec{f} = f_1 \hat{j} + f_2 \hat{k}$, so that the vector equation (1.2.17) may be written in component form as

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} - 2\omega \frac{\partial u_2}{\partial t} - \omega^2 u_1 &= c^2 \frac{\partial^2 u_1}{\partial x^2} + f_1, \\ \frac{\partial^2 u_2}{\partial t^2} + 2\omega \frac{\partial u_1}{\partial t} - \omega^2 u_2 &= c^2 \frac{\partial^2 u_2}{\partial x^2} + f_2. \end{aligned}$$

In particular, if there is no external force exerted on the tether, $\vec{f}$ must vanish, so that we obtain a coupled linear homogeneous system of equations, namely

$$\frac{\partial^2 u_1}{\partial t^2} - 2\omega \frac{\partial u_2}{\partial t} - \omega^2 u_1 = c^2 \frac{\partial^2 u_1}{\partial x^2}, \tag{1.2.18a}$$

$$\frac{\partial^2 u_2}{\partial t^2} + 2\omega \frac{\partial u_1}{\partial t} - \omega^2 u_2 = c^2 \frac{\partial^2 u_2}{\partial x^2}, \tag{1.2.18b}$$

where $u_1 = u_1(x, t)$ and $u_2 = u_2(x, t)$ denote the two components of the transverse displacement of the tether at position x at time t in the orthogonal coordinate system (x, u_1, u_2) and ω is an arbitrary constant representing the spin of the system about its longitudinal axis.

1.3 Summary of the Model

In the previous sections, the "rotational" time-derivative is introduced to incorporate the fact that the tether spins about its longitudinal axis. Some assumptions such as (i) the displacement of the tether is transverse, (ii) the angular velocity is constant, and (iii) the external force vanishes, simplify the introduced mathematical model to the following system of equations:

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} - 2\omega\frac{\partial u_2}{\partial t} - \omega^2 u_1 &= c^2\frac{\partial^2 u_1}{\partial x^2}, &\quad (1.3.1\text{a}) \\ \frac{\partial^2 u_2}{\partial t^2} + 2\omega\frac{\partial u_1}{\partial t} - \omega^2 u_2 &= c^2\frac{\partial^2 u_2}{\partial x^2}, &\quad (1.3.1\text{b}) \end{aligned}$$

which is the target system to be discussed in all subsequent chapters. We will solve the coupled system (1.3.1) subject to various types of initial and boundary conditions for $t \geq 0$ and x restricted to lie in various domains, such as

(a) $-\infty < x < \infty$, in the case of an "infinite string",

(b) $0 \leq x < \infty$, in the case of a "semi-infinite string",

(c) $0 \leq x \leq L$, in the case of a "finite string (of constant length)",

or

(d) $0 \leq x \leq S(t)$, in the case of a "finite string of variable length (with a moving endpoint)".

Remark: It should be noted that the motion of the subpayloads at the ends of the tether is incorporated into the model through the specification of the boundary conditions. That is, it is assumed that the end-body motions are known *a priori*. Thus, our model is concerned solely with the dynamics of the tether and incorporates end-body effects upon the tether, but does not take into account any influences that the tether motion may have on the subpayload dynamics. For our purposes the subpayloads are viewed simply as point masses whose motions are prescribed. In addition, external forces may be assumed to be negligible because, at least during the tether deployment phase of the mission, the whole system is in a state of "free fall" in a vacuum. Thus, gravitational and aerodynamic drag forces will both be absent.

CHAPTER 2

"Travelling Wave" Solutions for the Semi-infinite String

2.1 Introduction

For the usual one-dimensional wave equation, the so-called "d'Alembert form of the general solution" expresses the solution as the sum of two "travelling waves" (each of constant physical profile, but travelling in opposite directions) along the string [see section 2.2]. In this chapter, we employ a simple extension of this concept in order to determine the nature of travelling wave solutions of the coupled linear system (1.3.1) for a uniformly rotating semi-infinite string. Through the application of appropriate boundary conditions at the (single) end of a semi-infinite string, some special travelling wave solutions are determined which exhibit some very peculiar and interesting behavior. In particular, these special travelling wave solutions are shown to be simply (one or more) uniformly rotating "standing waves" (waves of constant physical profile that do not move along the string, but revolve uniformly about the rest position of the string). In some cases these standing waves are planar, while in others they are not. Not only are these solutions physically interesting, but will serve as test cases for more general procedures to be developed in subsequent chapters for the determination of solutions of the system (1.3.1).

2.2 d'Alembert's Solution of the 1-d Wave Equation

For the usual one-dimensional homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0, \tag{2.2.1}$$

the general solution is known to have form

$$u(x,t) = \varphi(\xi) + \psi(\eta), \tag{2.2.2}$$

where $\xi = x + ct,\ \eta = x - ct$ [7, 10].

For the case of an infinite string ($-\infty < x < \infty$), the initial conditions are given by

$$u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x), \qquad -\infty < x < \infty, \tag{2.2.3}$$

where $f(x)$ and $g(x)$ are specified functions for $-\infty < x < \infty$. By using the initial conditions (2.2.3), $\varphi(\xi)$ and $\psi(\eta)$ can be shown to have the form

$$\begin{aligned} \varphi(\xi) &= \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau + D, \\ \psi(\eta) &= \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - D, \end{aligned}$$

where D is any constant and $-\infty < \xi,\ \eta < \infty$, so that d'Alembert's solution of equation (2.2.1), subject to the initial conditions (2.2.3), can be written as

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, \tag{2.2.4}$$

$$-\infty < x < \infty,\ \ t \geq 0,$$

[7, pp. 17-19], [10, pp. 54-56].

The continuity or differentiability of the solution (2.2.4) depends on the properties of the functions $f(x)$ and $g(x)$, as shown below:

1) If $f(x) \in C^0$ and $g(x)$ is integrable, then $u(x,t) \in C^0$.

2) If $f(x) \in C^1$ and $g(x) \in C^0$, then $u(x,t) \in C^1$.

Note: C^0 is the space of all continuous functions and C^1 is the space of all continuously differentiable functions.

For the case of a semi-infinite string $(0 \leq x < \infty)$, with the initial and boundary conditions given by

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad x \geq 0 \tag{2.2.5a}$$

$$u(0,t) = p(t), \quad t \geq 0, \tag{2.2.5b}$$

the d'Alembert solution of equation (2.2.1) is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, \tag{2.2.6a}$$

$$\text{for } \; x > ct, \;\; t \geq 0,$$

while

$$u(x,t) = \frac{1}{2}[f(ct+x) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\tau)d\tau + p(t - \frac{x}{c}), \tag{2.2.6b}$$

$$\text{for } \; 0 \leq x < ct, \;\; t \geq 0,$$

[7, pp. 23-27], [10, pp. 68-69].

The continuity or differentiability of the solution (2.2.6) again depends on the properties of the functions $f(x)$, $g(x)$ and $p(t)$, as shown below:

1) If $f(x) \in C^0$, $p(t) \in C^0$, $g(x)$ is integrable and $f(0) = p(0)$, then $u(x,t) \in C^0$.

2) If $f(x) \in C^1$, $p(t) \in C^1$, $g(x) \in C^0$, $f(0) = p(0)$ and $g(0) = p'(0)$, then $u(x,t) \in C^1$.

Remark: The above statements include, not only continuity/differentiability properties of $f(x)$, $g(x)$ and $p(t)$ appearing in the specification of the initial/boundary conditions (2.2.6), but also "matching" conditions on $u(0,0)$ and $\frac{\partial u}{\partial t}(0,0)$ obtained from these conditions at the corner of the domain upon which we seek a solution of (2.2.1), subject to (2.2.6). For this reason, conditions such as 1) or 2), or extensions of them in the subsequent analysis, will be referred to as "compatability conditions" on the initial boundary value problem under discussion.

The following arguments confirm the validity of the above compatability conditions:

If $f(x) \in C^0$, $p(t) \in C^0$ and $g(x)$ is integrable, then it is immediately clear from (2.2.6) that $u(x,t)$ is continuous within either of the regions given by $x > ct$, $t \geq 0$ or $0 \leq x < ct$, $t \geq 0$. On the other hand, when (x,t) approaches the line $x = ct$ from arbitrary directions, we have

$$\begin{aligned}
\lim_{x \to ct} u(x,t) &= \lim_{x \to ct} \{\frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau\} \\
&= \frac{1}{2}[\lim_{x \to ct} f(x+ct) + \lim_{x \to ct} f(x-ct)] + \frac{1}{2c}\lim_{x \to ct}\int_{x-ct}^{x+ct} g(\tau)d\tau \\
&= \frac{1}{2}[f(2ct) + f(0)] + \frac{1}{2c}\int_0^{2ct} g(\tau)d\tau, \qquad \text{(2.2.7a)}
\end{aligned}$$

when (x,t) tends to the line $x = ct$ within the region $x > ct$, $t \geq 0$,

and

$$\begin{aligned}
\lim_{x\to ct} u(x,t) &= \lim_{x\to ct}\{\frac{1}{2}[f(ct+x)-f(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x} g(\tau)d\tau + p(t-\frac{x}{c})\} \\
&= \frac{1}{2}[\lim_{x\to ct} f(ct+x) - \lim_{x\to ct} f(ct-x)] + \frac{1}{2c}\lim_{x\to ct}\int_{ct-x}^{ct+x} g(\tau)d\tau \\
&\quad + \lim_{x\to ct} p(t-\frac{x}{c}) \\
&= \frac{1}{2}[f(2ct)-f(0)]+\frac{1}{2c}\int_0^{2ct} g(\tau)d\tau + p(0), \qquad (2.2.7b)
\end{aligned}$$

when (x,t) tends to the line $x=ct$ within the region $0 \le x < ct,\ t \ge 0$.

Moreover, if $f(0)=p(0)$, then it is evident that the two limits in (2.2.7) are equal, which indicates that $u(x,t)$ is continuous across the line $x=ct$. Thus, we conclude that $u(x,t)$ is continuous for all $x \ge 0$, $t \ge 0$.

More generally, if $f(x) \in C^1$, $p(t) \in C^1$ and $g(x) \in C^0$, then (2.2.6) confirm that $u(x,t)$ is differentiable within the region given by $x \ge 0$ and $t \ge 0$, except possibly along the line $x = ct$. In addition, the first order partial derivatives of $u(x,t)$ with respect to x and t are given by

$$\begin{aligned}
\frac{\partial u}{\partial x}(x,t) &= \frac{1}{2}[f'(x+ct)+f'(x-ct)] \\
&\quad + \frac{1}{2c}[g(x+ct)-g(x-ct)], \qquad (2.2.8a) \\
\frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}[f'(x+ct)-f'(x-ct)] \\
&\quad + \frac{1}{2}[g(x+ct)+g(x-ct)], \qquad (2.2.8b)
\end{aligned}$$

for $(x,t) \in$ region $x > ct,\ t \ge 0$,

and

$$\frac{\partial u}{\partial x}(x,t) = \frac{1}{2}[f'(ct+x)+f'(ct-x)]+\frac{1}{2c}[g(ct+x)+g(ct-x)]$$

$$
\begin{aligned}
& -\frac{1}{c}p'(t-\frac{x}{c}), && (2.2.9a) \\
\frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}[f'(ct+x)-f'(ct-x)]+\frac{1}{2}[g(ct+x)-g(ct-x)] \\
& +p'(t-\frac{x}{c}), && (2.2.9b)
\end{aligned}
$$

for $(x,t) \in$ region $0 \le x < ct,\ t \ge 0.$

The limits of the above partial derivatives, when (x,t) approaches the line $x = ct$ from arbitrary directions, are given by

$$
\begin{aligned}
\lim_{x\to ct}\frac{\partial u}{\partial x}(x,t) &= \lim_{x\to ct}\{\frac{1}{2}[f'(x+ct)+f'(x-ct)]+\frac{1}{2c}[g(x+ct)-g(x-ct)]\} \\
&= \frac{1}{2}[\lim_{x\to ct}f'(x+ct)+\lim_{x\to ct}f'(x-ct)] \\
&\quad +\frac{1}{2c}[\lim_{x\to ct}g(x+ct)-\lim_{x\to ct}g(x-ct)] \\
&= \frac{1}{2}[f'(2ct)+f'(0)]+\frac{1}{2c}[g(2ct)-g(0)], && (2.2.10a) \\
\lim_{x\to ct}\frac{\partial u}{\partial t}(x,t) &= \lim_{x\to ct}\{\frac{c}{2}[f'(x+ct)-f'(x-ct)]+\frac{1}{2}[g(x+ct)+g(x-ct)]\} \\
&= \frac{c}{2}[\lim_{x\to ct}f'(x+ct)-\lim_{x\to ct}f'(x-ct)] \\
&\quad +\frac{1}{2}[\lim_{x\to ct}g(x+ct)+\lim_{x\to ct}g(x-ct)] \\
&= \frac{c}{2}[f'(2ct)-f'(0)]+\frac{1}{2}[g(2ct)+g(0)], && (2.2.10b)
\end{aligned}
$$

when (x,t) tends to the line $x = ct$ within the region $x > ct,\ t \ge 0,$

and

$$
\begin{aligned}
\lim_{x\to ct}\frac{\partial u}{\partial x}(x,t) &= \lim_{x\to ct}\{\frac{1}{2}[f'(ct+x)+f'(ct-x)]+\frac{1}{2c}[g(ct+x)+g(ct-x)] \\
&\quad -\frac{1}{c}p'(t-\frac{x}{c})\} \\
&= \frac{1}{2}[\lim_{x\to ct}f'(ct+x)+\lim_{x\to ct}f'(ct-x)]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2c}[\lim_{x\to ct} g(ct+x) + \lim_{x\to ct} g(ct-x)] - \frac{1}{c}\lim_{x\to ct} p'(t-\frac{x}{c}) \\
&= \frac{1}{2}[f'(2ct)+f'(0)] + \frac{1}{2c}[g(2ct)+g(0)] - \frac{1}{c}p'(0), \qquad (2.2.11a)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to ct}\frac{\partial u}{\partial t}(x,t) &= \lim_{x\to ct}\{\frac{c}{2}[f'(ct+x)-f'(ct-x)] + \frac{1}{2}[g(ct+x)-g(ct-x)] \\
&\quad +p'(t-\frac{x}{c})\} \\
&= \frac{c}{2}[\lim_{x\to ct} f'(ct+x) - \lim_{x\to ct} f'(ct-x)] \\
&\quad +\frac{1}{2}[\lim_{x\to ct} g(ct+x) - \lim_{x\to ct} g(ct-x)] + \lim_{x\to ct} p'(t-\frac{x}{c}) \\
&= \frac{c}{2}[f'(2ct)-f'(0)] + \frac{1}{2}[g(2ct)-g(0)] + p'(0), \qquad (2.2.11b)
\end{aligned}
$$

when (x,t) tends to the line $x = ct$ within the region $0 \le x < ct,\ t \ge 0$.

Finally, if $g(0) = p'(0)$, then it is evident that the two limits in (2.2.10a) and (2.2.11a) are equal, and the two limits in (2.2.10b) and (2.2.11b) are equal. Thus, under conditions 2), we may conclude that $u(x,t)$ is continuous and differentiable across the line $x = ct$. Thus, we conclude that $u(x,t)$ is differentiable for all $x \ge 0$, $t \ge 0$.

2.3 Travelling Waves for the Semi-infinite String

We now turn our attention to the coupled linear system

$$
\begin{aligned}
\frac{\partial^2 u_1}{\partial t^2} - 2\omega\frac{\partial u_2}{\partial t} - \omega^2 u_1 &= c^2\frac{\partial^2 u_1}{\partial x^2}, \qquad (2.3.1a) \\
\frac{\partial^2 u_2}{\partial t^2} + 2\omega\frac{\partial u_1}{\partial t} - \omega^2 u_2 &= c^2\frac{\partial^2 u_2}{\partial x^2}, \qquad (2.3.1b)
\end{aligned}
$$

$$t \ge 0.$$

With the d'Alembert form of the solution (2.2.2) of (2.2.1) in mind, we then ask if there are any travelling wave solutions of (2.3.1) of the form

$$u_1(x,t) = \phi_1(\xi) + \psi_1(\eta), \tag{2.3.2a}$$

$$u_2(x,t) = \phi_2(\xi) + \psi_2(\eta), \tag{2.3.2b}$$

where, in the interest of generality, we have chosen $\xi = x + at$, $\eta = x - at$, in which the positive parameter a can be but need not necessarily be the same as c. First and second order partial derivatives of the above $u_1(x,t)$ and $u_2(x,t)$ are

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= a\phi_1'(\xi) - a\psi_1'(\eta), \quad \frac{\partial^2 u_1}{\partial t^2} = a^2\phi_1''(\xi) + a^2\psi_1''(\eta), \\
\frac{\partial u_2}{\partial t} &= a\phi_2'(\xi) - a\psi_2'(\eta), \quad \frac{\partial^2 u_1}{\partial t^2} = a^2\phi_2''(\xi) + a^2\psi_2''(\eta),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u_1}{\partial x} &= \phi_1'(\xi) + \psi_1'(\eta), \quad \frac{\partial^2 u_1}{\partial x^2} = \phi_1''(\xi) + \psi_1''(\eta), \\
\frac{\partial u_2}{\partial x} &= \phi_2'(\xi) + \psi_2'(\eta), \quad \frac{\partial^2 u_1}{\partial x^2} = \phi_2''(\xi) + \psi_2''(\eta).
\end{aligned}$$

Putting these partial derivatives into the equations (2.3.1) yields

$$\begin{aligned}
& (a^2 - c^2)\phi_1''(\xi) - 2\omega a\phi_2'(\xi) - \omega^2\phi_1(\xi) \\
= \; & -(a^2 - c^2)\psi_1''(\eta) - 2\omega a\psi_2'(\eta) + \omega^2\psi_1(\eta), \qquad (2.3.3a) \\
& (a^2 - c^2)\phi_2''(\xi) + 2\omega a\phi_1'(\xi) - \omega^2\phi_2(\xi) \\
= \; & -(a^2 - c^2)\psi_2''(\eta) + 2\omega a\psi_1'(\eta) + \omega^2\psi_2(\eta). \qquad (2.3.3b)
\end{aligned}$$

The left hand side of (2.3.3a) contains the variable ξ only, and the right hand side of (2.3.3a) contains the variable η only, so each side must be equal to the same constant, i.e.,

$$\begin{aligned}(a^2-c^2)\phi_1''(\xi)-2\omega a\phi_2'(\xi)-\omega^2\phi_1(\xi) &= l_1,\\(a^2-c^2)\psi_1''(\eta)+2\omega a\psi_2'(\eta)-\omega^2\psi_1(\eta) &= -l_1,\end{aligned}$$

where l_1 is an arbitrary constant. From (2.3.3b) we can similarly obtain

$$\begin{aligned}(a^2-c^2)\phi_2''(\xi)+2\omega a\phi_1'(\xi)-\omega^2\phi_2(\xi) &= l_2,\\(a^2-c^2)\psi_2''(\eta)-2\omega a\psi_1'(\eta)-\omega^2\psi_2(\eta) &= -l_2;\end{aligned}$$

where l_2 is an arbitrary constant. Thus, the above discussion yields two systems of pairs of ordinary differential equations, namely

$$(a^2-c^2)\phi_1''(\xi)-2\omega a\phi_2'(\xi)-\omega^2\phi_1(\xi) = l_1, \tag{2.3.4a}$$

$$(a^2-c^2)\phi_2''(\xi)+2\omega a\phi_1'(\xi)-\omega^2\phi_2(\xi) = l_2, \tag{2.3.4b}$$

and

$$(a^2-c^2)\psi_1''(\eta)+2\omega a\psi_2'(\eta)-\omega^2\psi_1(\eta) = -l_1, \tag{2.3.5a}$$

$$(a^2-c^2)\psi_2''(\eta)-2\omega a\psi_1'(\eta)-\omega^2\psi_2(\eta) = -l_2. \tag{2.3.5b}$$

In the following, we will divide the discussion of the solutions of the equations (2.3.4) and (2.3.5) into two different cases, depending on whether $a=c$ or $a\neq c$.

Case I: $a=c$

In this case (2.3.4) and (2.3.5) become

$$-2\omega c\phi_2'(\xi)-\omega^2\phi_1(\xi) = l_1, \tag{2.3.6a}$$

$$2\omega c\phi_1'(\xi)-\omega^2\phi_2(\xi) = l_2, \tag{2.3.6b}$$

and

$$2\omega c\psi_2'(\eta) - \omega^2\psi_1(\eta) = -l_1, \tag{2.3.7a}$$

$$-2\omega c\psi_1'(\eta) - \omega^2\psi_2(\eta) = -l_2. \tag{2.3.7b}$$

In the following discussion, we assume $\omega \neq 0$. Otherwise (2.3.1) will become two independent one-dimensional wave equations, each with the form (2.2.1), for which solutions are already known.

Differentiation of (2.3.6) gives

$$2c\phi_2''(\xi) + \omega\phi_1'(\xi) = 0, \tag{2.3.8a}$$

$$2c\phi_1''(\xi) - \omega\phi_2'(\xi) = 0. \tag{2.3.8b}$$

From (2.3.8b) we have

$$\phi_2'(\xi) = \frac{2c}{\omega}\phi_1''(\xi),$$

so that

$$\phi_2''(\xi) = \frac{2c}{\omega}\phi_1'''(\xi). \tag{2.3.9}$$

Substitution of (2.3.9) into (2.3.8a) gives

$$4c^2\phi_1'''(\xi) + \omega^2\phi_1'(\xi) = 0, \tag{2.3.10}$$

which is a homogeneous third order linear ordinary differential equation with constant coefficients. The three corresponding eigenvalues can be obtained from its auxiliary equation

$$4c^2\lambda^3 + \omega^2\lambda = 0,$$

and are given by

$$\lambda = 0, \ \lambda = \pm\frac{\omega}{2c}i,$$

where i denotes the usual imaginary unit, i.e., $i = \sqrt{-1}$. Therefore, the general solution of (2.3.10) is given by

$$\phi_1(\xi) = c_1 \cos\frac{\omega}{2c}\xi + c_2 \sin\frac{\omega}{2c}\xi + c_3, \tag{2.3.11}$$

where c_1, c_2 and c_3 are arbitrary constants. Then, by virtue of (2.3.6b), we obtain

$$\begin{aligned}\phi_2(\xi) &= \frac{2c}{\omega}\phi_1'(\xi) - \frac{l_2}{\omega^2} \\ &= -c_1 \sin\frac{\omega}{2c}\xi + c_2 \cos\frac{\omega}{2c}\xi - \frac{l_2}{\omega^2}.\end{aligned} \tag{2.3.12}$$

Now substitution of (2.3.11-12) into (2.3.6a) provides

$$c_3 = -\frac{l_1}{\omega^2}.$$

We can similarly solve (2.3.7) to get $\psi_1(\eta)$ and $\psi_2(\eta)$, as given by

$$\begin{aligned}\psi_1(\eta) &= d_1 \cos\frac{\omega}{2c}\eta + d_2 \sin\frac{\omega}{2c}\eta + \frac{l_1}{\omega^2}, \\ \psi_2(\eta) &= d_1 \sin\frac{\omega}{2c}\eta - d_2 \cos\frac{\omega}{2c}\eta + \frac{l_2}{\omega^2},\end{aligned}$$

where d_1 and d_2 are arbitrary constants.

In summary, by virtue of (2.3.2), we now have a special solution of the system (2.3.1) of the form

$$\begin{aligned}u_1(x,t) &= c_1 \cos\frac{\omega}{2c}(x+ct) + c_2 \sin\frac{\omega}{2c}(x+ct) \\ &\quad + d_1 \cos\frac{\omega}{2c}(x-ct) + d_2 \sin\frac{\omega}{2c}(x-ct),\end{aligned} \tag{2.3.13a}$$

$$\begin{aligned}u_2(x,t) &= -c_1 \sin\frac{\omega}{2c}(x+ct) + c_2 \cos\frac{\omega}{2c}(x+ct) \\ &\quad + d_1 \sin\frac{\omega}{2c}(x-ct) - d_2 \cos\frac{\omega}{2c}(x-ct).\end{aligned} \tag{2.3.13b}$$

We shall return to these expressions later in this section in order to investigate their physical significance and interpretation.

Case II: $a \neq c$

In this case, we solve the ordinary differential systems (2.3.4) and (2.3.5), namely

$$(a^2 - c^2)\phi_1''(\xi) - 2\omega a\phi_2'(\xi) - \omega^2\phi_1(\xi) = l_1, \quad (2.3.14a)$$

$$(a^2 - c^2)\phi_2''(\xi) + 2\omega a\phi_1'(\xi) - \omega^2\phi_2(\xi) = l_2, \quad (2.3.14b)$$

and

$$(a^2 - c^2)\psi_1''(\eta) + 2\omega a\psi_2'(\eta) - \omega^2\psi_1(\eta) = -l_1, \quad (2.3.15a)$$

$$(a^2 - c^2)\psi_2''(\eta) - 2\omega a\psi_1'(\eta) - \omega^2\psi_2(\eta) = -l_2. \quad (2.3.15b)$$

Differentiating both sides of (2.3.14a,b) yields

$$(a^2 - c^2)\phi_1'''(\xi) - 2\omega a\phi_2''(\xi) - \omega^2\phi_1'(\xi) = 0, \quad (2.3.16a)$$

$$(a^2 - c^2)\phi_2'''(\xi) + 2\omega a\phi_1''(\xi) - \omega^2\phi_2'(\xi) = 0. \quad (2.3.16b)$$

Moreover, from (2.3.16b), we may write

$$(a^2 - c^2)\phi_2^{(4)}(\xi) + 2\omega a\phi_1^{(3)}(\xi) - \omega^2\phi_2''(\xi) = 0, \quad (2.3.17)$$

while from (2.3.16a), we have

$$\phi_2''(\xi) = \frac{a^2 - c^2}{2\omega a}\phi_1'''(\xi) - \frac{\omega}{2a}\phi_1'(\xi). \quad (2.3.18)$$

Substitution of (2.3.18) into (2.3.17) yields the equation

$$(a^2 - c^2)\phi_1^{(5)}(\xi) + 2\omega^2(a^2 + c^2)\phi_1^{(3)}(\xi) + \omega^4\phi_1'(\xi) = 0, \quad (2.3.19)$$

whose five eigenvalues can be obtained from the auxiliary equation

$$(a^2-c^2)\lambda^5+2\omega^2(a^2+c^2)\lambda^3+\omega^4\lambda=0,$$

and are given by

$$\lambda=0,\ \lambda=\pm\frac{\omega}{a+c}i,\ \lambda=\pm\frac{\omega}{a-c}i.$$

Therefore, the general solution of (2.3.19) is given by

$$\begin{aligned}\phi_1(\xi) &= c_1\cos\frac{\omega}{a+c}\xi+c_2\sin\frac{\omega}{a+c}\xi\\ &\quad +c_3\cos\frac{\omega}{a-c}\xi+c_4\sin\frac{\omega}{a-c}\xi+c_5,\end{aligned}\tag{2.3.20}$$

from which we obtain

$$\begin{aligned}\phi_1''(\xi) &= -\frac{\omega^2}{(a+c)^2}c_1\cos\frac{\omega}{a+c}\xi-\frac{\omega^2}{(a+c)^2}c_2\sin\frac{\omega}{a+c}\xi\\ &\quad -\frac{\omega^2}{(a-c)^2}c_3\cos\frac{\omega}{a-c}\xi-\frac{\omega^2}{(a-c)^2}c_4\sin\frac{\omega}{a-c}\xi.\end{aligned}\tag{2.3.21}$$

Morever, by virtue of (2.3.14a), we have

$$\phi_2'(\xi)=\frac{a^2-c^2}{2\omega a}\phi_1''(\xi)-\frac{\omega}{2a}\phi_1(\xi)-\frac{l_1}{2\omega a}.\tag{2.3.22}$$

Substituting (2.3.20-21) into (2.3.22), we have

$$\begin{aligned}\phi_2'(\xi) &= -\frac{\omega}{a+c}(c_1\cos\frac{\omega}{a+c}\xi+c_2\sin\frac{\omega}{a+c}\xi)\\ &\quad -\frac{\omega}{a-c}(c_3\cos\frac{\omega}{a-c}\xi+c_4\sin\frac{\omega}{a-c}\xi)-\frac{\omega}{2a}c_5-\frac{l_1}{2\omega a},\end{aligned}$$

so that $\phi_2(\xi)$ is given by

$$\begin{aligned}\phi_2(\xi) &= -c_1\sin\frac{\omega}{a+c}\xi+c_2\cos\frac{\omega}{a+c}\xi\\ &\quad -c_3\sin\frac{\omega}{a-c}\xi+c_4\cos\frac{\omega}{a-c}\xi+c_6\xi+c_7,\end{aligned}\tag{2.3.23}$$

where

$$c_6 = -\frac{\omega}{2a}c_5 - \frac{l_1}{2\omega a},$$

and where c_1, c_2, c_3 and c_4 are arbitrary constants. Substitution of (2.3.20) and (2.3.23) into (2.3.14b) results in

$$c_5 = -\frac{l_1}{\omega^2},\ c_6 = 0,\ c_7 = -\frac{l_2}{\omega^2}.$$

We can similarly solve (2.3.15) to obtain $\psi_1(\eta)$ and $\psi_2(\eta)$, as given by

$$\begin{aligned}
\psi_1(\eta) &= d_1 \cos\frac{\omega}{a+c}\eta + d_2 \sin\frac{\omega}{a+c}\eta \\
&\quad + d_3 \cos\frac{\omega}{a-c}\eta + d_4 \sin\frac{\omega}{a-c}\eta + \frac{l_1}{\omega^2}, \\
\psi_2(\eta) &= d_1 \sin\frac{\omega}{a+c}\eta - d_2 \cos\frac{\omega}{a+c}\eta \\
&\quad + d_3 \sin\frac{\omega}{a-c}\eta - d_4 \cos\frac{\omega}{a-c}\eta + \frac{l_2}{\omega^2},
\end{aligned}$$

where d_1, d_2, d_3 and d_4 are arbitrary constants.

Thus, by virtue of (2.3.2), we obtain another special solution of (2.3.1), as given by

$$\begin{aligned}
u_1(x,t) &= c_1 \cos\frac{\omega}{a+c}(x+at) + c_2 \sin\frac{\omega}{a+c}(x+at) \\
&\quad + c_3 \cos\frac{\omega}{a-c}(x+at) + c_4 \sin\frac{\omega}{a-c}(x+at) \\
&\quad + d_1 \cos\frac{\omega}{a+c}(x-at) + d_2 \sin\frac{\omega}{a+c}(x-at) \\
&\quad + d_3 \cos\frac{\omega}{a-c}(x-at) + d_4 \sin\frac{\omega}{a-c}(x-at), \qquad (2.3.24\text{a}) \\
u_2(x,t) &= -c_1 \sin\frac{\omega}{a+c}(x+at) + c_2 \cos\frac{\omega}{a+c}(x+at) \\
&\quad - c_3 \sin\frac{\omega}{a-c}(x+at) + c_4 \cos\frac{\omega}{a-c}(x+at)
\end{aligned}$$

$$+d_1 \sin \frac{\omega}{a+c}(x-at) - d_2 \cos \frac{\omega}{a+c}(x-at)$$
$$+d_3 \sin \frac{\omega}{a-c}(x-at) - d_4 \cos \frac{\omega}{a-c}(x-at). \quad (2.3.24b)$$

To understand the physical nature of the solutions exhibited in the two cases (2.3.13) and (2.3.24), we consider the case of a semi-infinite string ($0 \le x < \infty$), and introduce two different sets of boundary conditions at the end of the string (i.e., at $x = 0$), namely,

(a) $u_1(0,t) = u_2(0,t) = 0$,

and

(b) $u_1(0,t) = \sin(kt)$, $u_2(0,t) = \cos(kt)$,

Note: the boundary conditions in (a) correspond to a fixed end at $x = 0$, while those in (b) correspond to a uniformly rotating "driver" of constant (unit) displacement at $x = 0$ revolving at the angular speed k.

Case I: $a = c$

If $x = 0$ in (2.3.13), then we have

$$\begin{aligned} u_1(0,t) &= (c_1 + d_1)\cos\frac{\omega t}{2} + (c_2 - d_2)\sin\frac{\omega t}{2}, && (2.3.25a) \\ u_2(0,t) &= -(c_1 + d_1)\sin\frac{\omega t}{2} + (c_2 - d_2)\cos\frac{\omega t}{2}. && (2.3.25b) \end{aligned}$$

Moreover, with the boundary conditions of case (a), (2.3.25) becomes

$$\begin{aligned} (c_1 + d_1)\cos\frac{\omega t}{2} + (c_2 - d_2)\sin\frac{\omega t}{2} &= 0, \\ -(c_1 + d_1)\sin\frac{\omega t}{2} + (c_2 - d_2)\cos\frac{\omega t}{2} &= 0. \end{aligned}$$

which are to be valid for any $t \geq 0$, so that must have

$$c_1 + d_1 = 0,\ c_2 - d_2 = 0,$$

$$\text{i.e., } d_1 = -c_1,\ d_2 = c_2.$$

Hence, the corresponding solution in this case is given by

$$\begin{aligned} u_1(x,t) &= c_1[\cos\frac{\omega}{2c}(x+ct) - \cos\frac{\omega}{2c}(x-ct)] \\ &\quad + c_2[\sin\frac{\omega}{2c}(x+ct) + \sin\frac{\omega}{2c}(x-ct)] \\ &= 2\sin\frac{\omega x}{2c}(-c_1\sin\frac{\omega t}{2} + c_2\cos\frac{\omega t}{2}), \qquad (2.3.26a) \\ u_2(x,t) &= -c_1[\sin\frac{\omega}{2c}(x+ct) + \sin\frac{\omega}{2c}(x-ct)] \\ &\quad + c_2[\cos\frac{\omega}{2c}(x+ct) - \cos\frac{\omega}{2c}(x-ct)] \\ &= 2\sin\frac{\omega x}{2c}(-c_1\cos\frac{\omega t}{2} - c_2\sin\frac{\omega t}{2}). \qquad (2.3.26b) \end{aligned}$$

The initial displacements, corresponding to this solution, can be obtained by letting $t = 0$ in (2.3.26), which yields

$$u_1(x,0) = 2c_2\sin\frac{\omega x}{2c},\ \ u_2(x,0) = -2c_1\sin\frac{\omega x}{2c}.$$

Now, taking advantage of the notation of matrix algebra, we may write

$$\overrightarrow{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \end{bmatrix}, \qquad (2.3.27)$$

so that

$$\overrightarrow{u}(x,0) = \begin{bmatrix} u_1(x,0) \\ u_2(x,0) \end{bmatrix} = 2\sin\frac{\omega x}{2c}\begin{bmatrix} c_2 \\ -c_1 \end{bmatrix},$$

and the solution (2.3.26) can be expressed in matrix form as

$$\vec{u}(x,t) = \begin{bmatrix} \cos\frac{\omega t}{2} & \sin\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{bmatrix} \vec{u}(x,0). \tag{2.3.28}$$

Since the matrix appearing in (2.3.28) is simply a rotation matrix through an angle $(-\frac{\omega t}{2})$ (at time t), the solution in (2.3.26) represents a uniformly rotating "standing" wave. Moreover, since $u_1(x,0)$ and $u_2(x,0)$ are merely constant multiples of each other, this uniformly rotating standing wave is *planar*. We also note that at the points $x = \frac{2m}{\omega}\pi$, $m = 0,1,2,\ldots$, $u_1(x,t) = u_2(x,t) = 0$, for any $t \geq 0$. That is to say, these points may be thought of as "nodes" on the tether in the sense that they remain motionless as the tether vibrates. In summary, the solution in (2.3.26) represents a uniformly rotating planar standing wave, with the angular velocity $-\frac{\omega}{2}$.

Similarly, with the boundary conditions of case (b), (2.3.25) yield

$$\begin{aligned} (c_1 + d_1)\cos\frac{\omega t}{2} + (c_2 - d_2)\sin\frac{\omega t}{2} &= \sin(kt), \\ -(c_1 + d_1)\sin\frac{\omega t}{2} + (c_2 - d_2)\cos\frac{\omega t}{2} &= \cos(kt). \end{aligned}$$

which are assumed to be valid for any $t \geq 0$, so that we must have

$$k = \frac{\omega}{2},\ c_1 + d_1 = 0,\ c_2 - d_2 = 1,$$

$$\text{i.e., } \omega = 2k,\ d_1 = -c_1,\ d_2 = c_2 - 1.$$

It is noted that the first of these conditions restricts the rotational speed of the "driver" of the tether to be half of the angular speed at which the tether revolves about its axis. Thus, we can expect to obtain travelling wave solutions, with a uniformly rotating end only under very special circumstances.

In this case, the corresponding solution is given by

$$\begin{aligned}
u_1(x,t) &= c_1[\cos\frac{\omega}{2c}(x+ct) - \cos\frac{\omega}{2c}(x-ct)] \\
&\quad + c_2[\sin\frac{\omega}{2c}(x+ct) + \sin\frac{\omega}{2c}(x-ct)] - \sin\frac{\omega}{2c}(x-ct) \\
&= (2c_2-1)\sin\frac{\omega x}{2c}\cos\frac{\omega t}{2} - 2c_1\sin\frac{\omega x}{2c}\sin\frac{\omega t}{2} \\
&\quad + \cos\frac{\omega x}{2c}\sin\frac{\omega t}{2}, \qquad (2.3.29a) \\
u_2(x,t) &= -c_1[\sin\frac{\omega}{2c}(x+ct) + \sin\frac{\omega}{2c}(x-ct)] \\
&\quad + c_2[\cos\frac{\omega}{2c}(x+ct) - \cos\frac{\omega}{2c}(x-ct)] + \cos\frac{\omega}{2c}(x-ct) \\
&= -2c_1\sin\frac{\omega x}{2c}\cos\frac{\omega t}{2} - (2c_2-1)\sin\frac{\omega x}{2c}\sin\frac{\omega t}{2} \\
&\quad + \cos\frac{\omega x}{2c}\cos\frac{\omega t}{2}. \qquad (2.3.29b)
\end{aligned}$$

The corresponding initial displacements can be obtained be letting $t = 0$ in (2.3.29) to obtain

$$u_1(x,0) = (2c_2-1)\sin\frac{\omega x}{2c}, \quad u_2(x,0) = -2c_1\sin\frac{\omega x}{2c} + \cos\frac{\omega x}{2c}.$$

If $\overrightarrow{u}(x,t)$ is defined as in (2.3.7), then we have

$$\begin{aligned}
\overrightarrow{u}(x,0) &= \begin{bmatrix} (2c_2-1)\sin\frac{\omega x}{2c} \\ -2c_1\sin\frac{\omega x}{2c} + \cos\frac{\omega x}{2c} \end{bmatrix} \\
&= \sin\frac{\omega x}{2c}\begin{bmatrix} 2c_2-1 \\ -2c_1 \end{bmatrix} + \cos\frac{\omega x}{2c}\begin{bmatrix} 0 \\ 1 \end{bmatrix},
\end{aligned}$$

from which we may deduce

$$\overrightarrow{u}(x,t) = \begin{bmatrix} \cos\frac{\omega t}{2} & \sin\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{bmatrix} \overrightarrow{u}(x,0), \tag{2.3.30}$$

as in (2.3.28). Once again, we may draw the conclusion that the solution (2.3.29) represents a uniformly rotating standing wave, with the angular velocity $-\frac{\omega}{2}$, but which is *not* in general a *planar* standing wave.

Case II: $a \neq c$

For $x = 0$, (2.3.24) becomes

$$\begin{aligned} u_1(0,t) &= (c_1+d_1)\cos\frac{\omega at}{a+c} + (c_2-d_2)\sin\frac{\omega at}{a+c} \\ &\quad +(c_3+d_3)\cos\frac{\omega at}{a-c} + (c_4-d_4)\sin\frac{\omega at}{a-c}, && (2.3.31a) \\ u_2(0,t) &= -(c_1+d_1)\sin\frac{\omega at}{a+c} + (c_2-d_2)\cos\frac{\omega at}{a+c} \\ &\quad -(c_3+d_3)\sin\frac{\omega at}{a-c} + (c_4-d_4)\cos\frac{\omega at}{a-c}. && (2.3.31b) \end{aligned}$$

Using the boundary conditions of case (a), each right-hand side of (2.3.31) is zero, and the resulting equations are assumed to be valid for any $t \geq 0$, so that we must have

$$c_1+d_1=0,\ c_2-d_2=0,\ c_3+d_3=0,\ c_4-d_4=0,$$

$$\text{i.e., } d_1=-c_1,\ d_2=c_2,\ d_3=-c_3,\ d_4=c_4.$$

Thus, the corresponding solution is given by

$$\begin{aligned} u_1(x,t) &= c_1[\cos\frac{\omega}{a+c}(x+at) - \cos\frac{\omega}{a+c}(x-at)] \\ &\quad +c_2[\sin\frac{\omega}{a+c}(x+at) + \sin\frac{\omega}{a+c}(x-at)] \end{aligned}$$

$$
\begin{aligned}
& +c_3[\cos\frac{\omega}{a-c}(x+at) - \cos\frac{\omega}{a-c}(x-at)] \\
& +c_4[\sin\frac{\omega}{a-c}(x+at) + \sin\frac{\omega}{a-c}(x-at)] \\
= \; & 2\sin\frac{\omega x}{a+c}(-c_1\sin\frac{\omega at}{a+c} + c_2\cos\frac{\omega at}{a+c}) \\
& +2\sin\frac{\omega x}{a-c}(-c_3\sin\frac{\omega at}{a-c} + c_4\cos\frac{\omega at}{a-c}),
\end{aligned} \tag{2.3.32a}
$$

$$
\begin{aligned}
u_2(x,t) = \; & -c_1[\sin\frac{\omega}{a+c}(x+at) + \sin\frac{\omega}{a+c}(x-at)] \\
& +c_2[\cos\frac{\omega}{a+c}(x+at) - \cos\frac{\omega}{a+c}(x-at)] \\
& -c_3[\sin\frac{\omega}{a-c}(x+at) + \sin\frac{\omega}{a-c}(x-at)] \\
& +c_4[\cos\frac{\omega}{a-c}(x+at) - \cos\frac{\omega}{a-c}(x-at)] \\
= \; & 2\sin\frac{\omega x}{a+c}(-c_1\cos\frac{\omega at}{a+c} - c_2\sin\frac{\omega at}{a+c}) \\
& +2\sin\frac{\omega x}{a-c}(-c_3\cos\frac{\omega at}{a-c} - c_4\sin\frac{\omega at}{a-c}).
\end{aligned} \tag{2.3.32b}
$$

Now, in analogy with preceding results, we let $\vec{u} = \vec{u}^* + \vec{u}^{**}$, where

$$
\vec{u}^*(x,t) = \begin{bmatrix} u_1^*(x,t) \\ u_2^*(x,t) \end{bmatrix} = 2\sin\frac{\omega x}{a+c}\begin{bmatrix} -c_1\sin\frac{\omega at}{a+c} + c_2\cos\frac{\omega at}{a+c} \\ -c_1\cos\frac{\omega at}{a+c} - c_2\sin\frac{\omega at}{a+c} \end{bmatrix}, \tag{2.3.33}
$$

and

$$
\vec{u}^{**}(x,t) = \begin{bmatrix} u_1^{**}(x,t) \\ u_2^{**}(x,t) \end{bmatrix} = 2\sin\frac{\omega x}{a-c}\begin{bmatrix} -c_3\sin\frac{\omega at}{a-c} + c_4\cos\frac{\omega at}{a-c} \\ -c_3\cos\frac{\omega at}{a-c} - c_4\sin\frac{\omega at}{a-c} \end{bmatrix}. \tag{2.3.34}
$$

Thus, we have

$$
\vec{u}^*(x,0) = \begin{bmatrix} u_1^*(x,0) \\ u_2^*(x,0) \end{bmatrix} = 2\sin\frac{\omega x}{a+c}\begin{bmatrix} c_2 \\ -c_1 \end{bmatrix}, \tag{2.3.35}
$$

and

$$
\vec{u}^{**}(x,0) = \begin{bmatrix} u_1^{**}(x,0) \\ u_2^{**}(x,0) \end{bmatrix} = 2\sin\frac{\omega x}{a-c}\begin{bmatrix} c_4 \\ -c_3 \end{bmatrix}. \tag{2.3.36}
$$

In addition, we note that $\overrightarrow{u}^*(x,t)$ and $\overrightarrow{u}^{**}(x,t)$ can be expressed as

$$\overrightarrow{u}^*(x,t) = \begin{bmatrix} \cos\frac{\omega at}{a+c} & \sin\frac{\omega at}{a+c} \\ -\sin\frac{\omega at}{a+c} & \cos\frac{\omega at}{a+c} \end{bmatrix} \overrightarrow{u}^*(x,0) \tag{2.3.37}$$

and

$$\overrightarrow{u}^{**}(x,t) = \begin{bmatrix} \cos\frac{\omega at}{a-c} & \sin\frac{\omega at}{a-c} \\ -\sin\frac{\omega at}{a-c} & \cos\frac{\omega at}{a-c} \end{bmatrix} \overrightarrow{u}^{**}(x,0). \tag{2.3.38}$$

The formulae in (2.3.33-38) indicate that $\overrightarrow{u}^*(x,t)$ and $\overrightarrow{u}^{**}(x,t)$ *each* represents a uniformly rotating *planar* standing wave (the nodes of $\overrightarrow{u}^*(x,t)$ and $\overrightarrow{u}^{**}(x,t)$ can be easily identified), with respective angular velocities $-\frac{\omega a}{a+c}$ and $-\frac{\omega a}{a-c}$. Finally since $\overrightarrow{u}(x,t) = \overrightarrow{u}^*(x,t) + \overrightarrow{u}^{**}(x,t)$, the solution in (2.3.32) represents the sum of two uniformly rotating planar standing waves with different angular velocities. Moreover, the signs of $\frac{\omega a}{a+c}$ and $\frac{\omega a}{a-c}$ determine whether these two planar standing waves rotate in the same (or opposite) directions.

Finally, we consider (2.3.31) in the case of the boundary conditions (b), which yield

$$\begin{aligned} u_1(0,t) &= (c_1+d_1)\cos\frac{\omega at}{a+c} + (c_2-d_2)\sin\frac{\omega at}{a+c} \\ &\quad +(c_3+d_3)\cos\frac{\omega at}{a-c} + (c_4-d_4)\sin\frac{\omega at}{a-c} \\ &= \sin(kt), \end{aligned} \tag{2.3.39a}$$

$$\begin{aligned} u_2(0,t) &= -(c_1+d_1)\sin\frac{\omega at}{a+c} + (c_2-d_2)\cos\frac{\omega at}{a+c} \\ &\quad -(c_3+d_3)\sin\frac{\omega at}{a-c} + (c_4-d_4)\cos\frac{\omega at}{a-c} \\ &= \cos(kt), \end{aligned} \tag{2.3.39b}$$

which are assumed to remain valid for any $t \geq 0$. From (2.3.39), we can see that there exists only two possibilities, depending on whether we choose $k = \frac{\omega a}{a+c}$ or $k = \frac{\omega a}{a-c}$, namely

$$k = \frac{\omega a}{a+c}: \; c_1 + d_1 = 0, \; c_2 - d_2 = 1, \; c_3 + d_3 = 0, \; c_4 - d_4 = 0,$$

$$\text{i.e., } d_1 = -c_1, \; d_2 = c_2 - 1, \; d_3 = -c_3, \; d_4 = c_4, \tag{2.3.40}$$

or

$$k = \frac{\omega a}{a-c}: \; c_1 + d_1 = 0, \; c_2 - d_2 = 0, \; c_3 + d_3 = 0, \; c_4 - d_4 = 1,$$

$$\text{i.e., } d_1 = -c_1, \; d_2 = c_2, \; d_3 = -c_3, \; d_4 = c_4 - 1. \tag{2.3.41}$$

In the following, we will base our discussion of the solution (2.3.24) on the choice given in (2.3.40) only. The discussion of the other solution, corresponding to the choice given in (2.3.41), is similar in nature.

As a result of (2.3.40), (2.3.24) become

$$\begin{aligned} u_1(x,t) &= c_1[\cos\frac{\omega}{a+c}(x+at) - \cos\frac{\omega}{a+c}(x-at)] \\ &\quad + c_2[\sin\frac{\omega}{a+c}(x+at) + \sin\frac{\omega}{a+c}(x-at)] \\ &\quad + c_3[\cos\frac{\omega}{a-c}(x+at) - \cos\frac{\omega}{a-c}(x-at)] \\ &\quad + c_4[\sin\frac{\omega}{a-c}(x+at) + \sin\frac{\omega}{a-c}(x-at)] \\ &\quad - \sin\frac{\omega}{a+c}(x-at) \\ &= 2\sin\frac{\omega x}{a+c}(-c_1\sin\frac{\omega at}{a+c} + c_2\cos\frac{\omega at}{a+c}) \\ &\quad + 2\sin\frac{\omega x}{a-c}(-c_3\sin\frac{\omega at}{a-c} + c_4\cos\frac{\omega at}{a-c}) \\ &\quad - \sin\frac{\omega}{a+c}(x-at), \end{aligned} \tag{2.3.42a}$$

$$
\begin{aligned}
u_2(x,t) &= -c_1[\sin\frac{\omega}{a+c}(x+at)+\sin\frac{\omega}{a+c}(x-at)] \\
&\quad +c_2[\cos\frac{\omega}{a+c}(x+at)-\cos\frac{\omega}{a+c}(x-at)] \\
&\quad -c_3[\sin\frac{\omega}{a-c}(x+at)+\sin\frac{\omega}{a-c}(x-at)] \\
&\quad +c_4[\cos\frac{\omega}{a-c}(x+at)-\cos\frac{\omega}{a-c}(x-at)] \\
&\quad +\cos\frac{\omega}{a+c}(x-at) \\
&= 2\sin\frac{\omega x}{a+c}(-c_1\cos\frac{\omega at}{a+c}-c_2\sin\frac{\omega at}{a+c}) \\
&\quad +2\sin\frac{\omega x}{a-c}(-c_3\cos\frac{\omega at}{a-c}-c_4\sin\frac{\omega at}{a-c}) \\
&\quad +\cos\frac{\omega}{a+c}(x-at). \qquad (2.3.42b)
\end{aligned}
$$

Now, as in previous cases, we let $\overrightarrow{u}=\overrightarrow{u}^*+\overrightarrow{u}^{**}$, where

$$
\begin{aligned}
\overrightarrow{u}^*(x,t) &= \begin{bmatrix} u_1^*(x,t) \\ u_2^*(x,t) \end{bmatrix} \\
&= 2\sin\frac{\omega x}{a+c}\begin{bmatrix} -c_1\sin\frac{\omega at}{a+c}+c_2\cos\frac{\omega at}{a+c} \\ -c_1\cos\frac{\omega at}{a+c}-c_2\sin\frac{\omega at}{a+c} \end{bmatrix} \\
&\quad +\begin{bmatrix} -\sin\frac{\omega}{a+c}(x-at) \\ \cos\frac{\omega}{a+c}(x-at) \end{bmatrix}, \qquad (2.3.43)
\end{aligned}
$$

and

$$
\begin{aligned}
\overrightarrow{u}^{**}(x,t) &= \begin{bmatrix} u_1^{**}(x,t) \\ u_2^{**}(x,t) \end{bmatrix} \\
&= 2\sin\frac{\omega x}{a-c}\begin{bmatrix} -c_3\sin\frac{\omega at}{a-c}+c_4\cos\frac{\omega at}{a-c} \\ -c_3\cos\frac{\omega at}{a-c}-c_4\sin\frac{\omega at}{a-c} \end{bmatrix}, \qquad (2.3.44)
\end{aligned}
$$

so that we have

$$\overrightarrow{u}^*(x,0) = \begin{bmatrix} u_1^*(x,0) \\ u_2^*(x,0) \end{bmatrix} = 2\sin\frac{\omega x}{a+c}\begin{bmatrix} c_2 \\ -c_1 \end{bmatrix} + \begin{bmatrix} -\sin\frac{\omega x}{a+c} \\ \cos\frac{\omega x}{a+c} \end{bmatrix}, \tag{2.3.45}$$

and

$$\overrightarrow{u}^{**}(x,0) = \begin{bmatrix} u_1^{**}(x,0) \\ u_2^{**}(x,0) \end{bmatrix} = 2\sin\frac{\omega x}{a-c}\begin{bmatrix} c_4 \\ -c_3 \end{bmatrix}. \tag{2.3.46}$$

Again, we note that $\overrightarrow{u}^*(x,t)$ and $\overrightarrow{u}^{**}(x,t)$ can be expressed as

$$\overrightarrow{u}^*(x,t) = \begin{bmatrix} \cos\frac{\omega at}{a+c} & \sin\frac{\omega at}{a+c} \\ -\sin\frac{\omega at}{a+c} & \cos\frac{\omega at}{a+c} \end{bmatrix}\overrightarrow{u}^*(x,0), \tag{2.3.47}$$

and

$$\overrightarrow{u}^{**}(x,t) = \begin{bmatrix} \cos\frac{\omega at}{a-c} & \sin\frac{\omega at}{a-c} \\ -\sin\frac{\omega at}{a-c} & \cos\frac{\omega at}{a-c} \end{bmatrix}\overrightarrow{u}^{**}(x,0). \tag{2.3.48}$$

The formulae in (2.3.43-48) indicate that $\overrightarrow{u}^*(x,t)$ represents a uniformly rotating *non-planar* standing wave (the nodes of $\overrightarrow{u}^*(x,t)$ can be easily identified), with the angular velocity $-\frac{\omega a}{a+c}$, while $\overrightarrow{u}^{**}(x,t)$ represents a uniformly rotating *planar* wave, with the angular velocity $-\frac{\omega a}{a-c}$. Finally, since $\overrightarrow{u}(x,t) = \overrightarrow{u}^*(x,t) + \overrightarrow{u}^{**}(x,t)$, the solution (2.3.42) represents the sum of two uniformly rotating standing waves with different angular velocities, only one of which is necessarily a planar standing wave. Again, these rotating standing waves may rotate in the same or opposite directions, depending on the signs of their respective angular speeds.

2.4 Examples of Standing Waves in the Case when a=c

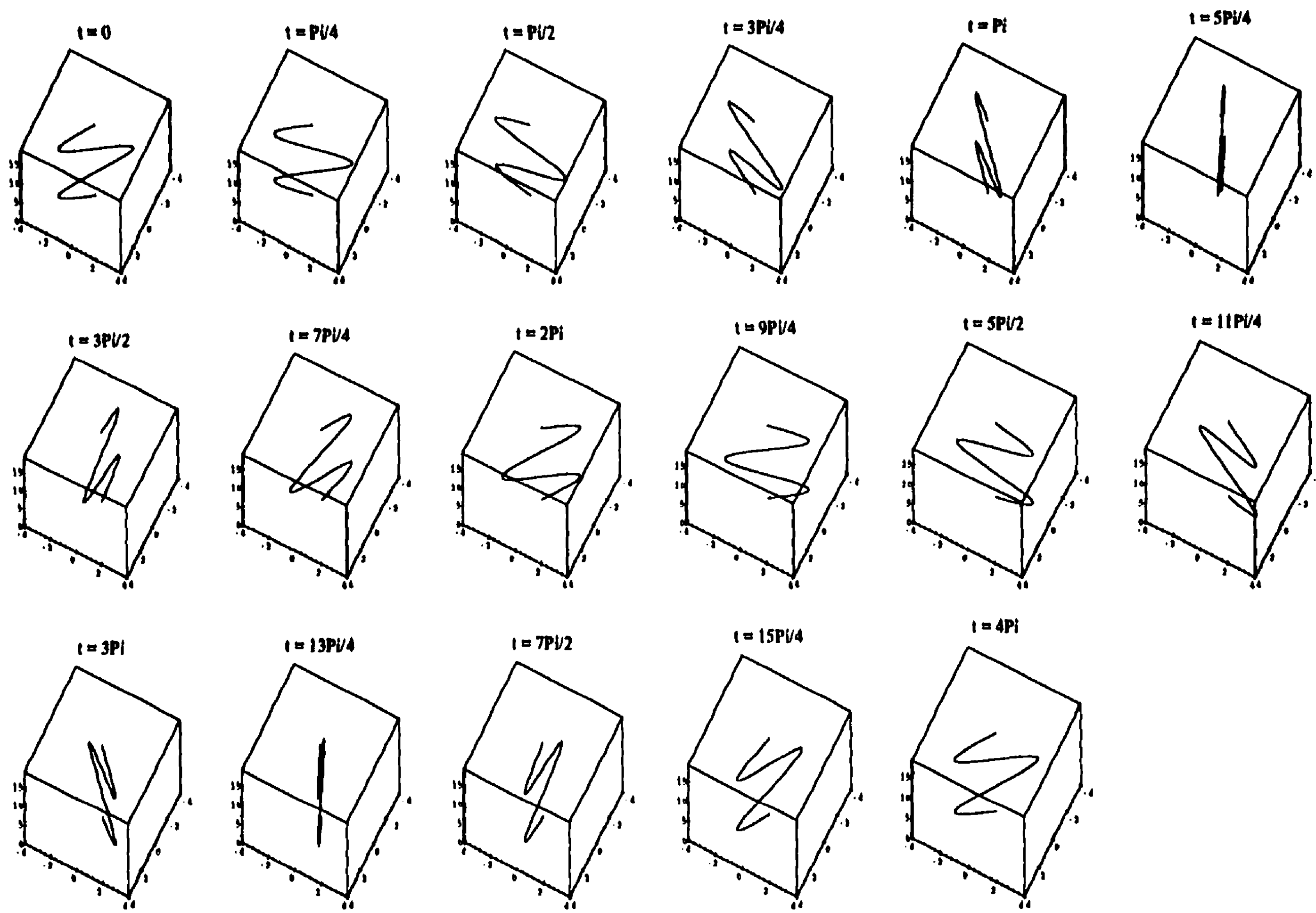
In this section, we will give two illustrative examples for the standing wave solutions (2.3.26) and (2.3.29) in the case when $a = c$, corresponding to homogeneous or nonhomogeneous boundary conditions, as discussed in section 2.3. Then we will display some graphs for each of these two solutions for illustration.

Example 2.4.1: $u_1(0,t) = u_2(0,t) = 0$, and we let $a = c = \omega = 1,\ c_1 = c_2 = 1$. Then the solution (2.3.26) becomes

$$\begin{aligned} u_1(x,t) &= \cos\frac{1}{2}(x+t) - \cos\frac{1}{2}(x-t) + \sin\frac{1}{2}(x+t) + \sin\frac{1}{2}(x-t) \\ &= 2\sin\frac{x}{2}(\cos\frac{t}{2} - \sin\frac{t}{2}), \qquad (2.4.1a) \\ u_2(x,t) &= -\sin\frac{1}{2}(x+t) - \sin\frac{1}{2}(x-t) + \cos\frac{1}{2}(x+t) - \cos\frac{1}{2}(x-t) \\ &= -2\sin\frac{x}{2}(\cos\frac{t}{2} + \sin\frac{t}{2}). \qquad (2.4.1b) \end{aligned}$$

A sequence of graphs illustrating the motion of the corresponding wave in (2.4.1) is presented in Figure 2.4.1. The time interval between "snapshots" is chosen to be $\frac{\pi}{4}$. The total time interval chosen is of duration 4π, and represents the time for the uniformly rotating planar standing wave to make one complete revolution.

Figure 2.4.1: graphic illustration of Example 2.4.1 where a=c=ω=1

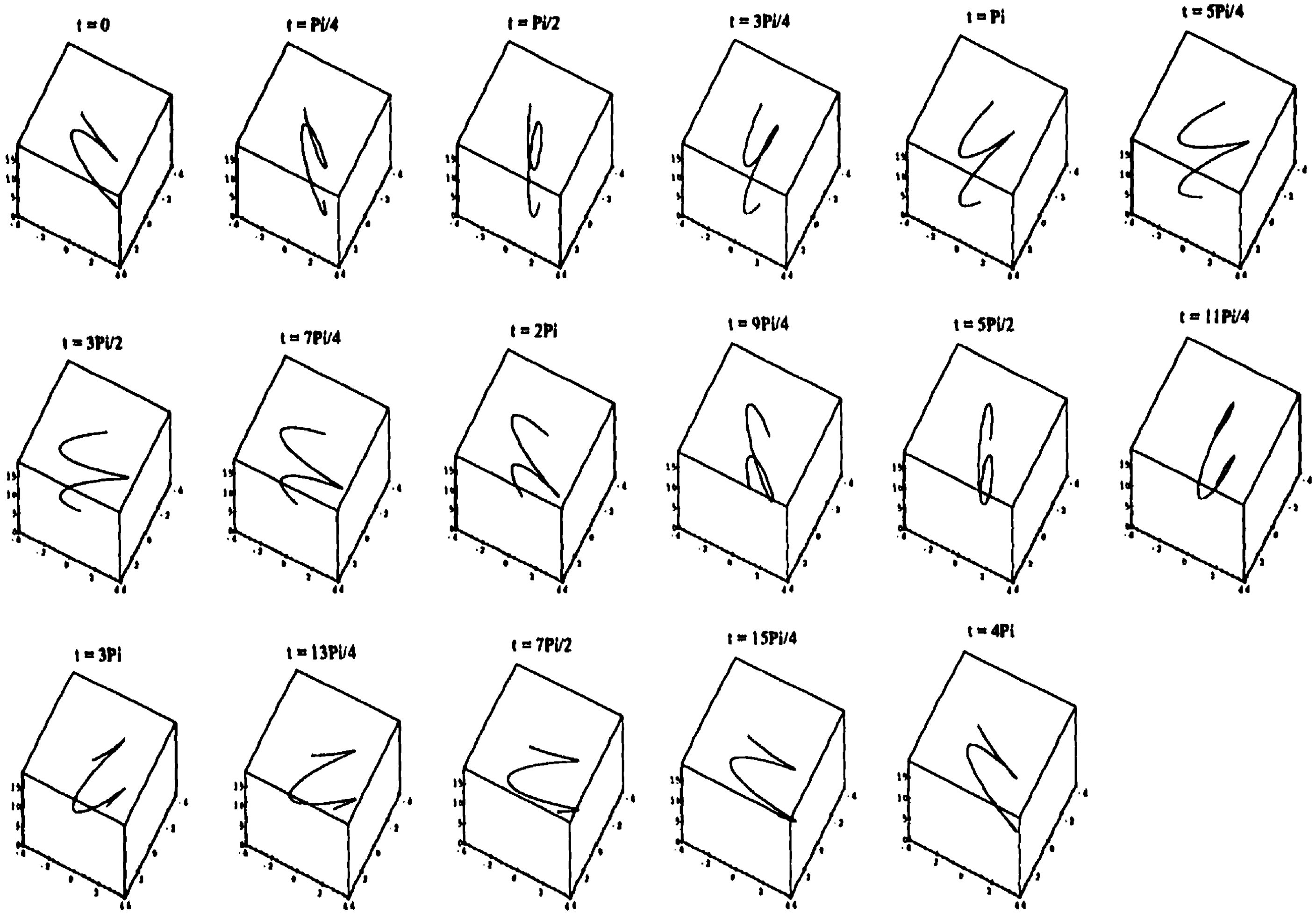


Example 2.4.2: $u_1(0,t) = \sin(kt)$, $u_2(0,t) = \cos(kt)$, and we let $a = c = \omega = 1$, $c_1 = -c_2 = -1$, so that $k = \frac{\omega}{2} = \frac{1}{2}$. Then the solution (2.3.29) becomes

$$\begin{aligned}
u_1(x,t) &= -\cos\frac{1}{2}(x+t) + \cos\frac{1}{2}(x-t) \\
&\quad + \sin\frac{1}{2}(x+t) + \sin\frac{1}{2}(x-t) - \sin\frac{1}{2}(x-t) \\
&= 2\sin\frac{x}{2}(\cos\frac{t}{2} + \sin\frac{t}{2}) - \sin\frac{1}{2}(x-t), \qquad (2.4.2a) \\
u_2(x,t) &= \sin\frac{1}{2}(x+t) + \sin\frac{1}{2}(x-t) \\
&\quad + \cos\frac{1}{2}(x+t) - \cos\frac{1}{2}(x-t) + \cos\frac{1}{2}(x-t) \\
&= 2\sin\frac{x}{2}(\cos\frac{t}{2} - \sin\frac{t}{2}) + \cos\frac{1}{2}(x-t). \qquad (2.4.2b)
\end{aligned}$$

A sequence of graphs describing the motion of the corresponding wave in (2.4.2) is presented in Figure 2.4.2, and other conditions remain as in the previous example. In these graphs it is evident that the uniformly rotating standing wave is *not planar*.

Figure 2.4.2: graphic illustration of Example 2.4.2 where a=c=ω=1



2.5 Examples of Standing Waves in the Case when a$\neq$c

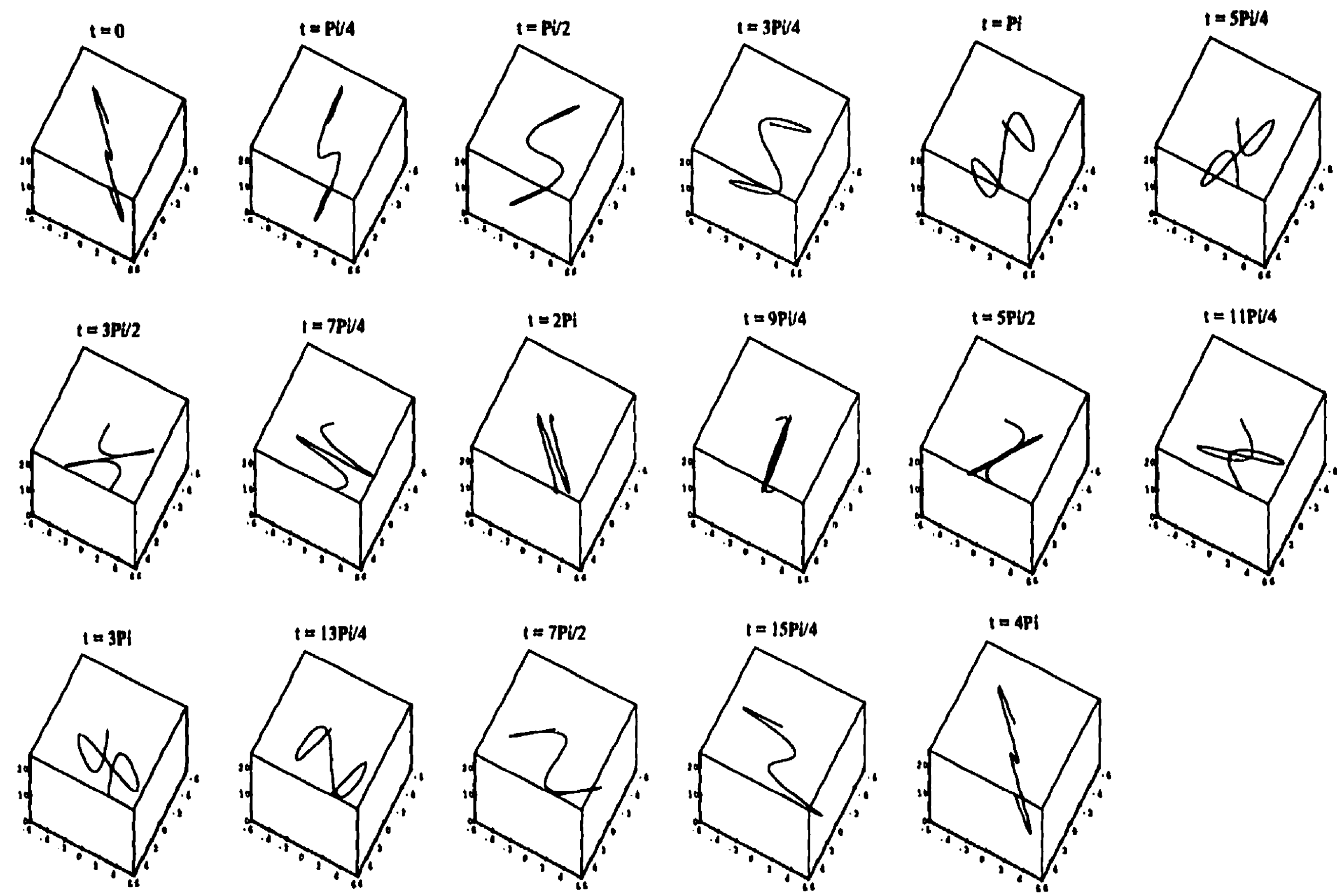
In this section, we will give two examples for the standing waves solutions (2.3.32) and (2.3.42) in the case when $a \neq c$, again corresponding to the homogeneous or nonhomogeneous boundary conditions of section 2.3. Illustrative sequences of graphs displaying the results are again given.

Example 2.5.1: $u_1(0,t) = u_2(0,t) = 0$, and we let $a = 2$, $c = \omega = \frac{2}{3}$, $c_1 = c_3 = -1$, $c_2 = c_4 = 1$. Then the solution (2.3.32) becomes

$$\begin{aligned}
u_1(x,t) &= -\cos\frac{1}{4}(x+2t) + \cos\frac{1}{4}(x-2t) + \sin\frac{1}{4}(x+2t) + \sin\frac{1}{4}(x-2t) \\
&\quad -\cos\frac{1}{2}(x+2t) + \cos\frac{1}{2}(x-2t) + \sin\frac{1}{2}(x+2t) + \sin\frac{1}{2}(x-2t) \\
&= 2\sin\frac{x}{4}(\cos\frac{t}{2} + \sin\frac{t}{2}) + 2\sin\frac{x}{2}(\cos t + \sin t), \qquad (2.5.1a) \\
u_2(x,t) &= \sin\frac{1}{4}(x+2t) + \sin\frac{1}{4}(x-2t) + \cos\frac{1}{4}(x+2t) - \cos\frac{1}{4}(x-2t) \\
&\quad +\sin\frac{1}{2}(x+2t) + \sin\frac{1}{2}(x-2t) + \cos\frac{1}{2}(x+2t) - \cos\frac{1}{2}(x-2t) \\
&= 2\sin\frac{x}{4}(\cos\frac{t}{2} - \sin\frac{t}{2}) + 2\sin\frac{x}{2}(\cos t - \sin t). \qquad (2.5.1b)
\end{aligned}$$

An illustrative sequence of graphs displaying the resulting motion is shown in Figure 2.5.1.

Figure 2.5.1: graphic illustration of Example 2.5.1 where a=2, c=ω=2/3

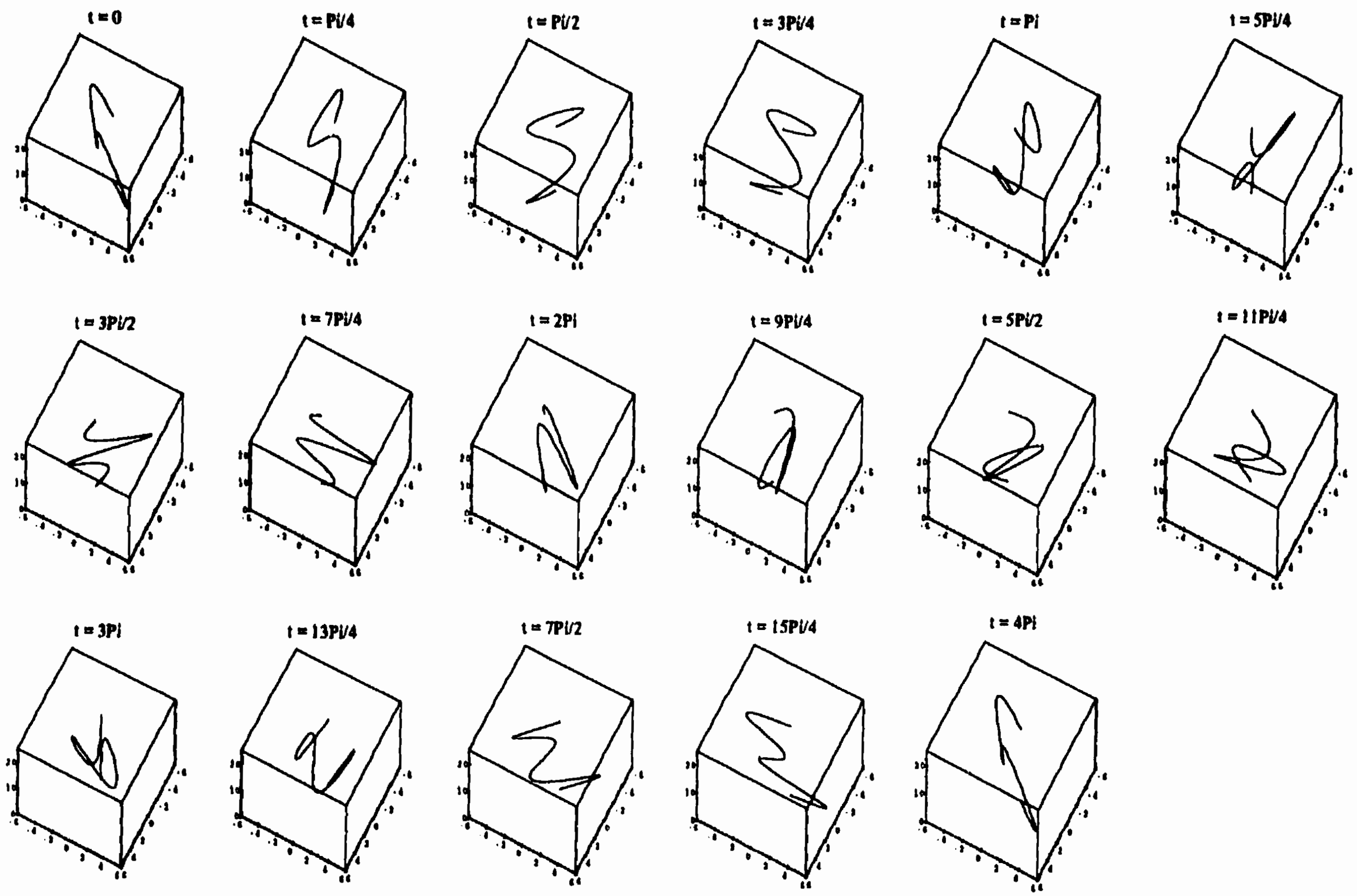


Example 2.5.2: $u_1(0,t) = \sin(kt)$, $u_2(0,t) = \cos(kt)$, and we let $a = 2$, $c = \omega = \frac{2}{3}$, $c_1 = c_3 = -1$, $c_2 = c_4 = 1$, so that $k = \frac{\omega a}{a+c} = \frac{1}{2}$. Then the solution (2.3.42) becomes

$$\begin{aligned}
u_1(x,t) &= -\cos\frac{1}{4}(x+2t) + \cos\frac{1}{4}(x-2t) + \sin\frac{1}{4}(x+2t) + \sin\frac{1}{4}(x-2t) \\
&\quad -\cos\frac{1}{2}(x+2t) + \cos\frac{1}{2}(x-2t) + \sin\frac{1}{2}(x+2t) + \sin\frac{1}{2}(x-2t) \\
&\quad -\sin\frac{1}{4}(x-2t) \\
&= 2\sin\frac{x}{4}(\cos\frac{t}{2} + \sin\frac{t}{2}) + 2\sin\frac{x}{2}(\cos t + \sin t) \\
&\quad -\sin\frac{1}{4}(x-2t), \qquad \text{(2.5.2a)} \\
u_2(x,t) &= \sin\frac{1}{4}(x+2t) + \sin\frac{1}{4}(x-2t) + \cos\frac{1}{4}(x+2t) - \cos\frac{1}{4}(x-2t) \\
&\quad +\sin\frac{1}{2}(x+2t) + \sin\frac{1}{2}(x-2t) + \cos\frac{1}{2}(x+2t) - \cos\frac{1}{2}(x-2t) \\
&\quad +\cos\frac{1}{4}(x-2t) \\
&= 2\sin\frac{x}{4}(\cos\frac{t}{2} - \sin\frac{t}{2}) + 2\sin\frac{x}{2}(\cos t - \sin t) \\
&\quad +\cos\frac{1}{4}(x-2t). \qquad \text{(2.5.2b)}
\end{aligned}$$

The resulting tether motion is displayed in Figure 2.5.2.

Figure 2.5.2: graphic illustration of Example 2.5.2 where a=2, c=ω=2/3



CHAPTER 3

The Complex Form of the Model

3.1 Introduction

In this chapter, we introduce a complex variable which enables us to write the coupled linear system (2.3.1) as a single partial differential equation (in a complex variable). In addition, by means of a second (complex) transformation, it is shown that this equation may be written in the form of the usual one-dimensional wave equation (in a complex variable). In subsequent chapters, this development allows us to construct solutions for both semi-infinite and (fixed or variable length) finite length strings, subject to general initial and boundary conditions.

3.2 Transformation of the Problem into Complex form

In the coupled system (2.3.1), namely,

$$\frac{\partial^2 u_1}{\partial t^2} - 2\omega\frac{\partial u_2}{\partial t} - \omega^2 u_1 = c^2\frac{\partial^2 u_1}{\partial x^2}, \tag{3.2.1a}$$

$$\frac{\partial^2 u_2}{\partial t^2} + 2\omega\frac{\partial u_1}{\partial t} - \omega^2 u_2 = c^2\frac{\partial^2 u_2}{\partial x^2}, \tag{3.2.1b}$$

$$t \geq 0,$$

we consider the (complex) transformation given by

$$V(x,t) = u_1(x,t) + iu_2(x,t), \tag{3.2.2}$$

where $i = \sqrt{-1}$. The system (3.2.1) is equivalent to the *single* equation

$$\frac{\partial^2 V}{\partial t^2} + 2\omega i\frac{\partial V}{\partial t} - \omega^2 V = c^2\frac{\partial^2 V}{\partial x^2}. \tag{3.2.3}$$

In order to write this equation in a more manageable form, we consider the effect of a second transformation of the form

$$W(x,t) = e^{(\alpha x+\beta t)}V(x,t),$$

or inversely,

$$V(x,t) = e^{-(\alpha x+\beta t)}W(x,t),$$

where α and β are unknown constants yet to be determined. Since

$$\begin{aligned}
\frac{\partial V}{\partial t} &= e^{-(\alpha x+\beta t)}(-\beta W + \frac{\partial W}{\partial t}), \\
\frac{\partial^2 V}{\partial t^2} &= e^{-(\alpha x+\beta t)}(\beta^2 W - 2\beta\frac{\partial W}{\partial t} + \frac{\partial^2 W}{\partial t^2}), \\
\frac{\partial V}{\partial x} &= e^{-(\alpha x+\beta t)}(-\alpha W + \frac{\partial W}{\partial x}) \\
\frac{\partial^2 V}{\partial x^2} &= e^{-(\alpha x+\beta t)}(\alpha^2 W - 2\alpha\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2}),
\end{aligned}$$

(3.2.3) becomes

$$\begin{aligned}
e^{-(\alpha x+\beta t)}\Bigg[&\frac{\partial^2 W}{\partial t^2} - c^2\frac{\partial^2 W}{\partial x^2} - 2\alpha c^2\frac{\partial W}{\partial x} \\
&+2(i\omega-\beta)\frac{\partial W}{\partial t} + (\beta^2 - 2i\omega\beta - \omega^2 - c^2\alpha^2)W\Bigg] = 0,
\end{aligned}$$

in which we desire to choose α and β in such a manner as to remove the terms involving the first order partial derivatives of $W(x,t)$. In particular, we now choose $\alpha = 0$ and $\beta = i\omega$, which not only removes the terms involving the first order partial derivatives of $W(x,t)$, but also removes the final term as well.

In summary, we make the transformation

$$W(x,t) = e^{i\omega t}V(x,t), \tag{3.2.4}$$

with inverse

$$V(x,t) = e^{-i\omega t}W(x,t), \tag{3.2.5}$$

so that

$$\begin{aligned}
\frac{\partial V}{\partial t} &= e^{-i\omega t}(-i\omega W + \frac{\partial W}{\partial t}), & (3.2.6a)\\
\frac{\partial^2 V}{\partial t^2} &= e^{-i\omega t}(-\omega^2 W - 2i\omega\frac{\partial W}{\partial t} + \frac{\partial^2 W}{\partial t^2}), & (3.2.6b)\\
\frac{\partial V}{\partial x} &= e^{-i\omega t}\frac{\partial W}{\partial x}, & (3.2.6c)\\
\frac{\partial^2 V}{\partial x^2} &= e^{-i\omega t}\frac{\partial^2 W}{\partial x^2}. & (3.2.6d)
\end{aligned}$$

Putting the above partial derivatives in the complex equation (3.2.3) reduces the latter simply to the one-dimensional wave equation

$$\frac{\partial^2 W}{\partial t^2} = c^2\frac{\partial^2 W}{\partial x^2}, \tag{3.2.7}$$

in the complex variable

$$W(x,t) = e^{i\omega t}[u_1(x,t) + iu_2(x,t)]. \tag{3.2.8}$$

The complex equation (3.2.7) is a one-dimensional wave equation, whose general solution is known to have the form

$$W(x,t) = F(\xi) + G(\eta), \tag{3.2.9}$$

where $\xi = x + ct$, $\eta = x - ct$, and $F(\xi)$ and $G(\eta)$ can be any twice differentiable complex functions, which are written in complex form as

$$\begin{aligned}
F(\xi) &= F_1(\xi) + iF_2(\xi), & (3.2.10a)\\
G(\eta) &= G_1(\eta) + iG_2(\eta), & (3.2.10b)
\end{aligned}$$

so that

$$W(x,t) = F_1(x+ct) + G_1(x-ct) + i[F_2(x+ct) + G_2(x-ct)]. \qquad (3.2.11)$$

For future reference, we may reverse the above process to obtain corresponding expressions in terms of the complex variable $V(x,t)$, or even the real variables $u_1(x,t)$ and $u_2(x,t)$. In particular, by virtue of Euler's identity, (3.2.5) may be written as

$$V(x,t) = [\cos(\omega t) - i\sin(\omega t)]W(x,t). \qquad (3.2.12)$$

Moreover, substitution of (3.2.11) into (3.2.12) gives

$$\begin{aligned} V(x,t) &= [\cos(\omega t) - i\sin(\omega t)] \cdot \\ &\quad \cdot \{F_1(x+ct) + G_1(x-ct) + i[F_2(x+ct) + G_2(x-ct)]\} \\ &= \cos(\omega t)[F_1(x+ct) + G_1(x-ct)] \\ &\quad + \sin(\omega t)[F_2(x+ct) + G_2(x-ct)] \\ &\quad + i\{-\sin(\omega t)[F_1(x+ct) + G_1(x-ct)] \\ &\quad + \cos(\omega t)[F_2(x+ct) + G_2(x-ct)]\}. \end{aligned}$$

Finally, with the aid of (3.2.2), we may obtain the general solution of the coupled system (3.2.1), as given by

$$\begin{aligned} u_1(x,t) &= \cos(\omega t)[F_1(x+ct) + G_1(x-ct)] \\ &\quad + \sin(\omega t)[F_2(x+ct) + G_2(x-ct)], \qquad (3.2.13a) \\ u_2(x,t) &= -\sin(\omega t)[F_1(x+ct) + G_1(x-ct)] \\ &\quad + \cos(\omega t)[F_2(x+ct) + G_2(x-ct)], \qquad (3.2.13b) \end{aligned}$$

where $F_1(\xi), F_2(\xi), G_1(\eta)$ and $G_2(\eta)$ are any twice differentiable real functions.

In subsequent chapters we shall investigate these solutions in the cases of semi-infinite or finite strings with either fixed or moving endpoints.

CHAPTER 4

Solutions for the Semi-infinite String

4.1 Introduction

In this chapter, we consider the coupled system (3.2.1) in the case of a semi-infinite string, by incorporating appropriate boundary and initial conditions. Based on the discussion in the previous chapter, we transform the system into a single one-dimensional (complex) wave equation, as shown in (3.2.6), with corresponding complex boundary and initial conditions. The latter problem is then solved using the usual d'Alembert form of the solution of the wave equation. The inverse transformations are finally applied to determine the solution of the original problem. Graphical illustrations of the solution are presented.

4.2 Transformation of Initial and Boundary Conditions

Throughout this chapter we assume that the coupled equations

$$\frac{\partial^2 u_1}{\partial t^2} - 2\omega\frac{\partial u_2}{\partial t} - \omega^2 u_1 = c^2\frac{\partial^2 u_1}{\partial x^2}, \tag{4.2.1a}$$

$$\frac{\partial^2 u_2}{\partial t^2} + 2\omega\frac{\partial u_1}{\partial t} - \omega^2 u_2 = c^2\frac{\partial^2 u_2}{\partial x^2}, \tag{4.2.1b}$$

$$0 \leq x < \infty, \quad t \geq 0,$$

have the following prescribed initial conditions

$$u_1(x,0) = f_1(x), \quad \frac{\partial u_1}{\partial t}(x,0) = g_1(x), \tag{4.2.2a}$$

$$u_2(x,0) = f_2(x), \;\; \frac{\partial u_2}{\partial t}(x,0) = g_2(x), \tag{4.2.2b}$$

$$x \geq 0,$$

and the following prescribed boundary conditions at $x = 0$

$$u_1(0,t) = p_1(t), \;\; u_2(0,t) = p_2(t), \;\; t \geq 0. \tag{4.2.2c}$$

Recall that $V = u_1 + iu_2$ and $W = e^{i\omega t}V$, as introduced in Chapter 3. By virtue of (3.2.2), we have

$$\begin{aligned} V(x,0) &= f_1(x) + if_2(x), \\ \frac{\partial V}{\partial t}(x,0) &= g_1(x) + ig_2(x), \\ V(0,t) &= p_1(t) + ip_2(t), \end{aligned}$$

so that, in light of (3.2.4) and (3.2.6), the initial and boundary conditions of the transformed complex equation

$$\frac{\partial^2 W}{\partial t^2} = c^2 \frac{\partial^2 W}{\partial x^2}, \;\; 0 \leq x < \infty, \;\; t \geq 0, \tag{4.2.3}$$

are respectively

$$W(x,0) = V(x,0) = f_1(x) + if_2(x),$$

$$\begin{aligned} \frac{\partial W}{\partial t}(x,0) &= \omega i V(x,0) + \frac{\partial V}{\partial t}(x,0) \\ &= \omega i[f_1(x) + if_2(x)] + g_1(x) + ig_2(x) \\ &= -\omega f_2(x) + g_1(x) + i[\omega f_1(x) + g_2(x)], \end{aligned}$$

$$W(0,t) = e^{i\omega t}V(0,t) = e^{i\omega t}[p_1(t) + ip_2(t)]$$

$$
\begin{aligned}
&= [\cos(\omega t) + i\sin(\omega t)][p_1(t) + ip_2(t)] \\
&= p_1(t)\cos(\omega t) - p_2(t)\sin(\omega t) \\
&\quad +i[p_1(t)\sin(\omega t) + p_2(t)\cos(\omega t)].
\end{aligned}
$$

We denote $W(x,0)$, $\frac{\partial W}{\partial t}(x,0)$ and $W(0,t)$ by $f(x)$, $g(x)$ and $p(t)$ respectively, i.e.,

$$W(x,0) = f(x), \;\; \frac{\partial W}{\partial t}(x,0) = g(x), \;\; W(0,t) = p(t), \tag{4.2.4}$$

where, for simplicity of notation, we have adopted the following definitions

$$f(x) = f_1(x) + if_2(x), \tag{4.2.5a}$$

$$g(x) = -\omega f_2(x) + g_1(x) + i[\omega f_1(x) + g_2(x)], \tag{4.2.5b}$$

$$
\begin{aligned}
p(t) &= p_1(t)\cos(\omega t) - p_2(t)\sin(\omega t) \\
&\quad +i[p_1(t)\sin(\omega t) + p_2(t)\cos(\omega t)].
\end{aligned} \tag{4.2.5c}
$$

According to equations (2.2.6), the one-dimensional wave equation (4.2.3), subject to the initial and boundary conditions (4.2.4), is known to have the following solution

$$W(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds; \tag{4.2.6}$$

$$\text{for } \; x > ct \geq 0,$$

and

$$W(x,t) = \frac{1}{2}[f(ct+x) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)ds + p(t-\frac{x}{c}), \tag{4.2.7}$$

$$\text{for } \; 0 \leq x < ct.$$

Upon substitution of (4.2.5) into (4.2.6) and (4.2.7), we have

$$\begin{aligned} W(x,t) &= \frac{1}{2}[f_1(x+ct)+if_2(x+ct)+f_1(x-ct)+if_2(x-ct)] \\ &\quad +\frac{1}{2c}\int_{x-ct}^{x+ct}\{-\omega f_2(s)+g_1(s)+i[\omega f_1(s)+g_2(s)]\}ds \\ &= \frac{1}{2}[f_1(x+ct)+f_1(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}[-\omega f_2(s)+g_1(s)]ds \\ &\quad +i\{\frac{1}{2}[f_2(x+ct)+f_2(x-ct)] \\ &\quad +\frac{1}{2c}\int_{x-ct}^{x+ct}[\omega f_1(s)+g_2(s)]ds\}, \end{aligned} \tag{4.2.8}$$

$$\text{for } x > ct \geq 0,$$

and

$$\begin{aligned} W(x,t) &= \frac{1}{2}[f_1(ct+x)+if_2(ct+x)-f_1(ct-x)-if_2(ct-x)] \\ &\quad +\frac{1}{2c}\int_{ct-x}^{ct+x}\{-\omega f_2(s)+g_1(s)+i[\omega f_1(s)+g_2(s)]\}ds \\ &\quad +p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})-p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c}) \\ &\quad +i[p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})+p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})] \\ &= \frac{1}{2}[f_1(ct+x)-f_1(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[-\omega f_2(s)+g_1(s)]ds \\ &\quad +p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})-p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c}) \\ &\quad +i\{\frac{1}{2}[f_2(ct+x)-f_2(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[\omega f_1(s)+g_2(s)]ds \\ &\quad +p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})+p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})\}, \end{aligned} \tag{4.2.9}$$

$$\text{for } 0 \leq x < ct.$$

Based on the compatability conditions given in Chapter 2, we can deduce that the continuity or differentiability of the solution in (4.2.8) and (4.2.9) depends on the properties of the functions $f_k(x)$, $g_k(x)$ and $p_k(t)$, $(k = 1, 2)$, as shown below:

1) If $f_k(x) \in C^0$, $p_k(t) \in C^0$, $g_k(x)$ are integrable and $f_k(0) = p_k(0)$, $(k = 1, 2)$, then $W(x,t) \in C^0$.

2) If $f_k(x) \in C^1$, $p_k(t) \in C^1$, $g_k(x) \in C^0$, $f_k(0) = p_k(0)$ and $g_k(0) = p'_k(0)$, $(k = 1, 2)$, then $W(x,t) \in C^1$.

Comparing (3.2.9-10) with (4.2.8) or (4.2.9), we have

$$\begin{aligned} F_1(x+ct) + G_1(x-ct) &= \frac{1}{2}[f_1(x+ct) + f_1(x-ct)] \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} [-\omega f_2(s) + g_1(s)]ds, \qquad (4.2.10) \\ F_2(x+ct) + G_2(x-ct) &= \frac{1}{2}[f_2(x+ct) + f_2(x-ct)] \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} [\omega f_1(s) + g_2(s)]ds, \qquad (4.2.11) \end{aligned}$$

$$\text{for } x > ct \geq 0,$$

and

$$\begin{aligned} F_1(x+ct) + G_1(x-ct) &= \frac{1}{2}[f_1(ct+x) - f_1(ct-x)] \\ &\quad + \frac{1}{2c}\int_{ct-x}^{ct+x} [-\omega f_2(s) + g_1(s)]ds \\ &\quad + p_1(t - \frac{x}{c}) \cos\omega(t - \frac{x}{c}) \\ &\quad - p_2(t - \frac{x}{c}) \sin\omega(t - \frac{x}{c}), \qquad (4.2.12) \\ F_2(x+ct) + G_2(x-ct) &= \frac{1}{2}[f_2(ct+x) - f_2(ct-x)] \\ &\quad + \frac{1}{2c}\int_{ct-x}^{ct+x} [\omega f_1(s) + g_2(s)]ds \\ &\quad + p_1(t - \frac{x}{c}) \sin\omega(t - \frac{x}{c}) \\ &\quad + p_2(t - \frac{x}{c}) \cos\omega(t - \frac{x}{c}), \qquad (4.2.13) \end{aligned}$$

$$\text{for } 0 \leq x < ct,$$

so that by virtue of (3.2.12) and (4.2.10-13) we may write the solution of the semi-infinite problem (4.2.1) and (4.2.2) as

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}\cos(\omega t)\{f_1(x+ct)+f_1(x-ct) \\ &\quad +\frac{1}{c}\int_{x-ct}^{x+ct}[-\omega f_2(s)+g_1(s)]ds\} \\ &\quad +\frac{1}{2}\sin(\omega t)\{f_2(x+ct)+f_2(x-ct) \\ &\quad +\frac{1}{c}\int_{x-ct}^{x+ct}[\omega f_1(s)+g_2(s)]ds\}, \qquad (4.2.14a) \\ u_2(x,t) &= -\frac{1}{2}\sin(\omega t)\{f_1(x+ct)+f_1(x-ct) \\ &\quad +\frac{1}{c}\int_{x-ct}^{x+ct}[-\omega f_2(s)+g_1(s)]ds\} \\ &\quad +\frac{1}{2}\cos(\omega t)\{f_2(x+ct)+f_2(x-ct) \\ &\quad +\frac{1}{c}\int_{x-ct}^{x+ct}[\omega f_1(s)+g_2(s)]ds\}, \qquad (4.2.14b) \end{aligned}$$

$$\text{for } x > ct \geq 0,$$

and

$$\begin{aligned} u_1(x,t) &= \cos(\omega t)\{\frac{1}{2}[f_1(ct+x)-f_1(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[-\omega f_2(s)+g_1(s)]ds \\ &\quad +p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})-p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})\} \\ &\quad +\sin(\omega t)\{\frac{1}{2}[f_2(ct+x)-f_2(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[\omega f_1(s)+g_2(s)]ds \\ &\quad +p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})+p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})\} \\ &= \cos(\omega t)\{\frac{1}{2}[f_1(ct+x)-f_1(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[-\omega f_2(s)+g_1(s)]ds\} \\ &\quad +\sin(\omega t)\{\frac{1}{2}[f_2(ct+x)-f_2(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[\omega f_1(s)+g_2(s)]ds\} \\ &\quad +p_1(t-\frac{x}{c})\cos[\omega t-\omega(t-\frac{x}{c})]+p_2(t-\frac{x}{c})\sin[\omega t-\omega(t-\frac{x}{c})] \\ &= \frac{1}{2}\cos(\omega t)\{f_1(ct+x)-f_1(ct-x)+\frac{1}{c}\int_{ct-x}^{ct+x}[-\omega f_2(s)+g_1(s)]ds\} \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\sin(\omega t)\{f_2(ct+x)-f_2(ct-x)+\frac{1}{c}\int_{ct-x}^{ct+x}[\omega f_1(s)+g_2(s)]ds\} \\
& +p_1(t-\frac{x}{c})\cos(\frac{\omega x}{c})+p_2(t-\frac{x}{c})\sin(\frac{\omega x}{c}), \qquad (4.2.15a)
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,t) \quad = \quad & -\sin(\omega t)\{\frac{1}{2}[f_1(ct+x)-f_1(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[-\omega f_2(s)+g_1(s)]ds \\
& +p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})-p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})\} \\
& +\cos(\omega t)\{\frac{1}{2}[f_2(ct+x)-f_2(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[\omega f_1(s)+g_2(s)]ds \\
& +p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})+p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})\} \\
= \quad & -\sin(\omega t)\{\frac{1}{2}[f_1(ct+x)-f_1(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[-\omega f_2(s)+g_1(s)]ds\} \\
& +\cos(\omega t)\{\frac{1}{2}[f_2(ct+x)-f_2(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}[\omega f_1(s)+g_2(s)]ds\} \\
& -p_1(t-\frac{x}{c})\sin[\omega t-\omega(t-\frac{x}{c})]+p_2(t-\frac{x}{c})\cos[\omega t-\omega(t-\frac{x}{c})] \\
= \quad & -\frac{1}{2}\sin(\omega t)\{f_1(ct+x)-f_1(ct-x)+\frac{1}{c}\int_{ct-x}^{ct+x}[-\omega f_2(s)+g_1(s)]ds\} \\
& +\frac{1}{2}\cos(\omega t)\{f_2(ct+x)-f_2(ct-x)+\frac{1}{c}\int_{ct-x}^{ct+x}[\omega f_1(s)+g_2(s)]ds\} \\
& -p_1(t-\frac{x}{c})\sin(\frac{\omega x}{c})+p_2(t-\frac{x}{c})\cos(\frac{\omega x}{c}), \qquad (4.2.15b)
\end{aligned}
$$

for $0 \le x < ct$.

In summary, equations (4.2.15) exhibit the "general" solution of the system (4.2.1) subject to the general initial conditions (4.2.2a,b) and general boundary conditions (4.2.2c) for a semi-infinite spinning string. The complicated nature of these expressions makes it difficult to explain the physical significance of this solution. Thus, in an effort to provide a physical understanding of this solution, we shall consider in the next section various illustrative examples, each of which may be provided a precise physical description.

4.3 Examples

Example 4.3.1: Our first example involves a derivation of the travelling wave solution discussed previously in example 2.4.1. In particular, we suppose that

$$c = \omega = 1,$$

and we consider a semi-infinite spinning string, with a fixed end at $x = 0$ (i.e., with boundary conditions $u_1(0,t) = p_1(t) = 0$, $u_2(0,t) = p_2(t) = 0$) and with initial conditions given by

$$\begin{aligned} u_1(x,0) &= f_1(x) = 2\sin\frac{x}{2}, \\ u_2(x,0) &= f_2(x) = -2\sin\frac{x}{2}, \\ \frac{\partial u_1}{\partial t}(x,0) &= g_1(x) = -\sin\frac{x}{2}, \\ \frac{\partial u_2}{\partial t}(x,0) &= g_2(x) = -\sin\frac{x}{2}. \end{aligned}$$

It should be noted that the last four conditions may be deduced from example 2.4.1.

In summary, the problem is to solve the system

$$\frac{\partial^2 u_1}{\partial t^2} - 2\frac{\partial u_2}{\partial t} - u_1 = \frac{\partial^2 u_1}{\partial x^2}, \tag{4.3.1a}$$

$$\frac{\partial^2 u_2}{\partial t^2} + 2\frac{\partial u_1}{\partial t} - u_2 = \frac{\partial^2 u_2}{\partial x^2}, \tag{4.3.1b}$$

$$0 \leq x < \infty, \ t \geq 0,$$

subject to the initial conditions

$$u_1(x,0) = 2\sin\frac{x}{2}, \quad \frac{\partial u_1}{\partial t}(x,0) = -\sin\frac{x}{2}, \tag{4.3.2a}$$

$$u_2(x,0) = -2\sin\frac{x}{2}, \quad \frac{\partial u_2}{\partial t}(x,0) = -\sin\frac{x}{2}, \tag{4.3.2b}$$

$$x \geq 0,$$

with "fixed end" boundary conditions at $x = 0$, i.e.,

$$u_1(0,t) = u_2(0,t) = 0, \; t \geq 0. \tag{4.3.2c}$$

According to (4.2.14-15), the solution of the problem (4.3.1-2) is given by

$$\begin{aligned}
u_1(x,t) &= \frac{1}{2}\cos t[2\sin\frac{1}{2}(x+t) + 2\sin\frac{1}{2}(x-t) + \int_{x-t}^{x+t}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&\quad +\frac{1}{2}\sin t[-2\sin\frac{1}{2}(x+t) - 2\sin\frac{1}{2}(x-t) + \int_{x-t}^{x+t}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&= 2\cos t\sin\frac{x}{2}\cos\frac{t}{2} - 2\sin t\sin\frac{x}{2}\cos\frac{t}{2} + \frac{1}{2}(\cos t + \sin t)\int_{x-t}^{x+t}\sin\frac{s}{2}ds \\
&= 2\sin\frac{x}{2}\cos\frac{t}{2}(\cos t - \sin t) + (\cos t + \sin t)[\cos\frac{1}{2}(x-t) - \cos\frac{1}{2}(x+t)] \\
&= 2\sin\frac{x}{2}\cos\frac{t}{2}(\cos t - \sin t) + 2\sin\frac{x}{2}\sin\frac{t}{2}(\cos t + \sin t) \\
&= 2\sin\frac{x}{2}[\cos(t - \frac{t}{2}) - \sin(t - \frac{t}{2})] \\
&= 2\sin\frac{x}{2}(\cos\frac{t}{2} - \sin\frac{t}{2}), \\
u_2(x,t) &= -\frac{1}{2}\sin t[2\sin\frac{1}{2}(x+t) + 2\sin\frac{1}{2}(x-t) + \int_{x-t}^{x+t}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&\quad +\frac{1}{2}\cos t[-2\sin\frac{1}{2}(x+t) - 2\sin\frac{1}{2}(x-t) + \int_{x-t}^{x+t}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&= -2\sin t\sin\frac{x}{2}\cos\frac{t}{2} - 2\cos t\sin\frac{x}{2}\cos\frac{t}{2} + \frac{1}{2}(\cos t - \sin t)\int_{x-t}^{x+t}\sin\frac{s}{2}ds \\
&= -2\sin\frac{x}{2}\cos\frac{t}{2}(\cos t + \sin t) + (\cos t - \sin t)[\cos\frac{1}{2}(x-t) - \cos\frac{1}{2}(x+t)] \\
&= -2\sin\frac{x}{2}\cos\frac{t}{2}(\cos t + \sin t) + 2\sin\frac{x}{2}\sin\frac{t}{2}(\cos t - \sin t) \\
&= -2\sin\frac{x}{2}[\cos(t - \frac{t}{2}) + \sin(t - \frac{t}{2})] \\
&= -2\sin\frac{x}{2}(\cos\frac{t}{2} + \sin\frac{t}{2}),
\end{aligned}$$

$$\text{for} \;\; x > t \geq 0,$$

and

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{2}\cos t[2\sin\frac{1}{2}(t+x) - 2\sin\frac{1}{2}(t-x) + \int_{t-x}^{t+x}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&\quad + \frac{1}{2}\sin t[-2\sin\frac{1}{2}(t+x) + 2\sin\frac{1}{2}(t-x) + \int_{t-x}^{t+x}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&= 2\cos t\cos\frac{t}{2}\sin\frac{x}{2} - 2\sin t\cos\frac{t}{2}\sin\frac{x}{2} + \frac{1}{2}(\cos t + \sin t)\int_{t-x}^{t+x}\sin\frac{s}{2}ds \\
&= 2\cos\frac{t}{2}\sin\frac{x}{2}(\cos t - \sin t) - (\cos t + \sin t)[\cos\frac{1}{2}(t+x) - \cos\frac{1}{2}(t-x)] \\
&= 2\cos\frac{t}{2}\sin\frac{x}{2}(\cos t - \sin t) + 2\sin\frac{t}{2}\sin\frac{x}{2}(\cos t + \sin t) \\
&= 2\sin\frac{x}{2}[\cos(t-\frac{t}{2}) - \sin(t-\frac{t}{2})] \\
&= 2\sin\frac{x}{2}(\cos\frac{t}{2} - \sin\frac{t}{2}), \\
u_2(x,t) &= -\frac{1}{2}\sin t[2\sin\frac{1}{2}(t+x) - 2\sin\frac{1}{2}(t-x) + \int_{t-x}^{t+x}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&\quad + \frac{1}{2}\cos t[-2\sin\frac{1}{2}(t+x) + 2\sin\frac{1}{2}(t-x) + \int_{t-x}^{t+x}(2\sin\frac{s}{2} - \sin\frac{s}{2})ds] \\
&= -2\sin t\cos\frac{t}{2}\sin\frac{x}{2} - 2\cos t\cos\frac{t}{2}\sin\frac{x}{2} + \frac{1}{2}(\cos t - \sin t)\int_{t-x}^{t+x}\sin\frac{s}{2}ds \\
&= -2\cos\frac{t}{2}\sin\frac{x}{2}(\cos t + \sin t) - (\cos t - \sin t)[\cos\frac{1}{2}(t+x) - \cos\frac{1}{2}(t-x)] \\
&= -2\cos\frac{t}{2}\sin\frac{x}{2}(\cos t + \sin t) + 2\sin\frac{t}{2}\sin\frac{x}{2}(\cos t - \sin t) \\
&= -2\sin\frac{x}{2}[\cos(t-\frac{t}{2}) + \sin(t-\frac{t}{2})] \\
&= -2\sin\frac{x}{2}(\cos\frac{t}{2} + \sin\frac{t}{2}),
\end{aligned}
$$

$$\text{for} \quad 0 \le x < t.$$

It should be noted that these expressions for $u_1(x,t)$ and $u_2(x,t)$ are precisely the same as those exhibited in the travelling wave solution of example 2.4.1. Thus, the graphs illustrating the motion of the corresponding wave of this example are the same as those presented in Figure 2.4.1.

Example 4.3.2: Our second example involves a derivation of the travelling wave solution discussed previously in example 2.5.2. We suppose that

$$c = \omega = \frac{2}{3},$$

and we consider a semi-infinite spinning string, with a uniformly rotating end of constant (unit) displacement at $x = 0$ revolving at the angular speed $\frac{1}{2}$, i.e., with boundary conditions

$$u_1(0,t) = p_1(t) = \sin\frac{t}{2}, \tag{4.3.3a}$$

$$u_2(0,t) = p_2(t) = \cos\frac{t}{2}, \tag{4.3.3b}$$

and with initial conditions given by

$$u_1(x,0) = f_1(x) = \sin\frac{x}{4} + 2\sin\frac{x}{2}, \tag{4.3.4a}$$

$$u_2(x,0) = f_2(x) = 2\sin\frac{x}{4} + 2\sin\frac{x}{2} + \cos\frac{x}{4}, \tag{4.3.4b}$$

$$\frac{\partial u_1}{\partial t}(x,0) = g_1(x) = \sin\frac{x}{4} + 2\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{4}, \tag{4.3.4c}$$

$$\frac{\partial u_2}{\partial t}(x,0) = g_2(x) = -\frac{1}{2}\sin\frac{x}{4} - 2\sin\frac{x}{2}. \tag{4.3.4d}$$

It is noted that the last four conditions may be deduced from example 2.5.2.

According to (4.2.14-15), the solution of the problem (4.2.1) with $c = \omega = \frac{2}{3}$, subject to (4.3.3-4), is given by

$$\begin{aligned} u_1(x,t) \;\; = \;\; & \frac{1}{2}\cos\frac{2t}{3}[\sin\frac{1}{4}(x+\frac{2t}{3}) + 2\sin\frac{1}{2}(x+\frac{2t}{3}) + \sin\frac{1}{4}(x-\frac{2t}{3}) \\ & +2\sin\frac{1}{2}(x-\frac{2t}{3}) + \frac{3}{2}\int_{x-\frac{2}{3}t}^{x+\frac{2}{3}t}(-\frac{1}{3}\sin\frac{s}{4} + \frac{2}{3}\sin\frac{s}{2} - \frac{1}{6}\cos\frac{s}{4})ds] \\ & +\frac{1}{2}\sin\frac{2t}{3}[2\sin\frac{1}{4}(x+\frac{2t}{3}) + 2\sin\frac{1}{2}(x+\frac{2t}{3}) + \cos\frac{1}{4}(x+\frac{2t}{3}) \end{aligned}$$

$$
\begin{aligned}
& +2\sin\frac{1}{4}(x-\frac{2t}{3})+2\sin\frac{1}{2}(x-\frac{2t}{3})+\cos\frac{1}{4}(x-\frac{2t}{3}) \\
& +\frac{3}{2}\int_{x-\frac{2}{3}t}^{x+\frac{2}{3}t}(\frac{1}{6}\sin\frac{s}{4}-\frac{2}{3}\sin\frac{s}{2})ds] \\
= \; & \frac{1}{2}\cos\frac{2t}{3}\{2\sin\frac{x}{4}\cos\frac{t}{6}+4\sin\frac{x}{2}\cos\frac{t}{3}+2[\cos\frac{1}{4}(x+\frac{2t}{3}) \\
& -\cos\frac{1}{4}(x-\frac{2t}{3})]-2[\cos\frac{1}{2}(x+\frac{2t}{3})-\cos\frac{1}{2}(x-\frac{2t}{3})] \\
& -[\sin\frac{1}{4}(x+\frac{2t}{3})-\sin\frac{1}{4}(x-\frac{2t}{3})]\} \\
& +\frac{1}{2}\sin\frac{2t}{3}\{4\sin\frac{x}{4}\cos\frac{t}{6}+4\sin\frac{x}{2}\cos\frac{t}{3}+2\cos\frac{x}{4}\cos\frac{t}{6} \\
& -[\cos\frac{1}{4}(x+\frac{2t}{3})-\cos\frac{1}{4}(x-\frac{2t}{3})]-2[\cos\frac{1}{2}(x+\frac{2t}{3})-\cos\frac{1}{2}(x-\frac{2t}{3})]\} \\
= \; & \cos\frac{2t}{3}[\sin\frac{x}{4}(\cos\frac{t}{6}-2\sin\frac{t}{6})+2\sin\frac{x}{2}(\cos\frac{t}{3}+\sin\frac{t}{3})-\cos\frac{x}{4}\sin\frac{t}{6}] \\
& +\sin\frac{2t}{3}[\sin\frac{x}{4}(2\cos\frac{t}{6}+\sin\frac{t}{6})+2\sin\frac{x}{2}(\cos\frac{t}{3}-\sin\frac{t}{3})+\cos\frac{x}{4}\cos\frac{t}{6}] \\
= \; & 2\sin\frac{x}{4}\sin(\frac{2t}{3}-\frac{t}{6})+2\sin\frac{x}{2}(\cos\frac{2t}{3}\cos\frac{t}{3}-\sin\frac{2t}{3}\sin\frac{t}{3} \\
& +\sin\frac{2t}{3}\cos\frac{t}{3}+\cos\frac{2t}{3}\sin\frac{t}{3})+\sin\frac{x}{4}\cos(\frac{2t}{3}-\frac{t}{6})+\cos\frac{x}{4}\sin(\frac{2t}{3}-\frac{t}{6}) \\
= \; & 2\sin\frac{x}{4}[\cos(\frac{2t}{3}-\frac{t}{6})+\sin(\frac{2t}{3}-\frac{t}{6})]+2\sin\frac{x}{2}[\cos(\frac{2t}{3}+\frac{t}{3})+\sin(\frac{2t}{3}+\frac{t}{3})] \\
& -\sin\frac{1}{4}(x-2t) \\
= \; & 2\sin\frac{x}{4}(\cos\frac{t}{2}+\sin\frac{t}{2})+2\sin\frac{x}{2}(\cos t+\sin t)-\sin\frac{1}{4}(x-2t),
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,t) \; = \; & -\frac{1}{2}\sin\frac{2t}{3}[\sin\frac{1}{4}(x+\frac{2t}{3})+2\sin\frac{1}{2}(x+\frac{2t}{3})+\sin\frac{1}{4}(x-\frac{2t}{3}) \\
& +2\sin\frac{1}{2}(x-\frac{2t}{3})+\frac{3}{2}\int_{x-\frac{2}{3}t}^{x+\frac{2}{3}t}(-\frac{1}{3}\sin\frac{s}{4}+\frac{2}{3}\sin\frac{s}{2}-\frac{1}{6}\cos\frac{s}{4})ds] \\
& +\frac{1}{2}\cos\frac{2t}{3}[2\sin\frac{1}{4}(x+\frac{2t}{3})+2\sin\frac{1}{2}(x+\frac{2t}{3})+\cos\frac{1}{4}(x+\frac{2t}{3}) \\
& +2\sin\frac{1}{4}(x-\frac{2t}{3})+2\sin\frac{1}{2}(x-\frac{2t}{3})+\cos\frac{1}{4}(x-\frac{2t}{3}) \\
& +\frac{3}{2}\int_{x-\frac{2}{3}t}^{x+\frac{2}{3}t}(\frac{1}{6}\sin\frac{s}{4}-\frac{2}{3}\sin\frac{s}{2})ds] \\
= \; & -\frac{1}{2}\sin\frac{2t}{3}\{2\sin\frac{x}{4}\cos\frac{t}{6}+4\sin\frac{x}{2}\cos\frac{t}{3}+2[\cos\frac{1}{4}(x+\frac{2t}{3})-\cos\frac{1}{4}(x-\frac{2t}{3})]
\end{aligned}
$$

$$
\begin{aligned}
& -2[\cos\frac{1}{2}(x+\frac{2t}{3})-\cos\frac{1}{2}(x-\frac{2t}{3})]-[\sin\frac{1}{4}(x+\frac{2t}{3})-\sin\frac{1}{4}(x-\frac{2t}{3})]\} \\
& +\frac{1}{2}\cos\frac{2t}{3}\{4\sin\frac{x}{4}\cos\frac{t}{6}+4\sin\frac{x}{2}\cos\frac{t}{3}+2\cos\frac{x}{4}\cos\frac{t}{6} \\
& -[\cos\frac{1}{4}(x+\frac{2t}{3})-\cos\frac{1}{4}(x-\frac{2t}{3})]-2[\cos\frac{1}{2}(x+\frac{2t}{3})-\cos\frac{1}{2}(x-\frac{2t}{3})]\} \\
= & -\sin\frac{2t}{3}[\sin\frac{x}{4}(\cos\frac{t}{6}-2\sin\frac{t}{6})+2\sin\frac{x}{2}(\cos\frac{t}{3}+\sin\frac{t}{3})-\cos\frac{x}{4}\sin\frac{t}{6}] \\
& +\cos\frac{2t}{3}[\sin\frac{x}{4}(2\cos\frac{t}{6}+\sin\frac{t}{6})+2\sin\frac{x}{2}(\cos\frac{t}{3}-\sin\frac{t}{3})+\cos\frac{x}{4}\cos\frac{t}{6}] \\
= & \ 2\sin\frac{x}{4}\cos(\frac{2t}{3}-\frac{t}{6})+2\sin\frac{x}{2}(\cos\frac{2t}{3}\cos\frac{t}{3}-\sin\frac{2t}{3}\sin\frac{t}{3} \\
& -\sin\frac{2t}{3}\cos\frac{t}{3}-\cos\frac{2t}{3}\sin\frac{t}{3})+\cos\frac{x}{4}\cos(\frac{2t}{3}-\frac{t}{6})-\sin\frac{x}{4}\sin(\frac{2t}{3}-\frac{t}{6}) \\
= & \ 2\sin\frac{x}{4}[\cos(\frac{2t}{3}-\frac{t}{6})-\sin(\frac{2t}{3}-\frac{t}{6})]+2\sin\frac{x}{2}[\cos(\frac{2t}{3}+\frac{t}{3})-\sin(\frac{2t}{3}+\frac{t}{3})] \\
& +\cos\frac{1}{4}(x-2t) \\
= & \ 2\sin\frac{x}{4}(\cos\frac{t}{2}-\sin\frac{t}{2})+2\sin\frac{x}{2}(\cos t-\sin t)+\cos\frac{1}{4}(x-2t),
\end{aligned}
$$

$$\text{for } \ x>\frac{2}{3}t\geq 0,$$

and

$$
\begin{aligned}
u_1(x,t) = & \ \frac{1}{2}\cos\frac{2t}{3}[\sin\frac{1}{4}(\frac{2t}{3}+x)+2\sin\frac{1}{2}(\frac{2t}{3}+x)-\sin\frac{1}{4}(\frac{2t}{3}-x) \\
& -2\sin\frac{1}{2}(\frac{2t}{3}-x)+\frac{3}{2}\int_{\frac{2t}{3}-x}^{\frac{2t}{3}+x}(-\frac{1}{3}\sin\frac{s}{4}+\frac{2}{3}\sin\frac{s}{2}-\frac{1}{6}\cos\frac{s}{4})ds] \\
& +\frac{1}{2}\sin\frac{2t}{3}[2\sin\frac{1}{4}(\frac{2t}{3}+x)+2\sin\frac{1}{2}(\frac{2t}{3}+x)+\cos\frac{1}{4}(\frac{2t}{3}+x) \\
& -2\sin\frac{1}{4}(\frac{2t}{3}-x)-2\sin\frac{1}{2}(\frac{2t}{3}-x)-\cos\frac{1}{4}(\frac{2t}{3}-x) \\
& +\frac{3}{2}\int_{\frac{2t}{3}-x}^{\frac{2t}{3}+x}(\frac{1}{6}\sin\frac{s}{4}-\frac{2}{3}\sin\frac{s}{2})ds] \\
& +\cos x\sin\frac{1}{2}(t-\frac{3x}{2})+\sin x\cos\frac{1}{2}(t-\frac{3x}{2}) \\
= & \ \cos\frac{2t}{3}\{\cos\frac{t}{6}\sin\frac{x}{4}+2\cos\frac{t}{3}\sin\frac{x}{2}+\cos\frac{1}{4}(\frac{2t}{3}+x) \\
& -\cos\frac{1}{4}(\frac{2t}{3}-x)-[\cos\frac{1}{2}(\frac{2t}{3}+x)-\cos\frac{1}{2}(\frac{2t}{3}-x)]
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2}[\sin\frac{1}{4}(\frac{2t}{3}+x)-\sin\frac{1}{4}(\frac{2t}{3}-x)]\} \\
& +\sin\frac{2t}{3}\{2\cos\frac{t}{6}\sin\frac{x}{4}+2\cos\frac{t}{3}\sin\frac{x}{2}-\sin\frac{t}{6}\sin\frac{x}{4} \\
& -\frac{1}{2}[\cos\frac{1}{4}(\frac{2t}{3}+x)-\cos\frac{1}{4}(\frac{2t}{3}-x)]+\cos\frac{1}{2}(\frac{2t}{3}+x) \\
& -\cos\frac{1}{2}(\frac{2t}{3}-x)\}+\sin[x+\frac{1}{2}(t-\frac{3x}{2})] \\
= \; & \cos\frac{2t}{3}(2\cos\frac{t}{3}\sin\frac{x}{2}-2\sin\frac{t}{6}\sin\frac{x}{4}+2\sin\frac{t}{3}\sin\frac{x}{2}) \\
& +\sin\frac{2t}{3}(2\cos\frac{t}{6}\sin\frac{x}{4}+2\cos\frac{t}{3}\sin\frac{x}{2}-2\sin\frac{t}{3}\sin\frac{x}{2}) \\
& +\sin(\frac{t}{2}+\frac{x}{4}) \\
= \; & 2\sin\frac{x}{4}(\sin\frac{2t}{3}\cos\frac{t}{6}-\cos\frac{2t}{3}\sin\frac{t}{6})+2\sin\frac{x}{2}(\cos\frac{2t}{3}\cos\frac{t}{3} \\
& -\sin\frac{2t}{3}\sin\frac{t}{3}+\sin\frac{2t}{3}\cos\frac{t}{3}+\sin\frac{2t}{3}\cos\frac{t}{3})+\sin(\frac{t}{2}+\frac{x}{4}) \\
= \; & 2\sin\frac{x}{4}\sin(\frac{2t}{3}-\frac{t}{6})+2\sin\frac{x}{2}[\cos(\frac{2t}{3}+\frac{t}{3})+\sin(\frac{2t}{3}+\frac{t}{3})] \\
& +\sin(\frac{t}{2}+\frac{x}{4}) \\
= \; & 2\sin\frac{x}{4}\sin\frac{t}{2}+2\sin\frac{x}{2}(\cos t+\sin t)+\sin\frac{1}{4}(x+2t) \\
= \; & 2\sin\frac{x}{4}(\cos\frac{t}{2}+\sin\frac{t}{2})+2\sin\frac{x}{2}(\cos t+\sin t)-\sin\frac{1}{4}(x-2t),
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,t) \; = \; & -\frac{1}{2}\sin\frac{2t}{3}[\sin\frac{1}{4}(\frac{2t}{3}+x)+2\sin\frac{1}{2}(\frac{2t}{3}+x)-\sin\frac{1}{4}(\frac{2t}{3}-x) \\
& -2\sin\frac{1}{2}(\frac{2t}{3}-x)+\frac{3}{2}\int_{\frac{2t}{3}-x}^{\frac{2t}{3}+x}(-\frac{1}{3}\sin\frac{s}{4}+\frac{2}{3}\sin\frac{s}{2}-\frac{1}{6}\cos\frac{s}{4})ds] \\
& +\frac{1}{2}\cos\frac{2t}{3}[2\sin\frac{1}{4}(\frac{2t}{3}+x)+2\sin\frac{1}{2}(\frac{2t}{3}+x)+\cos\frac{1}{4}(\frac{2t}{3}+x) \\
& -2\sin\frac{1}{4}(\frac{2t}{3}-x)-2\sin\frac{1}{2}(\frac{2t}{3}-x)-\cos\frac{1}{4}(\frac{2t}{3}-x) \\
& +\frac{3}{2}\int_{\frac{2t}{3}-x}^{\frac{2t}{3}+x}(\frac{1}{6}\sin\frac{s}{4}-\frac{2}{3}\sin\frac{s}{2})ds] \\
& -\sin x\sin\frac{1}{2}(t-\frac{3x}{2})+\cos x\cos\frac{1}{2}(t-\frac{3x}{2}) \\
= \; & -\sin\frac{2t}{3}\{\cos\frac{t}{6}\sin\frac{x}{4}+2\cos\frac{t}{3}\sin\frac{x}{2}+\cos\frac{1}{4}(\frac{2t}{3}+x)
\end{aligned}
$$

$$
\begin{aligned}
& -\cos\frac{1}{4}(\frac{2t}{3}-x) - [\cos\frac{1}{2}(\frac{2t}{3}+x) - \cos\frac{1}{2}(\frac{2t}{3}-x)] \\
& -\frac{1}{2}[\sin\frac{1}{4}(\frac{2t}{3}+x) - \sin\frac{1}{4}(\frac{2t}{3}-x)]\} \\
& +\cos\frac{2t}{3}\{2\cos\frac{t}{6}\sin\frac{x}{4} + 2\cos\frac{t}{3}\sin\frac{x}{2} - \sin\frac{t}{6}\sin\frac{x}{4} \\
& -\frac{1}{2}[\cos\frac{1}{4}(\frac{2t}{3}+x) - \cos\frac{1}{4}(\frac{2t}{3}-x)] + \cos\frac{1}{2}(\frac{2t}{3}+x) \\
& -\cos\frac{1}{2}(\frac{2t}{3}-x)\} + \cos[x+\frac{1}{2}(t-\frac{3x}{2})] \\
= \; & -\sin\frac{2t}{3}(2\cos\frac{t}{3}\sin\frac{x}{2} - 2\sin\frac{t}{6}\sin\frac{x}{4} + 2\sin\frac{t}{3}\sin\frac{x}{2}) \\
& +\cos\frac{2t}{3}(2\cos\frac{t}{6}\sin\frac{x}{4} + 2\cos\frac{t}{3}\sin\frac{x}{2} - 2\sin\frac{t}{3}\sin\frac{x}{2}) \\
& +\cos(\frac{t}{2}+\frac{x}{4}) \\
= \; & 2\sin\frac{x}{4}(\cos\frac{2t}{3}\cos\frac{t}{6} + \sin\frac{2t}{3}\sin\frac{t}{6}) + 2\sin\frac{x}{2}(\cos\frac{2t}{3}\cos\frac{t}{3} \\
& -\sin\frac{2t}{3}\sin\frac{t}{3} - \sin\frac{2t}{3}\cos\frac{t}{3} - \sin\frac{2t}{3}\cos\frac{t}{3}) + \cos(\frac{t}{2}+\frac{x}{4}) \\
= \; & 2\sin\frac{x}{4}\cos(\frac{2t}{3}-\frac{t}{6}) + 2\sin\frac{x}{2}[\cos(\frac{2t}{3}+\frac{t}{3}) - \sin(\frac{2t}{3}+\frac{t}{3})] \\
& +\cos(\frac{t}{2}+\frac{x}{4}) \\
= \; & 2\sin\frac{x}{4}\cos\frac{t}{2} + 2\sin\frac{x}{2}(\cos t - \sin t) + \cos(\frac{t}{2}+\frac{x}{4}) \\
= \; & 2\sin\frac{x}{4}(\cos\frac{t}{2} - \sin\frac{t}{2}) + 2\sin\frac{x}{2}(\cos t - \sin t) + \cos\frac{1}{4}(x-2t),
\end{aligned}
$$

$$
\text{for } \; 0 \le x < \frac{2}{3}t.
$$

Again, it is noted that these expressions for $u_1(x,t)$ and $u_2(x,t)$ are precisely the same as those exhibited in the travelling wave solution of example 2.5.2. The graphs describing the motion of the corresponding wave of this example again are the same as those presented in Figure 2.5.2.

The following three examples are chosen to illustrate the propagation of waves along a semi-infinite spinning string and the effect of boundary conditions at the end of this string. For these examples, it is convenient to introduce the Heaviside (unit step) function $H(x)$, as defined by

$$H(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}.$$

The Heaviside function allows us to include simple "pulse functions" in these models. An example of such a "sinusoidal pulse function" is

$$\begin{aligned} f(x) &= \begin{cases} 1 - \cos(2\pi x), & \text{for } 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases} \\ &= [H(x-2) - H(x-3)][1 - \cos(2\pi x)], \end{aligned}$$

whose graph is shown below:

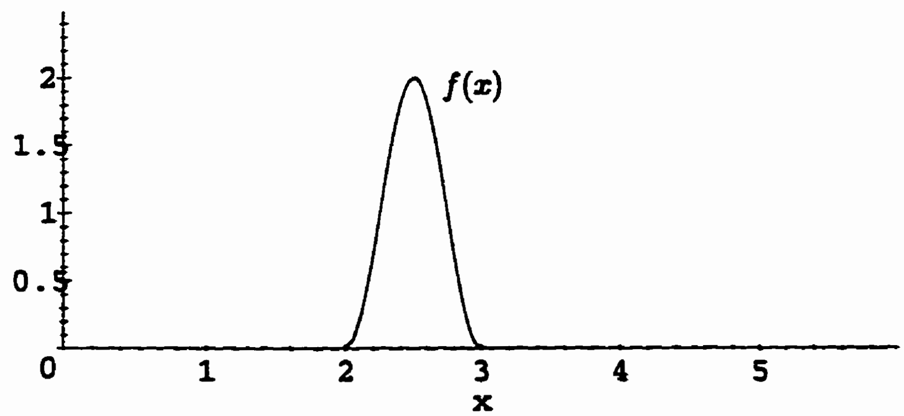


Diagram 4.3.1

Throughout the following examples, we shall choose

$$c = 1,\ \omega = \pi,$$

and employ various initial and boundary conditions.

Example 4.3.3: In the semi-infinite problem given by (4.2.1-2), we suppose

$$c = 1,\ \omega = \pi,$$

and

$$\begin{aligned} f_1(x) &= \begin{cases} 1 - \cos(2\pi x), & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases} \\ &= [H(x-2) - H(x-3)][1 - \cos(2\pi x)], \end{aligned}$$

$$f_2(x) = g_1(x) = g_2(x) = p_1(t) = p_2(t) = 0,$$

$$x \ge 0,\ t \ge 0.$$

These initial and boundary conditions physically describe a situation where the semi-infinite string has a fixed end at $x = 0$, while it is "plucked" with an initial displacement in the shape of a single sinusoidal pulse on $2 \le x \le 3$ and is given a zero initial velocity.

According to (4.2.14-15), the solution of this problem is given by

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}\cos(\pi t)[f_1(x+t) + f_1(x-t)] + \frac{\pi}{2}\sin(\pi t)\int_{x-t}^{x+t} f_1(s)ds \\ &= \frac{1}{2}\cos(\pi t)\{[H(x+t-2) - H(x+t-3)][1 - \cos 2\pi(x+t)] \\ &\quad +[H(x-t-2) - H(x-t-3)][1 - \cos 2\pi(x-t)]\} \\ &\quad +\frac{\pi}{2}\sin(\pi t)\int_{x-t}^{x+t} f_1(s)ds, \\ u_2(x,t) &= -\frac{1}{2}\sin(\pi t)[f_1(x+t) + f_1(x-t)] + \frac{\pi}{2}\cos(\pi t)\int_{x-t}^{x+t} f_1(s)ds \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sin(\pi t)\{[H(x+t-2)-H(x+t-3)][1-\cos 2\pi(x+t)] \\
&\quad +[H(x-t-2)-H(x-t-3)][1-\cos 2\pi(x-t)]\} \\
&\quad +\frac{\pi}{2}\cos(\pi t)\int_{x-t}^{x+t} f_1(s)ds;
\end{aligned}
$$

$$\text{for} \quad x > t \geq 0,$$

and

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{2}\cos(\pi t)[f_1(t+x)-f_1(t-x)]+\frac{\pi}{2}\sin(\pi t)\int_{t-x}^{t+x} f_1(s)ds \\
&= \frac{1}{2}\cos(\pi t)\{[H(t+x-2)-H(t+x-3)][1-\cos 2\pi(t+x)] \\
&\quad -[H(t-x-2)-H(t-x-3)][1-\cos 2\pi(t-x)]\} \\
&\quad +\frac{\pi}{2}\sin(\pi t)\int_{t-x}^{t+x} f_1(s)ds, \\
u_2(x,t) &= -\frac{1}{2}\sin(\pi t)[f_1(t+x)+f_1(t-x)]+\frac{\pi}{2}\cos(\pi t)\int_{t-x}^{t+x} f_1(s)ds \\
&= -\frac{1}{2}\sin(\pi t)\{[H(t+x-2)-H(t+x-3)][1-\cos 2\pi(t+x)] \\
&\quad -[H(t-x-2)-H(t-x-3)][1-\cos 2\pi(t-x)]\} \\
&\quad +\frac{\pi}{2}\cos(\pi t)\int_{t-x}^{t+x} f_1(s)ds,
\end{aligned}
$$

$$\text{for} \quad 0 \leq x < t.$$

Simplification of the above expressions for $u_1(x,t)$ and $u_2(x,t)$ involves evaluation of the integrals $\int_{x-t}^{x+t} f_1(s)ds$ and $\int_{t-x}^{t+x} f_1(s)ds$. To this end, we note that, for $x > t \geq 0$,

$$\int_{x-t}^{x+t} f_1(s)ds = \int_0^{x+t} f_1(s)ds - \int_0^{x-t} f_1(s)ds,$$

so we need only consider $\int_0^X f_1(s)ds$, where $X = x \pm t \geq 0$. In particular,

$$\int_0^X f_1(s)ds \quad = \quad \int_0^X [H(s-2)-H(s-3)][1-\cos(2\pi s)]ds$$

$$= \begin{cases} 0, & \text{for } 0 \leq X \leq 2 \\ \int_2^X [1-\cos(2\pi s)]ds, & \text{for } 2 \leq X \leq 3 \\ \int_2^3 [1-\cos(2\pi s)]ds, & \text{for } X \geq 3 \end{cases} .$$

Now

$$\int [1-\cos(2\pi s)]ds = s - \frac{\sin(2\pi s)}{2\pi},$$

so that

$$\int_0^X f_1(s)ds = \begin{cases} 0, & \text{for } 0 \leq X \leq 2 \\ X - 2 - \frac{1}{2\pi}\sin(2\pi X), & \text{for } 2 \leq X \leq 3 \\ 1, & \text{for } X \geq 3 \end{cases}$$

$$= [H(X-2) - H(X-3)][X - 2 - \frac{1}{2\pi}\sin(2\pi X)] + H(X-3).$$

Therefore,

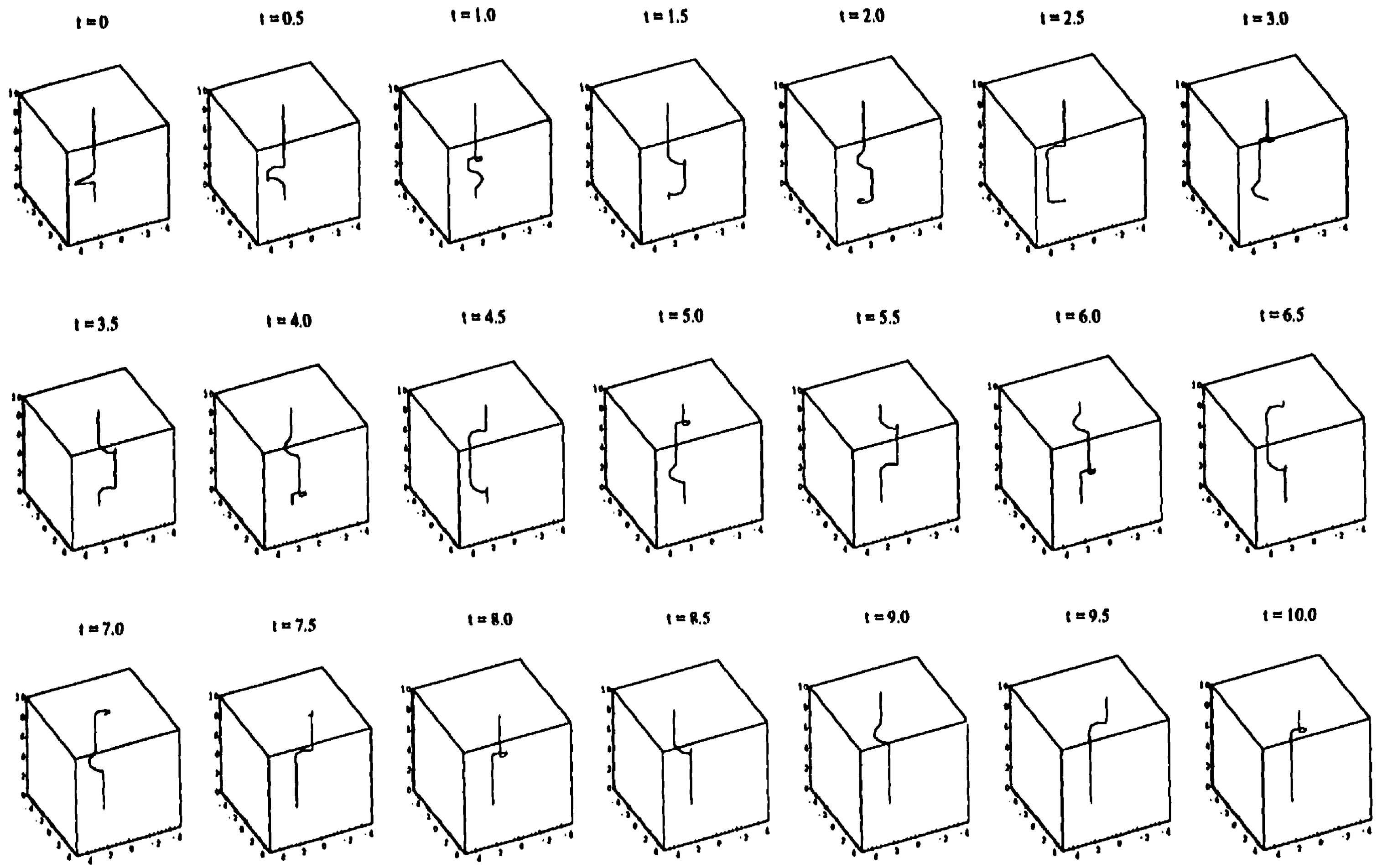
$$\begin{aligned} &\int_{x-t}^{x+t} f_1(s)ds \\ = \; &[H(x+t-2) - H(x+t-3)][x+t-2-\frac{1}{2\pi}\sin 2\pi(x+t)] + H(x+t-3) \\ &-[H(x-t-2) - H(x-t-3)][x-t-2-\frac{1}{2\pi}\sin 2\pi(x-t)] - H(x-t-3). \end{aligned}$$

Similarly, the evaluation of the integral $\int_{t-x}^{t+x} f_1(s)ds$ in the above solution, when $0 \leq x < t$, is given by

$$\begin{aligned} &\int_{t-x}^{t+x} f_1(s)ds \\ = \; &[H(t+x-2) - H(t+x-3)][t+x-2-\frac{1}{2\pi}\sin 2\pi(t+x)] + H(t+x-3) \\ &-[H(t-x-2) - H(t-x-3)][t-x-2-\frac{1}{2\pi}\sin 2\pi(t-x)] - H(t-x-3). \end{aligned}$$

The graphs describing the motion of the corresponding wave of this example are presented in Figure 4.3.3, where the time interval between consecutive graphs is chosen to be 0.5, and the total time of observation is from $t = 0$ to $t = 10.0$. It is seen that a "rotational lag" of the pulse occurs and the pulse broadens until the wave reflects from the fixed end. Thereafter the length of the non-planar pulse remains fixed, as it moves at constant velocity away from the fixed end. It is noted that upon the reflection from the fixed end, the displacement is inverted but the direction of rotation is preserved.

Figure 4.3.3: graphic illustration of Example 4.3.3 where c=1, $\omega=\pi$



Example 4.3.4: In this case we consider (4.2.1-2) with

$$c = 1,\ \omega = \pi,$$

and

$$\begin{aligned} f_1(x) &= \begin{cases} 1-\cos(2\pi x), & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases} \\ &= [H(x-2)-H(x-3)][1-\cos(2\pi x)], \\ g_2(x) &= -\omega f_1(x) \\ &= \begin{cases} \pi[\cos(2\pi x)-1], & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases} \\ &= \pi[H(x-2)-H(x-3)][\cos(2\pi x)-1], \end{aligned}$$

$$f_2(x) = 0,\ g_1(x) = \omega f_2(x) = 0,\ p_1(t) = p_2(t) = 0,$$

$$x \ge 0,\ \ t \ge 0.$$

These initial and boundary conditions physically mean that the semi-infinite string has a fixed end at $x = 0$, while it is "plucked" with an initial displacement in the shape of a single sinusoidal pulse on $2 \le x \le 3$ and has a non-zero initial velocity, chosen in such a way that the initial (complex) rotational velocity (4.2.5b) is zero.

By virtue of (4.2.14-15), the solution of the problem in this case is given by

$$u_1(x,t) = \frac{1}{2}\cos(\pi t)[f_1(x+t)+f_1(x-t)]$$

$$
\begin{aligned}
&= \frac{1}{2}\cos(\pi t)\{[H(x+t-2)-H(x+t-3)][1-\cos 2\pi(x+t)] \\
&\quad +[H(x-t-2)-H(x-t-3)][1-\cos 2\pi(x-t)]\}, \\
u_2(x,t) &= -\frac{1}{2}\sin(\pi t)[f_1(x+t)+f_1(x-t)] \\
&= -\frac{1}{2}\sin(\pi t)\{[H(x+t-2)-H(x+t-3)][1-\cos 2\pi(x+t)] \\
&\quad +[H(x-t-2)-H(x-t-3)][1-\cos 2\pi(x-t)]\};
\end{aligned}
$$

$$\text{for } \; x > t \geq 0,$$

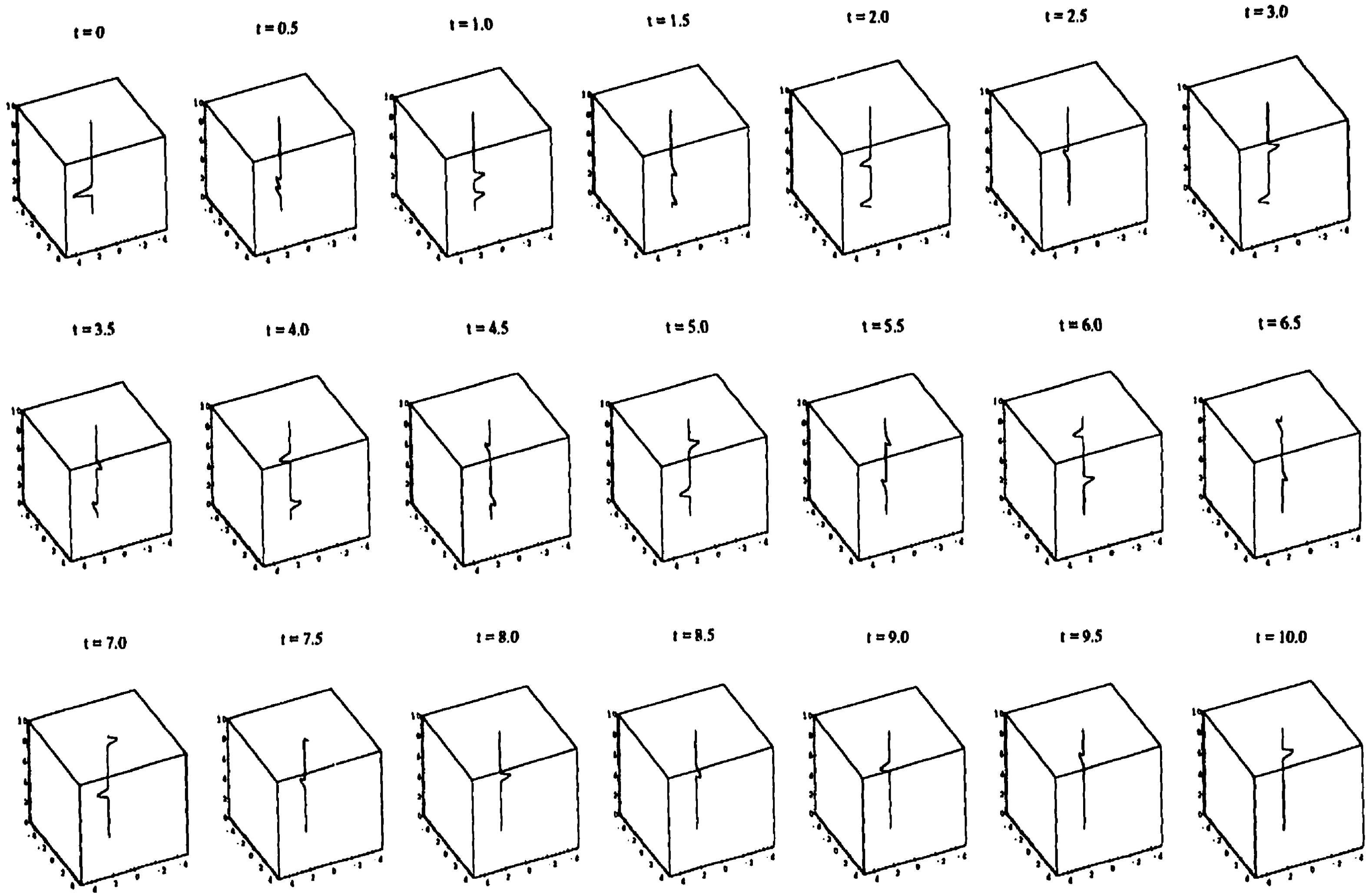
and

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{2}\cos(\pi t)[f_1(t+x)-f_1(t-x)] \\
&= \frac{1}{2}\cos(\pi t)\{[H(t+x-2)-H(t+x-3)][1-\cos 2\pi(t+x)] \\
&\quad -[H(t-x-2)-H(t-x-3)][1-\cos 2\pi(t-x)]\}, \\
u_2(x,t) &= -\frac{1}{2}\sin(\pi t)[f_1(t+x)-f_1(t-x)] \\
&= -\frac{1}{2}\sin(\pi t)\{[H(t+x-2)-H(t+x-3)][1-\cos 2\pi(t+x)] \\
&\quad -[H(t-x-2)-H(t-x-3)][1-\cos 2\pi(t-x)]\}.
\end{aligned}
$$

$$\text{for } \; 0 \leq x < t.$$

The graphs describing the motion of the corresponding wave of this example are presented in Figure 4.3.4. These graphs demonstrate that the rotational lag, exhibited in the previous example, is no longer present. Instead, the pulse simply splits into two planar pulses travelling in opposite directions at constant (unit) speed and rotating uniformly in the same direction at constant angular velocity, until one part reflects upside down from the fixed end. After the reflection from the fixed end, both pulses travel away from the fixed end at (unit) speed, and rotate in the same direction at constant angular velocity, with the distance between them remaining fixed.

Figure 4.3.4: graphic illustration of Example 4.3.4 where c=1, $\omega=\pi$



Example 4.3.5: Finally for (4.2.1-2), we assume

$$c = 1,\ \omega = \pi,$$

with

$$f_1(x) = f_2(x) = p_2(t) = 0,\ g_1(x) = \omega f_2(x) = 0,\ g_2(x) = -\omega f_1(x) = 0,$$

$$x \geq 0,\ t \geq 0.$$

$$\begin{aligned} p_1(t) &= \begin{cases} 1 - \cos(2\pi t), & \text{for } 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \\ &= [H(t) - H(t-1)][1 - \cos(2\pi t)], \end{aligned}$$

These conditions physically indicate that the semi-infinite string has zero initial displacement and zero initial velocity; it is simply "excited" with a single planar sinusoidal input at the end $x = 0$ for a unit interval of time.

By virtue of (4.2.14-15), the solution of this problem is given by

$$u_1(x,t) = 0,\ u_2(x,t) = 0,$$

$$\text{for } \ x > t \geq 0,$$

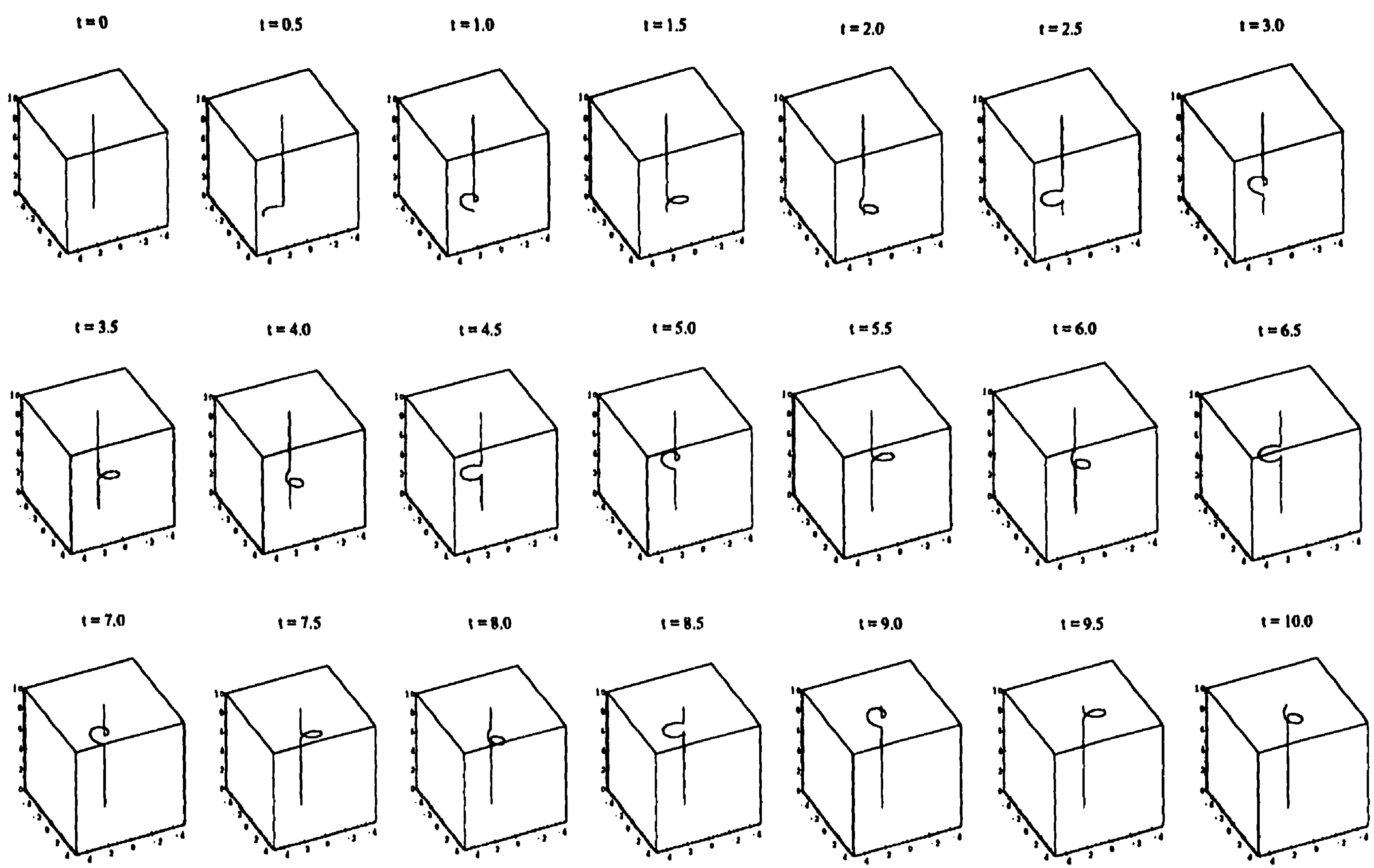
and

$$\begin{aligned} u_1(x,t) &= p_1(t-x)\cos(\pi x) \\ &= [H(t-x) - H(t-x-1)][1 - \cos 2\pi(t-x)]\cos(\pi x), \\ u_2(x,t) &= -p_1(t-x)\sin(\pi x) \\ &= -[H(t-x) - H(t-x-1)][1 - \cos 2\pi(t-x)]\sin(\pi x). \end{aligned}$$

$$\text{for } 0 \leq x < t.$$

The graphs describing the motion of the corresponding wave of this example are presented in Figure 4.3.5. These graphs demonstrate that the planar input signal at $x = 0$ gives rise to a non-planar "loop" travelling along the string at constant speed and with a constant angular velocity.

Figure 4.3.5: graphic illustration of Example 4.3.5 where c=1, $\omega=\pi$



CHAPTER 5

Solutions for the Fixed-Length String

5.1 Introduction

For the usual one-dimensional wave equation, in the case of a fixed-length string with fixed-end boundary conditions, the solution is typically determined in the form of a Fourier sine series. In this chapter, we will consider the coupled linear system (1.3.1) in the case of a fixed-length string ($0 \leq x \leq L$) with general (non-homogeneous) boundary and initial conditions. In order to allow Fourier analysis to be used to construct the solution of this problem, the complex transformations of Chapter 3 are used, along with other transformations, which reduce the problem to a single one-dimensional wave equation with a non-homogeneous forcing term, but with boundary conditions which are now homogeneous. At the end of this chapter, some examples are given which illustrate the technique developed herein. To illustrate this development, finite Fourier series approximations to the derived solutions are displayed graphically, and compared with results obtained in preceding chapters.

5.2 d'Alembert Solution and Compatability Conditions

For the usual one-dimensional homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0, \tag{5.2.1}$$

in the case of a fixed-length string ($0 \leq x \leq L$), the initial and boundary conditions are given by

$$u(x,0) = f(x), \ \frac{\partial u}{\partial t}(x,0) = g(x), \ 0 \leq x \leq L, \qquad (5.2.2a)$$

$$u(0,t) = p(t), \ u(L,t) = q(t), \quad t \geq 0, \qquad (5.2.2b)$$

where $f(x)$ and $g(x)$ are specified functions on $0 \leq x \leq L$, while $p(t)$ and $q(t)$ are specified functions on $t \geq 0$.

The fixed-length problem (5.2.1-2) is known to have a d'Alembert solution which is expressed in different forms in different subregions $R_{(i,j)}$ within the region $R = \{(x,t) \mid 0 \leq x \leq L, \ t \geq 0\}$ in the xt-plane [7, pp. 51-52], [10, pp. 70-73].

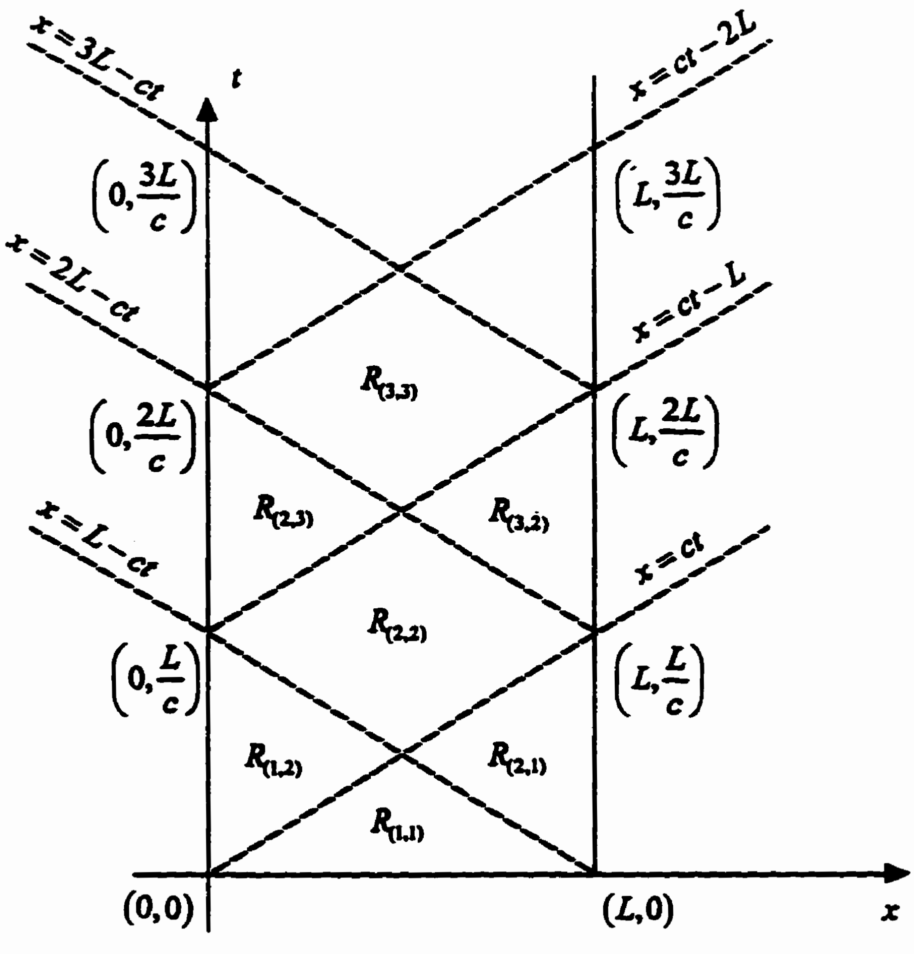


Diagram 5.2.1

We denote the solution in region $R_{(i,j)}$, of diagram 5.2.1, by

$$u(x,t) = \varphi_i(x+ct) + \psi_i(x-ct). \tag{5.2.3}$$

In $R_{(1,1)}$, we have the formulae

$$\varphi_1(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau + D,\ 0 \le \xi \le L, \tag{5.2.4a}$$

$$\psi_1(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - D,\ 0 \le \eta \le L, \tag{5.2.4b}$$

where $\xi = x + ct$, $\eta = x - ct$, and D is any constant.

To obtain formulae valid in $R_{(1,2)}$, we apply the first condition in (5.2.2b) to (5.2.3) with $i = 1$ and $j = 2$. This gives

$$\psi_2(\eta) = p(-\frac{\eta}{c}) - \varphi_1(-\eta),\ \ -L < \eta < 0,$$

and using (5.2.4a)

$$\psi_2(\eta) = p(-\frac{\eta}{c}) - \frac{1}{2}f(-\eta) - \frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau - D,\ \ -L < \eta < 0, \tag{5.2.5a}$$

Using the second condition in (5.2.2b) and (5.2.3), we obtain a similar result in $R_{(2,1)}$, namely,

$$\varphi_2(\xi) = q(\frac{\xi - L}{c}) - \psi_1(2L - \xi)$$

or equivalently, by virtue of (5.2.4b),

$$\varphi_2(\xi) = q(\frac{\xi - L}{c}) - \frac{1}{2}f(2L - \xi) + \frac{1}{2c}\int_0^{2L-\xi} g(\tau)d\tau + D, \tag{5.2.5b}$$

for $L < \xi < 2L$.

In general, for $i,\ j = 1, 2, 3...$, we have

$$\begin{aligned}\varphi_{i+1}(\xi) &= q(\frac{\xi - L}{c}) - \psi_i(2L - \xi), &(5.2.6a)\\ \psi_{j+1}(\eta) &= p(-\frac{\eta}{c}) - \varphi_j(-\eta), &(5.2.6b)\end{aligned}$$

$$i,\ j = 1, 2, 3...,$$

in which appropriate restrictions must be imposed on ξ and η, as illustrated in (5.2.5). In particular, we have

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau,$$

$$\text{for } (x,t) \in R_{(1,1)},$$

$$\begin{aligned}u(x,t) &= \frac{1}{2}[f(ct+x) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\tau)d\tau\\ &+p(t - \frac{x}{c}),\end{aligned}$$

$$\text{for } (x,t) \in R_{(1,2)},$$

$$\begin{aligned}u(x,t) &= \frac{1}{2}[f(x-ct) - f(2L-x-ct)] + \frac{1}{2c}\int_{x-ct}^{2L-x-ct} g(\tau)d\tau\\ &+q(t + \frac{x-L}{c}),\end{aligned}$$

$$\text{for } (x,t) \in R_{(2,1)},$$

$$\begin{aligned}u(x,t) &= -\frac{1}{2}[f(ct-x) + f(2L-x-ct)] + \frac{1}{2c}\int_{ct-x}^{2L-x-ct} g(\tau)d\tau\\ &+p(t - \frac{x}{c}) + q(t + \frac{x-L}{c}),\end{aligned}$$

$$\text{for } (x,t) \in R_{(2,2)}.$$

By an argument similar to that used to derive the compatability conditions in Chapter 2, we may show that the continuity or differentiability of the above solution depends on the following properties of the functions $f(x)$, $g(x)$, $p(t)$ and $q(t)$:

1) If $f(x)$, $p(t)$ and $q(t) \in C^0$, $g(x)$ is integrable, $f(0) = p(0)$ and $f(L) = q(0)$, then $u(x,t) \in C^0$.

2) If $f(x), p(t)$ and $q(t) \in C^1$, $g(x) \in C^0$, $f(0) = p(0)$, $f(L) = q(0)$, $g(0) = p'(0)$ and $g(L) = q'(0)$, then $u(x,t) \in C^1$.

5.3 Fourier Series Analysis

Now we consider the case of a spinning string of fixed length L, governed by the linear system

$$\frac{\partial^2 u_1}{\partial t^2} - 2\omega\frac{\partial u_2}{\partial t} - \omega^2 u_1 = c^2\frac{\partial^2 u_1}{\partial x^2}, \tag{5.3.1a}$$

$$\frac{\partial^2 u_2}{\partial t^2} + 2\omega\frac{\partial u_1}{\partial t} - \omega^2 u_2 = c^2\frac{\partial^2 u_2}{\partial x^2}, \tag{5.3.1b}$$

$$0 \le x \le L,\ t \ge 0,$$

subject to the following initial conditions:

$$u_1(x,0) = f_1(x),\ \ \frac{\partial u_1}{\partial t}(x,0) = g_1(x), \tag{5.3.2a}$$

$$u_2(x,0) = f_2(x),\ \ \frac{\partial u_2}{\partial t}(x,0) = g_2(x), \tag{5.3.2b}$$

$$\text{for } 0 \le x \le L,$$

and boundary conditions (at $x = 0$ and $x = L$):

$$u_1(0,t) = p_1(t),\ \ u_1(L,t) = q_1(t), \tag{5.3.2c}$$

$$u_2(0,t) = p_2(t), \quad u_2(L,t) = q_2(t), \tag{5.3.2d}$$

$$\text{for } t \geq 0.$$

As in Chapter 3, we let

$$V(x,t) = u_1(x,t) + iu_2(x,t), \tag{5.3.3}$$

where $i = \sqrt{-1}$, and

$$W(x,t) = e^{i\omega t}V(x,t), \tag{5.3.4}$$

so that (5.3.1) is transformed into

$$\frac{\partial^2 W}{\partial t^2} = c^2 \frac{\partial^2 W}{\partial x^2}, \tag{5.3.5}$$

$$0 \leq x \leq L, \ t \geq 0,$$

while the initial and boundary conditions (5.3.2) become

$$W(x,0) = f(x) = f_1(x) + if_2(x), \tag{5.3.6a}$$

$$\frac{\partial W}{\partial t}(x,0) = g(x) = -\omega f_2(x) + g_1(x) + i[\omega f_1(x) + g_2(x)], \tag{5.3.6b}$$

$$\begin{aligned} W(0,t) &= p(t) \\ &= p_1(t)\cos(\omega t) - p_2(t)\sin(\omega t) \\ &\quad + i[p_1(t)\sin(\omega t) + p_2(t)\cos(\omega t)], \end{aligned} \tag{5.3.6c}$$

$$\begin{aligned} W(L,t) &= q(t) \\ &= q_1(t)\cos(\omega t) - q_2(t)\sin(\omega t) \\ &\quad + i[q_1(t)\sin(\omega t) + q_2(t)\cos(\omega t)], \end{aligned} \tag{5.3.6d}$$

in analogy with the results in (4.2.4-5). In order to apply a Fourier sine series to solve the non-homogeneous problem (5.3.5-6), we must first transform the problem so that the boundary conditions become homogeneous. To this end, we let

$$Z(x,t) = W(x,t) + \frac{x}{L}[p(t) - q(t)] - p(t), \tag{5.3.7}$$

or equivalently

$$W(x,t) = Z(x,t) - \frac{x}{L}[p(t) - q(t)] + p(t), \tag{5.3.8}$$

so that

$$\begin{aligned}
\frac{\partial W}{\partial t} &= \frac{\partial Z}{\partial t} - \frac{x}{L}[p'(t) - q'(t)] + p'(t), \\
\frac{\partial^2 W}{\partial t^2} &= \frac{\partial^2 Z}{\partial t^2} - \frac{x}{L}[p''(t) - q''(t)] + p''(t), \\
\frac{\partial W}{\partial x} &= \frac{\partial Z}{\partial x} - \frac{1}{L}[p(t) - q(t)], \\
\frac{\partial^2 W}{\partial x^2} &= \frac{\partial^2 Z}{\partial x^2}.
\end{aligned}$$

Substitution of the above partial derivatives into (5.3.5) yields

$$\frac{\partial^2 Z}{\partial t^2} = c^2 \frac{\partial^2 Z}{\partial x^2} + A(x,t), \tag{5.3.9}$$

$$0 \le x \le L, \ t \ge 0,$$

where

$$A(x,t) = \frac{x}{L}[p''(t) - q''(t)] - p''(t), \tag{5.3.10}$$

while the boundary conditions become homogeneous, i.e., (5.3.6c,d) become

$$Z(0,t) = Z(L,t) = 0, \tag{5.3.11a}$$

$$\text{for } t \ge 0.$$

and the initial conditions of (5.3.6a,b) are given by

$$\begin{aligned} Z(x,0) &= f(x) + \frac{x}{L}[p(0) - q(0)] - p(0) \\ &\equiv \alpha(x), \qquad (5.3.11b) \\ \frac{\partial Z}{\partial t}(x,0) &= g(x) + \frac{x}{L}[p'(0) - q'(0)] - p'(0) \\ &\equiv \beta(x), \qquad (5.3.11c) \end{aligned}$$

$$\text{for } 0 \leq x \leq L.$$

If f, g, p and q satisfy the compatability conditions 2) of the previous section, so that $W(x,t) \in C^1$, then

$$\alpha(0) = \alpha(L) = \beta(0) = \beta(L) = 0, \qquad (5.3.12)$$

and the compatability conditions 2) for $Z(x,t)$ are valid, so that $Z(x,t) \in C^1$. Moreover, we note that (5.3.12) are necessary conditions for the uniform convergence of the Fourier sine series expansions for $\alpha(x)$ and $\beta(x)$ discussed below.

In order to obtain the compatability conditions 1) or 2) on f, g, p and q for $W(x,t)$, we require the following equivalent compatability conditions on the initial and boundary conditions for $u_1(x,t)$ and $u_2(x,t)$:

1) $f_j(x)$, $p_j(t)$, $q_j(t) \in C^0$, $g_j(x)$ are integrable, $f_j(0) = p_j(0)$, and $f_j(L) = q_j(0)$, for $j = 1, 2$. $\Longrightarrow W(x,t)$, $u_1(x,t)$, $u_2(x,t) \in C^0$.

2) $f_j(x)$, $p_j(t)$, $q_j(t) \in C^1$, $g_j(x) \in C^0$, $f_j(0) = p_j(0)$, $f_j(L) = q_j(0)$, $g_j(0) = p_j'(0)$, $g_j(L) = q_j'(0)$, for $j = 1, 2$. $\Longrightarrow W(x,t)$, $u_1(x,t)$, $u_2(x,t) \in C^1$.

Now we attempt to construct a Fourier sine series solution of the problem

(5.3.9-11), by assuming that the solution may be written in the form

$$Z(x,t) = \sum_{n=1}^{\infty} Z_n(t) \sin \frac{n\pi x}{L}, \tag{5.3.13}$$

so that

$$\frac{\partial Z}{\partial t} = \sum_{n=1}^{\infty} Z_n'(t) \sin \frac{n\pi x}{L}, \tag{5.3.14a}$$

$$\frac{\partial^2 Z}{\partial t^2} = \sum_{n=1}^{\infty} Z_n''(t) \sin \frac{n\pi x}{L}, \tag{5.3.14b}$$

$$\frac{\partial Z}{\partial x} = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) Z_n(t) \cos \frac{n\pi x}{L}, \tag{5.3.14c}$$

$$\frac{\partial^2 Z}{\partial x^2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 Z_n(t) \sin \frac{n\pi x}{L}. \tag{5.3.14d}$$

Substitution of (5.3.13-14) into (5.3.9) gives

$$\sum_{n=1}^{\infty} \left[Z_n''(t) + \left(\frac{n\pi c}{L}\right)^2 Z_n(t) \right] \sin \frac{n\pi x}{L} = A(x,t). \tag{5.3.15}$$

Now we multiply both sides of (5.3.15) by $\sin \frac{m\pi x}{L}$, and integrate both sides of the resulting equality from 0 to L. By virtue of the identities

$$\int_0^L \sin \frac{p\pi x}{L} \sin \frac{q\pi x}{L} dx = \begin{cases} \frac{L}{2}, & \text{for } p = q \\ 0, & \text{for } p \neq q \end{cases},$$

for p and q any positive integers, we obtain a sequence of ordinary differential equations for the functions $Z_n(t)$, namely

$$Z_n''(t) + \left(\frac{n\pi c}{L}\right)^2 Z_n(t) = A_n(t), \; n = 1, 2, 3..., \tag{5.3.16}$$

where

$$A_n(t) = \frac{2}{L} \int_0^L A(x,t) \sin \frac{n\pi x}{L} dx. \tag{5.3.17}$$

Substitution of (5.3.10) into (5.3.17) gives

$$\begin{aligned}
A_n(t) &= \frac{2}{L}\int_0^L \{\frac{x}{L}[p''(t)-q''(t)] - p''(t)\} \sin\frac{n\pi x}{L}dx \\
&= \frac{2}{L^2}[p''(t)-q''(t)]\int_0^L x\sin\frac{n\pi x}{L}dx - \frac{2}{L}p''(t)\int_0^L \sin\frac{n\pi x}{L}dx \\
&= \frac{2}{L^2}[p''(t)-q''(t)]\;[-\frac{L}{n\pi}(x\cos\frac{n\pi x}{L})\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}dx] \\
&\quad -\frac{2}{L}p''(t)\;(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L) \\
&= -\frac{2}{n\pi L}[p''(t)-q''(t)]\;[L\cos(n\pi) - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L] \\
&\quad +\frac{2}{n\pi}p''(t)[\cos(n\pi)-1] \\
&= -\frac{2}{n\pi}[p''(t)-q''(t)](-1)^n + \frac{2}{n\pi}p''(t)[(-1)^n-1] \\
&= \frac{2}{n\pi}[(-1)^n q''(t) - p''(t)]. \qquad (5.3.18)
\end{aligned}$$

The initial conditions for (5.3.16) may be easily determined through (5.3.11b,c), (5.3.13) and (5.3.14a) by noting that

$$Z(x,0) = \sum_{n=1}^{\infty} Z_n(0)\sin\frac{n\pi x}{L} = \alpha(x),$$

$$\frac{\partial Z}{\partial t}(x,0) = \sum_{n=1}^{\infty} Z_n'(0)\sin\frac{n\pi x}{L} = \beta(x),$$

and that the Fourier sine series expansions of $\alpha(x)$ and $\beta(x)$, namely

$$\alpha(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$$

and

$$\beta(x) = \sum_{n=1}^{\infty} \beta_n \sin\frac{n\pi x}{L}$$

(with Fourier coefficients $\alpha_n = \frac{2}{L}\int_0^L \alpha(x)\sin\frac{n\pi x}{L}dx$, $\beta_n = \frac{2}{L}\int_0^L \beta(x)\sin\frac{n\pi x}{L}dx$) converge uniformly on $0 \le x \le L$. Thus, we may conclude, by virtue of (5.3.11b,c), that

$$\begin{aligned} Z_n(0) &= \alpha_n \\ &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx + \frac{2}{n\pi}[(-1)^n q(0) - p(0)], \end{aligned} \qquad (5.3.19a)$$

and

$$\begin{aligned} Z_n'(0) &= \beta_n \\ &= \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx + \frac{2}{n\pi}[(-1)^n q'(0) - p'(0)], \end{aligned} \qquad (5.3.19b)$$

in which the final terms have been evaluated by the same method used in deriving (5.3.18).

In summary, the determination of the Fourier coefficients $Z_n(t)$ in (5.3.13) involves the solution of the sequence of non-homogeneous second order linear ordinary differential equations (5.3.16) and (5.3.18), subject to the initial conditions (5.3.19).

The corresponding homogeneous version of (5.3.16) is given by

$$\tilde{Z}_n''(t) + \left(\frac{n\pi c}{L}\right)^2 \tilde{Z}_n(t) = 0, \qquad (5.3.20)$$

and has the general solution

$$\tilde{Z}_n(t) = B_n \sin\left(\frac{n\pi c}{L}t\right) + C_n \cos\left(\frac{n\pi c}{L}t\right), \qquad (5.3.21)$$

where B_n and C_n are arbitrary (complex) constants yet to be determined. Moreover, a particular solution of equation (5.3.16) can be obtained, through the use of the method of variation of parameters [11, pp. 43-46], and is given by

$$Z_n^*(t) = \sin\left(\frac{n\pi c}{L}t\right)\int_0^t \frac{D_1(s)}{D(s)}A_n(s)ds + \cos\left(\frac{n\pi c}{L}t\right)\int_0^t \frac{D_2(s)}{D(s)}A_n(s)ds,$$

where

$$
\begin{aligned}
D(s) &= \begin{vmatrix} \sin\left(\frac{n\pi c}{L}s\right) & \cos\left(\frac{n\pi c}{L}s\right) \\ \frac{n\pi c}{L}\cos\left(\frac{n\pi c}{L}s\right) & -\frac{n\pi c}{L}\sin\left(\frac{n\pi c}{L}s\right) \end{vmatrix} \\
&= -\frac{n\pi c}{L}, \\
D_1(s) &= \begin{vmatrix} 0 & \cos\left(\frac{n\pi c}{L}s\right) \\ 1 & -\frac{n\pi c}{L}\sin\left(\frac{n\pi c}{L}s\right) \end{vmatrix} \\
&= -\cos\left(\frac{n\pi c}{L}s\right), \\
D_2(s) &= \begin{vmatrix} \sin\left(\frac{n\pi c}{L}s\right) & 0 \\ \frac{n\pi c}{L}\cos\left(\frac{n\pi c}{L}s\right) & 1 \end{vmatrix} \\
&= \sin\left(\frac{n\pi c}{L}s\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
Z_n^*(t) &= \frac{L}{n\pi c}\left[\sin\left(\frac{n\pi c}{L}t\right)\int_0^t A_n(s)\cos\left(\frac{n\pi c}{L}s\right)ds\right. \\
&\quad \left.-\cos\left(\frac{n\pi c}{L}t\right)\int_0^t A_n(s)\sin\left(\frac{n\pi c}{L}s\right)ds\right] \\
&= \frac{L}{n\pi c}\int_0^t A_n(s)\sin\left[\frac{n\pi c}{L}(t-s)\right]ds. \qquad (5.3.22)
\end{aligned}
$$

Therefore, by the superposition principle, the solution of (5.3.16-18) is simply

$$
\begin{aligned}
Z_n(t) &= \tilde{Z}_n(t) + Z_n^*(t) \\
&= B_n\sin\left(\frac{n\pi c}{L}t\right) + C_n\cos\left(\frac{n\pi c}{L}t\right) \\
&\quad + \frac{L}{n\pi c}\int_0^t A_n(s)\sin\left[\frac{n\pi c}{L}(t-s)\right]ds, \qquad (5.3.23)
\end{aligned}
$$

where the unknown coefficients B_n and C_n can be determined through the initial conditions (5.3.19), and are given by

$$B_n = \frac{L}{n\pi c} Z_n'(0) = \frac{L}{n\pi c}\beta_n, \tag{5.3.24a}$$

$$C_n = Z_n(0) = \alpha_n, \tag{5.3.24b}$$

which are derived from the fact that

$$Z_n^*(0) = Z_n^{*\prime}(0) = 0,$$

[11]. In particular, in view of (5.3.18), we have

$$\begin{aligned}
& \int_0^t A_n(s) \sin\left[\frac{n\pi c}{L}(t-s)\right] ds \\
= \; & \frac{2}{n\pi}\int_0^t [(-1)^n q''(s) - p''(s)] \sin\left[\frac{n\pi c}{L}(t-s)\right] ds,
\end{aligned}$$

which, by the method of integration by parts, becomes

$$\begin{aligned}
& \int_0^t A_n(s) \sin\left[\frac{n\pi c}{L}(t-s)\right] ds \\
= \; & \frac{2}{n\pi}\int_0^t \sin\left[\frac{n\pi c}{L}(t-s)\right] d[(-1)^n q'(s) - p'(s)] \\
= \; & \frac{2}{n\pi}[p'(0) - (-1)^n q'(0)] \sin\left(\frac{n\pi c}{L}t\right) \\
& + \frac{2c}{L}\int_0^t [(-1)^n q'(s) - p'(s)] \cos\left[\frac{n\pi c}{L}(t-s)\right] ds \\
= \; & \frac{2}{n\pi}[p'(0) - (-1)^n q'(0)] \sin\left(\frac{n\pi c}{L}t\right) \\
& + \frac{2c}{L}\int_0^t \cos\left[\frac{n\pi c}{L}(t-s)\right] d[(-1)^n q(s) - p(s)] \\
= \; & \frac{2}{n\pi}[p'(0) - (-1)^n q'(0)] \sin\left(\frac{n\pi c}{L}t\right) \\
& + \frac{2c}{L}[p(0) - (-1)^n q(0)] \cos\left(\frac{n\pi c}{L}t\right) + \frac{2c}{L}[(-1)^n q(t) - p(t)] \\
& + \frac{2n\pi c^2}{L^2}\int_0^t [p(s) - (-1)^n q(s)] \sin\left[\frac{n\pi c}{L}(t-s)\right] ds.
\end{aligned} \tag{5.3.25}$$

Moreover, substitution of (5.3.19) into (5.3.24) gives

$$
\begin{aligned}
B_n &= \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx + \frac{2L}{(n\pi)^2 c}[(-1)^n q'(0) - p'(0)], && (5.3.26a)\\
C_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx + \frac{2}{n\pi}[(-1)^n q(0) - p(0)]. && (5.3.26b)
\end{aligned}
$$

Thus, upon substitution of (5.3.25-26) into (5.3.23), we have that the solution of (5.3.16), (5.3.18) and (5.3.19) is given by

$$
\begin{aligned}
Z_n(t) &= \left\{\frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx + \frac{2L}{(n\pi)^2 c}[(-1)^n q'(0) - p'(0)]\right\}\sin\left(\frac{n\pi c}{L}t\right)\\
&\quad + \left\{\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx + \frac{2}{n\pi}[(-1)^n q(0) - p(0)]\right\}\cos\left(\frac{n\pi c}{L}t\right)\\
&\quad + \frac{L}{n\pi c}\left\{\frac{2}{n\pi}[p'(0) - (-1)^n q'(0)]\sin\left(\frac{n\pi c}{L}t\right)\right.\\
&\quad + \frac{2c}{L}[p(0) - (-1)^n q(0)]\cos\left(\frac{n\pi c}{L}t\right) + \frac{2c}{L}[(-1)^n q(t) - p(t)]\\
&\quad \left. + \frac{2n\pi c^2}{L^2}\int_0^t [p(s) - (-1)^n q(s)]\sin\left[\frac{n\pi c}{L}(t-s)\right]ds\right\}\\
&= \frac{2}{L}\cos\left(\frac{n\pi c}{L}t\right)\int_0^L f(x)\sin\frac{n\pi x}{L}dx\\
&\quad + \frac{2}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right)\int_0^L g(x)\sin\frac{n\pi x}{L}dx\\
&\quad + \frac{2c}{L}\int_0^t [p(s) - (-1)^n q(s)]\sin\left[\frac{n\pi c}{L}(t-s)\right]ds\\
&\quad + \frac{2}{n\pi}[(-1)^n q(t) - p(t)]. && (5.3.27)
\end{aligned}
$$

It should be noted that the $Z_n(t)$, as given by (5.3.27), are complex functions. Thus the real and imaginary parts of $Z_n(t)$ are given respectively by

$$
\begin{aligned}
\mathrm{Re}\, Z_n(t) &= \frac{2}{L}\cos\left(\frac{n\pi c}{L}t\right)\int_0^L \mathrm{Re}\, f(x)\sin\frac{n\pi x}{L}dx\\
&\quad + \frac{2}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right)\int_0^L \mathrm{Re}\, g(x)\sin\frac{n\pi x}{L}dx\\
&\quad + \frac{2c}{L}\int_0^t \mathrm{Re}[p(s) - (-1)^n q(s)]\sin\left[\frac{n\pi c}{L}(t-s)\right]ds\\
&\quad + \frac{2}{n\pi}\mathrm{Re}[(-1)^n q(t) - p(t)], && (5.3.28a)
\end{aligned}
$$

$$\begin{aligned}
\operatorname{Im} Z_n(t) &= \frac{2}{L}\cos\left(\frac{n\pi c}{L}t\right)\int_0^L \operatorname{Im} f(x)\sin\frac{n\pi x}{L}dx \\
&+\frac{2}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right)\int_0^L \operatorname{Im} g(x)\sin\frac{n\pi x}{L}dx \\
&+\frac{2c}{L}\int_0^t \operatorname{Im}[p(s)-(-1)^n q(s)]\sin\left[\frac{n\pi c}{L}(t-s)\right]ds \\
&+\frac{2}{n\pi}\operatorname{Im}[(-1)^n q(t)-p(t)]. \qquad (5.3.28b)
\end{aligned}$$

But, from (5.3.6), we have

$$\begin{aligned}
\operatorname{Re} f(x) &= f_1(x), \\
\operatorname{Im} f(x) &= f_2(x), \\
\operatorname{Re} g(x) &= -\omega f_2(x) + g_1(x), \\
\operatorname{Im} g(x) &= \omega f_1(x) + g_2(x), \\
\operatorname{Re}[p(s)-(-1)^n q(s)] &= (-1)^n[-q_1(s)\cos(\omega s) + q_2(s)\sin(\omega s)] \\
&\quad +p_1(s)\cos(\omega s) - p_2(s)\sin(\omega s), \\
\operatorname{Im}[p(s)-(-1)^n q(s)] &= (-1)^n[-q_1(s)\sin(\omega s) - q_2(s)\cos(\omega s)] \\
&\quad +p_1(s)\sin(\omega s) + p_2(s)\cos(\omega s), \\
\operatorname{Re}[(-1)^n q(t)-p(t)] &= (-1)^n[q_1(t)\cos(\omega t) - q_2(t)\sin(\omega t)] \\
&\quad -p_1(t)\cos(\omega t) + p_2(t)\sin(\omega t), \\
\operatorname{Im}[(-1)^n q(t)-p(t)] &= (-1)^n[q_1(t)\sin(\omega t) + q_2(t)\cos(\omega t)] \\
&\quad -p_1(t)\sin(\omega t) - p_2(t)\cos(\omega t).
\end{aligned}$$

Therefore, (5.3.28) become

$$\begin{aligned}
\operatorname{Re} Z_n(t) &= \frac{2}{L}\cos\left(\frac{n\pi c}{L}t\right)\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx \\
&+\frac{2}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right)\int_0^L [-\omega f_2(x)+g_1(x)]\sin\frac{n\pi x}{L}dx
\end{aligned}$$

$$
\begin{aligned}
& +\frac{2c}{L}\int_0^t \{(-1)^n[-q_1(s)\cos(\omega s)+q_2(s)\sin(\omega s)] \\
& +p_1(s)\cos(\omega s)-p_2(s)\sin(\omega s)\}\sin\left[\frac{n\pi c}{L}(t-s)\right]ds \\
& +\frac{2}{n\pi}\{(-1)^n[q_1(t)\cos(\omega t)-q_2(t)\sin(\omega t)] \\
& -p_1(t)\cos(\omega t)+p_2(t)\sin(\omega t)\}, \qquad (5.3.29a)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Im} Z_n(t) \;=\; & \frac{2}{L}\cos\left(\frac{n\pi c}{L}t\right)\int_0^L f_2(x)\sin\frac{n\pi x}{L}dx \\
& +\frac{2}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right)\int_0^L[\omega f_1(x)+g_2(x)]\sin\frac{n\pi x}{L}dx \\
& +\frac{2c}{L}\int_0^t \{(-1)^n[-q_1(s)\sin(\omega s)-q_2(s)\cos(\omega s)] \\
& +p_1(s)\sin(\omega s)+p_2(s)\cos(\omega s)\}\sin\left[\frac{n\pi c}{L}(t-s)\right]ds \\
& +\frac{2}{n\pi}\{(-1)^n[q_1(t)\sin(\omega t)+q_2(t)\cos(\omega t)] \\
& -p_1(t)\sin(\omega t)-p_2(t)\cos(\omega t)\}. \qquad (5.3.29b)
\end{aligned}
$$

From (5.3.6), we also have

$$
\begin{aligned}
\operatorname{Re} p(t) &= p_1(t)\cos(\omega t)-p_2(t)\sin(\omega t), \\
\operatorname{Re}[p(t)-q(t)] &= \cos(\omega t)[p_1(t)-q_1(t)] \\
& \quad -\sin(\omega t)[p_2(t)-q_2(t)], \\
\operatorname{Im} p(t) &= p_1(t)\sin(\omega t)+p_2(t)\cos(\omega t), \\
\operatorname{Im}[p(t)-q(t)] &= \sin(\omega t)[p_1(t)-q_1(t)] \\
& \quad +\cos(\omega t)[p_2(t)-q_2(t)].
\end{aligned}
$$

However, we recall that by definition

$$
W(x,t)=Z(x,t)-\frac{x}{L}[p(t)-q(t)]+p(t),
$$

so the real and imaginary parts of $W(x,t)$ are given respectively by

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \operatorname{Re} Z(x,t) - \frac{x}{L}\operatorname{Re}[p(t)-q(t)] + \operatorname{Re} p(t) \\
&= \sum_{n=1}^{\infty} \operatorname{Re} Z_n(t) \sin\frac{n\pi x}{L} \\
&\quad -\frac{x}{L}\{\cos(\omega t)[p_1(t)-q_1(t)] \\
&\quad -\sin(\omega t)[p_2(t)-q_2(t)]\} \\
&\quad +p_1(t)\cos(\omega t) - p_2(t)\sin(\omega t), \qquad (5.3.30a)
\end{aligned}$$

$$\begin{aligned}
\operatorname{Im} W(x,t) &= \operatorname{Im} Z(x,t) - \frac{x}{L}\operatorname{Im}[p(t)-q(t)] + \operatorname{Im} p(t) \\
&= \sum_{n=1}^{\infty} \operatorname{Im} Z_n(t) \sin\frac{n\pi x}{L} \\
&\quad -\frac{x}{L}\{\sin(\omega t)[p_1(t)-q_1(t)] \\
&\quad +\cos(\omega t)[p_2(t)-q_2(t)]\} \\
&\quad +p_1(t)\sin(\omega t) + p_2(t)\cos(\omega t). \qquad (5.3.30b)
\end{aligned}$$

Finally, in terms of Euler's identity, (5.3.4) becomes

$$\begin{aligned}
V(x,t) &= e^{-i\omega t}W(x,t) \\
&= [\cos(\omega t) - i\sin(\omega t)][\operatorname{Re} W(x,t) + i\operatorname{Im} W(x,t)] \\
&= \cos(\omega t)\operatorname{Re} W(x,t) + \sin(\omega t)\operatorname{Im} W(x,t) \\
&\quad +i[-\sin(\omega t)\operatorname{Re} W(x,t) + \cos(\omega t)\operatorname{Im} W(x,t)]. \qquad (5.3.31)
\end{aligned}$$

But, in terms of our original notation

$$V(x,t) = u_1(x,t) + iu_2(x,t), \qquad (5.3.32)$$

so that the orthogonal transverse displacements $u_1(x,t)$ and $u_2(x,t)$ of the string at position x at time t are given by

$$u_1(x,t) = \cos(\omega t)\,\mathrm{Re}\,W(x,t) + \sin(\omega t)\,\mathrm{Im}\,W(x,t), \tag{5.3.33a}$$

and

$$u_2(x,t) = -\sin(\omega t)\,\mathrm{Re}\,W(x,t) + \cos(\omega t)\,\mathrm{Im}\,W(x,t), \tag{5.3.33b}$$

respectively. Substitution of (5.3.30) into (5.3.33) yields

$$\begin{aligned} u_1(x,t) &= \cos(\omega t)\sum_{n=1}^{\infty}\mathrm{Re}\,Z_n(t)\sin\frac{n\pi x}{L} \\ &\quad +\sin(\omega t)\sum_{n=1}^{\infty}\mathrm{Im}\,Z_n(t)\sin\frac{n\pi x}{L} \\ &\quad -\frac{x}{L}[p_1(t)-q_1(t)]+p_1(t), \qquad (5.3.34a) \\ u_2(x,t) &= -\sin(\omega t)\sum_{n=1}^{\infty}\mathrm{Re}\,Z_n(t)\sin\frac{n\pi x}{L} \\ &\quad +\cos(\omega t)\sum_{n=1}^{\infty}\mathrm{Im}\,Z_n(t)\sin\frac{n\pi x}{L} \\ &\quad -\frac{x}{L}[p_2(t)-q_2(t)]+p_2(t), \qquad (5.3.34b) \end{aligned}$$

where $\mathrm{Re}\,Z_n(t)$ and $\mathrm{Im}\,Z_n(t)$ are given in (5.3.29). The Fourier sine series solution of the fixed-length spinning string problem (5.3.1-2) is determined by the results displayed in (5.3.34) and (5.3.29).

In the following section, we will exhibit several illustrative examples of the solution of this problem for various sets of initial and boundary conditions.

5.4 Examples

In this section we illustrate the theory of the previous section by means of several examples. Indeed, the first four examples illustrate how the Fourier series analysis of the previous section may be applied to find the rotating standing wave solutions of sections 2.4 and 2.5 but on a finite string of length L. Thus, the appropriate initial and boundary conditions are obtained directly from the examples of sections 2.4 and 2.5 and are used to specify the choices of the functions appearing in (5.3.2). Other parameter values are chosen to coincide with the values in the corresponding examples from sections 2.4 and 2.5.

The remaining three examples are chosen to illustrate how the Fourier series analysis of the previous section may be applied to find the "pulse wave" solutions of section 4.3 but again on a finite string of length L. As in the previous cases, other parameter values are chosen to be consistent with the values from the corresponding examples of section 4.3.

In all cases, the Fourier series solution is truncated after a finite number of terms in order to produce graphs corresponding to those of the previous sections. It should be noted that small differences between the graphs in this section and those of the corresponding examples of the previous sections occur, as a result of this truncation and the fact that the previous graphs illustrate the behaviour of waves on a finite section of a semi-infinite string, while the current graphs illustrate the corresponding behaviour on a finite string. Nonetheless, the similarity of the corresponding graphs is strikingly evident!

Example 5.4.1: In the fixed-length string problem (5.3.1-2), let

$$c = \omega = 1, \;\; L = 7\pi,$$

and assume initial and boundary conditions which are consistent with example 2.4.1, namely

$$f_1(x) = 2\sin\frac{x}{2},\ g_1(x) = -\sin\frac{x}{2},$$

$$f_2(x) = -2\sin\frac{x}{2},\ g_2(x) = -\sin\frac{x}{2},$$

$$0 \le x \le 7\pi,$$

$$p_1(t) = p_2(t) = 0,\ (\text{at } x = 0)$$

$$q_1(t) = 2(\sin\frac{t}{2} - \cos\frac{t}{2}),\ q_2(t) = 2(\sin\frac{t}{2} + \cos\frac{t}{2}),\ (\text{at } x = 7\pi)$$

$$t \ge 0.$$

Putting the above functions in (5.3.29) and (5.3.34), we obtain the corresponding Fourier series solution, as given by

$$\begin{aligned}
u_1(x,t) &= \cos t\sum_{n=1}^{\infty}\operatorname{Re} Z_n(t)\sin\frac{nx}{7} + \sin t\sum_{n=1}^{\infty}\operatorname{Im} Z_n(t)\sin\frac{nx}{7} \\
&\quad + \frac{2x}{7\pi}(\sin\frac{t}{2} - \cos\frac{t}{2}), \\
u_2(x,t) &= -\sin t\sum_{n=1}^{\infty}\operatorname{Re} Z_n(t)\sin\frac{nx}{7} + \cos t\sum_{n=1}^{\infty}\operatorname{Im} Z_n(t)\sin\frac{nx}{7} \\
&\quad + \frac{2x}{7\pi}(\sin\frac{t}{2} + \cos\frac{t}{2}),
\end{aligned}$$

in which

$$\begin{aligned}
\operatorname{Re} Z_n(t) &= \left[\frac{4}{7\pi}\cos\left(\frac{n}{7}t\right) + \frac{2}{n\pi}\sin\left(\frac{n}{7}t\right)\right]\int_0^{7\pi}\sin\frac{x}{2}\sin\frac{nx}{7}dx \\
&\quad + (-1)^n\frac{4}{7\pi}\left\{\int_0^t \sin\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds + \int_0^t \cos\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds\right\} \\
&\quad - (-1)^n\frac{4}{n\pi}(\sin\frac{t}{2} + \cos\frac{t}{2}), \\
\operatorname{Im} Z_n(t) &= \left[-\frac{4}{7\pi}\cos\left(\frac{n}{7}t\right) + \frac{2}{n\pi}\sin\left(\frac{n}{7}t\right)\right]\int_0^{7\pi}\sin\frac{x}{2}\sin\frac{nx}{7}dx
\end{aligned}$$

$$+(-1)^n \frac{4}{7\pi}\left\{\int_0^t \sin\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds - \int_0^t \cos\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds\right\}$$
$$-(-1)^n \frac{4}{n\pi}(\sin\frac{t}{2} - \cos\frac{t}{2}).$$

Simplification of the above results requires the evaluation of the integrals $\int_0^{7\pi}\sin\frac{x}{2}\sin\frac{nx}{7}dx$, $\int_0^t \sin\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds$ and $\int_0^t \cos\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds$.

Since

$$\begin{aligned}\int \sin\frac{x}{2}\sin\frac{nx}{7}dx &= \frac{\sin(\frac{1}{2}-\frac{n}{7})x}{1-\frac{2n}{7}} - \frac{\sin(\frac{1}{2}+\frac{n}{7})x}{1+\frac{2n}{7}}\\ &= \frac{7}{7-2n}\sin\frac{(7-2n)x}{14} - \frac{7}{7+2n}\sin\frac{(7+2n)x}{14},\end{aligned}$$

$$\begin{aligned}\int \sin\frac{x}{2}\cos\frac{nx}{7}dx &= -\frac{\cos(\frac{1}{2}-\frac{n}{7})x}{1-\frac{2n}{7}} - \frac{\cos(\frac{1}{2}+\frac{n}{7})x}{1+\frac{2n}{7}}\\ &= -\frac{7}{7-2n}\cos\frac{(7-2n)x}{14} - \frac{7}{7+2n}\cos\frac{(7+2n)x}{14},\end{aligned}$$

$$\begin{aligned}\int \cos\frac{x}{2}\sin\frac{nx}{7}dx &= \frac{\cos(\frac{1}{2}-\frac{n}{7})x}{1-\frac{2n}{7}} - \frac{\cos(\frac{1}{2}+\frac{n}{7})x}{1+\frac{2n}{7}}\\ &= \frac{7}{7-2n}\cos\frac{(7-2n)x}{14} - \frac{7}{7+2n}\cos\frac{(7+2n)x}{14},\end{aligned}$$

and

$$\begin{aligned}\int \cos\frac{x}{2}\cos\frac{nx}{7}dx &= \frac{\sin(\frac{1}{2}-\frac{n}{7})x}{1-\frac{2n}{7}} + \frac{\sin(\frac{1}{2}+\frac{n}{7})x}{1+\frac{2n}{7}}\\ &= \frac{7}{7-2n}\sin\frac{(7-2n)x}{14} + \frac{7}{7+2n}\sin\frac{(7+2n)x}{14},\end{aligned}$$

we have

$$\begin{aligned}&\int_0^{7\pi}\sin\frac{x}{2}\sin\frac{nx}{7}dx\\ =\ &\left[\frac{7}{7-2n}\sin\frac{(7-2n)x}{14} - \frac{7}{7+2n}\sin\frac{(7+2n)x}{14}\right]_0^{7\pi}\end{aligned}$$

$$
\begin{aligned}
&= \frac{7}{7-2n}\sin(\frac{7\pi}{2}-n\pi) - \frac{7}{7+2n}\sin(\frac{7\pi}{2}+n\pi) \\
&= -\frac{7}{7-2n}(-1)^n + \frac{7}{7+2n}(-1)^n \\
&= (-1)^n \frac{28n}{4n^2-49},
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t \sin\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds \\
= {}& \sin\frac{nt}{7}\int_0^t \sin\frac{s}{2}\cos\frac{ns}{7}ds - \cos\frac{nt}{7}\int_0^t \sin\frac{s}{2}\sin\frac{ns}{7}ds \\
= {}& \sin\frac{nt}{7}\left[-\frac{7}{7-2n}\cos\frac{(7-2n)s}{14} - \frac{7}{7+2n}\cos\frac{(7+2n)s}{14}\right]_0^t \\
& -\cos\frac{nt}{7}\left[\frac{7}{7-2n}\sin\frac{(7-2n)s}{14} - \frac{7}{7+2n}\sin\frac{(7+2n)s}{14}\right]_0^t \\
= {}& \sin\frac{nt}{7}\left[-\frac{7}{7-2n}\cos\frac{(7-2n)t}{14} - \frac{7}{7+2n}\cos\frac{(7+2n)t}{14} + \frac{7}{7-2n} + \frac{7}{7+2n}\right] \\
& -\cos\frac{nt}{7}\left[\frac{7}{7-2n}\sin\frac{(7-2n)t}{14} - \frac{7}{7+2n}\sin\frac{(7+2n)t}{14}\right] \\
= {}& -\frac{7}{7-2n}\sin\left[\frac{nt}{7}+\frac{(7-2n)t}{14}\right] - \frac{7}{7+2n}\sin\left[\frac{nt}{7}-\frac{(7+2n)t}{14}\right] + \frac{98}{49-4n^2}\sin\frac{nt}{7} \\
= {}& -\frac{7}{7-2n}\sin\frac{t}{2} + \frac{7}{7+2n}\sin\frac{t}{2} + \frac{98}{49-4n^2}\sin\frac{nt}{7} \\
= {}& \frac{14}{49-4n^2}\left[7\sin\frac{nt}{7} - 2n\sin\frac{t}{2}\right],
\end{aligned}
$$

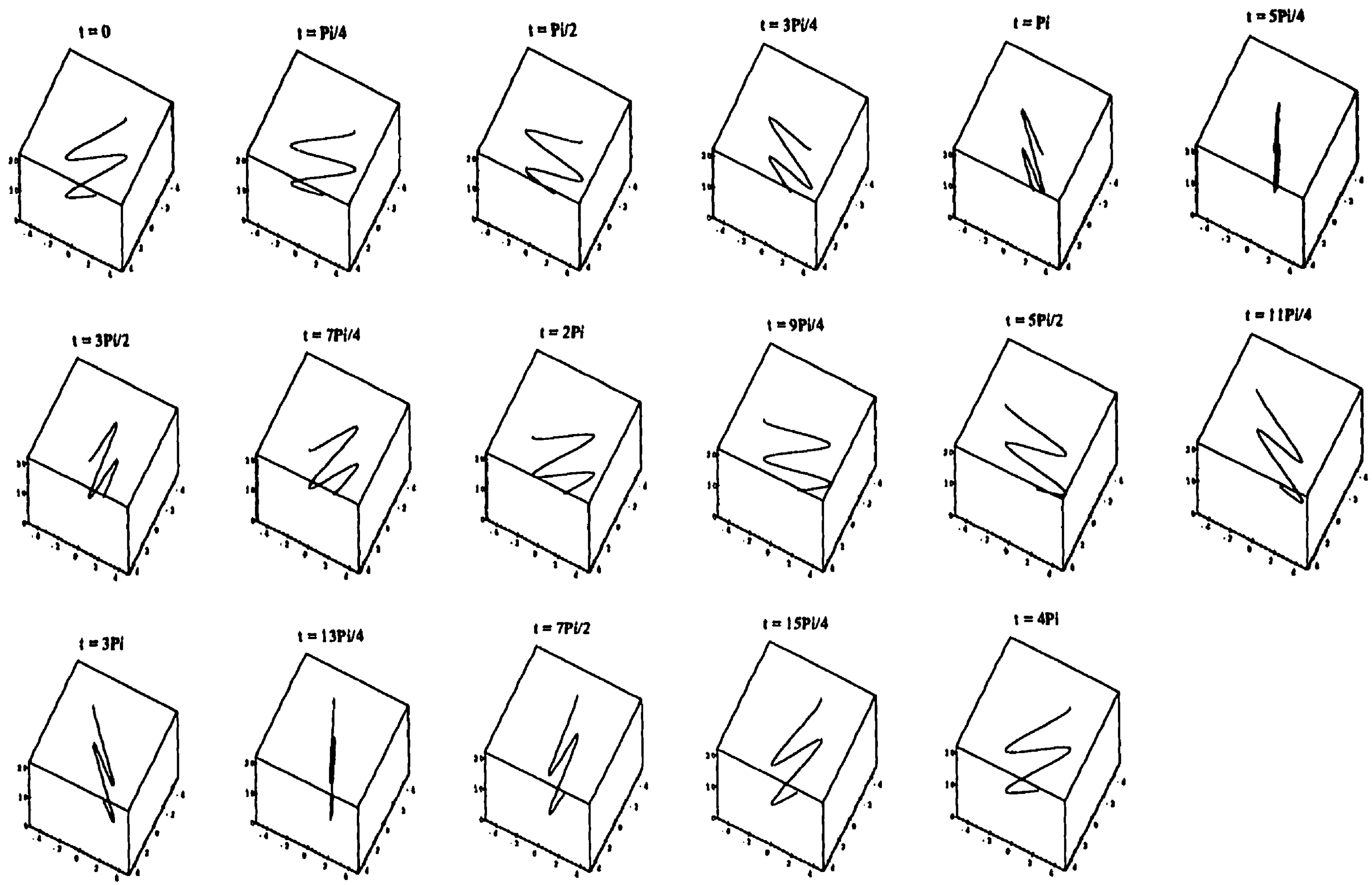
and

$$
\begin{aligned}
&\int_0^t \cos\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds \\
= {}& \sin\frac{nt}{7}\int_0^t \cos\frac{s}{2}\cos\frac{ns}{7}ds - \cos\frac{nt}{7}\int_0^t \cos\frac{s}{2}\sin\frac{ns}{7}ds \\
= {}& \sin\frac{nt}{7}\left[\frac{7}{7-2n}\sin\frac{(7-2n)s}{14} + \frac{7}{7+2n}\sin\frac{(7+2n)s}{14}\right]_0^t \\
& -\cos\frac{nt}{7}\left[\frac{7}{7-2n}\cos\frac{(7-2n)s}{14} - \frac{7}{7+2n}\cos\frac{(7+2n)s}{14}\right]_0^t \\
= {}& \sin\frac{nt}{7}\left[\frac{7}{7-2n}\sin\frac{(7-2n)t}{14} + \frac{7}{7+2n}\sin\frac{(7+2n)t}{14}\right] \\
& -\cos\frac{nt}{7}\left[\frac{7}{7-2n}\cos\frac{(7-2n)t}{14} - \frac{7}{7+2n}\cos\frac{(7+2n)t}{14} - \frac{7}{7-2n} + \frac{7}{7+2n}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{7}{7-2n}\cos\left[\frac{nt}{7}+\frac{(7-2n)t}{14}\right]+\frac{7}{7+2n}\cos\left[\frac{nt}{7}-\frac{(7+2n)t}{14}\right]+\frac{28n}{49-4n^2}\cos\frac{nt}{7} \\
&= -\frac{7}{7-2n}\cos\frac{t}{2}+\frac{7}{7+2n}\cos\frac{t}{2}+\frac{28n}{49-4n^2}\cos\frac{nt}{7} \\
&= \frac{28n}{49-4n^2}\left[\cos\frac{nt}{7}-\cos\frac{t}{2}\right].
\end{aligned}
$$

For the purpose of graphing the solution in this case, we truncate the Fourier series after 20 terms, as illustrated in Figure 5.4.1. A comparison of Figure 5.4.1 with Figure 2.4.1 shows that the truncated Fourier series provides a very reasonable approximation to the standing wave solution of example 2.4.1, even though the spatial dimension ($0 \leq x \leq 7\pi$) of the "graphical window" is somewhat larger than in Figure 2.4.1.

Figure 5.4.1: graphic illustration of Example 5.4.1 (20 term Fourier series approximation) where c=ω=1, L=7π



Example 5.4.2: For the fixed-length string problem (5.3.1-2), we let

$$c = \omega = 1,\ L = 7\pi,$$

with initial and boundary conditions determined by example 2.4.2, i.e.,

$$f_1(x) = \sin\frac{x}{2},\ g_1(x) = \sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2},$$

$$f_2(x) = 2\sin\frac{x}{2} + \cos\frac{x}{2},\ g_2(x) = -\frac{1}{2}\sin\frac{x}{2},$$

$$0 \le x \le 7\pi,$$

$$p_1(t) = \sin\frac{t}{2},\ p_2(t) = \cos\frac{t}{2},\ (\text{at } x = 0)$$

$$q_1(t) = -2\sin\frac{t}{2} - \cos\frac{t}{2},\ q_2(t) = -2\cos\frac{t}{2} + \sin\frac{t}{2},\ (\text{at } x = 7\pi)$$

$$t \ge 0.$$

Putting the above functions in (5.3.29) and (5.3.34), we obtain the Fourier series solution, as given by

$$\begin{aligned} u_1(x,t) &= \cos t\sum_{n=1}^{\infty}\operatorname{Re} Z_n(t)\sin\frac{nx}{7} + \sin t\sum_{n=1}^{\infty}\operatorname{Im} Z_n(t)\sin\frac{nx}{7} \\ &\quad -\frac{x}{7\pi}(3\sin\frac{t}{2} + \cos\frac{t}{2}) + \sin\frac{t}{2}, \\ u_2(x,t) &= -\sin t\sum_{n=1}^{\infty}\operatorname{Re} Z_n(t)\sin\frac{nx}{7} + \cos t\sum_{n=1}^{\infty}\operatorname{Im} Z_n(t)\sin\frac{nx}{7} \\ &\quad -\frac{x}{7\pi}(3\cos\frac{t}{2} - \sin\frac{t}{2}) + \cos\frac{t}{2}, \end{aligned}$$

in which

$$\begin{aligned} \operatorname{Re} Z_n(t) &= \left[\frac{2}{7\pi}\cos\left(\frac{n}{7}t\right) - \frac{2}{n\pi}\sin\left(\frac{n}{7}t\right)\right]\int_0^{7\pi}\sin\frac{x}{2}\sin\frac{nx}{7}dx \\ &\quad -\frac{1}{n\pi}\sin\left(\frac{n}{7}t\right)\int_0^{7\pi}\cos\frac{x}{2}\sin\frac{nx}{7}dx \end{aligned}$$

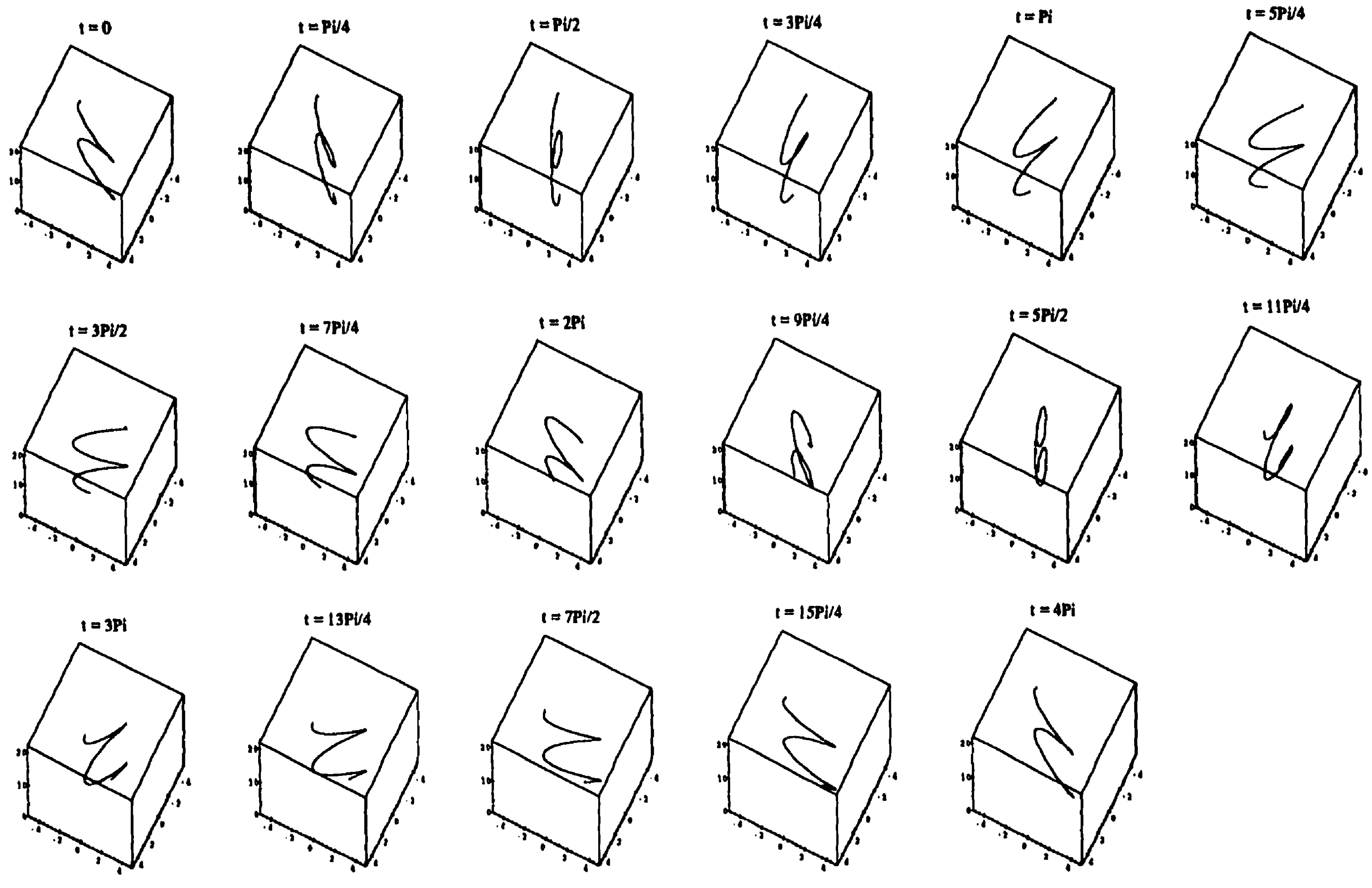
$$
\begin{aligned}
& +\frac{2}{7\pi}\left\{-[1+2(-1)^n]\int_0^t \sin\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds\right. \\
& \left.+(-1)^n\int_0^t \cos\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds\right\} \\
& +\frac{2}{n\pi}[(-1)^n(2\sin\frac{t}{2}-\cos\frac{t}{2})+\sin\frac{t}{2}], \\
\operatorname{Im} Z_n(t) \;=\; & \left[\frac{4}{7\pi}\cos\left(\frac{n}{7}t\right)+\frac{1}{n\pi}\sin\left(\frac{n}{7}t\right)\right]\int_0^{7\pi}\sin\frac{x}{2}\sin\frac{nx}{7}dx \\
& +\frac{2}{7\pi}\cos\left(\frac{n}{7}t\right)\int_0^{7\pi}\cos\frac{x}{2}\sin\frac{nx}{7}dx \\
& +\frac{2}{7\pi}\left\{[1+2(-1)^n]\int_0^t \cos\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds\right. \\
& \left.+(-1)^n\int_0^t \sin\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds\right\} \\
& -\frac{2}{n\pi}[(-1)^n(2\cos\frac{t}{2}+\sin\frac{t}{2})+\cos\frac{t}{2}].
\end{aligned}
$$

The evaluation of the integrals $\int_0^{7\pi}\sin\frac{x}{2}\sin\frac{nx}{7}dx$, $\int_0^t\sin\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds$ and $\int_0^t\cos\frac{s}{2}\sin\left[\frac{n}{7}(t-s)\right]ds$ was given in example 5.4.1. The evaluation of the integral $\int_0^{7\pi}\cos\frac{x}{2}\sin\frac{nx}{7}dx$ is given by

$$
\begin{aligned}
& \int_0^{7\pi}\cos\frac{x}{2}\sin\frac{nx}{7}dx \\
= & \left[\frac{7}{7-2n}\cos\frac{(7-2n)x}{14}-\frac{7}{7+2n}\cos\frac{(7+2n)x}{14}\right]_0^{7\pi} \\
= & \frac{7}{7-2n}\cos(\frac{7\pi}{2}-n\pi)-\frac{7}{7+2n}\cos(\frac{7\pi}{2}+n\pi)-\frac{7}{7-2n}+\frac{7}{7+2n} \\
= & -\frac{28n}{49-4n^2}.
\end{aligned}
$$

Again a 20-term Fourier approximation to the solution for $0\leq x\leq 7\pi$ is plotted for various times in Figure 5.4.2, results of which compare favourably with those in Figure 2.4.2, at least for $0\leq x\leq 20$.

Figure 5.4.2: graphic illustration of Example 5.4.2 (20 term Fourier series approximation) where c=ω=1, L=7π



Example 5.4.3: In the fixed-length string problem (5.3.1-2), we let

$$c = \omega = \frac{2}{3},\ L = 9\pi,$$

with the initial and boundary conditions prescribed by example 2.5.1, namely

$$f_1(x) = f_2(x) = 2(\sin\frac{x}{4} + \sin\frac{x}{2}),$$

$$g_1(x) = -\,g_2(x) = \sin\frac{x}{4} + 2\sin\frac{x}{2},$$

$$0 \le x \le 9\pi,$$

$$p_1(t) = p_2(t) = 0,\ (\text{at } x = 0)$$

$$\begin{aligned} q_1(t) &= \sqrt{2}(\cos\frac{t}{2} + \sin\frac{t}{2}) + 2(\cos t + \sin t),\ (\text{at } x = 9\pi) \\ q_2(t) &= \sqrt{2}(\cos\frac{t}{2} - \sin\frac{t}{2}) + 2(\cos t - \sin t),\ (\text{at } x = 9\pi) \end{aligned}$$

$$t \ge 0.$$

Putting the above functions in (5.3.29) and (5.3.34), we obtain the Fourier series solution, as given by

$$\begin{aligned} u_1(x,t) &= \cos\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\operatorname{Re} Z_n(t)\sin\frac{nx}{9} + \sin\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\operatorname{Im} Z_n(t)\sin\frac{nx}{9} \\ &\quad + \frac{x}{9\pi}\left[\sqrt{2}(\cos\frac{t}{2} + \sin\frac{t}{2}) + 2(\cos t + \sin t)\right], \\ u_2(x,t) &= -\sin\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\operatorname{Re} Z_n(t)\sin\frac{nx}{9} + \cos\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\operatorname{Im} Z_n(t)\sin\frac{nx}{9} \\ &\quad + \frac{x}{9\pi}\left[\sqrt{2}(\cos\frac{t}{2} - \sin\frac{t}{2}) + 2(\cos t - \sin t)\right], \end{aligned}$$

in which

$$\operatorname{Re} Z_n(t) = \left[\frac{4}{9\pi}\cos\left(\frac{2n}{27}t\right) + \frac{2}{n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{2}\sin\frac{nx}{9}dx$$

$$
\begin{aligned}
& +\left[\frac{4}{9\pi}\cos\left(\frac{2n}{27}t\right)-\frac{1}{n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{4}\sin\frac{nx}{9}dx \\
& -(-1)^n\frac{4}{27\pi}\left\{\sqrt{2}\int_0^t\cos\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds\right. \\
& -\sqrt{2}\int_0^t\sin\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds+2\int_0^t\cos\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
& \left.+2\int_0^t\sin\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds\right\} \\
& +(-1)^n\frac{2}{n\pi}\left[\sqrt{2}(\cos\frac{t}{6}-\sin\frac{t}{6})+2(\cos\frac{t}{3}+\sin\frac{t}{3})\right], \\
\operatorname{Im} Z_n(t) \;=\; & \left[\frac{4}{9\pi}\cos\left(\frac{2n}{27}t\right)-\frac{2}{n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{2}\sin\frac{nx}{9}dx \\
& +\left[\frac{4}{9\pi}\cos\left(\frac{2n}{27}t\right)+\frac{1}{n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{4}\sin\frac{nx}{9}dx \\
& -(-1)^n\frac{4}{27\pi}\left\{\sqrt{2}\int_0^t\cos\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds\right. \\
& +\sqrt{2}\int_0^t\sin\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds+2\int_0^t\cos\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
& \left.-2\int_0^t\sin\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds\right\} \\
& +(-1)^n\frac{2}{n\pi}\left[\sqrt{2}(\cos\frac{t}{6}+\sin\frac{t}{6})+2(\cos\frac{t}{3}-\sin\frac{t}{3})\right].
\end{aligned}
$$

The evaluation of the integrals in the above two expressions may be accomplished as in the previous examples.

Since

$$
\begin{aligned}
\int\sin\frac{x}{2}\sin\frac{nx}{9}dx &= \frac{\sin(\frac{1}{2}-\frac{n}{9})x}{1-\frac{2n}{9}}-\frac{\sin(\frac{1}{2}+\frac{n}{9})x}{1+\frac{2n}{9}} \\
&= \frac{9}{9-2n}\sin\frac{(9-2n)x}{18}-\frac{9}{9+2n}\sin\frac{(9+2n)x}{18}
\end{aligned}
$$

$$
\begin{aligned}
\int\sin\frac{x}{4}\sin\frac{nx}{9}dx &= \frac{\sin(\frac{1}{4}-\frac{n}{9})x}{\frac{1}{2}-\frac{2n}{9}}-\frac{\sin(\frac{1}{4}+\frac{n}{9})x}{\frac{1}{2}+\frac{2n}{9}} \\
&= \frac{18}{9-4n}\sin\frac{(9-4n)x}{36}-\frac{18}{9+4n}\sin\frac{(9+4n)x}{36},
\end{aligned}
$$

$$
\int\sin\frac{s}{6}\sin\frac{2ns}{27}ds = \frac{\sin(\frac{1}{6}-\frac{2n}{27})x}{\frac{1}{3}-\frac{4n}{27}}-\frac{\sin(\frac{1}{6}+\frac{2n}{27})x}{\frac{1}{3}+\frac{4n}{27}}
$$

$$= \frac{27}{9-4n}\sin\frac{(9-4n)x}{54} - \frac{27}{9+4n}\sin\frac{(9+4n)x}{54},$$

$$\begin{aligned}
\int \sin\frac{s}{6}\cos\frac{2ns}{27}ds &= -\frac{\cos(\frac{1}{6}-\frac{2n}{27})x}{\frac{1}{3}-\frac{4n}{27}} - \frac{\cos(\frac{1}{6}+\frac{2n}{27})x}{\frac{1}{3}+\frac{4n}{27}} \\
&= -\frac{27}{9-4n}\cos\frac{(9-4n)x}{54} - \frac{27}{9+4n}\cos\frac{(9+4n)x}{54},
\end{aligned}$$

$$\begin{aligned}
\int \cos\frac{s}{6}\sin\frac{2ns}{27}ds &= \frac{\cos(\frac{1}{6}-\frac{2n}{27})x}{\frac{1}{3}-\frac{4n}{27}} - \frac{\cos(\frac{1}{6}+\frac{2n}{27})x}{\frac{1}{3}+\frac{4n}{27}} \\
&= \frac{27}{9-4n}\cos\frac{(9-4n)x}{54} - \frac{27}{9+4n}\cos\frac{(9+4n)x}{54},
\end{aligned}$$

$$\begin{aligned}
\int \cos\frac{s}{6}\cos\frac{2ns}{27}ds &= \frac{\sin(\frac{1}{6}-\frac{2n}{27})x}{\frac{1}{3}-\frac{4n}{27}} + \frac{\sin(\frac{1}{6}+\frac{2n}{27})x}{\frac{1}{3}+\frac{4n}{27}} \\
&= \frac{27}{9-4n}\sin\frac{(9-4n)x}{54} + \frac{27}{9+4n}\sin\frac{(9+4n)x}{54},
\end{aligned}$$

$$\begin{aligned}
\int \sin\frac{s}{3}\sin\frac{2ns}{27}ds &= \frac{\sin(\frac{1}{3}-\frac{2n}{27})x}{\frac{2}{3}-\frac{4n}{27}} - \frac{\sin(\frac{1}{3}+\frac{2n}{27})x}{\frac{2}{3}+\frac{4n}{27}} \\
&= \frac{27}{18-4n}\sin\frac{(9-2n)x}{27} - \frac{27}{18+4n}\sin\frac{(9+2n)x}{27},
\end{aligned}$$

$$\begin{aligned}
\int \sin\frac{s}{3}\cos\frac{2ns}{27}ds &= -\frac{\cos(\frac{1}{3}-\frac{2n}{27})x}{\frac{2}{3}-\frac{4n}{27}} - \frac{\cos(\frac{1}{3}+\frac{2n}{27})x}{\frac{2}{3}+\frac{4n}{27}} \\
&= -\frac{27}{18-4n}\cos\frac{(9-2n)x}{27} - \frac{27}{18+4n}\cos\frac{(9+2n)x}{27},
\end{aligned}$$

$$\begin{aligned}
\int \cos\frac{s}{3}\sin\frac{2ns}{27}ds &= \frac{\cos(\frac{1}{3}-\frac{2n}{27})x}{\frac{2}{3}-\frac{4n}{27}} - \frac{\cos(\frac{1}{3}+\frac{2n}{27})x}{\frac{2}{3}+\frac{4n}{27}} \\
&= \frac{27}{18-4n}\cos\frac{(9-2n)x}{27} - \frac{27}{18+4n}\cos\frac{(9+2n)x}{27},
\end{aligned}$$

and

$$\begin{aligned}
\int \cos\frac{s}{3}\cos\frac{2ns}{27}ds &= \frac{\sin(\frac{1}{3}-\frac{2n}{27})x}{\frac{2}{3}-\frac{4n}{27}} + \frac{\sin(\frac{1}{3}+\frac{2n}{27})x}{\frac{2}{3}+\frac{4n}{27}} \\
&= \frac{27}{18-4n}\sin\frac{(9-2n)x}{27} + \frac{27}{18+4n}\sin\frac{(9+2n)x}{27},
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^{9\pi} \sin\frac{x}{2}\sin\frac{nx}{9}dx \\
= &\left[\frac{9}{9-2n}\sin\frac{(9-2n)x}{18} - \frac{9}{9+2n}\sin\frac{(9+2n)x}{18}\right]_0^{9\pi} \\
= &\frac{9}{9-2n}\sin(\frac{9\pi}{2}-n\pi) - \frac{9}{9+2n}\sin(\frac{9\pi}{2}+n\pi) \\
= &\frac{9}{9-2n}(-1)^n - \frac{9}{9+2n}(-1)^n \\
= &(-1)^n\frac{36n}{81-4n^2},
\end{aligned}$$

$$\begin{aligned}
&\int_0^{9\pi} \sin\frac{x}{4}\sin\frac{nx}{9}dx \\
= &\left[\frac{18}{9-4n}\sin\frac{(9-4n)x}{36} - \frac{18}{9+4n}\sin\frac{(9+4n)x}{36}\right]_0^{9\pi} \\
= &\frac{18}{9-4n}\sin(\frac{9\pi}{4}-n\pi) - \frac{18}{9+4n}\sin(\frac{9\pi}{4}+n\pi) \\
= &(-1)^n\frac{18}{9-4n}\cos\frac{\pi}{4} - (-1)^n\frac{18}{9+4n}\cos\frac{\pi}{4} \\
= &(-1)^n\frac{72\sqrt{2}n}{81-16n^2},
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \cos\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
= &\sin\frac{2nt}{27}\int_0^t \cos\frac{s}{6}\cos\frac{2ns}{27}ds - \cos\frac{2nt}{27}\int_0^t \cos\frac{s}{6}\sin\frac{2ns}{27}ds \\
= &\sin\frac{2nt}{27}\left[\frac{27}{9-4n}\sin\frac{(9-4n)s}{54} + \frac{27}{9+4n}\sin\frac{(9+4n)s}{54}\right]_0^t \\
&-\cos\frac{2nt}{27}\left[\frac{27}{9-4n}\cos\frac{(9-4n)s}{54} - \frac{27}{9+4n}\cos\frac{(9+4n)s}{54}\right]_0^t \\
= &\sin\frac{2nt}{27}\left[\frac{27}{9-4n}\sin\frac{(9-4n)t}{54} + \frac{27}{9+4n}\sin\frac{(9+4n)t}{54}\right] \\
&-\cos\frac{2nt}{27}\left[\frac{27}{9-4n}\cos\frac{(9-4n)t}{54} - \frac{27}{9+4n}\cos\frac{(9+4n)t}{54} - \frac{27}{9-4n} + \frac{27}{9+4n}\right] \\
= &-\frac{27}{9-4n}\cos\left[\frac{2nt}{27} + \frac{(9-4n)t}{54}\right] + \frac{27}{9+4n}\cos\left[\frac{2nt}{27} - \frac{(9+4n)t}{54}\right] \\
&+\frac{216n}{81-16n^2}\cos\frac{2nt}{27}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{27}{9-4n}\cos\frac{t}{6}+\frac{27}{9+4n}\cos\frac{t}{6}+\frac{216n}{81-16n^2}\cos\frac{2nt}{27} \\
&= \frac{216n}{81-16n^2}\left[\cos\frac{2nt}{27}-\cos\frac{t}{6}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t \sin\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
&= \sin\frac{2nt}{27}\int_0^t \sin\frac{s}{6}\cos\frac{2ns}{27}ds-\cos\frac{2nt}{27}\int_0^t \sin\frac{s}{6}\sin\frac{2ns}{27}ds \\
&= \sin\frac{2nt}{27}\left[-\frac{27}{9-4n}\cos\frac{(9-4n)s}{54}-\frac{27}{9+4n}\cos\frac{(9+4n)s}{54}\right]_0^t \\
&\quad -\cos\frac{2nt}{27}\left[\frac{27}{9-4n}\sin\frac{(9-4n)s}{54}-\frac{27}{9+4n}\sin\frac{(9+4n)s}{54}\right]_0^t \\
&= \sin\frac{2nt}{27}\left[-\frac{27}{9-4n}\cos\frac{(9-4n)t}{54}-\frac{27}{9+4n}\cos\frac{(9+4n)t}{54}+\frac{27}{9-4n}+\frac{27}{9+4n}\right] \\
&\quad -\cos\frac{2nt}{27}\left[\frac{27}{9-4n}\sin\frac{(9-4n)t}{54}-\frac{27}{9+4n}\sin\frac{(9+4n)t}{54}\right] \\
&= -\frac{27}{9-4n}\sin\left[\frac{2nt}{27}+\frac{(9-4n)t}{54}\right]-\frac{27}{9+4n}\sin\left[\frac{2nt}{27}-\frac{(9+4n)t}{54}\right] \\
&\quad +\frac{486}{81-16n^2}\sin\frac{2nt}{27} \\
&= -\frac{27}{9-4n}\sin\frac{t}{6}+\frac{27}{9+4n}\sin\frac{t}{6}+\frac{486}{81-16n^2}\sin\frac{2nt}{27} \\
&= \frac{54}{81-16n^2}\left[9\sin\frac{2nt}{27}-4n\sin\frac{t}{6}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t \cos\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
&= \sin\frac{2nt}{27}\int_0^t \cos\frac{s}{3}\cos\frac{2ns}{27}ds-\cos\frac{2nt}{27}\int_0^t \cos\frac{s}{3}\sin\frac{2ns}{27}ds \\
&= \sin\frac{2nt}{27}\left[\frac{27}{18-4n}\sin\frac{(9-2n)s}{27}+\frac{27}{18+4n}\sin\frac{(9+2n)s}{27}\right]_0^t \\
&\quad -\cos\frac{2nt}{27}\left[\frac{27}{18-4n}\cos\frac{(9-2n)s}{27}-\frac{27}{18+4n}\cos\frac{(9+2n)s}{27}\right]_0^t \\
&= \sin\frac{2nt}{27}\left[\frac{27}{18-4n}\sin\frac{(9-2n)t}{27}+\frac{27}{18+4n}\sin\frac{(9+2n)t}{27}\right] \\
&\quad -\cos\frac{2nt}{27}\left[\frac{27}{18-4n}\cos\frac{(9-2n)t}{27}-\frac{27}{18+4n}\cos\frac{(9+2n)t}{27}-\frac{27}{18-4n}+\frac{27}{18+4n}\right] \\
&= -\frac{27}{18-4n}\cos\left[\frac{2nt}{27}+\frac{(9-2n)t}{27}\right]+\frac{27}{18+4n}\cos\left[\frac{2nt}{27}-\frac{(9+2n)t}{27}\right]
\end{aligned}
$$

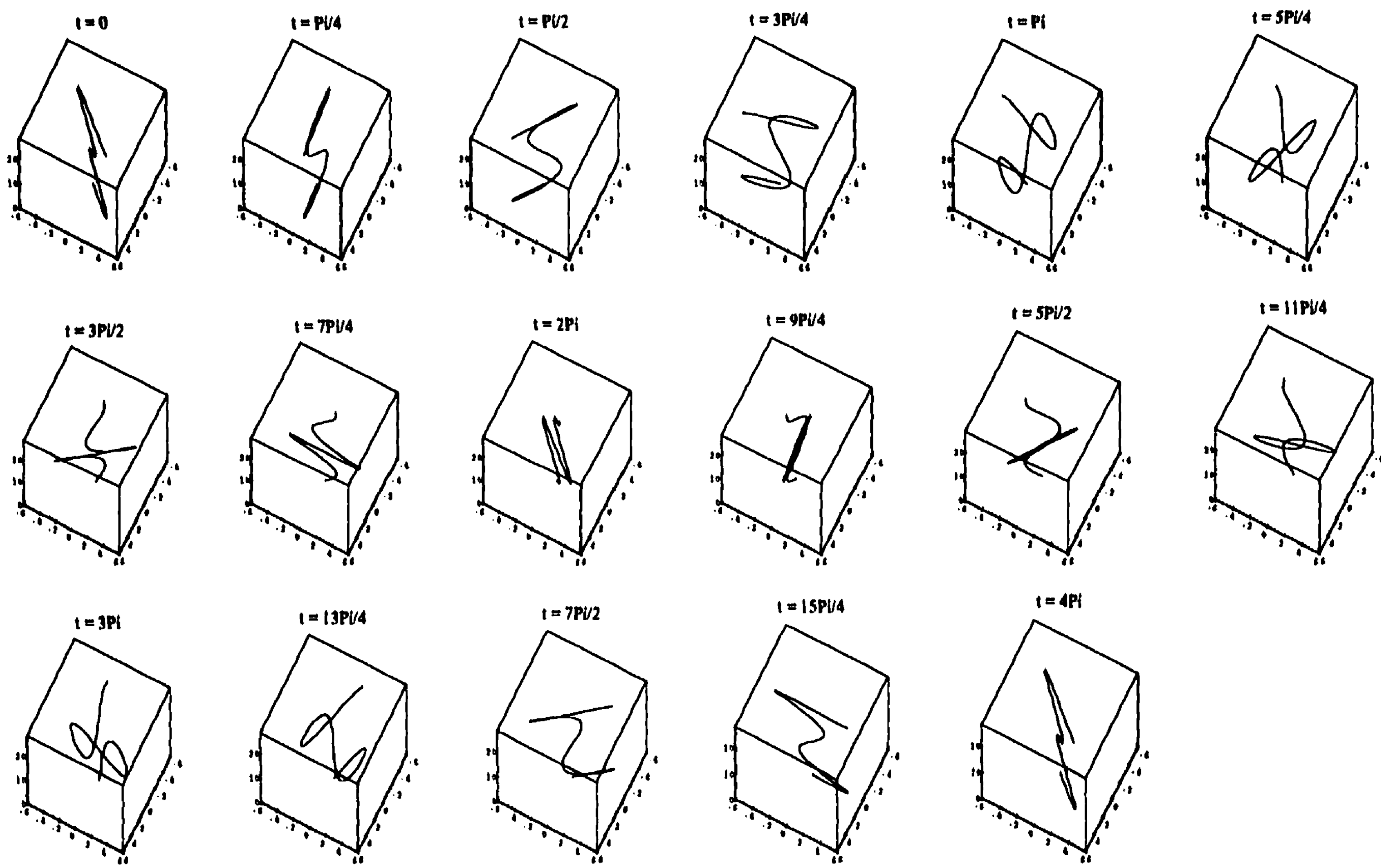
$$
\begin{aligned}
&+\frac{216n}{324-16n^2}\cos\frac{2nt}{27} \\
=\ & -\frac{27}{18-4n}\cos\frac{t}{3}+\frac{27}{18+4n}\cos\frac{t}{3}+\frac{216n}{324-16n^2}\cos\frac{2nt}{27} \\
=\ & \frac{216n}{324-16n^2}\left[\cos\frac{2nt}{27}-\cos\frac{t}{3}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \sin\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
=\ & \sin\frac{2nt}{27}\int_0^t \sin\frac{s}{3}\cos\frac{2ns}{27}ds-\cos\frac{2nt}{27}\int_0^t \sin\frac{s}{3}\sin\frac{2ns}{27}ds \\
=\ & \sin\frac{2nt}{27}\left[-\frac{27}{18-4n}\cos\frac{(9-2n)s}{27}-\frac{27}{18+4n}\cos\frac{(9+2n)s}{27}\right]_0^t \\
& -\cos\frac{2nt}{27}\left[\frac{27}{18-4n}\sin\frac{(9-2n)s}{27}-\frac{27}{18+4n}\sin\frac{(9+2n)s}{27}\right]_0^t \\
=\ & \sin\frac{2nt}{27}\left[-\frac{27}{18-4n}\cos\frac{(9-2n)t}{27}-\frac{27}{18+4n}\cos\frac{(9+2n)t}{27}+\frac{27}{18-4n}+\frac{27}{18+4n}\right] \\
& -\cos\frac{2nt}{27}\left[\frac{27}{18-4n}\sin\frac{(9-2n)t}{27}-\frac{27}{18+4n}\sin\frac{(9+2n)t}{27}\right] \\
=\ & -\frac{27}{18-4n}\sin\left[\frac{2nt}{27}+\frac{(9-2n)t}{27}\right]-\frac{27}{18+4n}\sin\left[\frac{2nt}{27}-\frac{(9+2n)t}{27}\right] \\
& +\frac{972}{324-16n^2}\sin\frac{2nt}{27} \\
=\ & -\frac{27}{18-4n}\sin\frac{t}{3}+\frac{27}{18+4n}\sin\frac{t}{3}+\frac{972}{324-16n^2}\sin\frac{2nt}{27} \\
=\ & \frac{108}{324-16n^2}\left[9\sin\frac{2nt}{27}-2n\sin\frac{t}{3}\right].
\end{aligned}
$$

The 20-term Fourier approximation to the solution in this case is shown for various times in Figure 5.4.3. Comparison with Figure 2.5.1 is favourable, in spite of the differences in the spatial dimension of the plot "windows" used.

Figure 5.4.3: graphic illustration of Example 5.4.3 (20 term Fourier series approximation) where $c=\omega=2/3$, $L=9\pi$



Example 5.4.4: For the fixed-length string problem (5.3.1-2), we let

$$c = \omega = \frac{2}{3}, \; L = 9\pi,$$

with initial and boundary conditions chosen to agree with those of example 2.5.2, namely

$$\begin{aligned}
f_1(x) &= \sin\frac{x}{4} + 2\sin\frac{x}{2}, \\
f_2(x) &= 2(\sin\frac{x}{4} + \sin\frac{x}{2}) + \cos\frac{x}{4}, \\
g_1(x) &= \sin\frac{x}{4} + 2\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{4}, \\
g_2(x) &= -\frac{1}{2}\sin\frac{x}{4} - 2\sin\frac{x}{2},
\end{aligned}$$

$$0 \le x \le 9\pi,$$

$$p_1(t) = \sin\frac{t}{2}, \; p_2(t) = \cos\frac{t}{2}, \; (\text{at } x = 0)$$

$$\begin{aligned}
q_1(t) &= 2(\cos t + \sin t) + \frac{\sqrt{2}}{2}(\cos\frac{t}{2} + 3\sin\frac{t}{2}), \; (\text{at } x = 9\pi) \\
q_2(t) &= 2(\cos t - \sin t) + \frac{\sqrt{2}}{2}(3\cos\frac{t}{2} - \sin\frac{t}{2}), \; (\text{at } x = 9\pi)
\end{aligned}$$

$$t \ge 0.$$

Putting the above functions in (5.3.29) and (5.3.34), we obtain the Fourier series solution, as given by

$$\begin{aligned}
u_1(x,t) &= \cos\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\mathrm{Re}\, Z_n(t)\sin\frac{nx}{9} + \sin\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\mathrm{Im}\, Z_n(t)\sin\frac{nx}{9} \\
&\quad -\frac{x}{9\pi}\left[(1-\frac{3\sqrt{2}}{2})\sin\frac{t}{2} - \frac{\sqrt{2}}{2}\cos\frac{t}{2} - 2(\cos t + \sin t)\right] + \sin\frac{t}{2}, \\
u_2(x,t) &= -\sin\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\mathrm{Re}\, Z_n(t)\sin\frac{nx}{9} + \cos\left(\frac{2}{3}t\right)\sum_{n=1}^{\infty}\mathrm{Im}\, Z_n(t)\sin\frac{nx}{9} \\
&\quad -\frac{x}{9\pi}\left[(1-\frac{3\sqrt{2}}{2})\cos\frac{t}{2} + \frac{\sqrt{2}}{2}\sin\frac{t}{2} - 2(\cos t - \sin t)\right] + \cos\frac{t}{2},
\end{aligned}$$

in which

$$\begin{aligned}
\operatorname{Re} Z_n(t) &= \left[\frac{4}{9\pi}\cos\left(\frac{2n}{27}t\right)+\frac{2}{n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{2}\sin\frac{nx}{9}dx \\
&+\left[\frac{2}{9\pi}\cos\left(\frac{2n}{27}t\right)-\frac{1}{n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{4}\sin\frac{nx}{9}dx \\
&-\frac{1}{2n\pi}\sin\left(\frac{2n}{27}t\right)\int_0^{9\pi}\cos\frac{x}{2}\sin\frac{nx}{9}dx \\
&-\frac{4}{27\pi}\left\{\left[1-\frac{3\sqrt{2}}{2}(-1)^n\right]\int_0^t \sin\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds\right. \\
&+\frac{\sqrt{2}}{2}(-1)^n\int_0^t\cos\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
&+2(-1)^n\int_0^t\cos\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
&\left.+2(-1)^n\int_0^t\sin\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds\right\} \\
&+\frac{2}{n\pi}\left\{\sin\frac{t}{6}+(-1)^n\left[\frac{\sqrt{2}}{2}(\cos\frac{t}{6}-3\sin\frac{t}{6})+2(\cos\frac{t}{3}+\sin\frac{t}{3})\right]\right\}, \\
\operatorname{Im} Z_n(t) &= \left[\frac{4}{9\pi}\cos\left(\frac{2n}{27}t\right)-\frac{2}{n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{2}\sin\frac{nx}{9}dx \\
&+\left[\frac{4}{9\pi}\cos\left(\frac{2n}{27}t\right)+\frac{1}{2n\pi}\sin\left(\frac{2n}{27}t\right)\right]\int_0^{9\pi}\sin\frac{x}{4}\sin\frac{nx}{9}dx \\
&+\frac{2}{9\pi}\cos\left(\frac{2n}{27}t\right)\int_0^{9\pi}\cos\frac{x}{4}\sin\frac{nx}{9}dx \\
&-\frac{4}{27\pi}\left\{\left[-1+\frac{3\sqrt{2}}{2}(-1)^n\right]\int_0^t \cos\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds\right. \\
&+\frac{\sqrt{2}}{2}(-1)^n\int_0^t\sin\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
&+2(-1)^n\int_0^t\cos\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds \\
&\left.-2(-1)^n\int_0^t\sin\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds\right\} \\
&+\frac{2}{n\pi}\left\{-\cos\frac{t}{6}+(-1)^n\left[\frac{\sqrt{2}}{2}(3\cos\frac{t}{6}+\sin\frac{t}{6})+2(\cos\frac{t}{3}-\sin\frac{t}{3})\right]\right\}.
\end{aligned}$$

The evaluation of the integrals $\int_0^{9\pi}\sin\frac{x}{2}\sin\frac{nx}{9}dx$, $\int_0^{9\pi}\sin\frac{x}{4}\sin\frac{nx}{9}dx$, $\int_0^t\cos\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds$, $\int_0^t\sin\frac{s}{6}\sin\left[\frac{2n}{27}(t-s)\right]ds$, $\int_0^t\cos\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds$ and $\int_0^t\sin\frac{s}{3}\sin\left[\frac{2n}{27}(t-s)\right]ds$ was given in example 5.4.3. The evaluation of the integrals

$\int_0^{9\pi} \cos\frac{x}{2}\sin\frac{nx}{9}dx$ and $\int_0^{9\pi} \cos\frac{x}{4}\sin\frac{nx}{9}dx$ is given by

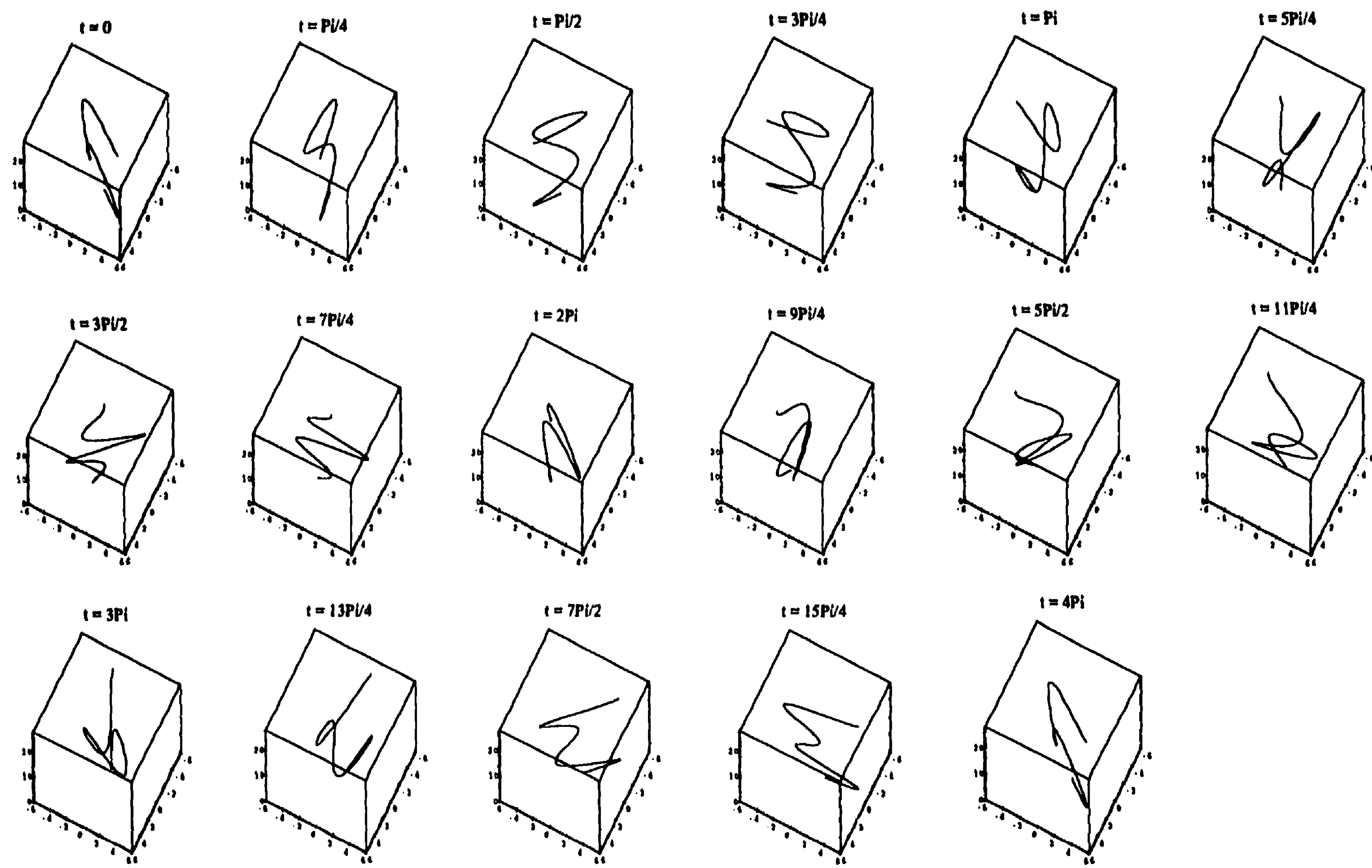
$$\begin{aligned}
& \int_0^{9\pi} \cos\frac{x}{2}\sin\frac{nx}{9}dx \\
= & \left[\frac{9}{9-2n}\cos\frac{(9-2n)x}{18} - \frac{9}{9+2n}\cos\frac{(9+2n)x}{18}\right]_0^{9\pi} \\
= & \frac{9}{9-2n}\cos(\frac{9\pi}{2} - n\pi) - \frac{9}{9+2n}\cos(\frac{9\pi}{2} + n\pi) - \frac{9}{9-2n} + \frac{9}{9+2n} \\
= & -\frac{36n}{81-4n^2},
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{9\pi} \cos\frac{x}{4}\sin\frac{nx}{9}dx \\
= & \left[\frac{18}{9-4n}\cos\frac{(9-4n)x}{36} - \frac{18}{9+4n}\cos\frac{(9+4n)x}{36}\right]_0^{9\pi} \\
= & \frac{18}{9-4n}\cos(\frac{9\pi}{4} - n\pi) - \frac{18}{9+4n}\cos(\frac{9\pi}{4} + n\pi) - \frac{18}{9-4n} + \frac{18}{9+4n} \\
= & (-1)^n\frac{\sqrt{2}}{2}\frac{18}{9-4n} - (-1)^n\frac{\sqrt{2}}{2}\frac{18}{9+4n} - \frac{144n}{81-16n^2} \\
= & \frac{144n}{2(81-16n^2)}[(-1)^n\sqrt{2} - 2].
\end{aligned}$$

The 50-term Fourier approximation to the solution in this case is shown for various times in Figure 5.4.4. Comparison with Figure 2.5.2 is favourable, in spite of the differences in the spatial dimension of the plot "windows" used.

Figure 5.4.4: graphic illustration of Example 5.4.4 (50 term Fourier series approximation) where c=ω=2/3, L=9π



The remaining examples are based on the "pulse wave" examples of section 4.3.

Example 5.4.5: In the fixed-length string problem (5.3.1-2), we let

$$c = 1, \; \omega = \pi, \; L = 10,$$

and adopt initial and boundary conditions consistent with those of example 4.3.3, namely

$$\begin{aligned} f_1(x) &= \begin{cases} 1 - \cos(2\pi x) & 2 \le x \le 3 \\ 0 & otherwise \end{cases} \\ &= [H(x-2) - H(x-3)][1 - \cos(2\pi x)], \end{aligned}$$

$$f_2(x) = g_1(x) = g_2(x) = 0,$$

$$\text{for } 0 \le x \le 10,$$

$$p_1(t) = p_2(t) = q_1(t) = q_2(t) = 0,$$

$$\text{for } t \ge 0.$$

Putting the above functions in (5.3.29) and (5.3.34), we obtain the Fourier series solution, as given by

$$\begin{aligned} u_1(x,t) &= \cos(\pi t) \sum_{n=1}^{\infty} \operatorname{Re} Z_n(t) \sin\frac{n\pi x}{10} + \sin(\pi t) \sum_{n=1}^{\infty} \operatorname{Im} Z_n(t) \sin\frac{n\pi x}{10}, \\ u_2(x,t) &= -\sin(\pi t) \sum_{n=1}^{\infty} \operatorname{Re} Z_n(t) \sin\frac{n\pi x}{10} + \cos(\pi t) \sum_{n=1}^{\infty} \operatorname{Im} Z_n(t) \sin\frac{n\pi x}{10}, \end{aligned}$$

in which

$$\begin{aligned} \operatorname{Re} Z_n(t) &= \frac{1}{5} \cos\left(\frac{n\pi}{10} t\right) \int_2^3 [1 - \cos(2\pi x)] \sin\frac{n\pi x}{10} dx, \\ \operatorname{Im} Z_n(t) &= \frac{2}{n} \sin\left(\frac{n\pi}{10} t\right) \int_2^3 [1 - \cos(2\pi x)] \sin\frac{n\pi x}{10} dx. \end{aligned}$$

In addition, we have that for $n \neq 20$,

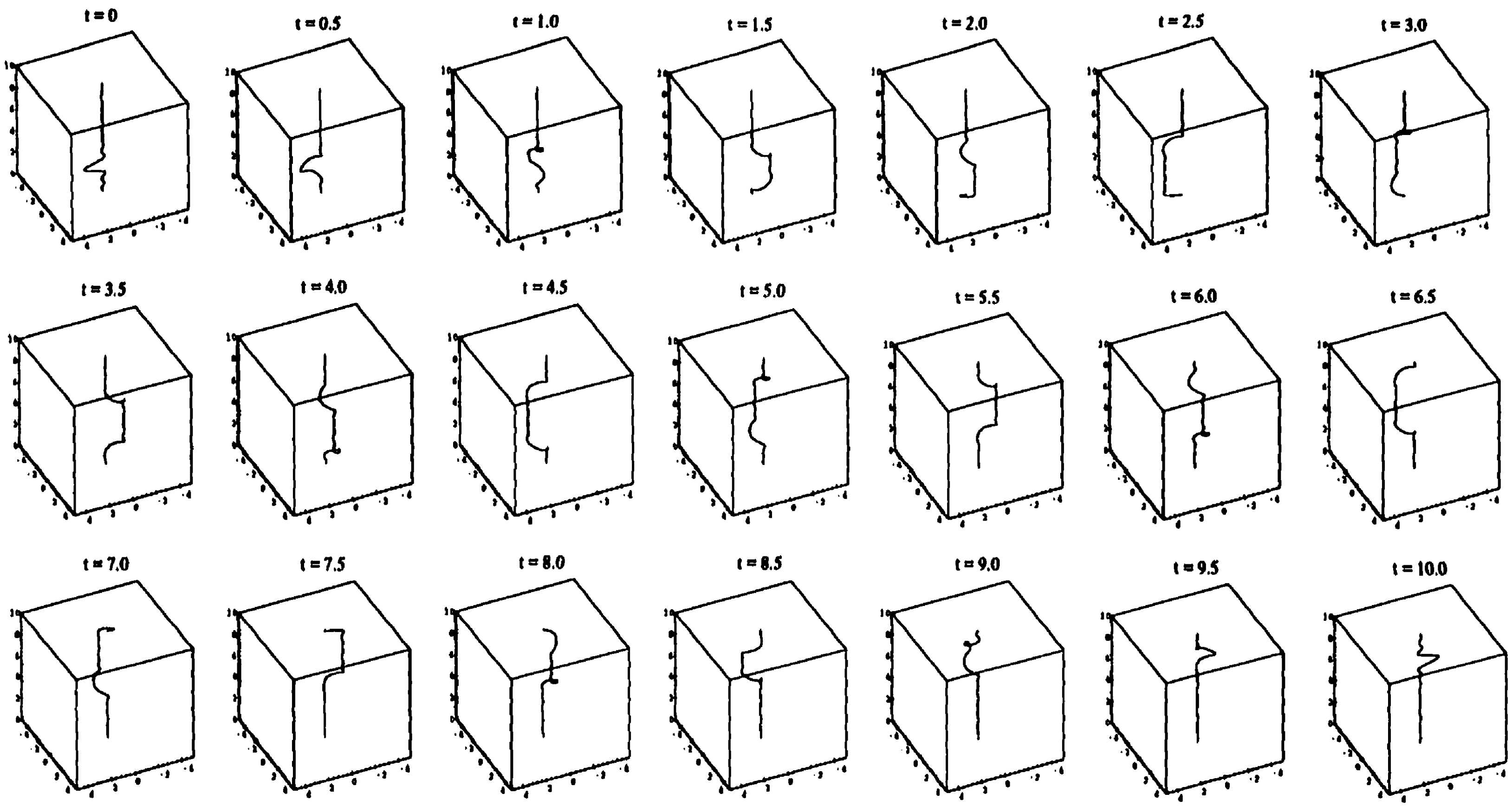
$$\begin{aligned}
&\int_2^3 [1-\cos(2\pi x)] \sin\frac{n\pi x}{10} dx \\
= &\left[\frac{5}{(n-20)\pi}\cos\frac{(n-20)\pi x}{10} + \frac{5}{(n+20)\pi}\cos\frac{(n+20)\pi x}{10} - \frac{10}{n\pi}\cos\frac{n\pi x}{10}\right]_2^3 \\
= &\frac{10}{\pi}\cos\frac{3n\pi}{10}\left(\frac{n}{n^2-400}-\frac{1}{n}\right) - \frac{10}{\pi}\cos\frac{n\pi}{5}\left(\frac{n}{n^2-400}-\frac{1}{n}\right) \\
= &\frac{10}{\pi}\left(\frac{n}{n^2-400}-\frac{1}{n}\right)\left(\cos\frac{3n\pi}{10}-\cos\frac{n\pi}{5}\right),
\end{aligned}$$

and for $n = 20$,

$$\begin{aligned}
&\int_2^3 [1-\cos(2\pi x)] \sin\frac{n\pi x}{10} dx \\
= &\int_2^3 [1-\cos(2\pi x)] \sin(2\pi x) dx \\
= &-\frac{1}{2\pi}\left[\cos(2\pi x) + \frac{\sin^2(2\pi x)}{2}\right]_2^3 \\
= &\ 0.
\end{aligned}$$

Graphical illustration of the 20-term Fourier series approximation, displayed in Figure 5.4.5, compares very well with that of Figure 4.3.3, although the typical oscillatory behaviour (i.e., the "wiggles") in the Fourier series approximation is more evident than it has been in previous examples. Of course we would expect the approximation to be more accurate if more terms were included in the Fourier series approximation.

Figure 5.4.5: graphic illustration of Example 5.4.5 (20 term Fourier series approximation) where c=1, ω=π, L=10



Example 5.4.6: We now consider the fixed-length string problem (5.3.1-2), with conditions determined by example 4.3.4. Thus, we choose

$$c = 1,\ \omega = \pi,\ L = 10,$$

and initial and boundary conditions given by

$$\begin{aligned} f_1(x) &= \begin{cases} 1 - \cos(2\pi x) & 2 \le x \le 3 \\ 0 & \textit{otherwise} \end{cases} \\ &= [H(x-2) - H(x-3)][1 - \cos(2\pi x)], \end{aligned}$$

$$f_2(x) = g_1(x) = 0,\ \text{for } 0 \le x \le 10,$$

$$\begin{aligned} g_2(x) &= \begin{cases} \pi[\cos(2\pi x) - 1] & 2 \le x \le 3 \\ 0 & \textit{otherwise} \end{cases} \\ &= \pi[H(x-2) - H(x-3)][\cos(2\pi x) - 1], \end{aligned}$$

$$p_1(t) = p_2(t) = q_1(t) = q_2(t) = 0,$$

$$\text{for } t \ge 0.$$

Substitution of the above functions into (5.3.29) and (5.3.34) provides the Fourier series solution, as given by

$$\begin{aligned} u_1(x,t) &= \cos(\pi t) \sum_{n=1}^{\infty} \operatorname{Re} Z_n(t) \sin\frac{n\pi x}{10}, \\ u_2(x,t) &= -\sin(\pi t) \sum_{n=1}^{\infty} \operatorname{Re} Z_n(t) \sin\frac{n\pi x}{10}, \end{aligned}$$

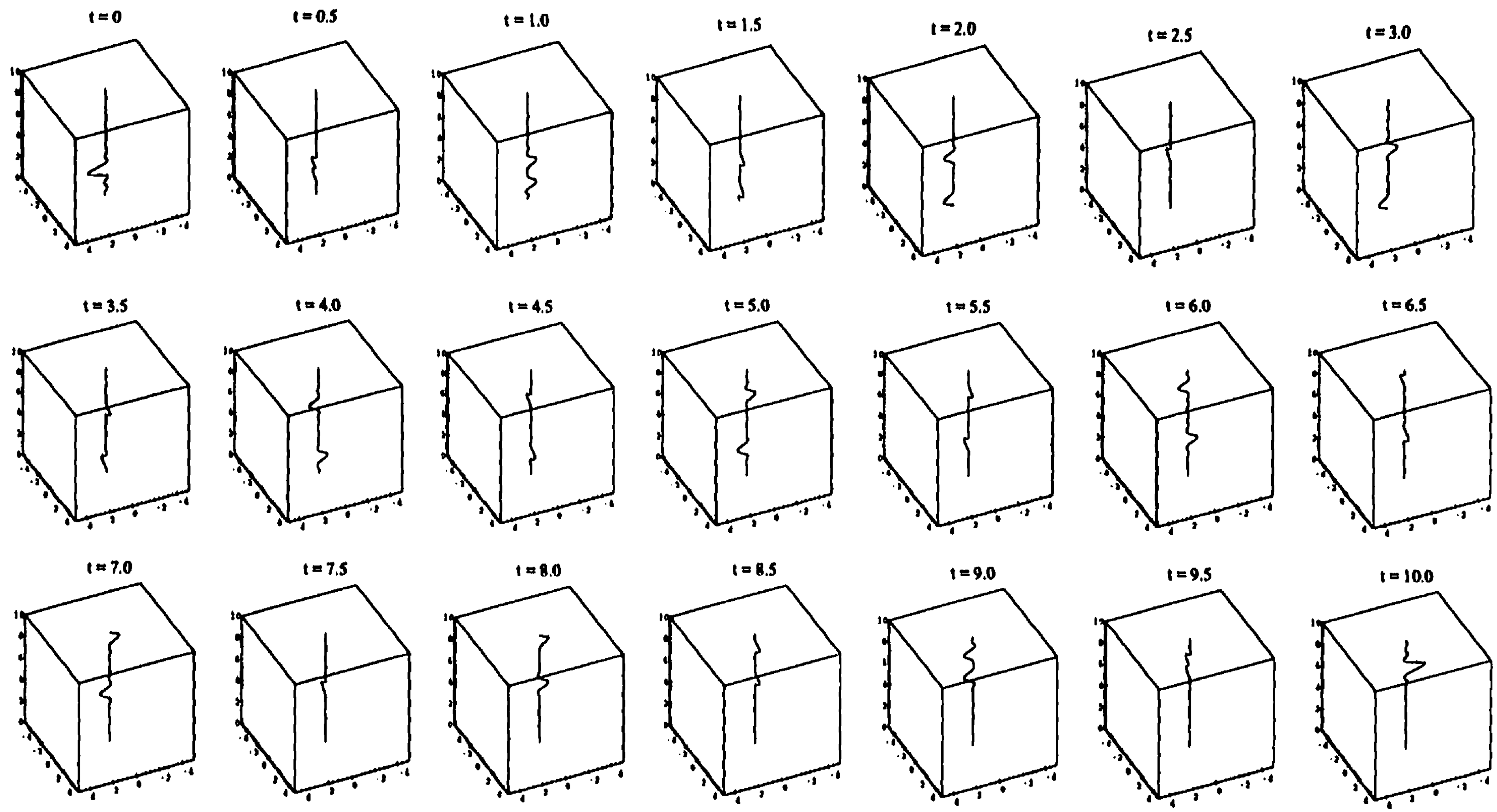
in which

$$
\begin{aligned}
\operatorname{Re} Z_n(t) &= \frac{1}{5}\cos\left(\frac{n\pi}{10}t\right)\int_2^3 [1-\cos(2\pi x)]\sin\frac{n\pi x}{10}dx, \\
\operatorname{Im} Z_n(t) &= 0.
\end{aligned}
$$

The evaluation of the integral $\int_2^3 [1-\cos(2\pi x)]\sin\frac{n\pi x}{10}dx$ was given in example 5.4.5.

Graphical illustration of the 20-term Fourier series approximation compares very well with that of Figure 4.3.4, although the typical oscillatory behaviour of the Fourier series approximation is again evident, and the approximation would be more accurate if more terms were included in the Fourier series approximation.

Figure 5.4.6: graphic illustration of Example 5.4.6 (20 term Fourier series approximation) where c=1, ω=π, L=10



Example 5.4.7: The final example involves the fixed-length string problem (5.3.1-2) previously considered in example 4.3.5. Thus, we choose

$$c = 1,\ \omega = \pi,\ L = 10,$$

and initial and boundary conditions given by

$$f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0,$$

$$\text{for } 0 \le x \le 10,$$

$$\begin{aligned} p_1(t) &= \begin{cases} 1 - \cos(2\pi t) & 0 \le t \le 1 \\ 0 & \textit{otherwise} \end{cases} \\ &= [H(t) - H(t-1)][1 - \cos(2\pi t)], \end{aligned}$$

$$p_2(t) = q_1(t) = q_2(t) = 0,$$

$$\text{for } t \ge 0.$$

Putting the above functions in (5.3.29) and (5.3.34), we obtain the Fourier series solution, as given by

$$\begin{aligned} u_1(x,t) &= \cos(\pi t)\sum_{n=1}^{\infty} \operatorname{Re} Z_n(t) \sin\frac{n\pi x}{10} + \sin(\pi t)\sum_{n=1}^{\infty} \operatorname{Im} Z_n(t) \sin\frac{n\pi x}{10} \\ & \quad (1 - \frac{x}{10})[H(t) - H(t-1)][1 - \cos(2\pi t)], \\ u_2(x,t) &= -\sin(\pi t)\sum_{n=1}^{\infty} \operatorname{Re} Z_n(t) \sin\frac{n\pi x}{10} + \cos(\pi t)\sum_{n=1}^{\infty} \operatorname{Im} Z_n(t) \sin\frac{n\pi x}{10}, \end{aligned}$$

in which

$$\operatorname{Re} Z_n(t) = -\frac{2}{n\pi}[H(t) - H(t-1)]\cos(\pi t)[1 - \cos(2\pi t)]$$

$$
\begin{aligned}
& +\frac{1}{5}\int_0^t [H(s)-H(s-1)]\cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds, \\
\operatorname{Im} Z_n(t) \quad = \quad & -\frac{2}{n\pi}[H(t)-H(t-1)]\sin(\pi t)[1-\cos(2\pi t)] \\
& +\frac{1}{5}\int_0^t [H(s)-H(s-1)]\sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds.
\end{aligned}
$$

The evaluation of the integrals $\int_0^t [H(s)-H(s-1)]\cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds$ and $\int_0^t [H(s)-H(s-1)]\sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds$ is performed in the usual manner, illustrated previously in section 4.3.

$$
\int_0^t [H(s)-H(s-1)]\cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds
$$

$$
= \begin{cases} \int_0^t \cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds, & \text{for } 0 \le t \le 1 \\ \int_0^1 \cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds & \text{for } t \ge 1 \end{cases},
$$

and

$$
\int_0^t [H(s)-H(s-1)]\sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds
$$

$$
= \begin{cases} \int_0^t \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds, & \text{for } 0 \le t \le 1 \\ \int_0^1 \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right] ds & \text{for } t \ge 1 \end{cases}.
$$

For $n \neq 10, 30$,

$$
\begin{aligned}
& \int \cos(\pi s)\sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= \; & \sin\frac{n\pi t}{10}\int \cos(\pi s)\cos\frac{n\pi s}{10} ds - \cos\frac{n\pi t}{10}\int \cos(\pi s)\sin\frac{n\pi s}{10} ds \\
= \; & \sin\frac{n\pi t}{10}\left[\frac{\sin(\frac{n\pi}{10}-\pi)s}{2(\frac{n\pi}{10}-\pi)} + \frac{\sin(\frac{n\pi}{10}+\pi)s}{2(\frac{n\pi}{10}+\pi)}\right] \\
& -\cos\frac{n\pi t}{10}\left[-\frac{\cos(\frac{n\pi}{10}-\pi)s}{2(\frac{n\pi}{10}-\pi)} - \frac{\cos(\frac{n\pi}{10}+\pi)s}{2(\frac{n\pi}{10}+\pi)}\right] \\
= \; & \frac{5}{(n-10)\pi}\left[\cos\frac{n\pi t}{10}\cos(\frac{n\pi}{10}-\pi)s + \sin\frac{n\pi t}{10}\sin(\frac{n\pi}{10}-\pi)s\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5}{(n+10)\pi}\left[\cos\frac{n\pi t}{10}\cos(\frac{n\pi}{10}+\pi)s+\sin\frac{n\pi t}{10}\sin(\frac{n\pi}{10}+\pi)s\right]\\
=\ &\frac{5}{(n-10)\pi}\cos\left[\frac{n\pi t}{10}-(\frac{n\pi}{10}-\pi)s\right]+\frac{5}{(n+10)\pi}\cos\left[\frac{n\pi t}{10}-(\frac{n\pi}{10}+\pi)s\right]\\
=\ &\frac{5}{(n-10)\pi}\cos\pi\left[\frac{n}{10}(t-s)+s\right]+\frac{5}{(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\cos(\pi s)\cos(2\pi s)\sin\left[\frac{n\pi}{10}(t-s)\right]ds\\
=\ &\frac{1}{2}\int[\cos(\pi s+2\pi s)+\cos(\pi s-2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds\\
=\ &\frac{1}{2}\int\cos(3\pi s)\sin\left[\frac{n\pi}{10}(t-s)\right]ds+\frac{1}{2}\int\cos(\pi s)\sin\left[\frac{n\pi}{10}(t-s)\right]ds\\
=\ &\frac{1}{2}\left[\sin\frac{n\pi t}{10}\int\cos(3\pi s)\cos\frac{n\pi s}{10}ds-\cos\frac{n\pi t}{10}\int\cos(3\pi s)\sin\frac{n\pi s}{10}ds\right]\\
&+\frac{5}{2(n-10)\pi}\cos\pi\left[\frac{n}{10}(t-s)+s\right]+\frac{5}{2(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right]\\
=\ &\frac{1}{2}\sin\frac{n\pi t}{10}\left[\frac{\sin(\frac{n\pi}{10}-3\pi)s}{2(\frac{n\pi}{10}-3\pi)}+\frac{\sin(\frac{n\pi}{10}+3\pi)s}{2(\frac{n\pi}{10}+3\pi)}\right]\\
&-\frac{1}{2}\cos\frac{n\pi t}{10}\left[-\frac{\cos(\frac{n\pi}{10}-3\pi)s}{2(\frac{n\pi}{10}-3\pi)}-\frac{\cos(\frac{n\pi}{10}+3\pi)s}{2(\frac{n\pi}{10}+3\pi)}\right]\\
&+\frac{5}{2(n-10)\pi}\cos\pi\left[\frac{n}{10}(t-s)+s\right]+\frac{5}{2(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right]\\
=\ &\frac{5}{2(n-30)\pi}\left[\cos\frac{n\pi t}{10}\cos(\frac{n\pi}{10}-3\pi)s+\sin\frac{n\pi t}{10}\sin(\frac{n\pi}{10}-3\pi)s\right]\\
&+\frac{5}{2(n+30)\pi}\left[\cos\frac{n\pi t}{10}\cos(\frac{n\pi}{10}+3\pi)s+\sin\frac{n\pi t}{10}\sin(\frac{n\pi}{10}+3\pi)s\right]\\
&+\frac{5}{2(n-10)\pi}\cos\pi\left[\frac{n}{10}(t-s)+s\right]+\frac{5}{2(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right]\\
=\ &\frac{5}{2(n-30)\pi}\cos\left[\frac{n\pi t}{10}-(\frac{n\pi}{10}-3\pi)s\right]+\frac{5}{2(n+30)\pi}\cos\left[\frac{n\pi t}{10}-(\frac{n\pi}{10}+3\pi)s\right]\\
&+\frac{5}{2(n-10)\pi}\cos\pi\left[\frac{n}{10}(t-s)+s\right]+\frac{5}{2(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right]\\
=\ &\frac{5}{2(n-30)\pi}\cos\pi\left[\frac{n}{10}(t-s)+3s\right]+\frac{5}{2(n+30)\pi}\cos\pi\left[\frac{n}{10}(t-s)-3s\right]\\
&+\frac{5}{2(n-10)\pi}\cos\pi\left[\frac{n}{10}(t-s)+s\right]+\frac{5}{2(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right].
\end{aligned}
$$

For $n = 10$,

$$\begin{aligned}
&\int \cos(\pi s) \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= &\int \cos(\pi s) \sin \pi(t-s) ds \\
= &\sin(\pi t) \int \cos^2(\pi s) ds - \cos(\pi t) \int \cos(\pi s) \sin(\pi s) ds \\
= &\sin(\pi t)\left[\frac{s}{2} + \frac{1}{4\pi}\sin(2\pi s)\right] - \cos(\pi t)\frac{1}{2\pi}\sin^2(\pi s) \\
= &\frac{s}{2}\sin(\pi t) + \frac{1}{2\pi}\sin(\pi s)\sin\pi(t-s),
\end{aligned}$$

and

$$\begin{aligned}
&\int \cos(\pi s)\cos(2\pi s) \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= &\int \cos(\pi s)\cos(2\pi s) \sin \pi(t-s) ds \\
= &\sin(\pi t) \int \cos^2(\pi s)\cos(2\pi s) ds - \cos(\pi t) \int \sin(\pi s)\cos(\pi s)\cos(2\pi s) ds \\
= &\frac{1}{4}\sin(\pi t)\left[s + \frac{1}{\pi}\sin(2\pi s) + \frac{1}{4\pi}\sin(4\pi s)\right] - \frac{1}{8\pi}\cos(\pi t)\sin^2(2\pi s) \\
= &\frac{1}{4}\sin(\pi t)\left[s + \frac{1}{\pi}\sin(2\pi s)\right] + \frac{1}{8\pi}\sin(2\pi s)\sin\pi(t-2s).
\end{aligned}$$

For $n = 30$,

$$\begin{aligned}
&\int \cos(\pi s) \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= &\int \cos(\pi s) \sin 3\pi(t-s) ds \\
= &\sin(3\pi t) \int \cos(\pi s)\cos(3\pi s) ds - \cos(3\pi t) \int \cos(\pi s)\sin(3\pi s) ds \\
= &\frac{1}{2}\sin(3\pi t)\int [\cos(2\pi s) + \cos(4\pi s)] ds \\
&-\frac{1}{2}\cos(3\pi t)\int [\sin(2\pi s) + \sin(4\pi s)] ds \\
= &\frac{1}{2}\sin(3\pi t)\left[\frac{1}{2\pi}\sin(2\pi s) + \frac{1}{4\pi}\sin(4\pi s)\right] \\
&-\frac{1}{2}\cos(3\pi t)\left[-\frac{1}{2\pi}\cos(2\pi s) - \frac{1}{4\pi}\cos(4\pi s)\right]
\end{aligned}$$

$$= \frac{1}{4\pi}\cos\pi(3t-2s) + \frac{1}{8\pi}\cos\pi(3t-4s),$$

and

$$\begin{aligned}
&\int \cos(\pi s)\cos(2\pi s)\sin\left[\frac{n\pi}{10}(t-s)\right]ds \\
=& \int \cos(\pi s)\cos(2\pi s)\sin 3\pi(t-s)ds \\
=& \sin(3\pi t)\int \cos(\pi s)\cos(2\pi s)\cos(3\pi s)ds - \cos(3\pi t)\int \cos(\pi s)\cos(2\pi s)\sin(3\pi s)ds \\
=& \frac{1}{2}\sin(3\pi t)\int \cos(2\pi s)[\cos(2\pi s)+\cos(4\pi s)]ds \\
&-\frac{1}{2}\cos(3\pi t)\int \cos(2\pi s)[\sin(2\pi s)+\sin(4\pi s)]ds \\
=& \frac{1}{2}\sin(3\pi t)\left[\frac{s}{2}+\frac{1}{8\pi}\sin(4\pi s)+\frac{1}{2\pi}\sin(2\pi s)-\frac{1}{3\pi}\sin^3(2\pi s)\right] \\
&+\frac{1}{2\pi}\cos(3\pi t)\left[\frac{1}{8}\cos(4\pi s)+\frac{1}{3}\cos^3(2\pi s)\right] \\
=& \frac{1}{4}\sin(3\pi t)\left[s+\frac{1}{\pi}\sin(2\pi s)\right]+\frac{1}{16\pi}\cos\pi(3t-4s)+\frac{1}{6\pi}\cos(4\pi s).
\end{aligned}$$

Then for $n \neq 10$, 30, we have

$$\begin{aligned}
&\int_0^t \cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds \\
=& \int_0^t \cos(\pi s)\sin\left[\frac{n\pi}{10}(t-s)\right]ds \\
&-\int_0^t \cos(\pi s)\cos(2\pi s)\sin\left[\frac{n\pi}{10}(t-s)\right]ds \\
=& \left\{\frac{5}{2(n-10)\pi}\cos\pi\left[\frac{n}{10}(t-s)+s\right]+\frac{5}{2(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right]\right\}_0^t \\
&-\left\{\frac{5}{2(n-30)\pi}\cos\pi\left[\frac{n}{10}(t-s)+3s\right]+\frac{5}{2(n+30)\pi}\cos\pi\left[\frac{n}{10}(t-s)-3s\right]\right\}_0^t \\
=& \frac{5}{2(n-10)\pi}\left[\cos(\pi t)-\cos\frac{n\pi t}{10}\right]+\frac{5}{2(n+10)\pi}\left[\cos(\pi t)-\cos\frac{n\pi t}{10}\right] \\
&-\frac{5}{2(n-30)\pi}\left[\cos(3\pi t)-\cos\frac{n\pi t}{10}\right]-\frac{5}{2(n+30)\pi}\left[\cos(3\pi t)-\cos\frac{n\pi t}{10}\right] \\
=& \frac{5n}{\pi(n^2-100)}\left[\cos(\pi t)-\cos\frac{n\pi t}{10}\right]-\frac{5n}{\pi(n^2-900)}\left[\cos(3\pi t)-\cos\frac{n\pi t}{10}\right],
\end{aligned}$$

and

$$
\begin{aligned}
& \int_0^1 \cos(\pi s)[1-\cos(2\pi s)] \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= \; & \int_0^1 \cos(\pi s) \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
& - \int_0^1 \cos(\pi s)\cos(2\pi s) \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= \; & \left\{ \frac{5}{2(n-10)\pi} \cos\pi\left[\frac{n}{10}(t-s)+s\right] + \frac{5}{2(n+10)\pi}\cos\pi\left[\frac{n}{10}(t-s)-s\right] \right\}_0^1 \\
& - \left\{ \frac{5}{2(n-30)\pi} \cos\pi\left[\frac{n}{10}(t-s)+3s\right] + \frac{5}{2(n+30)\pi}\cos\pi\left[\frac{n}{10}(t-s)-3s\right] \right\}_0^1 \\
= \; & -\frac{5}{2(n-10)\pi}\left[\cos\frac{n\pi}{10}(t-1) + \cos\frac{n\pi t}{10}\right] - \frac{5}{2(n+10)\pi}\left[\cos\frac{n\pi}{10}(t-1) + \cos\frac{n\pi t}{10}\right] \\
& + \frac{5}{2(n-30)\pi}\left[\cos\frac{n\pi}{10}(t-1) + \cos\frac{n\pi t}{10}\right] + \frac{5}{2(n+30)\pi}\left[\cos\frac{n\pi}{10}(t-1) + \cos\frac{n\pi t}{10}\right] \\
= \; & \frac{5n}{\pi}\frac{1000-2n^2}{(n^2-100)(n^2-900)}\left[\cos\frac{n\pi}{10}(t-1) + \cos\frac{n\pi t}{10}\right].
\end{aligned}
$$

For $n = 10$, we have

$$
\begin{aligned}
& \int_0^t \cos(\pi s)[1-\cos(2\pi s)] \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= \; & \int_0^t \cos(\pi s) \sin\pi(t-s) ds \\
& - \int_0^t \cos(\pi s)\cos(2\pi s) \sin\pi(t-s) ds \\
= \; & \left[\frac{s}{2}\sin(\pi t) + \frac{1}{2\pi}\sin(\pi s)\sin\pi(t-s)\right]_0^t \\
& - \left\{\frac{1}{4}\sin(\pi t)\left[s + \frac{1}{\pi}\sin(2\pi s)\right] + \frac{1}{8\pi}\sin(2\pi s)\sin\pi(t-2s)\right\}_0^t \\
= \; & \frac{t}{4}\sin(\pi t) - \frac{1}{8\pi}\sin(\pi t)\sin(2\pi t),
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1 \cos(\pi s)[1-\cos(2\pi s)] \sin\left[\frac{n\pi}{10}(t-s)\right] ds \\
= \; & \int_0^1 \cos(\pi s) \sin\pi(t-s) ds
\end{aligned}
$$

$$
\begin{aligned}
& -\int_0^1 \cos(\pi s)\cos(2\pi s)\sin\pi(t-s)ds \\
= & \left[\frac{s}{2}\sin(\pi t) + \frac{1}{2\pi}\sin(\pi s)\sin\pi(t-s)\right]_0^1 \\
& -\left\{\frac{1}{4}\sin(\pi t)\left[s + \frac{1}{\pi}\sin(2\pi s)\right] + \frac{1}{8\pi}\sin(2\pi s)\sin\pi(t-2s)\right\}_0^1 \\
= & \frac{1}{4}\sin(\pi t).
\end{aligned}
$$

For $n = 30$, we have

$$
\begin{aligned}
& \int_0^t \cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds \\
= & \int_0^t \cos(\pi s)\sin 3\pi(t-s)ds \\
& -\int_0^t \cos(\pi s)\cos(2\pi s)\sin 3\pi(t-s)ds \\
= & \left[\frac{1}{4\pi}\cos\pi(3t-2s) + \frac{1}{8\pi}\cos\pi(3t-4s)\right]_0^t \\
& -\left\{\frac{1}{4}\sin(3\pi t)\left[s + \frac{1}{\pi}\sin(2\pi s)\right] + \frac{1}{16\pi}\cos\pi(3t-4s) + \frac{1}{6\pi}\cos(4\pi s)\right\}_0^t \\
= & \frac{3}{4\pi}\cos(\pi t)\sin^2(\pi t) - \frac{1}{4}t\sin(3\pi t),
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1 \cos(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds \\
= & \int_0^1 \cos(\pi s)\sin 3\pi(t-s)ds \\
& -\int_0^1 \cos(\pi s)\cos(2\pi s)\sin 3\pi(t-s)ds \\
= & \left[\frac{1}{4\pi}\cos\pi(3t-2s) + \frac{1}{8\pi}\cos\pi(3t-4s)\right]_0^1 \\
& -\left\{\frac{1}{4}\sin(3\pi t)\left[s + \frac{1}{\pi}\sin(2\pi s)\right] + \frac{1}{16\pi}\cos\pi(3t-4s) + \frac{1}{6\pi}\cos(4\pi s)\right\}_0^1 \\
= & -\frac{1}{4}\sin(3\pi t).
\end{aligned}
$$

Similar calculations give, for $n \neq 10, 30$,

$$
\int_0^t \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds
$$

$$= \frac{15}{\pi(n^2-100)}\left[n\sin(\pi t) - 10\sin\frac{n\pi t}{10}\right] - \frac{5}{\pi(n^2-900)}\left[n\sin(3\pi t) - 30\sin\frac{n\pi t}{10}\right],$$

and

$$\int_0^1 \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds$$
$$= \frac{12\times 10^4}{\pi(n^2-100)(n^2-900)}\left[\sin\frac{n\pi}{10}(t-1) + \sin\frac{n\pi t}{10}\right].$$

For $n = 10$,

$$\int_0^t \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds$$
$$= \frac{9}{16\pi}\sin(\pi t) + \frac{1}{16\pi}\sin(3\pi t) - \frac{3}{4}t\cos(\pi t),$$

and

$$\int_0^1 \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds$$
$$= -\frac{3}{4}\cos(\pi t).$$

For $n = 30$,

$$\int_0^t \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds$$
$$= \frac{9}{16\pi}\sin(\pi t) - \frac{13}{48\pi}\sin(3\pi t) + \frac{1}{4}t\cos(3\pi t),$$

and

$$\int_0^1 \sin(\pi s)[1-\cos(2\pi s)]\sin\left[\frac{n\pi}{10}(t-s)\right]ds$$
$$= \frac{1}{4}\cos(3\pi t).$$

As in previous examples, graphical illustration of the 20-term Fourier series approximation to the solution of this problem is shown in Figure 5.4.7a. To illustrate the convergence of the Fourier series approximation to the solution, Figure

5.4.7b displays corresponding results for the 50-term Fourier series approximation to the solution. As one would expect, the accuracy of the approximation increases as the number of terms in the approximation increases.

Figure 5.4.7a: graphic illustration of Example 5.4.7 (20 term Fourier series approximation) where c=1, $\omega=\pi$, L=10

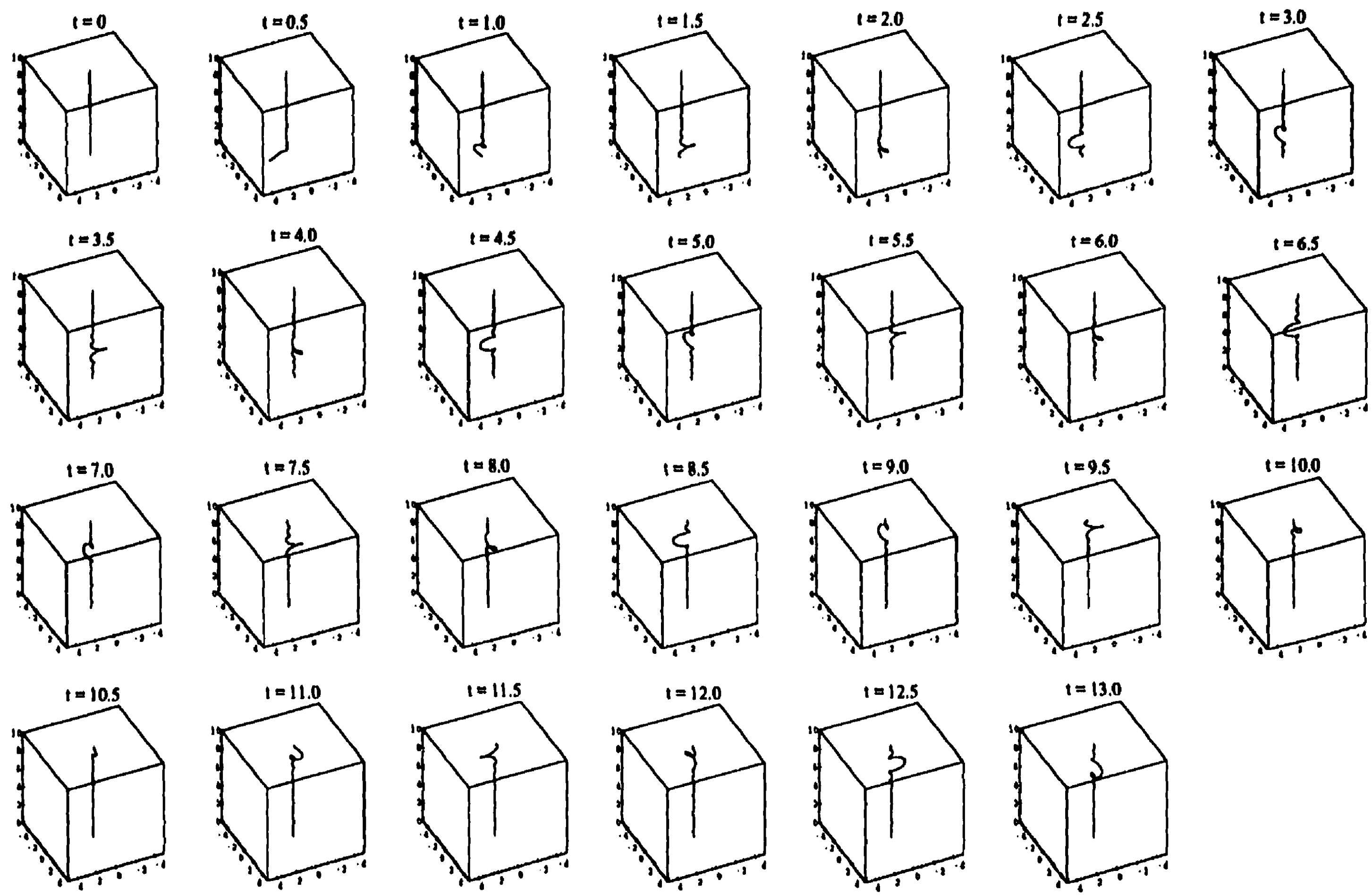
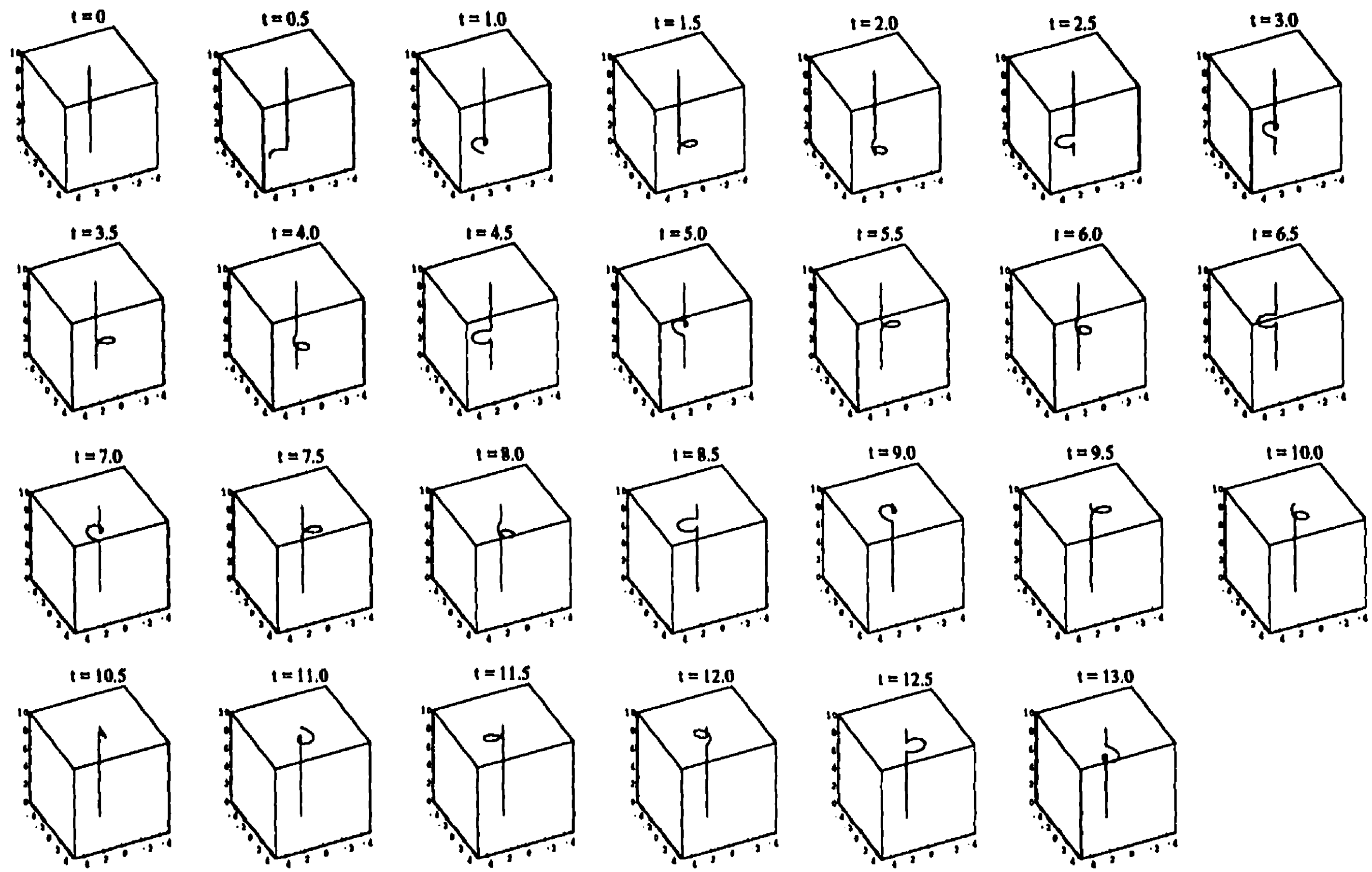


Figure 5.4.7b: graphic illustration of Example 5.4.7 (50 term Fourier series approximation) where c=1, ω=π, L=10



CHAPTER 6

Solutions for the Finite String with One Linearly-Moving Endpoint

6.1 Introduction

In this chapter, we solve the linear system (1.3.1) in the case of a finite string with one linearly-moving endpoint, subject to general initial and boundary conditions. As shown in in Chapter 3, we may transform the system into a single one-dimensional (complex) wave equation, on the region in the xt-plane with bounds given by $0 \leq x \leq mt + L$ and $t \geq 0$, with corresponding (complex) initial and boundary conditions. The latter problem is solved using the usual d'Alembert form of the solution of the wave equation subject to the specified initial and boundary conditions, using results developed by Corbett [9]. Finally we apply the inverse transformations to determine the solution of the original problem. Several examples and corresponding graphical illustrations of the solution are presented.

6.2 d'Alembert Solution and Compatability Conditions

We consider the usual one-dimensional homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0, \tag{6.2.1}$$

in the case of a finite string with one linearly-moving endpoint, so the domain of (6.2.1) is $0 \leq x \leq mt + L$, $t \geq 0$, with $|m| < c$. The parameter m represents the velocity at which the right-hand end of the string is moving along the x-axis;

the condition $|m| < c$ is imposed to avoid the difficulties associated with the introduction of "supersonic" motion of the endpoint [7, p. 29].

The initial conditions are given by

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad 0 \le x \le L, \tag{6.2.2a}$$

while the boundary conditions (at $x = 0$ and $x = mt + L$, respectively) are given by

$$u(0,t) = p(t), \quad u(mt + L, t) = q(t), \quad t \ge 0, \tag{6.2.2b}$$

where $f(x)$ and $g(x)$ are specified functions on $0 \le x \le L$, while $p(t)$ and $q(t)$ are specified functions on $t \ge 0$.

In order to solve the above initial moving-boundary value problem, we introduce the dimensionless variables

$$x = Lx^*, \quad t = \frac{L}{c}t^*, \quad m^* = \frac{m}{c}, \tag{6.2.3}$$

and let

$$u^*(x^*, t^*) = u(x,t) = u(Lx^*, \frac{L}{c}t^*), \tag{6.2.4}$$

so that (6.2.1) becomes

$$\frac{\partial^2 u^*}{\partial t^{*2}} = \frac{\partial^2 u^*}{\partial x^{*2}}, \quad 0 \le x^* \le m^* t^* + 1, \quad t^* \ge 0, \quad |m^*| < 1, \tag{6.2.5}$$

while the initial conditions (6.2.2a) become

$$u^*(x^*, 0) = u(x,0) = f(x) = f(Lx^*), \tag{6.2.6a}$$

$$\frac{\partial u^*}{\partial t^*}(x^*, 0) = \frac{L}{c}\frac{\partial u}{\partial t}(x,0) = \frac{L}{c}g(Lx^*), \tag{6.2.6b}$$

$$0 \leq x^* \leq 1,$$

and the boundary conditions (6.2.2b) become

$$u^*(0,t^*) = u(0,t) = p(t) = p(\frac{L}{c}t^*), \tag{6.2.6c}$$

$$u^*(m^*t^*+1,t^*) = u(L[m^*t^*+1], \frac{L}{c}t^*) = u(mt+L,t) = q(t) = q(\frac{L}{c}t^*). \tag{6.2.6d}$$

The problem (6.2.5-6) was considered in detail by Corbett [7, pp. 22-28]. In [7, pp. 22-28], it is shown that the form of the d'Alembert solution depends upon the region $D_{(i,j)}$ under consideration in the domain $D = \{(x^*,t^*)\colon\ 0 \leq x^* \leq m^*t^*+1,\ t^* \geq 0\}$, as shown in the following diagram.

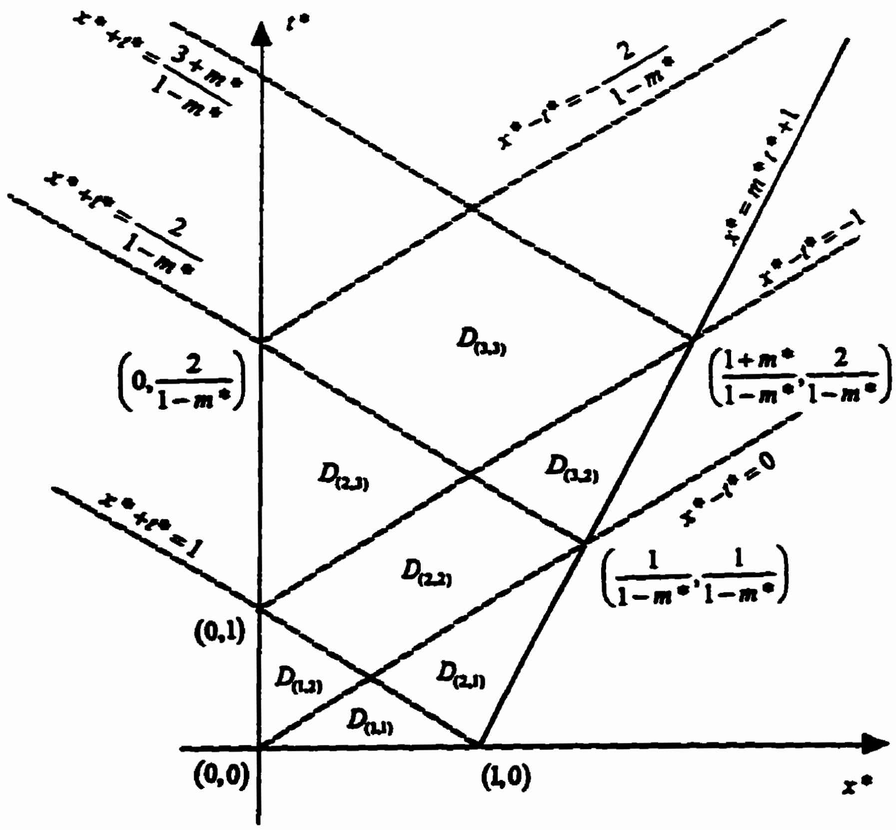


Diagram 6.2.1

We denote the solution of (6.2.5-6) in region $D_{(i,j)}$, of diagram 6.2.1, by

$$u^*(x^*,t^*) = \varphi_i(x^*+t^*) + \psi_j(x^*-t^*). \tag{6.2.7}$$

By the usual procedures, we may show that

$$\begin{aligned}\varphi_1(\xi^*) &= \frac{1}{2}f(L\xi^*) + \frac{1}{2}\int_0^{\xi^*}\frac{L}{c}g(L\tau)d\tau + F,\\ &= \frac{1}{2}f(L\xi^*) + \frac{1}{2c}\int_0^{L\xi^*} g(\tau)d\tau + F,\end{aligned} \tag{6.2.8a}$$

$$\text{for } 0 \le \xi^* \le 1,$$

and

$$\begin{aligned}\psi_1(\eta^*) &= \frac{1}{2}f(L\eta^*) - \frac{1}{2}\int_0^{\eta^*}\frac{L}{c}g(L\tau)d\tau - F\\ &= \frac{1}{2}f(L\eta^*) - \frac{1}{2c}\int_0^{L\eta^*} g(\tau)d\tau - F\end{aligned} \tag{6.2.8b}$$

$$\text{for } 0 \le \eta^* \le 1,$$

where $\xi^* = x^* + t^*$, $\eta^* = x^* - t^*$, and F is any constant. In general, we have

$$\varphi_{i+1}(\xi^*) = q(\frac{L}{c}\frac{\xi^*-1}{m^*+1}) - \psi_i([\frac{m^*-1}{m^*+1}][\xi^*-1]+1), \tag{6.2.9a}$$

and

$$\psi_{j+1}(\eta^*) = p(-\frac{L}{c}\eta^*) - \varphi_j(-\eta^*), \tag{6.2.9b}$$

for $i = 1, 2, 3....$ As in section 5.2, it is noted that these recurrence relations carry appropriate restrictions on ξ^* and η^* as a result of those appearing in (6.2.8).

In particular, we have

$$\begin{aligned} u^*(x^*,t^*) &= \varphi_1(x^*+t^*)+\psi_1(x^*-t^*) \\ &= \frac{1}{2}[f(Lx^*+Lt^*)+f(Lx^*-Lt^*)]+\frac{1}{2c}\int_{Lx^*-Lt^*}^{Lx^*+Lt^*} g(\tau)d\tau, \end{aligned} \tag{6.2.10}$$

for $(x^*,t^*)\in D_{(1,1)}$

$$\begin{aligned} u^*(x^*,t^*) &= \varphi_1(x^*+t^*)+\psi_2(x^*-t^*) \\ &= \frac{1}{2}[f(Lt^*+Lx^*)-f(Lt^*-Lx^*)]+\frac{1}{2c}\int_{Lt^*-Lx^*}^{Lt^*+Lx^*} g(\tau)d\tau \\ &\quad +p(\frac{L}{c}t^*-\frac{L}{c}x^*), \end{aligned} \tag{6.2.11}$$

for $(x^*,t^*)\in D_{(1,2)}$,

$$\begin{aligned} u^*(x^*,t^*) &= \varphi_2(x^*+t^*)+\psi_1(x^*-t^*) \\ &= \frac{1}{2}\{f(Lx^*-Lt^*)-f(\frac{m^*-1}{m^*+1}[Lx^*+Lt^*-L]+L)\} \\ &\quad +\frac{1}{2c}\int_{Lx^*-Lt^*}^{\frac{m^*-1}{m^*+1}(Lx^*+Lt^*-L)+L} g(\tau)d\tau \\ &\quad +q(\frac{Lx^*+Lt^*-L}{cm^*+c}), \end{aligned} \tag{6.2.12}$$

for $(x^*,t^*)\in D_{(2,1)}$,

$$\begin{aligned} u^*(x^*,t^*) &= \varphi_2(x^*+t^*)+\psi_2(x^*-t^*) \\ &= -\frac{1}{2}\{f(\frac{m^*-1}{m^*+1}[Lt^*+Lx^*-L]+L)+f(Lt^*-Lx^*)\} \\ &\quad +\frac{1}{2c}\int_{Lt^*-Lx^*}^{\frac{m^*-1}{m^*+1}(Lt^*+Lx^*-L)+L} g(\tau)d\tau+p(\frac{L}{c}t^*-\frac{L}{c}x^*) \\ &\quad +q(\frac{Lt^*+Lx^*-L}{cm^*+c}), \end{aligned} \tag{6.2.13}$$

for $(x,t)\in D_{(2,2)}$,

$$
\begin{aligned}
u^*(x^*,t^*) &= \varphi_2(x^*+t^*)+\psi_3(x^*-t^*) \\
&= \frac{1}{2}\{f(\frac{m^*-1}{m^*+1}[Lt^*-Lx^*-L]+L)-f(\frac{m^*-1}{m^*+1}[Lt^*+Lx^*-L]+L)\} \\
&\quad +\frac{1}{2c}\int_{\frac{m^*-1}{m^*+1}(Lt^*-Lx^*-L)+L}^{\frac{m^*-1}{m^*+1}(Lt^*+Lx^*-L)+L} g(\tau)d\tau+p(\frac{L}{c}t^*-\frac{L}{c}x^*) \\
&\quad +q(\frac{Lt^*+Lx^*-L}{cm^*+c})-q(\frac{Lt^*-Lx^*-L}{cm^*+c}), \qquad (6.2.14)
\end{aligned}
$$

for $(x,t)\in D_{(2,3)}$,

$$
\begin{aligned}
u^*(x^*,t^*) &= \varphi_3(x^*+t^*)+\psi_2(x^*-t^*) \\
&= \frac{1}{2}\{f(-\frac{m^*-1}{m^*+1}[Lt^*+Lx^*-L]-L)-f(Lt^*-Lx^*)\} \\
&\quad +\frac{1}{2c}\int_{Lt^*-Lx^*}^{-\frac{m^*-1}{m^*+1}(Lt^*+Lx^*-L)-L} g(\tau)d\tau+p(\frac{L}{c}t^*-\frac{L}{c}x^*) \\
&\quad -p(-\frac{m^*-1}{cm^*+c}[Lt^*+Lx^*-L]-\frac{L}{c}) \\
&\quad +q(\frac{Lt^*+Lx^*-L}{cm^*+c}), \qquad (6.2.15)
\end{aligned}
$$

for $(x,t)\in D_{(3,2)}$,

and

$$
\begin{aligned}
u^*(x^*,t^*) &= \varphi_3(x^*+t^*)+\psi_3(x^*-t^*) \\
&= \frac{1}{2}\{f(-\frac{m^*-1}{m^*+1}[Lt^*+Lx^*-L]-L)+f(\frac{m^*-1}{m^*+1}[Lt^*-Lx^*-L]+L)\} \\
&\quad +\frac{1}{2c}\int_{\frac{m^*-1}{m^*+1}(Lt^*-Lx^*-L)+L}^{-\frac{m^*-1}{m^*+1}(Lt^*+Lx^*-L)-L} g(\tau)d\tau+p(\frac{L}{c}t^*-\frac{L}{c}x^*) \\
&\quad -p(-\frac{m^*-1}{cm^*+c}[Lt^*+Lx^*-L]-\frac{L}{c})+q(\frac{Lt^*+Lx^*-L}{cm^*+c}) \\
&\quad -q(\frac{Lt^*-Lx^*-L}{cm^*+c}), \qquad (6.2.16)
\end{aligned}
$$

for $(x,t)\in D_{(3,3)}$.

Continuation of the solution into other subregions may be accomplished by repeated use of (6.2.7-9).

Using (6.2.3-4), (6.2.10-16) give the corresponding solution of the problem (6.2.1-2) in the subregions $R_{(i,j)}$ of region $R = \{(x,t)\colon\ 0 \le x \le mt + L,\ t \ge 0\}$, as illustrated in the following diagram in the xt-plane.

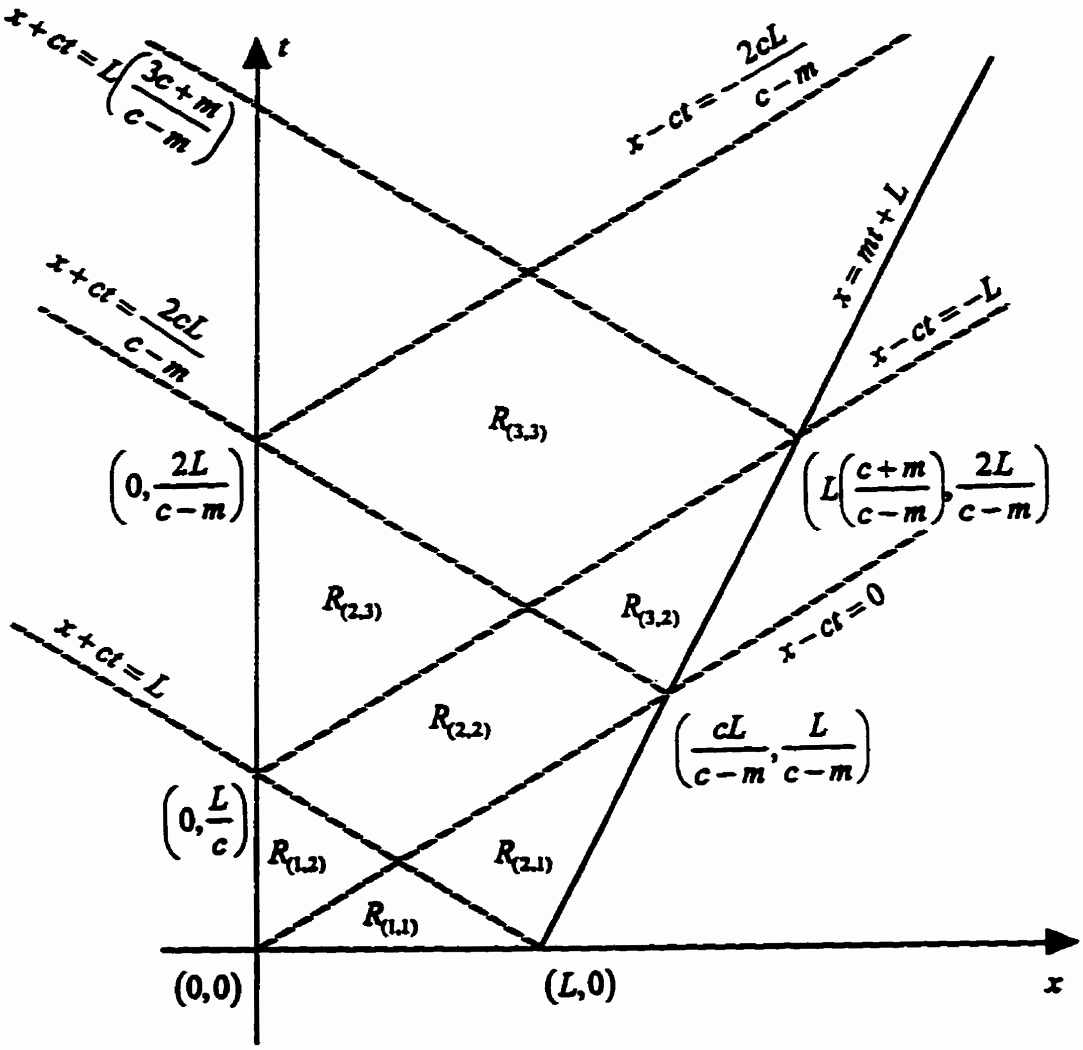


Diagram 6.2.2

Moreover, in analogy with (6.2.7), we write the solution $u(x,t)$ of (6.2.1-2) in subregion $R_{(i,j)}$, of diagram 6.2.2, as

$$u(x,t) = \varphi_j\left(\frac{x+ct}{L}\right) + \psi_j\left(\frac{x-ct}{L}\right), \tag{6.2.17}$$

where the functions $\varphi_i(\xi^*)$ and $\psi_j(\eta^*)$ are defined as in (6.2.8-9). In particular, we have

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, \tag{6.2.18}$$

$$\text{for } (x,t) \in R_{(1,1)},$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(ct+x) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\tau)d\tau \\ &+ p(t-\frac{x}{c}), \end{aligned} \tag{6.2.19}$$

$$\text{for } (x,t) \in R_{(1,2)},$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\{f(x-ct) - f(\frac{m-c}{m+c}[x+ct-L]+L)\} \\ &+ \frac{1}{2c}\int_{x-ct}^{\frac{m-c}{m+c}(x+ct-L)+L} g(\tau)d\tau + q(\frac{x+ct-L}{m+c}), \end{aligned} \tag{6.2.20}$$

$$\text{for } (x,t) \in R_{(2,1)},$$

$$\begin{aligned} u(x,t) &= -\frac{1}{2}\{f(\frac{m-c}{m+c}[ct+x-L]+L) + f(ct-x)\} \\ &+ \frac{1}{2c}\int_{ct-x}^{\frac{m-c}{m+c}(ct+x-L)+L} g(\tau)d\tau + p(t-\frac{x}{c}) \\ &+ q(\frac{ct+x-L}{m+c}), \end{aligned} \tag{6.2.21}$$

$$\text{for } (x,t) \in R_{(2,2)},$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\{f(\frac{m-c}{m+c}[ct-x-L]+L) - f(\frac{m-c}{m+c}[ct+x-L]+L)\} \\ &+ \frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{\frac{m-c}{m+c}(ct+x-L)+L} g(\tau)d\tau + p(t-\frac{x}{c}) \\ &+ q(\frac{ct+x-L}{m+c}) - q(\frac{ct-x-L}{m+c}), \end{aligned} \tag{6.2.22}$$

$$\text{for } (x,t) \in R_{(2,3)},$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\{f(-\frac{m-c}{m+c}[ct+x-L]-L)-f(ct-x)\} \\ &+\frac{1}{2c}\int_{ct-x}^{-\frac{m-c}{m+c}(ct+x-L)-L} g(\tau)d\tau + p(t-\frac{x}{c}) \\ &-p(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})+q(\frac{ct+x-L}{m+c}), \end{aligned} \tag{6.2.23}$$

$$\text{for } (x,t)\in R_{(3,2)},$$

and

$$\begin{aligned} u(x,t) &= \frac{1}{2}\{f(-\frac{m-c}{m+c}[ct+x-L]-L)+f(\frac{m-c}{m+c}[ct-x-L]+L)\} \\ &+\frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{-\frac{m-c}{m+c}(ct+x-L)-L} g(\tau)d\tau + p(t-\frac{x}{c}x) \\ &-p(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})+q(\frac{ct+x-L}{m+c}) \\ &-q(\frac{ct-x-L}{m+c}), \end{aligned} \tag{6.2.24}$$

$$\text{for } (x,t)\in R_{(3,3)}..$$

As indicated earlier, the continuation of the solution in additional subregions involves repeated use of (6.2.7-9). Alternatively, it is noted that, since the horizontal line $t=\frac{2L}{c-m}$ passes through the two points $(0,\frac{2L}{c-m})$ and $(\frac{(c+m)L}{c-m},\frac{2L}{c-m})$, we could simply treat the continuation of our solution as a new initial value problem having $u(x,\frac{2L}{c-m})$ and $\frac{\partial}{\partial t}(x,\frac{2L}{c-m})$ as evaluated from (6.2.24) along $t=\frac{2L}{c-m}$ as the new initial conditions. Then a simple linear change of variable of the from

$$\tilde{t}=t-\frac{2L}{c-m}$$

reduces this new problem to one having the original form, namely (6.2.1-2).

Finally to complete this section, we note the following compatability conditions on the functions which determine the initial and boundary conditions for problem (6.2.1-2):

1) If $f(x)$, $p(t)$ and $q(t) \in C^0$, $g(x)$ is integrable, $f(0) = p(0)$ and $f(L) = q(0)$, then $u(x,t) \in C^0$.

2) If $f(x)$, $p(t)$ and $q(t) \in C^1$, $g(x) \in C^0$, $f(0) = p(0)$ and $f(L) = q(0)$, $g(0) = p'(0)$ and $mf'(L) + g(L) = q'(0)$, then $u(x,t) \in C^1$.

Once again these conditions follow immediately from (6.2.18-24), with the last condition in 2) coming from differentiating $u(mt + L, t) = q(t)$ with respect to t, and then setting $t = 0$.

6.3 Transformation of Results

We now turn our attention to the coupled linear system

$$\frac{\partial^2 u_1}{\partial t^2} - 2\omega\frac{\partial u_2}{\partial t} - \omega^2 u_1 = c^2\frac{\partial^2 u_1}{\partial x^2}, \tag{6.3.1a}$$

$$\frac{\partial^2 u_2}{\partial t^2} + 2\omega\frac{\partial u_1}{\partial t} - \omega^2 u_2 = c^2\frac{\partial^2 u_2}{\partial x^2}, \tag{6.3.1b}$$

$$0 \le x \le mt + L, \quad t \ge 0, \quad |m| < c,$$

with initial conditions given by

$$u_1(x,0) = f_1(x), \quad \frac{\partial u_1}{\partial t}(x,0) = g_1(x), \tag{6.3.2a}$$

$$u_2(x,0) = f_2(x), \quad \frac{\partial u_2}{\partial t}(x,0) = g_2(x), \tag{6.3.2b}$$

$$0 \le x \le L,$$

boundary conditions at $x = 0$ given by

$$u_1(0,t) = p_1(t), \quad u_2(0,t) = p_2(t), \tag{6.3.2c}$$

and boundary conditions, at the linearly-moving endpoint $x = mt + L$, given by

$$u_1(mt+L,t) = q_1(t), \quad u_2(mt+L,t) = q_2(t). \tag{6.3.2d}$$

As in Chapter 3, 4 and 5, we again let

$$V(x,t) = u_1(x,t) + iu_2(x,t), \tag{6.3.3}$$

where $i = \sqrt{-1}$, and

$$W(x,t) = e^{i\omega t}V(x,t). \tag{6.3.4}$$

so that (6.3.1) is transformed into

$$\frac{\partial^2 W}{\partial t^2} = c^2 \frac{\partial^2 W}{\partial x^2}, \tag{6.3.5}$$

$$0 \le x \le mt + L, \ t \ge 0,$$

while the initial and boundary conditions (6.3.2) become respectively

$$W(x,0) = f(x) = f_1(x) + if_2(x), \tag{6.3.6a}$$

$$\frac{\partial W}{\partial t}(x,0) = g(x) = -\omega f_2(x) + g_1(x) + i[\omega f_1(x) + g_2(x)], \tag{6.3.6b}$$

$$\begin{aligned} W(0,t) &= p(t) \\ &= p_1(t)\cos(\omega t) - p_2(t)\sin(\omega t) \\ &\quad + i[p_1(t)\sin(\omega t) + p_2(t)\cos(\omega t)], \end{aligned} \tag{6.3.6c}$$

$$\begin{aligned} W(mt+L,t) &= q(t) \\ &= q_1(t)\cos(\omega t) - q_2(t)\sin(\omega t) \\ &\quad + i[q_1(t)\sin(\omega t) + q_2(t)\cos(\omega t)]. \end{aligned} \tag{6.3.6d}$$

In analogy with (6.2.17), we may write the solution of this problem for $(x,t) \in R_{(i,j)}$, i, $j = 1, 2, 3...$, in the form

$$W(x,t) = \varphi_i(\frac{x+ct}{L}) + \psi_j(\frac{x-ct}{L}), \tag{6.3.7}$$

where the functions $\varphi_i(\xi^*)$ and $\psi_j(\eta^*)$ are defined as in (6.2.8-9). Specifically, we have

$$W(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, \tag{6.3.8}$$

for $(x,t) \in R_{(1,1)}$,

$$\begin{aligned} W(x,t) &= \frac{1}{2}[f(ct+x) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\tau)d\tau \\ &\quad + p(t - \frac{x}{c}), \end{aligned} \tag{6.3.9}$$

for $(x,t) \in R_{(1,2)}$,

$$\begin{aligned} W(x,t) &= \frac{1}{2}\{f(x-ct) - f(\frac{m-c}{m+c}[x+ct-L] + L)\} \\ &\quad + \frac{1}{2c}\int_{x-ct}^{\frac{m-c}{m+c}(x+ct-L)+L} g(\tau)d\tau + q(\frac{x+ct-L}{m+c}), \end{aligned} \tag{6.3.10}$$

for $(x,t) \in R_{(2,1)}$,

$$\begin{aligned} W(x,t) &= -\frac{1}{2}\{f(\frac{m-c}{m+c}[ct+x-L] + L) + f(ct-x)\} \\ &\quad + \frac{1}{2c}\int_{ct-x}^{\frac{m-c}{m+c}(ct+x-L)+L} g(\tau)d\tau + p(t - \frac{x}{c}) \\ &\quad + q(\frac{ct+x-L}{m+c}), \end{aligned} \tag{6.3.11}$$

for $(x,t) \in R_{(2,2)}$,

$$\begin{aligned} W(x,t) &= \frac{1}{2}\{f(\frac{m-c}{m+c}[ct-x-L]+L) - f(\frac{m-c}{m+c}[ct+x-L]+L)\} \\ &+\frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{\frac{m-c}{m+c}(ct+x-L)+L} g(\tau)d\tau + p(t-\frac{x}{c}) \\ &+q(\frac{ct+x-L}{m+c}) - q(\frac{ct-x-L}{m+c}), \end{aligned} \tag{6.3.12}$$

for $(x,t) \in R_{(2,3)}$,

$$\begin{aligned} W(x,t) &= \frac{1}{2}\{f(-\frac{m-c}{m+c}[ct+x-L]-L) - f(ct-x)\} \\ &+\frac{1}{2c}\int_{ct-x}^{-\frac{m-c}{m+c}(ct+x-L)-L} g(\tau)d\tau + p(t-\frac{x}{c}) \\ &-p(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}) + q(\frac{ct+x-L}{m+c}), \end{aligned} \tag{6.3.13}$$

for $(x,t) \in R_{(3,2)}$,

and

$$\begin{aligned} W(x,t) &= \frac{1}{2}\{f(-\frac{m-c}{m+c}[ct+x-L]-L) + f(\frac{m-c}{m+c}[ct-x-L]+L)\} \\ &+\frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{-\frac{m-c}{m+c}(ct+x-L)-L} g(\tau)d\tau + p(t-\frac{x}{c}x) \\ &-p(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}) + q(\frac{ct+x-L}{m+c}) \\ &-q(\frac{ct-x-L}{m+c}), \end{aligned} \tag{6.3.14}$$

for $(x,t) \in R_{(3,3)}$.

By virtue of (6.3.6), the above results confirm that at any point $W(x,t)$ is complex. However, (6.3.3) and (6.3.4) imply that

$$u_1(x,t) = \cos(\omega t)\,\mathrm{Re}\,W(x,t) + \sin(\omega t)\,\mathrm{Im}\,W(x,t), \tag{6.3.15a}$$

$$u_2(x,t) = -\sin(\omega t)\,\mathrm{Re}\,W(x,t) + \cos(\omega t)\,\mathrm{Im}\,W(x,t). \tag{6.3.15b}$$

Substitution of (6.3.6) into (6.3.8-14) gives

$$\begin{aligned}\operatorname{Re} W(x,t) &= \frac{1}{2}[f_1(x+ct)+f_1(x-ct)] \\ &\quad +\frac{1}{2c}\int_{x-ct}^{x+ct}[-\omega f_2(\tau)+g_1(\tau)]d\tau, \qquad (6.3.16a)\\ \operatorname{Im} W(x,t) &= \frac{1}{2}[f_2(x+ct)+f_2(x-ct)] \\ &\quad +\frac{1}{2c}\int_{x-ct}^{x+ct}[\omega f_1(\tau)+g_2(\tau)]d\tau, \qquad (6.3.16b)\end{aligned}$$

for $(x,t)\in R_{(1,1)}$,

$$\begin{aligned}\operatorname{Re} W(x,t) &= \frac{1}{2}[f_1(ct+x)-f_1(ct-x)] \\ &\quad +\frac{1}{2c}\int_{ct-x}^{ct+x}[-\omega f_2(\tau)+g_1(\tau)]d\tau \\ &\quad +p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c}) \\ &\quad -p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c}), \qquad (6.3.17a)\\ \operatorname{Im} W(x,t) &= \frac{1}{2}[f_2(ct+x)-f_2(ct-x] \\ &\quad +\frac{1}{2c}\int_{ct-x}^{ct+x}[\omega f_1(\tau)+g_2(\tau)]d\tau \\ &\quad +p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c}) \\ &\quad +p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c}), \qquad (6.3.17b)\end{aligned}$$

for $(x,t)\in R_{(1,2)}$,

$$\begin{aligned}\operatorname{Re} W(x,t) &= \frac{1}{2}\{f_1(x-ct)-f_1(\frac{m-c}{m+c}[x+ct-L]+L)\} \\ &\quad +\frac{1}{2c}\int_{x-ct}^{\frac{m-c}{m+c}(x+ct-L)+L}[-\omega f_2(\tau)+g_1(\tau)]d\tau \\ &\quad +q_1(\frac{x+ct-L}{m+c})\cos\frac{\omega(x+ct-L)}{m+c} \\ &\quad -q_2(\frac{x+ct-L}{m+c})\sin\frac{\omega(x+ct-L)}{m+c}, \qquad (6.3.18a)\end{aligned}$$

$$
\begin{aligned}
\operatorname{Im} W(x,t) &= \frac{1}{2}\{f_2(x-ct) - f_2(\frac{m-c}{m+c}[x+ct-L]+L)\} \\
&+\frac{1}{2c}\int_{x-ct}^{\frac{m-c}{m+c}(x+ct-L)+L}[\omega f_1(\tau)+g_2(\tau)]d\tau \\
&+q_1(\frac{x+ct-L}{m+c})\sin\frac{\omega(x+ct-L)}{m+c} \\
&+q_2(\frac{x+ct-L}{m+c})\cos\frac{\omega(x+ct-L)}{m+c}, \qquad (6.3.18b)
\end{aligned}
$$

for $(x,t) \in R_{(2,1)}$,

$$
\begin{aligned}
\operatorname{Re} W(x,t) &= -\frac{1}{2}\{f_1(\frac{m-c}{m+c}[ct+x-L]+L) + f_1(ct-x)\} \\
&+\frac{1}{2c}\int_{ct-x}^{\frac{m-c}{m+c}(ct+x-L)+L}[-\omega f_2(\tau)+g_1(\tau)]d\tau \\
&+p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c}) - p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c}) \\
&+q_1(\frac{ct+x-L}{m+c})\cos\frac{\omega(ct+x-L)}{m+c} \\
&-q_2(\frac{ct+x-L}{m+c})\sin\frac{\omega(ct+x-L)}{m+c}, \qquad (6.3.19a) \\
\operatorname{Im} W(x,t) &= -\frac{1}{2}\{f_2(\frac{m-c}{m+c}[ct+x-L]+L) + f_2(ct-x)\} \\
&+\frac{1}{2c}\int_{ct-x}^{\frac{m-c}{m+c}(ct+x-L)+L}[\omega f_1(\tau)+g_2(\tau)]d\tau \\
&+p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c}) + p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c}) \\
&+q_1(\frac{ct+x-L}{m+c})\sin\frac{\omega(ct+x-L)}{m+c} \\
&+q_2(\frac{ct+x-L}{m+c})\cos\frac{\omega(ct+x-L)}{m+c}, \qquad (6.3.19b)
\end{aligned}
$$

for $(x,t) \in R_{(2,2)}$,

$$
\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{f_1(\frac{m-c}{m+c}[ct-x-L]+L) - f_1(\frac{m-c}{m+c}[ct+x-L]+L)\} \\
&+\frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{\frac{m-c}{m+c}(ct+x-L)+L}[-\omega f_2(\tau)+g_1(\tau)]d\tau \\
&+p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c}) - p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})
\end{aligned}
$$

$$
\begin{aligned}
&+q_1(\frac{ct+x-L}{m+c})\cos\frac{\omega(ct+x-L)}{m+c}\\
&-q_2(\frac{ct+x-L}{m+c})\sin\frac{\omega(ct+x-L)}{m+c}\\
&-q_1(\frac{ct-x-L}{m+c})\cos\frac{\omega(ct-x-L)}{m+c}\\
&+q_2(\frac{ct-x-L}{m+c})\sin\frac{\omega(ct-x-L)}{m+c}, \qquad (6.3.20a)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Im} W(x,t) &= \frac{1}{2}\{f_2(\frac{m-c}{m+c}[ct-x-L]+L)-f_2(\frac{m-c}{m+c}[ct+x-L]+L)\}\\
&+\frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{\frac{m-c}{m+c}(ct+x-L)+L}[\omega f_1(\tau)+g_2(\tau)]d\tau\\
&+p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})+p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})\\
&+q_1(\frac{ct+x-L}{m+c})\sin\frac{\omega(ct+x-L)}{m+c}\\
&+q_2(\frac{ct+x-L}{m+c})\cos\frac{\omega(ct+x-L)}{m+c}\\
&-q_1(\frac{ct-x-L}{m+c})\sin\frac{\omega(ct-x-L)}{m+c}\\
&-q_2(\frac{ct-x-L}{m+c})\cos\frac{\omega(ct-x-L)}{m+c}, \qquad (6.3.20b)
\end{aligned}
$$

for $(x,t)\in R_{(2,3)}$,

$$
\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{f_1(-\frac{m-c}{m+c}[ct+x-L]-L)-f_1(ct-x)\}\\
&+\frac{1}{2c}\int_{ct-x}^{-\frac{m-c}{m+c}(ct+x-L)-L}[-\omega f_2(\tau)+g_1(\tau)]d\tau\\
&+p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})-p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})\\
&-p_1(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\cos\omega(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\\
&+p_2(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\sin\omega(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\\
&+q_1(\frac{ct+x-L}{m+c})\cos\frac{\omega(ct+x-L)}{m+c}\\
&-q_2(\frac{ct+x-L}{m+c})\sin\frac{\omega(ct+x-L)}{m+c}, \qquad (6.3.21a)
\end{aligned}
$$

$$
\operatorname{Im} W(x,t) = \frac{1}{2}\{f_2(-\frac{m-c}{m+c}[ct+x-L]-L)-f_2(ct-x)\}
$$

$$
\begin{aligned}
&+\frac{1}{2c}\int_{ct-x}^{-\frac{m-c}{m+c}(ct+x-L)-L}[\omega f_1(\tau)+g_2(\tau)]d\tau \\
&+p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})+p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c}) \\
&-p_1(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\sin\omega(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}) \\
&-p_2(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\cos\omega(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}) \\
&+q_1(\frac{ct+x-L}{m+c})\sin\frac{\omega(ct+x-L)}{m+c} \\
&+q_2(\frac{ct+x-L}{m+c})\cos\frac{\omega(ct+x-L)}{m+c}, \qquad (6.3.21b)
\end{aligned}
$$

$$
\text{for } (x,t)\in R_{(3,2)},
$$

and

$$
\begin{aligned}
\operatorname{Re}W(x,t) &= \frac{1}{2}\{f_1(-\frac{m-c}{m+c}[ct+x-L]-L)+f_1(\frac{m-c}{m+c}[ct-x-L]+L)\} \\
&+\frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{-\frac{m-c}{m+c}(ct+x-L)-L}[-\omega f_2(\tau)+g_1(\tau)]d\tau \\
&+p_1(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})-p_2(t-\frac{x}{c})\sin\omega(t-\frac{x}{c}) \\
&-p_1(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\cos\omega(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}) \\
&+p_2(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c})\sin\omega(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}) \\
&+q_1(\frac{ct+x-L}{m+c})\cos\frac{\omega(ct+x-L)}{m+c} \\
&-q_2(\frac{ct+x-L}{m+c})\sin\frac{\omega(ct+x-L)}{m+c} \\
&-q_1(\frac{ct-x-L}{m+c})\cos\frac{\omega(ct-x-L)}{m+c} \\
&+q_2(\frac{ct-x-L}{m+c})\sin\frac{\omega(ct-x-L)}{m+c}, \qquad (6.3.22a) \\
\operatorname{Im}W(x,t) &= \frac{1}{2}\{f_2(-\frac{m-c}{m+c}[ct+x-L]-L)+f_2(\frac{m-c}{m+c}[ct-x-L]+L)\} \\
&+\frac{1}{2c}\int_{\frac{m-c}{m+c}(ct-x-L)+L}^{-\frac{m-c}{m+c}(ct+x-L)-L}[\omega f_1(\tau)+g_2(\tau)]d\tau \\
&+p_1(t-\frac{x}{c})\sin\omega(t-\frac{x}{c})+p_2(t-\frac{x}{c})\cos\omega(t-\frac{x}{c})
\end{aligned}
$$

$$
\begin{aligned}
&-p_1\left(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}\right)\sin\omega\left(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}\right)\\
&-p_2\left(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}\right)\cos\omega\left(-\frac{m-c}{cm+c^2}[ct+x-L]-\frac{L}{c}\right)\\
&+q_1\left(\frac{ct+x-L}{m+c}\right)\sin\frac{\omega(ct+x-L)}{m+c}\\
&+q_2\left(\frac{ct+x-L}{m+c}\right)\cos\frac{\omega(ct+x-L)}{m+c}\\
&-q_1\left(\frac{ct-x-L}{m+c}\right)\sin\frac{\omega(ct-x-L)}{m+c}\\
&-q_2\left(\frac{ct-x-L}{m+c}\right)\cos\frac{\omega(ct-x-L)}{m+c},
\end{aligned}
\tag{6.3.22b}
$$

for $(x,t)\in R_{(3,3)}$.

The solution of the problem (6.2.1-2) in the first seven subregions $R_{(1,1)}$, $R_{(1,2)}$, $R_{(2,1)}$, $R_{(2,2)}$, $R_{(2,3)}$, $R_{(3,2)}$ and $R_{(3,3)}$ is determined by (6.3.15-22). The solution in any other subregion within region R can be similarly determined by first substituting (6.2.8) and (6.2.9) into the formulae (6.3.7) to obtain the real and imaginary parts of the corresponding $W(x,t)$, and then using (6.3.15).

6.4 Examples

The examples which follow are chosen to illustrate results corresponding to the "pulse wave" examples of section 4.3 and 5.4, but in the case of a spinning string of finite length with a linearly-moving endpoint. Parameter values, as well as functions specifying the initial and boundary conditions, are chosen (so far as possible) to coincide with those of the corresponding examples of the previous chapters.

Examples 6.4.1: In the moving-boundary problem (6.2.1-2), we suppose

$$c = 1,\ \omega = \frac{\pi}{2},\ m = 0.5,\ L = 4,$$

and

$$\begin{aligned} f_1(x) &= \begin{cases} 1 - \cos(2\pi x), & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases} \\ &= [H(x-2) - H(x-3)][1 - \cos(2\pi x)], \end{aligned}$$

$$f_2(x) = g_1(x) = g_2(x) = 0,\ \text{for } 0 \le x \le 4,$$

$$p_1(t) = p_2(t) = q_1(t) = q_2(t) = 0,\ \text{for } t \ge 0.$$

These conditions state that the string has a fixed end at $x = 0$, and that the other end is moving away from the fixed end at a constant speed, which is precisely one half of the wave propagation speed (c) along the string. Morever, initially the string has length 4 and has been "plucked" in the shape of a single planar sinusoidal pulse on $2 \le x \le 3$, and is released from rest (with zero initial velocity).

According to (6.3.16-22), we may write

$$\begin{aligned} \operatorname{Re} W(x,t) &= \frac{1}{2}\{[1 - \cos 2\pi(x+t)][H(x+t-2) - H(x+t-3)] \\ &\quad +[1 - \cos 2\pi(x-t)][H(x-t-2) - H(x-t-3)]\}, \\ \operatorname{Im} W(x,t) &= \frac{\pi}{4}\int_{x-t}^{x+t}[H(\tau-2) - H(\tau-3)][1 - \cos(2\pi\tau)]d\tau, \end{aligned}$$

$$\text{for } (x,t) \in R_{(1,1)},$$

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos 2\pi(t+x)][H(t+x-2)-H(t+x-3)] \\
&\quad -[1-\cos 2\pi(t-x)][H(t-x-2)-H(t-x-3)]\}, \\
\operatorname{Im} W(x,t) &= \frac{\pi}{4}\int_{t-x}^{t+x}[H(\tau-2)-H(\tau-3)][1-\cos(2\pi\tau)]d\tau,
\end{aligned}$$

for $(x,t) \in R_{(1,2)}$,

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos 2\pi(x-t)][H(x-t-2)-H(x-t-3)] \\
&\quad -[1-\cos\frac{2\pi(x+t-16)}{3}] \\
&\quad \cdot[H(-\frac{x+t-16}{3}-2)-H(-\frac{x+t-16}{3}-3)]\}, \\
\operatorname{Im} W(x,t) &= \frac{\pi}{4}\int_{x-t}^{-\frac{x+t-16}{3}}[H(\tau-2)-H(\tau-3)][1-\cos(2\pi\tau)]d\tau,
\end{aligned}$$

for $(x,t) \in R_{(2,1)}$,

$$\begin{aligned}
\operatorname{Re} W(x,t) &= -\frac{1}{2}\{[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\quad \cdot[H(-\frac{t+x-16}{3}-2)-H(-\frac{t+x-16}{3}-3)] \\
&\quad +[1-\cos 2\pi(t-x)][H(t-x-2)-H(t-x-3)]\}, \\
\operatorname{Im} W(x,t) &= \frac{\pi}{4}\int_{t-x}^{-\frac{t+x-16}{3}}[H(\tau-2)-H(\tau-3)][1-\cos(2\pi\tau)]d\tau,
\end{aligned}$$

for $(x,t) \in R_{(2,2)}$,

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos\frac{2\pi(t-x-16)}{3}] \\
&\quad \cdot[H(-\frac{t-x-16}{3}-2)-H(-\frac{t-x-16}{3}-3)] \\
&\quad -[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\quad \cdot[H(-\frac{t+x-16}{3}-2)-H(-\frac{t+x-16}{3}-3)]\},
\end{aligned}$$

$$\operatorname{Im} W(x,t) = \frac{\pi}{4}\int_{-\frac{t-x-16}{3}}^{-\frac{t+x-16}{3}} [H(\tau-2)-H(\tau-3)][1-\cos(2\pi\tau)]d\tau,$$

$$\text{for } (x,t)\in R_{(2,3)},$$

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\quad \cdot[H(\frac{t+x-16}{3}-2)-H(\frac{t+x-16}{3}-3)] \\
&\quad -[1-\cos 2\pi(t-x)][H(t-x-2)-H(t-x-3)]\}, \\
\operatorname{Im} W(x,t) &= \frac{\pi}{4}\int_{t-x}^{\frac{t+x-16}{3}} [H(\tau-2)-H(\tau-3)][1-\cos(2\pi\tau)]d\tau,
\end{aligned}$$

$$\text{for } (x,t)\in R_{(3,2)},$$

and

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\quad \cdot[H(\frac{t+x-16}{3}-2)-H(\frac{t+x-16}{3}-3)] \\
&\quad +[1-\cos\frac{2\pi(t-x-16)}{3}] \\
&\quad \cdot[H(-\frac{t-x-16}{3}-2)-H(-\frac{t-x-16}{3}-3)]\}, \\
\operatorname{Im} W(x,t) &= \frac{\pi}{4}\int_{-\frac{t-x-16}{3}}^{\frac{t+x-16}{3}} [H(\tau-2)-H(\tau-3)][1-\cos(2\pi\tau)]d\tau,
\end{aligned}$$

$$\text{for } (x,t)\in R_{(3,3)}.$$

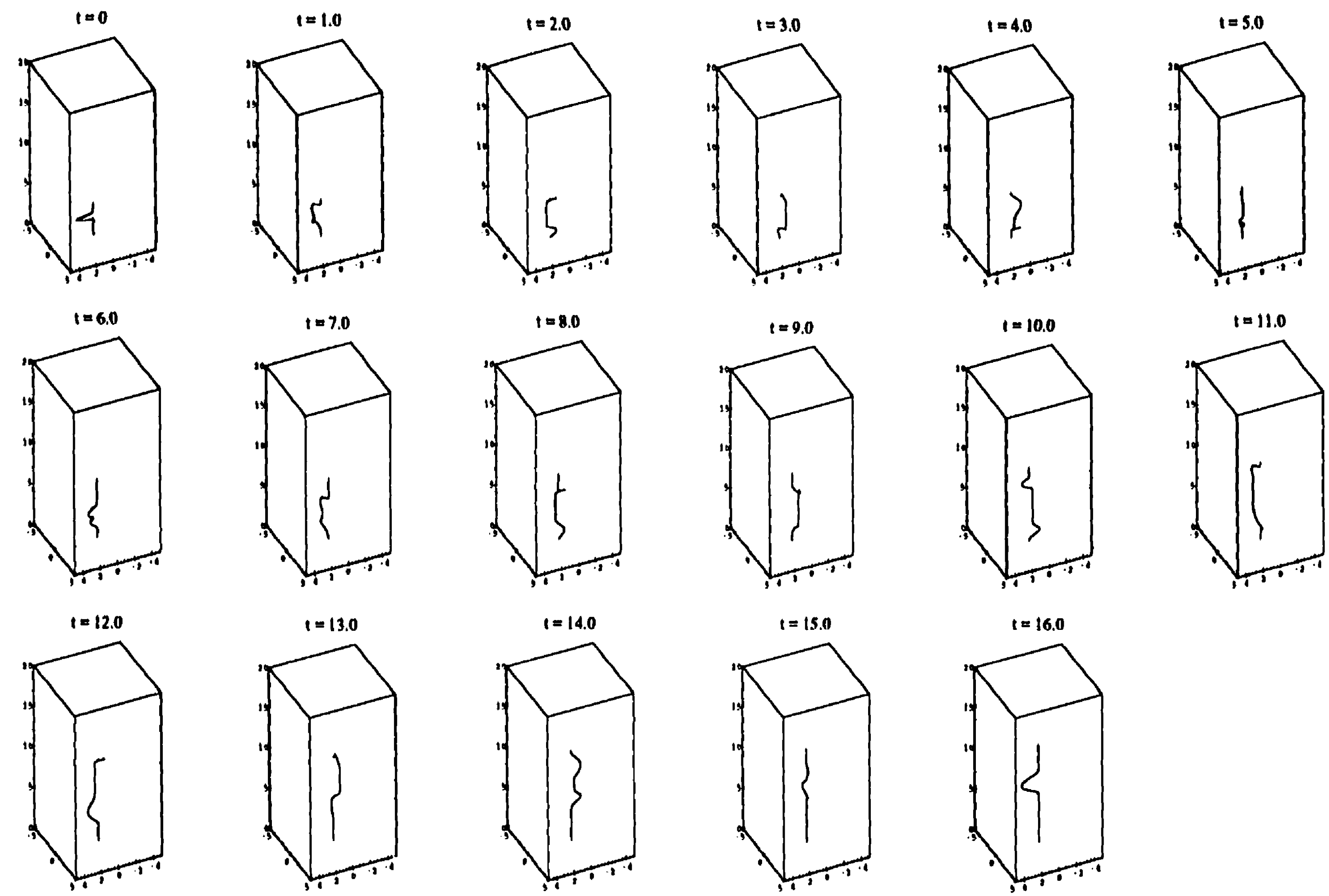
Finally, the solution of problem (6.2.1-2) as described in example 6.4.1 is given, by virtue of (6.3.15), by

$$u_1(x,t) = \cos(\frac{\pi}{2}t)\operatorname{Re} W(x,t) + \sin(\frac{\pi}{2}t)\operatorname{Im} W(x,t), \tag{6.4.1a}$$

$$u_2(x,t) = -\sin(\frac{\pi}{2}t)\operatorname{Re} W(x,t) + \cos(\frac{\pi}{2}t)\operatorname{Im} W(x,t). \tag{6.4.1b}$$

A sequence of graphs describing the motion of the corresponding wave of this example is presented in Figure 6.4.1. In this case, the time interval between successive graphs is 1.0 and the total elapsed time is of duration 16.0. The graphs in Figure 6.4.1 demonstrate that the initial pulse does not remain planar; instead it splits into two nonplanar pulses which travel away from each other, with one pulse reflecting from the fixed end, the other pulse reflecting from the moving end a very short time later. Upon these first reflections, the original rotational directions of these pulses are maintained and the two pulses travel toward each other, pass through each other and then travel in opposite directions until these pulses reflect from the fixed end and from the moving end at a later time. After these second reflections, the two pulses again travel toward each other and meet at time $t = 16$, at which time they merge into a single pulse similar to, but larger than, the initial pulse. It is noted in this example that reflection off of a linearly-receding endpoint causes a magnification in the length of the reflected pulse.

Figure 6.4.1: graphic illustration of Example 6.4.1 where $c=1$, $\omega=\pi/2$, $m=0.5$, $L=4$



Example 6.4.2: In the moving-boundary problem (6.2.1-2), we suppose

$$c = 1,\ \omega = \frac{\pi}{2},\ m = 0.5,\ L = 4,$$

and

$$\begin{aligned} f_1(x) &= \begin{cases} 1 - \cos(2\pi x), & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases} \\ &= [H(x-2) - H(x-3)][1 - \cos(2\pi x)], \end{aligned}$$

$$f_2(x) = 0,\ g_1(x) = \omega f_2(x) = 0,\ \text{for } 0 \le x \le 4,$$

$$\begin{aligned} g_2(x) &= -\omega f_1(x) \\ &= \begin{cases} \pi[\cos(2\pi x) - 1], & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases} \\ &= \pi[H(x-2) - H(x-3)][\cos(2\pi x) - 1], \end{aligned}$$

$$p_1(t) = p_2(t) = q_1(t) = q_2(t) = 0,\ \text{for } t\ \ge 0.$$

Physically, this example describes a finite spinning string with a fixed end at $x = 0$ and a linearly-moving end at $x = \frac{1}{2}t + 4$. The initial "plucked" displacement is in the form of a simple planar sinusoidal pulse on $2 \le x \le 3$ and the (complex) rotational velocity is chosen to eliminate rotational lag, as in examples 4.3.4 and 5.4.6.

Equations (6.3.16-22) provide

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos 2\pi(x+t)][H(x+t-2)-H(x+t-3)] \\
&\quad +[1-\cos 2\pi(x-t)][H(x-t-2)-H(x-t-3)]\}, \\
\operatorname{Im} W(x,t) &= 0,
\end{aligned}$$

$$\text{for } (x,t)\in R_{(1,1)},$$

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos 2\pi(t+x)][H(t+x-2)-H(t+x-3)] \\
&\quad -[1-\cos 2\pi(t-x)][H(t-x-2)-H(t-x-3)]\}, \\
\operatorname{Im} W(x,t) &= 0,
\end{aligned}$$

$$\text{for } (x,t)\in R_{(1,2)},$$

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos 2\pi(x-t)][H(x-t-2)-H(x-t-3)] \\
&\quad -[1-\cos\frac{2\pi(x+t-16)}{3}] \\
&\quad \cdot[H(-\frac{x+t-16}{3}-2)-H(-\frac{x+t-16}{3}-3)]\}, \\
\operatorname{Im} W(x,t) &= 0,
\end{aligned}$$

$$\text{for } (x,t)\in R_{(2,1)},$$

$$\begin{aligned}
\operatorname{Re} W(x,t) &= -\frac{1}{2}\{[H(-\frac{t+x-16}{3}-2)-H(-\frac{t+x-16}{3}-3)] \\
&\quad \cdot[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\quad +[H(t-x-2)-H(t-x-3)][1-\cos 2\pi(t-x)]\}, \\
\operatorname{Im} W(x,t) &= 0,
\end{aligned}$$

for $(x,t) \in R_{(2,2)}$,

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos\frac{2\pi(t-x-16)}{3}] \\
&\cdot[H(-\frac{t-x-16}{3}-2) - H(-\frac{t-x-16}{3}-3)] \\
&-[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\cdot[H(-\frac{t+x-16}{3}-2) - H(-\frac{t+x-16}{3}-3)]\}, \\
\operatorname{Im} W(x,t) &= 0,
\end{aligned}$$

for $(x,t) \in R_{(2,3)}$,

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\cdot[H(\frac{t+x-16}{3}-2) - H(\frac{t+x-16}{3}-3)] \\
&-[1-\cos 2\pi(t-x)][H(t-x-2) - H(t-x-3)]\}, \\
\operatorname{Im} W(x,t) &= 0,
\end{aligned}$$

for $(x,t) \in R_{(3,2)}$,

and

$$\begin{aligned}
\operatorname{Re} W(x,t) &= \frac{1}{2}\{[1-\cos\frac{2\pi(t+x-16)}{3}] \\
&\cdot[H(\frac{t+x-16}{3}-2) - H(\frac{t+x-16}{3}-3)] \\
&+[1-\cos\frac{2\pi(t-x-16)}{3}] \\
&\cdot[H(-\frac{t-x-16}{3}-2) - H(-\frac{t-x-16}{3}-3)]\}, \\
\operatorname{Im} W(x,t) &= 0,
\end{aligned}$$

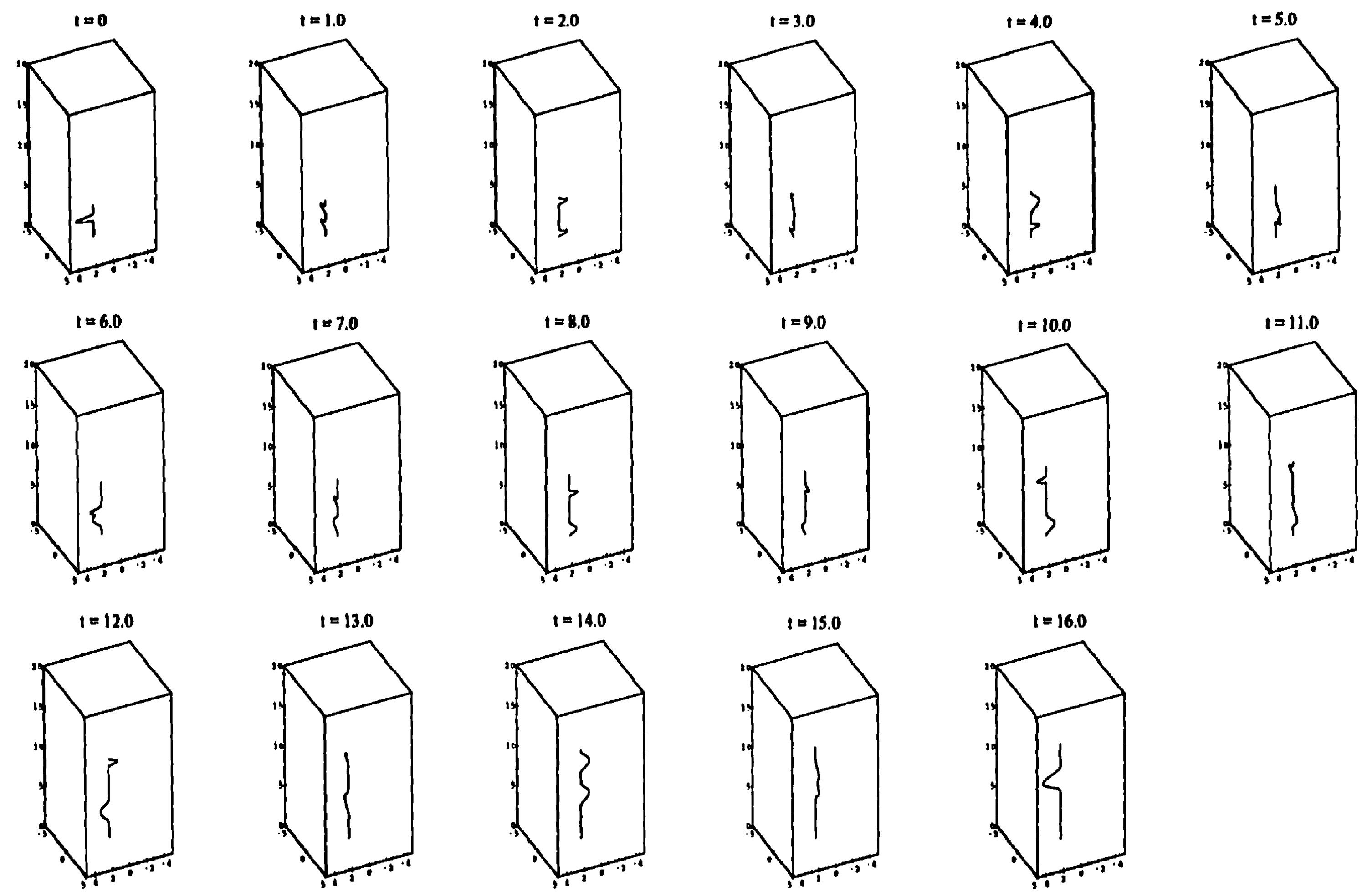
for $(x,t) \in R_{(3,3)}$,

so that the solution of problem (6.2.1-2), with the prescribed information of example 6.4.2, is once again given by equations (6.4.1); namely,

$$\begin{aligned} u_1(x,t) &= \cos(\frac{\pi}{2}t)\,\mathrm{Re}\,W(x,t) + \sin(\frac{\pi}{2}t)\,\mathrm{Im}\,W(x,t), \\ u_2(x,t) &= -\sin(\frac{\pi}{2}t)\,\mathrm{Re}\,W(x,t) + \cos(\frac{\pi}{2}t)\,\mathrm{Im}\,W(x,t). \end{aligned}$$

The graphs describing the motion of the corresponding wave of this example are presented in Figure 6.4.2. The choice of zero initial (complex) rotational velocity once again ensures that the travelling pulses remain planar and rotate at uniform angular velocity, while travelling along the string at the wave propagation speed (c). The elongation of a pulse upon reflection from the linearly-receding endpoint is more evident in this example than in the preceding example, and the "inversion" of a reflected pulse at either end is also evident. As in the previous example, by time $t = 16$, the two travelling pulses have each reflected off both ends, and have merged to produce an enlarged version of the original pulse.

Figure 6.4.2: graphic illustration of Example 6.4.2 where c=1, $\omega=\pi/2$, m=0.5, L=4



Examples 6.4.3: In the moving-boundary problem (6.2.1-2), we suppose

$$c = 1,\ \omega = \frac{\pi}{2},\ m = 0.8,\ L = 3,$$

and

$$f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0,\ \text{for } 0 \le x \le 3,$$

$$\begin{aligned} p_1(t) &= \begin{cases} 1 - \cos(2\pi t), & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \\ &= [H(t) - H(t-1)][1 - \cos(2\pi t)], \end{aligned}$$

$$p_2(t) = q_1(t) = q_2(t) = 0,\ \text{for } t\ \ge 0.$$

The physical set-up of this problem is similar to that of examples 4.3.5 and 5.4.7. The spinning string, in this case, has an initial length of 3, a fixed end at $x = 0$ and a linearly-moving end at $x = \frac{8}{10}t + 3$. The initial displacements and velocities are both assumed to be zero, and a simple planar sinusoidal pulse is used to excite the string at $x = 0$, while the string is attached to the moving end, and thus, its displacement is zero there.

According to (6.3.16-22), we have

$$\operatorname{Re} W(x,t) = 0,\ \operatorname{Im} W(x,t) = 0,$$

$$\text{for } (x,t) \in R_{(1,1)} \cup\ R_{(2,1)},$$

$$\operatorname{Re} W(x,t) = \cos\frac{\pi}{2}(t-x)[1 - \cos 2\pi(t-x)][H(t-x) - H(t-x-1)],$$

$$\operatorname{Im} W(x,t) = \sin\frac{\pi}{2}(t-x)[1-\cos 2\pi(t-x)][H(t-x)-H(t-x-1)],$$

$$\text{for } (x,t) \in R_{(1,2)} \cup R_{(2,2)} \cup R_{(2,3)},$$

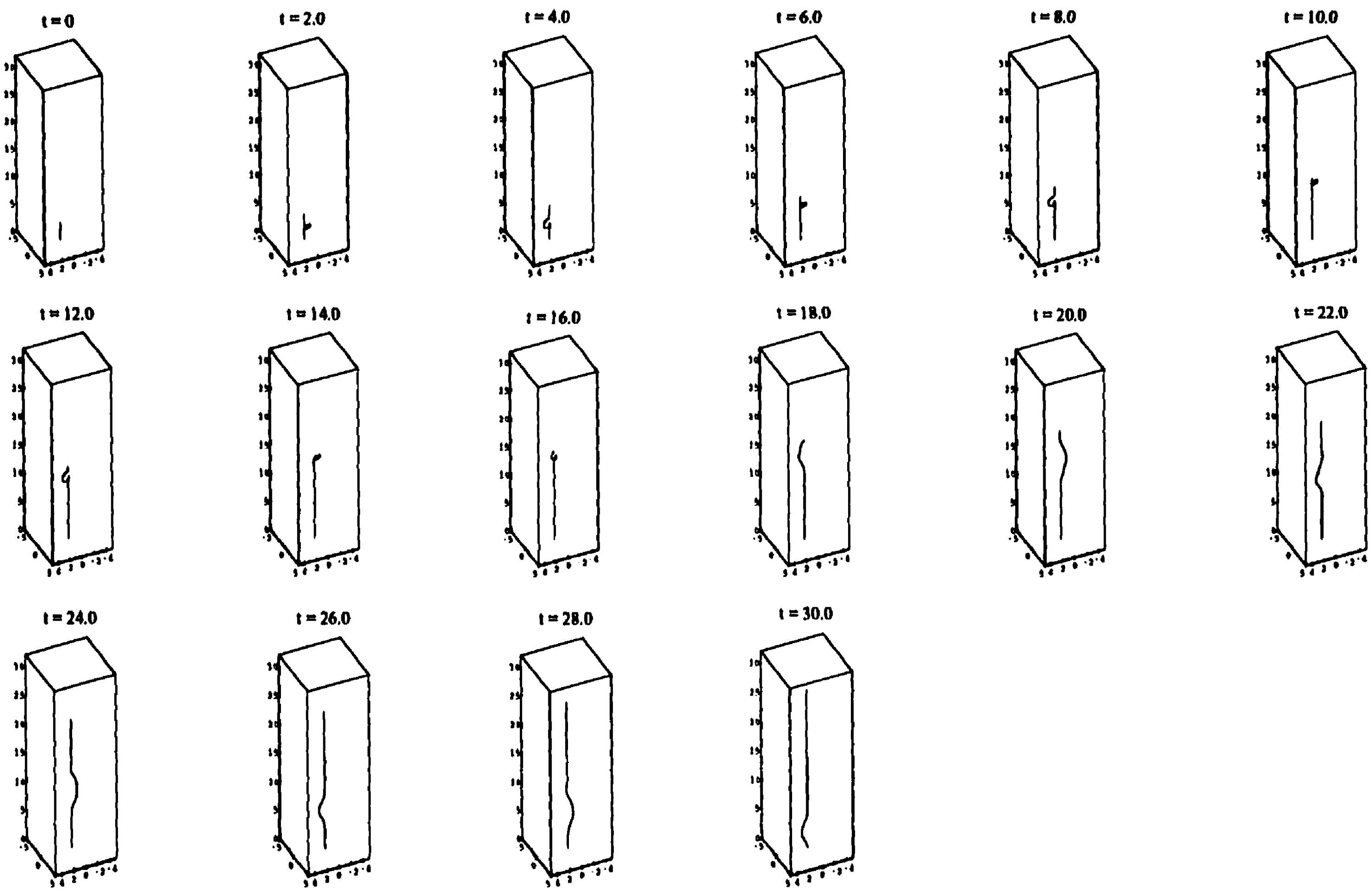
$$\begin{aligned}
\operatorname{Re} W(x,t) &= \cos\frac{\pi}{2}(t-x)[1-\cos 2\pi(t-x)][H(t-x)-H(t-x-1)] \\
&\quad -\cos\frac{\pi}{18}(t+x-30)[1-\cos\frac{2\pi}{9}(t+x-30)] \\
&\quad \cdot[H(\frac{t+x-30}{9})-H(\frac{t+x-39}{9})], \\
\operatorname{Im} W(x,t) &= \sin\frac{\pi}{2}(t-x)[1-\cos 2\pi(t-x)][H(t-x)-H(t-x-1)] \\
&\quad -\sin\frac{\pi}{18}(t+x-30)[1-\cos\frac{2\pi}{9}(t+x-30)] \\
&\quad \cdot[H(\frac{t+x-30}{9})-H(\frac{t+x-39}{9})],
\end{aligned}$$

$$\text{for } (x,t) \in R_{(3,2)} \cup R_{(3,3)},$$

and the solution of (6.2.1-2), with the data prescribed in example 6.4.3, is given by (6.4.1).

The graphs describing the motion of the corresponding wave of this example are presented in Figure 6.4.3. The non-planar nature of the travelling "loop" is evident as in example 4.3.5. In addition, the elongation of this loop upon reflection from the moving endpoint is also evident as in example 6.4.2, as is the "inversion" of the reflected loop at either end of the spinning string. Finally, the uniform rotation of this loop is also noted.

Figure 6.4.3: graphic illustration of Example 6.4.3 where $c=1$, $\omega=\pi/2$, $m=0.8$, $L=3$



Summary

Solutions of the model for a spinning tether have been determined for a variety of initial and boundary conditions, corresponding to tethers of semi-infinite or finite (constant or linearly-increasing) length. A variety of techniques, based on the usual d'Alembert form of the solution and Fourier analysis, have been used, and illustrative examples have been developed for each case studied. Solutions from the various cases have been used as the basis for illustrative examples for other cases, thereby confirming the validity of the analysis in each case.

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